Sommersemester 2025 Homework 1

1. Questions

(1) By the time you submit this homework, you will have at least one lecture and one exercise class. Write one thing that you learned that you had not known before. (If you already knew everything, write one thing you know about Diophantine equations that was not covered but you find interesting).

(2) In class, you have learned about the Diophantine equation $x^2+y^2 = 1$. You attempted to solve it as follows:
   (a) Assume (x, y) is a solution.
   (b) Write the equation as $-y^2 = x^2 - 1$.
   (c) Factorize to get $-y^2 = (x + 1)(x - 1)$.
   (d) The integers $x - 1, x + 1$ are either both even or they are both odd. So, $\gcd(x - 1, x + 1) = 2$ or $\gcd(x - 1, x + 1) = 1$.
   (e) Why can they not have a common divisor bigger than 2?
   (f) In class, you discussed the case $\gcd(x - 1, x + 1) = 1$. You wrote prime factorizations of $x - 1, x + 1$ and y. Then, after some discussion, you concluded that $x = 0$ and $y = -1$ or $y = 1$.

In this question, you will consider the case $\gcd(x - 1, x + 1) = 2$ and **your task is to give a complete solution** including both the $\gcd(x - 1, x + 1) = 1$ case and $\gcd(x - 1, x + 1) = 2$ case.