Sommersemester 2025 Blatt 7

1. Questions

(1) If m, n are positive integers which are sums of two squares, then show that their product mn is also a sum of two squares.
(2) Show that a positive integer n is a difference of two squares if and only if $n \neq 4k+2$ for some $k \in \mathbb{N}$.
(3) Show that the equation

$$a^2 + b^2 + c^2 + a + b + c = 1$$

does not have any integer solutions.

2. Comments

(1) (a) Write $m = a^2 + b^2$ and $n = c^2 + d^2$ and compute mn.
(b) Add and subtract $2abcd$.
(2) (a) For the "only if" part, assume that $n = a^2 - b^2$ and check the four cases where a, b are even or odd.
(b) For the "if" part, start with showing that

$$(n+4)^2 - (n-4)^2 = 16n$$

for all positive integers n. Argue that this proves the $n = 0 \mod 4$ case.
(c) For the case where $n = 1 \mod 4$ or $n = 3 \mod 4$ what can you say about the parity of n? What can you say about the parity of $n-1$ and $n+1$?