Sommersemester 2025 Blatt 4

1. Questions

(1) Consider the generalised Fibonacci sequence $H^{p,q}$ with p, q positive integers. Denote by H_n the nth term $H_n^{p,q}$ of this sequence.
  (a) Show that every Fibonacci triple satisfies the Pythagorean equation. That is, show that the triple
$$(H_nH_{n+3}\ ,\ 2H_{n+1}H_{n+2}\ ,\ 2H_{n+1}H_{n+2} + H_n^2)$$
  satisfies the Pythagorean equation. Is this always a Pythagorean triple?
  (b) Show that if (x, y, z) is a Pythagorean triple, then it is a Fibonacci triple. That is, there are positive integers p, q and N such that
$$\begin{aligned} x &= H_NH_{N+3} \\ y &= 2H_{N+1}H_{N+2} \\ z &= 2H_{N+1}H_{N+2} + H_N^2 \end{aligned}$$

(2) Find all positive integers x, n, m such that
$$(x + 1)^n = x^m + 1.$$

(3) Show that if F is a degenerate quadratic equation in two variables with integer coefficients, then it describes either a point or a union of two lines.

2. Comments

(1) For the first question, you need to recall the definition of $H^{p,q}$. Recall that we have the Fibonacci sequence $F_n = 0, 1, 1, 2, 3, 5, \ldots$ given by the rule $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. We can define similarly a generalised Fibonacci equation for any two positive integers p, q by the rule
$$H_n^{p,q} = pF_n + qF_{n-1}$$
for any $n \geq 1$.
  (a) Before you continue, make sure that you understand the sequence. Try it for $p = 2, q = 3$.
  (b) Is it true that $H_n = H_{n-1} + H_{n-2}$ for any choices of p, q?
  (c) There are two approaches to part a. You can try brute force. Just plug in
$$\begin{aligned} x &= H_nH_{n+3} \\ y &= 2H_{n+1}H_{n+2} \end{aligned}$$
  and compute $x^2 + y^2$. Then, compute $2H_{n+1}H_{n+2} + H_n^2$. Then, show that these two are equal.
  (d) Brute force method would work but could be nasty. You can also try the following:
    (i) First show that for any n, we have
$$(H_{n+2} - H_{n+1})^2 = H_n^2.$$

(ii) Then show that

$$(H_{n+2} + H_{n+1})^2 = H_n^2 + ???.$$

(iii) Multiply both sides by H_{n+3}^2.
(iv) Add something to both sides.

(e) For the second part, you need to remember the definition of a Pythagorean triple. You have seen this in class and in the homework. A Pythagorean triple is of the form (x, y, z) where $x = a^2 - b^2$, $y = 2ab$ and $z = a^2 + b^2$ and a, b should satisfy some conditions. What are those conditions?

(f) Show that if a, b as above (giving a Pythagorean triple), then $b < a < 2b$. That is, both $p = a - b$ and $q = 2b - a$ are positive numbers.

(2) Another way to state this question is you are solving infinitely many Diophantine equations: $(x + 1)^n = x^m + 1$ where m, n some positive integers.

(a) Before you start to attempt solving infinitely many of them, try solving some examples. Start with the smallest possible choices for m, n.
  (i) If $m = n = 1$, what happens? Is every positive integer x in the solution set?
  (ii) If $m = 1, n = 2$, how many positive integer x are there in the solution set?
  (iii) If $m = 1$ and n any positive integer, what happens?
  (iv) If $m = 2, m = 1$, what happens?
  (v) If m is any positive integer and $n = 1$, what happens?
  (vi) Try $m = n = 2$. Remember we are interested in solutions where x is a positive integer.
  (vii) Try $m = 2, n = 3$.

(b) Now, attempt to solve the problem. The main idea is similar to what Magdelana and Antonia did in Week 3: to figure out the parity of m, n and x. (Actually, it is easier if you do this in the order m, x and n.)

(c) Fix $m, n > 1$ two positive integers. You already have seen what happens if one of them is 1, so we can assume they are bigger than 1. Assume that you have a solution x.

(d) Reduce the original equation modulo $x + 1$ to get

$$x^m + 1 = 0 \mod x + 1.$$

Then, use $(-1)^m + 1 = 0 \mod x + 1$ to show that m must be odd (so at least 3). So, we can write $m = 2c + 1$.

(e) Write $m = 2c + 1$ in the original equation. And factorize. Then, reduce modulo 2 to find the parity of x. So either $x = 2a$ or $x = 2a + 1$ for some a. Find which one.

(f) Then plug this into the original equation and factorize again. And reduce modulo 2 again to find the parity of n. It should be an even number: $n = 2b$.

(g) Now, you have the equation

$$(x + 1)^{2b} = x^{2c+1} + 1$$

which you can rewrite as $x^{2c+1} = (x + 1)^{2b} - 1$. This will allow you to use the identity $U^2 - V^2 = (U + V)(U - V)$.

(h) You are getting closer to the solution. Now, you are in a situation where you have been before. You have two numbers $(x+1)^b+1$ and $(x+1)^b-1$. What are the possibilies for their greatest common divisor?

(i) Now, use the parity of x to figure out the only possibility for the greatest common divisor (of $(x+1)^b+1$ and $(x+1)^b-1$). From now on you should write $x=2a$ or $x=2a+1$ depending on your finding in part (e).

(j) Go back to part (g). You wrote $(x+1)^{2b}$ as a product of two things, you also know the greatest common divisor of these two things. What are possibilities for these two things?

(3) No comments for the last question, we have already started thinking about it in Week 3 as we had spare time.