Sommersemester 2025 Blatt 3

1. Questions

(1) Consider the generalised Fermat equation

$$x^\ell + y^m = z^n$$

with ℓ, m, n, x, y, z positive integers.

(a) Show that if $\gcd(\ell, m, n) = d > 2$, then there are no solutions to the generalised Fermat equation.

(b) Let us restrict now to the case $\ell = m$ with $\gcd(m, n) = 1$. Find infinitely many solutions to the generalised Fermat equation with this restriction.

(2) (a) Show that there are no integer valued solutions to the equation

$$x^2 + y^2 + z^2 = 2xyz.$$

(b) Are there any integer valued solutions to the equation

$$x^2 + y^2 + z^2 = 4xyz?$$

(c) Are there any integer valued solutions to the equation

$$x^2 + y^2 + z^2 = 8xyz?$$

(d) Are there any integer valued solutions to the equation

$$x^2 + y^2 + z^2 = 16xyz?$$

(e) Are there any integer valued solutions to the equation

$$x^2 + y^2 + z^2 = 2^r xyz?$$

where r is a positive integer?

2. Comments

(1) (a) Read Fermat's Last Theorem on Wikipedia.

(b) Use Fermat's Last Theorem to prove the first part.

(c) For the second part, use Euclid's algorithm to show that there are **positive** integers u, v such that $nv - mu = 1$. How can this be utilised?

(2) (a) First argue that if there is a solution (x, y, z), we can not have that all x, y, z are odd. Why?

(b) So, you can assume without loss of generality that x is even. Then, you can rewrite $x = 2x'$.

(c) Now, argue using modulo 4, that y, z also must be even.

(d) How can you use this to finish the problem?