Ausgewählte Kapital der Mathematik 2 **Özgür Esentepe**

Sommersemester 2025 Blatt 2

1. Questions

(1) Find the solution set to the linear equation
$$2x + 3y + 5z = 10.$$
(2) Consider the Diophantine equation
$$x^2 + y^2 = z^2.$$
   (a) Show that for any positive integer k, the triple $(2k+1, 2k^2+2k, 2k^2+2k+1)$ gives a solution to this equation.
   (b) Are these all of the solutions?

(3) (*) Find all integer solutions for the Diophantine equation
$$y^3 = x^3 + x^2 + x + 1.$$

2. Comments

(1) Before attempting to solve Question 1, first make sure you understand what the question is asking.
   (a) One way to approach this problem is to solve *two easier equations* instead of one more difficult question. How can we do this?
   (b) What is difficult with this question is that we have three variables. We know from first week how to find the solution set to a linear Diophantine equation. So, we can try to reduce the problem to that.
   (c) If we have $ax + by + cz = d$ with $a, b, c \neq 0$ and $\gcd(a, b, c)|d$, then we can let $E = \gcd(b, c)$ and solve for the system
$$\begin{aligned} ax + Ew &= d \\ by + cz &= Ew \end{aligned}$$
   where w is now a new variable.
   (d) Did you notice that the second system here is homogeneous? How is this related to the first exercise from Blatt 1?
   (e) Why does this method give us solutions to the original equation?
   (f) Why does this method give us *all* solutions to the original equation?

(2) (a) The (English) Wikipedia page for Pythogerean triples contain 46 links as notes, 12 references and 17 external links. In your free time, you can look at all of them!
   (b) Is $(12, 17, 46)$ a Pythogerean triple?

(3) (a) This question is a starred question. What does that mean? First, let me tell you what it does not mean: it does not mean that it is more difficult. It means that the question requires a method that does not align with the methods you learn in the lecture.
   (b) In particular, you will solve this question using some "calculus".
   (c) Go to `https://www.desmos.com/calculator`.

   (i) In the first line, type in $a = (x-1)^3$. It will draw the graph for you.
   (ii) In the second line, type in $b = x^3 + x^2 + x + 1$.
   (iii) In the third line, type in $b - a$.

   By looking at the graph of $b - a$, conclude that $(x-1)^3 \leq y^3$.

(d) Similarly, conclude that $y^3 \leq (x+1)^3$.
(e) Now, you have $x^3 \leq y^3 \leq (x+1)^3$. Both x, y are integers. What does this tell you?