Causal Structure of Spacetime

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Abstract

In this project, I studied the causal structure of Spacetime. From some exact solutions of the Einstein field equation, we know that the solutions may not correspond to a physical spacetime due to causality violation. I studied how we approach causal structure mathematically and discussed the implications that arise in one of the spacetimes that violate this structure. It is concluded that no theory establishes the physically logical global causal structure, so there must be a correction in the Einstein field equations such that all the exact solutions for it must be physical.

Introduction

In 1949, famous mathematician and logician Kurt Gödel founded an exact solution for the Einstein Field Equations which has the metric as follows:

$$ds^2 = -dt^2 + dx^2 - \frac{1}{2}e^{2\sqrt{2}\omega x}\cdot dy^2 + dz^2 - 2e^{\sqrt{2}\omega x}dt\,dy$$

What is important is that this spacetime allows Closed Timelike Curves, which enables a physical particle to travel through spacetime and come to its own past event at the end of its journey. This establishes an enormous problem in physics since it violates the causal structure.

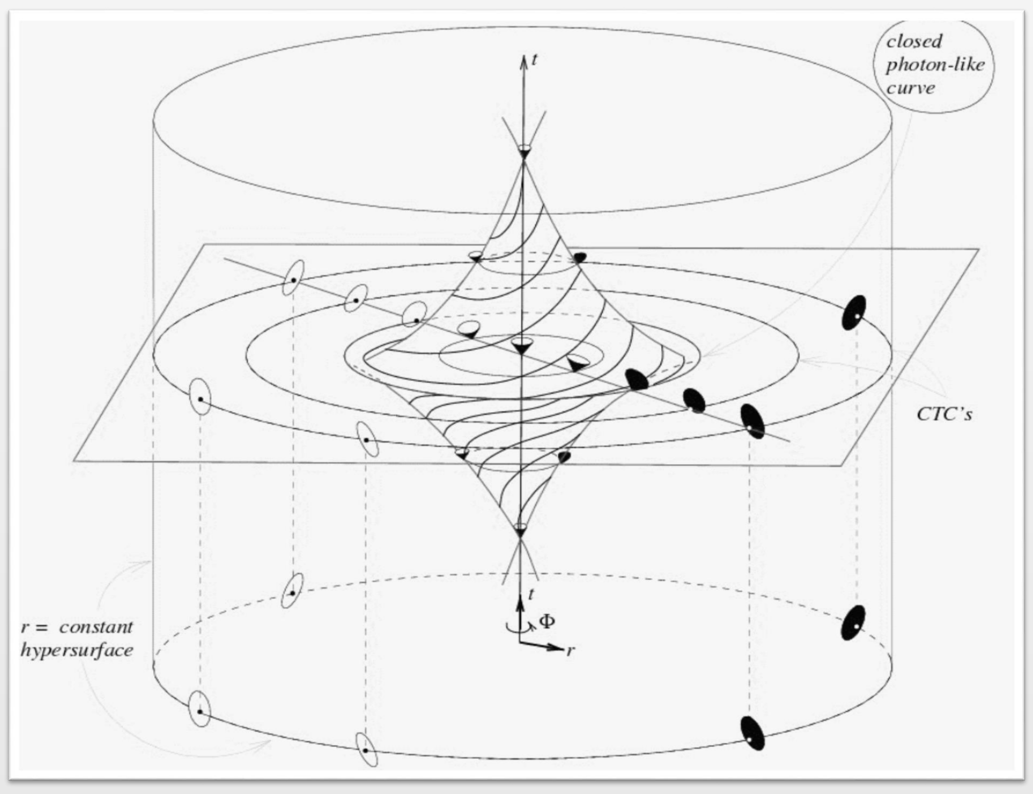


Fig.1. Gödel Spacetime

Gödel Metric is not the only solution that allows closed timelike curves and closed timelike curves are not the only reason for causal structure to be violated. In order to establish a physical causal structure, one has to find relation between the spacetime events such that the resultant relations will emerge to a global causal structure of spacetime.

Since we assume that the spacetime manifold M is a collection of subsets, one has to define the relationship between points topologically. By defining these relations, we want to establish the difference between past and future, chronologically, and causally.

The purpose of this project is to understand how we can establish a topological structure on the spacetime manifold to construct a global causal structure.

Methodology

Definitions:

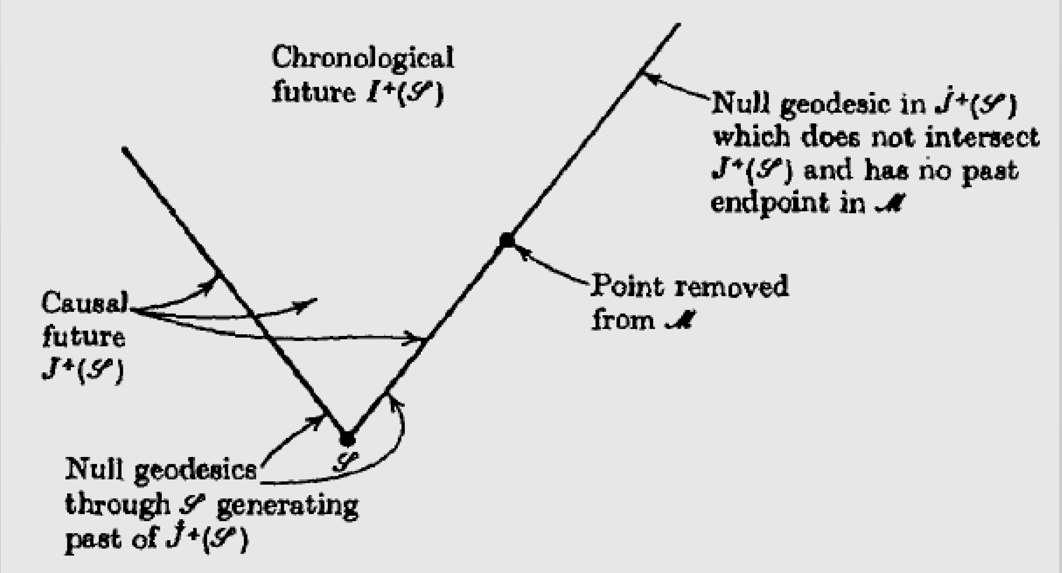
Chronological Future:

$$I^+(p) = \{q \in M \mid \text{There exists a future directed timelike curve } \lambda(t) \text{ from } p \text{ to } q$$

Causal Future:

$$J^+(p) = \{q \in M \mid \text{There exists a future directed causal curve } \lambda(t) \text{ from } p \text{ to } q$$

Fig.2. Showing the chronological and causal future sets



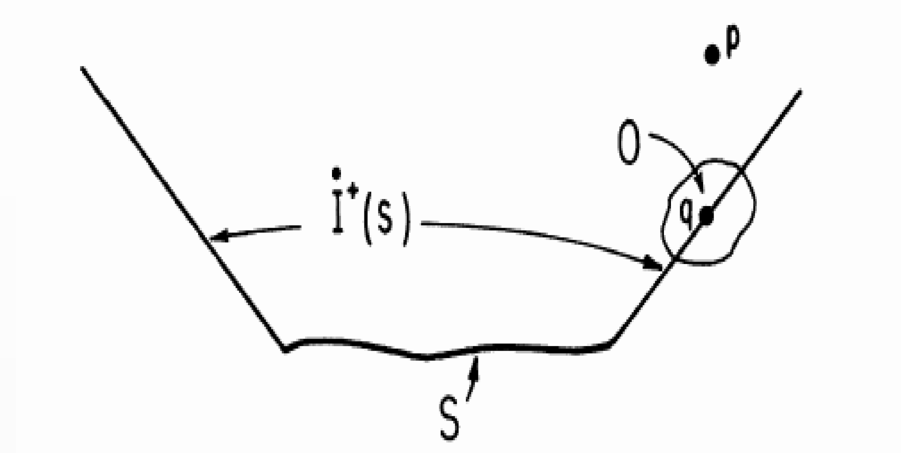
For any subset S of M, one can define

$$I^+(S) = \bigcup_{p\in S} I^+(p)$$

$$J^+(S) = \bigcup_{p\in S} J^+(p)$$

Analogous definitions for $I^-(p)$, $I^-(S)$, $J^-(p)$, $J^-(S)$

Fig.3. Boundary of the Chronological future



Lemma: If the (M, g_{ab}) is time orientable, than there exists a nonvanishing timelike vector field t^a on M. The converse is also true.

Corollary: If $q \in J^+(p) - I^+(p)$, then any curve that connects p and q must be a null curve.

Theorem: Let (M, g_{ab}) be a time orientable spacetime. Than the boundary of the chronological future: $\dot{I}^+(S)$ will be an achronal subset.

Definition: If there is a point p on the manifold such that for a future directed causal curve λ(t), there always a value t_0 such that the intersection of the open neigbourhood O of point p, $\lambda(t) \in O$.

Definition: A spacetime (M, g_{ab}) is said to be strongly causal if for all point p on M, and for every neighborhood O of p, there exists a neigborhood which is contained in V such that no causal curve intersects V more than once.

Definition: A subset S of M is said to be achronal if any two points on S are not chronologically related, which can be stated as: $I^+(S) \cap S = \emptyset$.

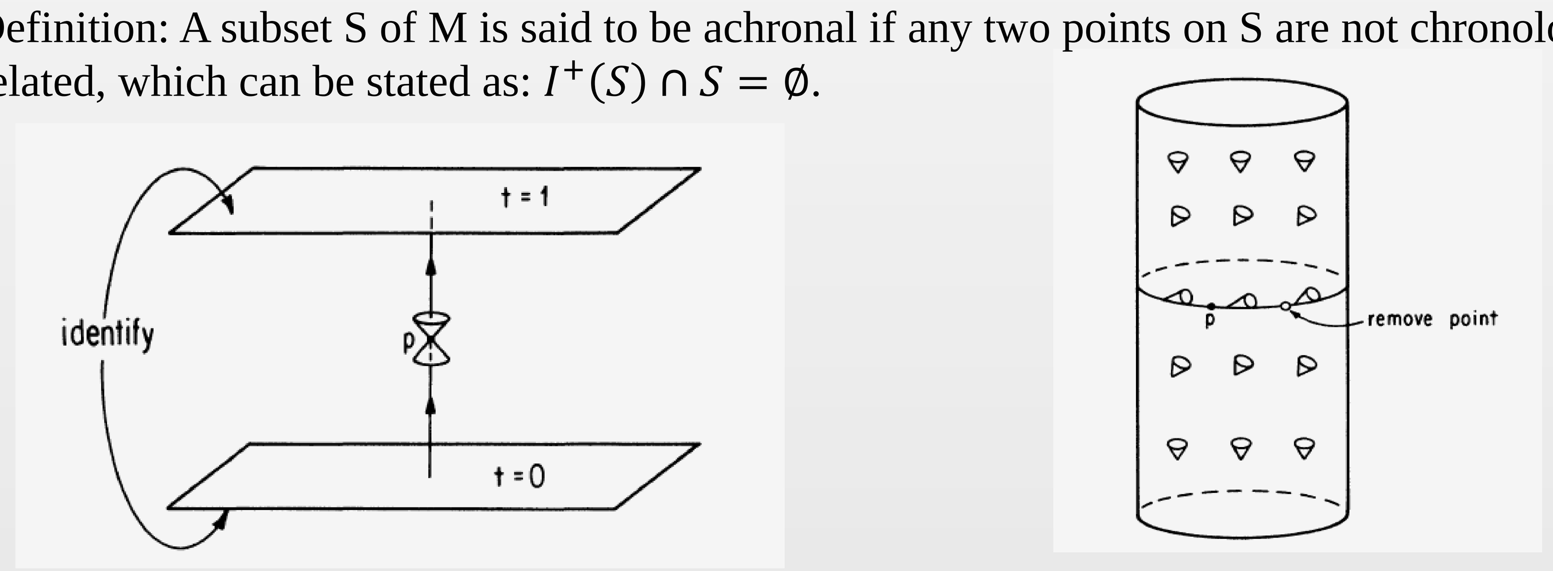


Fig.4.5. Spacetime Diagrams that show how Causality can violated and how it artificially reduced

For a spacetime (M, g_{ab}) , one can define $\tilde{g}_{ab} = g_{ab} - t_a t_b$ at p in which t_a is a timelike vector. The light cone is now strictly larger than that of g_{ab}. If there is a null path in g_{ab} that is closed, than it will become a timelike curve in $\tilde{g}_{ab}$.

Definition: If is said to be stably causal if there is no $\tilde{g}_{ab}$ that can be defined as given to give out closed null curves to become closed timelike curves.

Than from these, A Spacetime is said to have a Closed Timelike curve if it satisfies:

$$I^+(p) \cap I^-(p) \neq \emptyset$$

Results

One can 'artificially' get rid of the CTC's in Gödel Spacetime by taking out a branch cut for every corresponding constant time surfaces. This does not solve the main problem that underlies within the context of causality, which should be manifested explicitly by being the solution of the Einstein Field Equation. Alternatively, we have to accept the fact that EFE does not always give physical solutions.

Conclusion

The question of whether there is a well established global causal structure in spacetime or not is left open. Can there be any closed timelike curves in physically reasonable spacetimes if we arrange the matter as such? We do not have any answers to this either, but as we see in the Gödel Universe, Einstein Field Equations allow that kind of solution. So, the question of ‘’completeness’’ of the Einstein Field Equation can be asked.

References

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