Sommersemester 2025 Blatt 12

1. Questions

(1) Let

$$\omega = \frac{-1+\sqrt{-3}}{2}.$$

Show that $\mathbb{Z}[\omega]$ is a ring.
(2) Show that there is an inclusion $\mathbb{Z}[\sqrt{-3}] \subseteq \mathbb{Z}[\omega]$.
(3) Show that every natural number has a *unique* factorisation into primes.

2. Comments

(1) For the first question, it might be useful to show that ω is a root of the polynomial x^2+x+1. So, first prove this.
(2) For the second question, it might be useful to compute ω^2 first. You can use the previous comment to compute this.
(3) The last question is something that you have already known. The task is to re-prove the uniqueness part. The reason you are doing this task is because in the next few weeks, you will learn about other rings where unique factorisation may or may not exist. You should be comfortable with this proof.