Sommersemester 2025 Blatt 13

1. Questions

(1) Consider the ring of Gaussian integers $R = \mathbb{Z}[\sqrt{-1}]$. Show that

$$1 + \sqrt{-1} \text{ and } 1 - \sqrt{-1}$$

are prime elements in R.

(2) Let $R \subseteq \mathbb{C}$ be a ring and $\tilde{N}\colon R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ be a function with the following property

N2 For all $a, b \in R$ with $b \neq 0$, there exist $q, r \in R$ such that

$$a = bq + r \text{ with } r = 0 \text{ or } \tilde{N}(r) < \tilde{N}(b).$$

Show that the function $N\colon R \setminus \{0\} \to \mathbb{Z}_{\geq 0}$ defined by

$$N(a) = \min_{b \neq 0} \tilde{N}(ab)$$

satisfies the property **N2** and also it holds that for any $a, b \in R \setminus \{0\}$ we have $N(a) \leq N(ab)$.

(3) Let

$$\omega = \frac{-1 + \sqrt{-3}}{2}$$

and consider $R = \mathbb{Z}[\omega]$. Show that R is a Euclidean ring with the norm

$$N(a + b\omega) = a^2 - ab + b^2.$$