Mathematical Foundations of Machine Learning

Exercise Sheet 1

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Vector Spaces

1.1 Verify that $\mathbb{R}^n$ with componentwise addition and scalar multiplication satisfies all vector space axioms.
1.2 Show that the set of all polynomials of degree $\leq d$ with real coefficients forms a vector space.
1.3 Prove that the set of 2×2 real matrices is a vector space.
1.4 Give an example of a subset of $\mathbb{R}^3$ that is closed under addition but not under scalar multiplication.
1.5 Give an example of a subset of $\mathbb{R}^3$ that is closed under scalar multiplication but not under addition.
1.6 Find two different bases of $\mathbb{R}^2$ and two different bases of $\mathbb{R}^3$.
1.7 Show that if $\mathcal{B}_1$ and $\mathcal{B}_2$ are two bases of a finite-dimensional vector space V, then $|B_1| = |B_2|$.
1.8 Compute the dimension of the vector space of polynomials of degree ≤ 5.
1.9 Give an example of an infinite-dimensional vector space.
1.10 Let A be an $n \times n$ matrix with coefficients in $\mathbb{R}$ and $x \in \mathbb{R}^n$. Show that the set of solutions to the homogeneous system of linear equations determined by A (that is $Ax = 0$) is a subspace of $\mathbb{R}^n$.
1.11 For $V = C([0,1])$, prove that the set $\{1, t, t^2\}$ is linearly independent.

Inner Products, Norms, Orthogonality

2.1 Find the angle between $u = (1,0,1,0)^\top$ and $v = (1,1,0,1)^\top$.
2.2 Determine whether $u = (1,2,3,4)$ and $v = (2,-1,-2,1)$ are orthogonal.
2.3 Prove the Pythagorean theorem in inner product spaces: if $u \perp v$, then $\|u+v\|^2 = \|u\|^2 + \|v\|^2$.
2.4 Show that the projection of $v = (1,2,3,4)$ onto $u = (1,0,0,0)$ is $(1,0,0,0)$.
2.5 Find the projection of $v = (1,2,3,4)$ onto $u = (0,1,1,0)$.
2.6 Compute the projection of $g(x) = x$ onto $f(x) = \sin(x)$ in $C([0,\pi])$.
2.7 Show that $\sin(x)$ and $\cos(x)$ are orthogonal on $[-\pi,\pi]$. Find two functions that are orthogonal to each other on $[-1.1]$.

Cauchy–Schwarz, Subspaces, Span, Dimension

3.1 Describe all subspaces of $\mathbb{R}^3$ geometrically. What about $\mathbb{R}^n$
3.2 Show that $\operatorname{span}\{(1,0,0),(0,1,0),(1,1,0)\} = \{(x,y,0) \mid x,y \in \mathbb{R}\}$.
3.3 Prove that dimension of $\operatorname{span}\{1, t, t^2, t+t^2, 1+t-t^2\} \subset C([0,1])$ is 3.
3.4 Show that the set of all polynomials is an infinite-dimensional vector space.
3.5 Show that $W = \{(x,y,z,w) \in \mathbb{R}^4 \mid x+y+z=0\}$ is a subspace of $\mathbb{R}^3$ and find a basis and state the dimension of W.
3.6 Prove that $W^\perp$ is a subspace for any W.
3.7 Verify the Projection Theorem in $\mathbb{R}^2$ with $u = (1,1)$ and $v = (2,1)$.

Gram–Schmidt and Orthonormal Bases

4.1 Apply Gram–Schmidt to $\{(1,1),(1,-1)\}$ in $\mathbb{R}^2$.
4.2 Apply Gram–Schmidt to $\{(1,0,0),(1,1,0),(1,1,1)\}$ in $\mathbb{R}^3$.
4.3 Show that the plane $P = \{(x,y,z) \in \mathbb{R}^3 \mid x+y+z=0\}$ is a vector subspace. Show that the vectors $v_1 = (1,-1,0)^\top$ and $v_2 = (0,1,-1)^\top$ generate P, that is $P = \operatorname{span}(v_1, v_2)$. Find an orhonormal basis for P.
4.4 Show that the Gram–Schmidt process always produces orthogonal vectors.
4.5 Construct an orthonormal set starting from $\{1, t, t^2\}$ with respect to the inner product : $\langle f, g\rangle = \int_0^1 f(t)g(t)\mathrm{d}t$

Norms and Dual Norms

5.1 Compute $\|x\|_1, \|x\|_2, \|x\|_\infty$ for $x = (3,-4,1)$.
5.2 Sketch the unit balls of $\ell^1, \ell^2, \ell^\infty$ norms in $\mathbb{R}^3$.
5.3 Show that every norm induces a metric.

Linear Transformations and Matrices

6.1 Show that $T : \mathbb{R}^2 \to \mathbb{R}^2$, $T(x, y) = (2x, 3y)$ is linear.
6.2 Prove that the derivative operator $D : C^1([0, 1]) \to C([0, 1])$ is linear.
6.3 Find the matrix of $T(x, y) = (x + y, x - y)$ with respect to the basis $\mathcal{B} = \{(1, 1)^\top, (1, -1)^\top\}$.
6.4 Find the matrix of $T(x, y) = (2x + y, x - 2y)$ with respect to the basis $\mathcal{B} = \{(1, 2)^\top, (2, -1)^\top\}$.
6.5 Compute the matrix of $T(x, y, z) = (x + 2y, y + z, x - z)$ in standard basis.
6.6 Compute the matrix of $T(x, y, z) = (x-y+z, y+2z, x+2y-z)$ in basis $\mathcal{B} = \{(1, 1, 0)^\top, (1, 0, 1)^\top, (0, 1, 1)^\top\}$.
6.7 Give examples of linear transformations corresponding to scaling, rotation, projection.
6.8 Prove that the composition of linear transformations is linear. That is, if $T : V \to W$ and $S : W \to Y$ are two linear transformations then their composition $S \circ T : V \to Y$ is also linear.
6.9 Is the sum of two linear transformations again linear? That is, if $T, T' : V \to W$ are two linear transformations, then is $T + T' : V \to W$ again linear?

Dual Space

7.1 Define the dual space V^* for $V = \mathbb{R}^2$ and give examples of functionals.
7.2 Show that $f(x, y) = 3x - 2y$ is linear.
7.3 Prove that V^* is a vector space.
7.4 Define the dual basis for $\mathbb{R}^3$ and verify the Kronecker delta property.
7.5 Compute the dual map T^* for $T(x, y) = (x + 2y, y)$.
7.6 Show that the matrix of T^* is the transpose of the matrix of T.
7.7 Explain geometrically what linear functionals represent.
7.8 Show that $\mathrm{Hom}(V, W)$ is a vector space.
7.9 Prove that $\dim(V^*) = \dim(V)$ when V is finite-dimensional.
7.10 Compute the dual basis of $B = \{(1, 1, 0)^\top, (1, 0, 1)^\top, (0, 1, 1)^\top\}$ in $\mathbb{R}^3$.