Ausgewählte Kapital der Mathematik 2 **Özgür Esentepe**

Sommersemester 2025 Blatt 8

1. Questions

(1) Show that 2^n is a sum of two squares for every positive integer n.
(2) Let p be a prime. Show that if $p = 3 \mod 9$ or $p = 6 \mod 9$, then p is not a sum of two squares.
(3) Every Fermat number $F_n = 2^{(2^n)} + 1$ (with $n \geq 1$) can be expressed as a sum of two squares.
(4) Show that every prime p of the form $8k + 1$ or $8k + 3$ can be written as

$$p = a^2 + 2b^2$$

for some integers a, b.

2. Comments

(1) (a) Try $n = 1, 2, 3, 4, 5$ and verify that the statement is correct for them.
(b) Use a result from last week and prove by induction.
(2) (a) Find the first three primes of the form $9k + 3$ and convince yourself that they are not a sum of two squares.
(b) Find the first three primes of the form $9k + 6$ and convince yourself that they are not a sum of two squares.
(c) Use a result that you should know from the lecture: p is a sum of two squares if and only if $p = 1 \mod 4$.
(3) Do not overthink.
(4) (a) Find the first three primes of the form $8k + 1$ and verify that the statement is correct for those primes.
(b) Find the first three primes of the form $8k + 3$ and verify that the statement is correct for those primes.
(c) The proof will mimic the proof of Fermat's Theorem (See Comment 2c). You are going to need the following fact: If p is of the form $8k+1$ or $8k+3$ for some positive integer k, then there exists an integer a such that $a^2 = -2 \mod p$.
(d) You can use this fact without proving but if we have time, we will give the proof of this fact as another exercise.