Sommersemester 2025 Blatt 9

1. Questions

(1) Show that a positive integer n is a sum of two squares if and only if $n = 2^m a^2 b$ where $m \geq 0$, a an odd integer, and every prime divisor of b is of the form $4k+1$.
(2) Write the integer 3185 as a sum of two squares.
(3) If a positive integer n is not a sum of squares of two integers, show that n can not be written as a sum of squares of two rational numbers.
(4) Prove that every odd integer is the sum of four squares, two of which are consecutive.

2. Comments

(1) Before doing this problem, look at the homework first.
(2) Look at the prime factorisation of the number, use a result you know from two weeks ago.
(3) (a) Theorem 5.8 tells you that you can write $n = N^2 m$ where m is square free and m has a prime factor p of the form $4k+3$.
  (b) Assume for a contradiction that
  $$n = \left(\frac{a}{b}\right)^2 + \left(\frac{c}{d}\right)^2$$
  where a, b, c, d are integers with b, d nonzero. This implies that
  $$nb^2d^2 = \cdots .$$
  (c) You will use this equality and count how many times the p from part (a) appears on both sides to reach a contradiction.
(4) (a) You can use without proof that a positive integer n is a sum of three squares if and only if $n \neq 4^m(8m+7)$ $(m \geq 0\ )$.
  (b) Show that for any positive integer n, we can write $4n+1 = a^2+b^2+c^2$ for some integers a, b, c.
  (c) Show that two of a, b, c are even and one of them is odd.
  (d) Deduce that $2n+1$ can be written as a sum of four squares.