

Robot Navigation in Complex Workspaces Using Conformal Navigation Transformations

Li Fan , Jianchang Liu , Wenle Zhang , and Peng Xu 

Abstract—Navigation functions provide both path and motion planning, which can be used to ensure obstacle avoidance and convergence in the sphere world. When dealing with complex and realistic scenarios, constructing a transformation to the sphere world is essential and, at the same time, challenging. This work proposes a novel transformation termed the conformal navigation transformation to achieve collision-free navigation of a robot in a workspace populated with obstacles of arbitrary shape. The properties of the conformal navigation transformation, including uniqueness, invariance of navigation properties, and no angular deformation, are investigated, which contribute to the solution of the robot navigation problem in complex environments. Based on navigation functions and the proposed transformations, feedback controllers are derived for the automatic guidance and motion control of kinematic and dynamic mobile robots. Moreover, an iterative method is proposed to construct the conformal navigation transformation in a multiply-connected workspace, which transforms the multiply-connected problem into multiple simply connected problems to achieve fast convergence. In addition to the analytic guarantees, simulation studies verify the effectiveness of the proposed methodology in workspaces with non-trivial obstacles.

Index Terms—Motion and path planning, collision avoidance, navigation function, conformal map.

I. INTRODUCTION

THE autonomous navigation of robots in cluttered environments is an actively studied topic of robotics research. Control laws based on artificial potential fields (APFs) constitute a widely researched tool for solving the robot navigation problem due to their intuitive design and ability to simultaneously solve path planning and motion planning subproblems [1]. On the other hand, this control method suffers from the presence of local minima and the inability to cope well with arbitrary non-convex obstacles.

The navigation function [2], [3] method proposed by Rimon and Koditschek is an efficient method for constructing a

class of local minima free APFs in the sphere world, which can be extended to environments with star-shaped obstacles by constructing a diffeomorphism. The negative gradient of the navigation function can be served as a control input to guarantee collision-free motion of the kinematic mobile robot and convergence to the goal point from almost all initial points. A large number of successful applications have appeared in the literature, such as multi-robot systems [4], [5], highly dynamic or partially known environments [6], [7], uncertain dynamic systems [8], and constrained stabilization problems [9]. In a recent work [10], the construction of navigation functions was extended to non-spherical convex obstacles; however, the construction of navigation functions in geometrically complex spaces is still a challenging problem.

Transforming geometrically complex spaces to geometrically simple spaces is an important and elegant way to implement navigation functions in complex workspaces. Techniques based on diffeomorphism and navigation transformation are a few of the promising methods found in the literature to achieve the above objective. In [3], a family of analytic diffeomorphisms was constructed for mapping any star world to an appropriate sphere world, requiring appropriate tuning of a certain parameter. To cope with the complexity of the workspace, the authors in [11] proposed a diffeomorphism based on the harmonic map, which maps the workspace to a punctured disc by solving multiple boundary problems. By applying a two-step navigation transformation, the author in [12] achieved a tuning-free solution by pulling back a trivial solution in the point world to the initial star world. In a subsequent work [13], the authors proposed a single-step navigation transformation candidate that directly maps a known star world to a point world.

The transformation candidates in most of the aforementioned works are only suitable for star worlds rather than arbitrary workspaces, which hinders the application of navigation functions in realistic environments. In this work, we address the problem of navigating kinematic and dynamic mobile robot systems in a workspace with arbitrary connectedness and shape by constructing a novel transformation termed the conformal navigation transformation. Unlike the transformations mentioned above, the conformal navigation transformation requires neither tuning a parameter nor decomposing the workspace into trees of stars, and it is unique in a given workspace. Besides, it can pull back the navigation function in the sphere world to the initial workspace, as well as the trivial solution in the sphere world determined by the vector field, so that the navigation problem in the workspace can be solved elegantly. Further,

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we present a fast iterative method for constructing the proposed transformations in any complex workspace. To the best of the authors' knowledge, this is the first work to provide a tuning-free transformation candidate that maps any workspace with non-star-shaped obstacles to a sphere world. The innovations and contributions of this work are summarized in the following.

- 1) The conformal navigation transformation is proposed to solve the navigation problem for a workspace with arbitrary shape and countably many boundary components. The proposed transformation has no angular distortion, no tuning, and is unique for a given workspace.
- 2) The provably correct feedback control laws for kinematic and dynamic mobile robots are designed to achieve convergence from almost all initial points to the goal point.
- 3) An iterative method is presented to rapidly construct the conformal navigation transformation, which is suitable for non-star-shaped obstacles.

The rest of this letter is organized as follows. Section II introduces the necessary preliminaries. Section III gives the definition of the conformal navigation transformation and analyzes its properties. Section IV presents two mobile robot controllers designed using conformal navigation transformations and navigation functions. Section V proposes the construction method of conformal navigation transformations. Section VI provides simulation studies to demonstrate the effectiveness of the proposed control scheme. Section VII concludes this letter.

II. PRELIMINARIES

This section introduces sphere worlds and workspaces, the navigation function, the mobile robot models, and the conformal map.

A. Sphere Worlds and Workspaces

The n -dimensional sphere world \mathcal{M}^n as defined in [2] is a compact connected subset of \mathbb{R}^n whose boundary is formed by the disjoint union of an external $(n-1)$ -sphere $\tilde{\mathcal{O}}_0$ and other internal $(n-1)$ -spheres $\tilde{\mathcal{O}}_i$ that define obstacles in \mathbb{R}^n .

Clearly, the sphere world is geometrically simple and can serve as a topological model for geometrically complex spaces. We will restrict our attention to geometrically complex robot workspaces, defined as follows:

Definition 1: The n -dimensional robot workspace $\mathcal{N}^n \subset \mathbb{R}^n$ is a connected and compact n -dimensional manifold with boundary such that \mathcal{N}^n is homeomorphic to \mathcal{M}^n .

For a finite $M \in \mathbb{N}$, the internal obstacle \mathcal{O}_i of \mathcal{N}^n is the interior of a connected and compact n -dimensional subset of \mathbb{R}^n such that $\mathcal{O}_i \cap \mathcal{O}_j = \emptyset$, $i \neq j$, $i, j \in \{1, \dots, M\}$. It is convenient to regard the unbounded component of the workspace's complement as the zeroth obstacle. Each obstacle can be represented by an obstacle function $\beta_i : \mathbb{R}^n \rightarrow \mathbb{R}$ in the following form:

$$\mathcal{O}_i = \{q \in \mathbb{R}^n : \beta_i(q) < 0\}, \quad i \in \{0, \dots, M\}. \quad (1)$$

Then, the n -dimensional workspace with M internal obstacles is represented as:

$$\mathcal{N}^n = \{q \in \mathbb{R}^n : \beta_0(q) \geq 0, \dots, \beta_M(q) \geq 0\}. \quad (2)$$

B. Navigation Function

Definition 2 [2]: Let $\mathcal{F} \subset \mathbb{R}^n$ be a compact connected n -dimensional analytic manifold with boundary, and let x_d be a goal point in the interior of \mathcal{F} . A map $\phi : \mathcal{F} \rightarrow [0, 1]$, is a navigation function if it is:

- 1) Smooth: at least twice continuously differentiable on \mathcal{F} .
- 2) Admissible: uniformly maximal on $\partial\mathcal{F}$.
- 3) Polar at x_d : has a unique minimum at x_d .
- 4) Morse: has only non-degenerate critical points on \mathcal{F} .

The Koditschek-Rimon navigation function $\phi_{kr} = \frac{\|q - q_d\|}{(\|q - q_d\|^k + \beta)^{1/k}}$, is a navigation function constructed in the sphere world, so long as k exceeds some threshold k_{min} , where $\beta = \prod_{i=0}^M \beta_i$ is the aggregate obstacle function.

C. Mobile Robot Models

This work considers two types of mobile robots that navigate in a 2-dimensional workspace $\mathcal{N}^2 \subset \mathbb{R}^2$.

The motion of the first type of mobile robot is described by the trivial kinematic integrator model:

$$\dot{x}(t) = u(t), \quad (3)$$

where $x \in \mathbb{R}^2$ is the robot position and $u \in \mathbb{R}^2$ is the corresponding control input vector. The second model is the second-order dynamic integrator for a mobile robot of mass m :

$$m \cdot \ddot{x}(t) = \tau(t), \quad (4)$$

where (x, \dot{x}) are the robot position and velocity. The control input $\tau \in \mathbb{R}^2$ represents the torque applied to the robot. The initial position and goal position of the mobile robots are denoted as $x_0 \in \mathcal{N}^2$ and $x_d \in \mathcal{N}^2$, respectively.

D. Conformal Map

Let U be a connected open set, and let $z = x + iy$ be a complex variable. A complex-valued function $f(z) : U \subset \mathbb{C} \rightarrow \mathbb{C}$ is holomorphic on U if it is complex differentiable at every point of U .

Definition 3 [14]: Let $f(x, y) = (u(x, y), v(x, y)) : U \subset \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be of class C^1 with non-vanishing Jacobian, then f is a conformal map if and only if $f(z) = u(x, y) + iv(x, y) : U \subset \mathbb{C} \rightarrow \mathbb{C}$, as a function of z , is holomorphic.

III. CONFORMAL NAVIGATION TRANSFORMATION

In this section, we define the conformal navigation transformation and investigate the existence and uniqueness of the proposed transformation.

Formally, the conformal navigation transformation is defined as follows.

Definition 4: A conformal navigation transformation is a map $T : \mathcal{N}^2 \rightarrow \mathcal{M}^2$ which:

- 1) maps the interior of the workspace conformally to the interior of a 2-dimensional sphere world.
- 2) maps the external boundary of the workspace homeomorphically to the unit circle.
- 3) maps the boundaries of M internal obstacles homeomorphically to M disjoint circles inside the unit circle.

This definition completely captures the topology of 2-dimensional workspaces. According to the classification theorem of compact surfaces with boundary [15], two compact surfaces with boundary are topologically equivalent if and only if they agree in character of orientability, number of boundaries, and Euler characteristic. Obviously, \mathcal{N}^2 and \mathcal{M}^2 have the same orientability and number of boundaries. Moreover, the Euler characteristic of a workspace \mathcal{N}^2 with M internal obstacles is $1 - M$, which is the same as the Euler characteristic of the sphere world \mathcal{M}^2 . Therefore, \mathcal{N}^2 and \mathcal{M}^2 are topologically equivalent.

The conformal navigation transformation has several properties that enable it to be applied to robot navigation problems in complex workspaces. The first property is that it exists in all workspaces in \mathbb{R}^2 , implying that the proposed transformation can be constructed in a 2D environment with arbitrarily shaped obstacles. The following results describe the first property in detail.

Lemma 1 [16]: Let Ω be a domain in Riemann sphere $\bar{\mathbb{C}}$ and let $B_{nc}(\Omega)$ be the collection of all boundary components of Ω which are not circles or points. If the closure of $B_{nc}(\Omega)$ in the space $B(\Omega)$ of all boundary components of Ω is (at most) countable, then Ω is conformally equivalent to a domain in $\bar{\mathbb{C}}$ whose boundary components are all circles or points.

The 2-dimensional workspace \mathcal{N}^2 is a closed set in Euclidean space containing all its $M + 1$ boundaries, which is different from the multiply-connected open set in the Riemann sphere $\bar{\mathbb{C}}$. The following proposition generalizes Lemma 1 to \mathcal{N}^2 .

Proposition 1 (Existence): If \mathcal{N}^2 is a 2-dimensional workspace with M internal obstacles, then there exists a 2-dimensional sphere world \mathcal{M}^2 with M internal obstacles and a conformal navigation transformation T , such that $T(\mathcal{N}^2) = \mathcal{M}^2$.

Proof: See the proof of Proposition 1 in [17]. ■

After concluding that conformal navigation transformations are ubiquitous in 2-dimensional workspaces, it is natural to wonder how many conformal navigation transformations exist in a given 2-dimensional workspace. Proposition 2 answers this question and leads to the second property of the conformal navigational transformation: uniqueness.

Proposition 2 (Uniqueness): The conformal navigation transformation $T : \mathcal{N}^2 \rightarrow \mathcal{M}^2$ and \mathcal{M}^2 are uniquely determined by \mathcal{N}^2 if the images of the prescribed three points on the external boundary $\partial\mathcal{O}_0$ are fixed on the unit circle.

Proof: See the proof of Proposition 2 in [17]. ■

The uniqueness of conformal navigation transformations has two advantages for robot navigation problems. Firstly, no further parameter tuning is required after the construction of the conformal navigation transformation, that is, the resulting solution is correct-by-construction. Secondly, the conformal navigation transformation is easy to determine. In fact, it can be uniquely determined by fixing only three points on the unit circle.

IV. CONTROLLER DESIGN USING THE CONFORMAL NAVIGATION TRANSFORMATION

In this section, the third property, i.e., the invariance of the navigation function properties, is given, then two controllers based on the conformal navigation transformation are presented and analyzed to illustrate its application to robot navigation problems in complex 2D environments.

The next proposition describes the third property in detail.

Proposition 3 (invariance): Let \mathcal{N}^2 be a workspace with smooth boundaries, and $\phi : \mathcal{M}^2 \rightarrow [0, 1]$ be a navigation function on \mathcal{M}^2 . If $T : \mathcal{N}^2 \rightarrow \mathcal{M}^2$ is the conformal navigation transformation determined by \mathcal{N}^2 , then $\tilde{\phi} = \phi \circ T$ is a navigation function on \mathcal{N}^2 .

Proof: Since each $\partial\mathcal{O}_i$ is a smooth Jordan curve, according to the Kellogg-Warschawski theorem [18], T is a smooth map on \mathcal{N}^2 . Then, the function $\tilde{\phi}$ is smooth on \mathcal{N}^2 .

From the proof of Proposition 1, we know that the map T is a conformal equivalence from \mathcal{N}^2 onto \mathcal{M}^2 . Therefore, T is a diffeomorphism from \mathcal{N}^2 onto \mathcal{M}^2 and the Jacobian of T is non-singular in \mathcal{N}^2 . It is sufficient to study the influence of T on the critical points of the navigation function in \mathcal{N}^2 because the critical points of the navigation function ϕ are all in \mathcal{N}^2 . Applying the chain rule yields $\nabla\tilde{\phi} = [J_T]^T \nabla(\phi \circ T)$. As T is diffeomorphic in \mathcal{N}^2 , T is a bijection from the set of critical points of $\tilde{\phi}$, $\mathcal{C}_{\tilde{\phi}}$, to the set of critical points of ϕ , \mathcal{C}_{ϕ} . Furthermore, the Hessian matrix of $\tilde{\phi}$ at the critical points satisfies $H(\tilde{\phi}) = [J_T]^T H(\phi \circ T) [J_T]$. Since J_T is non-singular in \mathcal{N}^2 , $H(\tilde{\phi})$ is non-singular at the critical points. Hence, $\tilde{\phi}$ is a Morse function.

Next, consider the minimum of T in \mathcal{N}^2 . At the critical points, we have that: $H(\tilde{\phi}) = [J_T]^T H(\phi \circ T) [J_T]$. Since T is a bijection between $\mathcal{C}_{\tilde{\phi}}$ and \mathcal{C}_{ϕ} , the Morse index of $\tilde{\phi}$ matches the Morse index of ϕ at each of the critical points. Furthermore, ϕ has a unique minimum at x_d . Hence, $\tilde{\phi}$ has exactly one minimum at $p_d = T(x_d)$.

Last consider the image of $\partial\mathcal{O}_i$ under $\tilde{\phi}$. Since T is a diffeomorphism on \mathcal{N}^2 and a homeomorphism on each smooth boundary $\partial\mathcal{O}_i$, the image of $\partial\mathcal{O}_i$ under T is exactly $\partial\tilde{\mathcal{O}}_i$. Moreover, ϕ is uniformly maximal on $\partial\mathcal{O}_i$. Hence, $\tilde{\phi}$ is admissible on \mathcal{N}^2 . ■

Proposition 3 shows that the conformal navigation transformation between \mathcal{M}^2 and \mathcal{N}^2 with smooth boundaries does not change the navigation function properties. This property provides an approach for constructing navigation functions in complex 2D environments, as shown below.

Proposition 4: The mobile robot system (3) under the feedback control law

$$\begin{aligned} u(x) &= -K(J_T(x))^T \nabla_{T(x)} \phi_{kr}(T(x)) \\ &= -K\|J_T(x)\| (J_T(x))^{-1} \nabla_{T(x)} \phi_{kr}(T(x)), \end{aligned} \quad (5)$$

where K is a positive gain, is globally asymptotically stable at the goal point $x_d \in \mathcal{N}^2$ from almost every point in \mathcal{N}^2 with smooth boundaries.

Proof: Since T is a conformal map on \mathcal{N}^2 , according to the Cauchy-Riemann equations, the Jacobian $J_T(x) =$

$\begin{bmatrix} u_{x_1} & -v_{x_1} \\ v_{x_1} & u_{x_1} \end{bmatrix}$ at $x = (x_1, x_2) \in \mathcal{N}^2$. Moreover, the Jacobian $J_T(x) = (u_{x_1}^2 + v_{x_1}^2)^{\frac{1}{2}} R = \|J_T(x)\|^{\frac{1}{2}} R$, where R is a rotation matrix. Hence, for T on \mathcal{N}^2 , we have that: $(J_T(x))^T = \|J_T(x)\| (J_T(x))^{-1}$.

Choosing $V(x) = \phi_{kr}(T(x))$ as a candidate Lyapunov function. Differentiating V along the system's trajectories yields: $\dot{V}(x) = -K \|J_T(x)\| \|\nabla_{T(x)} \phi_{kr}(T(x))\|^2 \leq 0$. Since $J_T(x)$ is non-singular, the set for $\dot{V}(x) = 0$ consists only of the critical points of $\phi_{kr}(T(x))$. The critical points of $\phi_{kr}(T(x))$ are the unique minimum point x_d and the isolated saddle points. Lasalle's Invariance Theorem dictates that the closed loop system will converge either to the goal or the saddle points. Since the saddle point has a stable manifold of 1-dimension in \mathcal{N}^2 , the isolated saddle points have attractive basins of zero measure. Hence $\dot{V}(x) < 0$ holds for almost everywhere, which implies global asymptotic stability for the system from almost all initial points. ■

The velocity of the robot at each point x in \mathcal{N}^2 is a rotation and scaling of the velocity of the corresponding point $T(x)$ in \mathcal{M}^2 . To see this, observe that $(J_T(x))^T$ is a positive scalar $\|J_T(x)\|^{\frac{1}{2}}$ times a rotation matrix R^T . This implies that the conformal navigation transformation preserves both angle and local shape. Therefore, the conformal navigation transformation can completely eliminate the angular distortion when mapping from a geometrically complex workspace to a sphere world, which is its fourth property.

For the mobile robot with mass m and second-order integrator dynamics, a minor modification can be used, as stated in the following proposition.

Proposition 5: The mobile robot system (4) under the feedback control law

$$\begin{aligned} u(x) &= -K(J_T(x))^T \nabla_{T(x)} \phi_{kr}(T(x)) + d(x, \dot{x}) \\ &= -K \|J_T(x)\| (J_T(x))^{-1} \nabla_{T(x)} \phi_{kr}(T(x)) + d(x, \dot{x}), \end{aligned} \quad (6)$$

where K is a positive gain and $d(x, \dot{x})$ is a dissipative field, i.e., $\dot{x}^T d(x, \dot{x}) < 0$ for $\dot{x} \neq 0$, is globally asymptotically stable at the goal point $x_d \in \mathcal{N}^2$ from almost every point with zero initial velocity in \mathcal{N}^2 with smooth boundaries.

Proof: Since $\phi_{kr}(T(x))$ is a navigation function on \mathcal{N}^2 and $\dot{x}_0(t) = 0$ at $t = 0$, the trajectory of the mobile robot system (4) under the feedback control law (6) stay inside \mathcal{N}^2 [2]. Consider the candidate Lyapunov function:

$$V(x) = K \phi_{kr}(T(x)) + \frac{1}{2} m \dot{x}^T \dot{x}.$$

Its time derivative is provided as:

$$\dot{V}(x) = K [\nabla_{T(x)} \phi_{kr}(T(x))]^T J_T(x) \dot{x} + m \dot{x}^T \dot{x}.$$

Following the trajectory of the closed-loop system, we get $\dot{V}(x) = [d(x, \dot{x})]^T \dot{x} \leq 0$. Noting that the set where $\dot{V}(x) = [d(x, \dot{x})]^T \dot{x} = 0$ is equal to the set where $\dot{x} = 0$. Hence, the largest invariant set consists only of the goal point x_d and the saddle points. Since only a set of measure zero of initial

points is attracted to the saddles, then according to the Lasalle's Invariance Theorem, $\dot{V}(x) < 0$ holds for almost everywhere. The control law (6) is thus globally asymptotically stable at x_d from almost every point. ■

Since T is a diffeomorphism on \mathcal{N}^2 , there is an alternative way of solving the above problem: pulling back the solution determined by a vector field on \mathcal{M}^2 to \mathcal{N}^2 by J_T^{-1} . It is worth mentioning that the special property of T , i.e., $[J_T]^T = \|J_T\| [J_T]^{-1}$, establishes a close connection between the above two methods.

V. CONSTRUCTION OF THE CONFORMAL NAVIGATION TRANSFORMATION

In this section, we present a construction method of the conformal navigation transformation without tuning parameters, which operates directly in the 2D environment with non-star-shaped obstacles. We first construct T in workspaces without obstacles and then use them successively to obtain T in workspaces with M internal obstacles. The machinery of complex variables facilitates the construction of conformal navigation transformations in \mathcal{N}^2 , which will be used in this section. In fact, $T(x)$ on \mathcal{N}^2 and $T(z)$ on the complex plane satisfy the following relation: $T(x) = [\text{Re}(T(z)), \text{Im}(T(z))]$.

A. Conformal Navigation Transformations for Simply Connected Workspaces

Let U^+ and U^- be a bounded and an unbounded simply connected workspace in the extended complex plane $\mathbb{C} \cup \{\infty\}$, respectively. For U^+ , assume that z_c is a given point in U^+ . For U^- , assume that z_c is a given point in the complement of U^- and $\infty \in U^-$. The boundary Γ is a closed smooth Jordan curve which is parametrized by a 2π -periodic twice continuously differentiable complex function $\gamma(s)$ with $\gamma'(s) \neq 0$ for $s \in [0, 2\pi]$. The orientation of Γ is counterclockwise for U^+ and clockwise for U^- .

According to Definition 3 and Definition 4, the conformal navigational transformation T in the extended complex plane is a holomorphic function $T(z)$ with a non-zero derivative and has a continuous extension to the boundary Γ . Then, from the Cauchy integral formula we have that:

$$T(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{T(\gamma)}{\gamma - z} d\gamma, \quad z \in U^+, \quad (7)$$

and

$$T(z) = T(\infty) - \frac{1}{2\pi i} \int_{\Gamma} \frac{T(\gamma)}{\gamma - z} d\gamma, \quad z \in U^-. \quad (8)$$

Equations (7) and (8) imply that the conformal navigation transformations on U^+ and U^- are completely determined by their boundary values $T(\gamma)$, which is the fifth property of T . This property reduces the construction of T on the entire workspace to the construction of T on the boundary. Therefore, we focus on the solution of $T(\gamma(s))$.

For a bounded simply connected workspace U^+ , its boundary Γ is mapped onto the unit circle ∂D and $T(\gamma(s))$ is given by:

$$T(\gamma(s)) = e^{i\theta(s)}, \quad s \in [0, 2\pi]. \quad (9)$$

The proof of Proposition 2 shows that $T(\gamma(s))$ is uniquely determined by a Möbius transformation. Then, the normalization condition in Proposition 2 is equivalent to:

$$T(z_c) = 0, \arg(T'(z_c)) = 0. \quad (10)$$

The function $T(z)$ can be written as $T(z) = a(z - z_c)h(z)$, where $h(z)$ is a holomorphic function on U^+ and $a \in \mathbb{R}$ is a constant. Since $|T(\gamma(s))| = |e^{i\theta(s)}| = 1$ and the modulus of the complex exponential function is determined by its real part, $h(z)$ can be expressed as $h(z) = e^{g(z)}$, where $g(z)$ is a holomorphic function on U^+ . Then, in view of $T'(z_c)$ being a positive real number, we obtain:

$$T(z) = a(z - z_c)e^{(z - z_c)f(z)}, \quad (11)$$

where $f(z)$ is a holomorphic function on U^+ and $a = T'(z_c) > 0$. Taking the logarithm of both sides of (11) and substituting (9), we have that:

$$\begin{aligned} (\gamma(s) - z_c)f(\gamma(s)) &= -\log(|\gamma(s) - z_c|) - \log(T'(z_c)) \\ &\quad + i(\theta(s) - \arg(\gamma(s) - z_c)). \end{aligned} \quad (12)$$

(12) can be written as:

$$G(s)f(\gamma(s)) = \mu(s) + c(s) + iv(s), \quad s \in [0, 2\pi], \quad (13)$$

where $G(s) = \gamma(s) - z_c$, $\mu(s) = -\log|\gamma(s) - z_c|$, $c(s) = -\log(T'(z_c))$, $v(s) = \theta(s) - \arg(\gamma(s) - z_c)$. Note that $\mu(s)$ is a given real-valued function and $c(s)$ is a constant. Although $f(\gamma(s))$, $c(s)$, and $v(s)$ are all unknown, (13) is still solvable since $\mu(s)$ is a known real-valued function and $f(z)$ is a holomorphic function on U^+ . The following lemma is derived to support the above statement. The various variables mentioned in the sequel are defined in Appendix.

Lemma 2: For the given real-valued function μ , there exists a unique constant c and a unique function v , such that (13) are boundary values of a holomorphic function f in U^+ . The function v is the unique solution of the equation

$$v - \mathbf{R}v = -\mathbf{H}\mu \quad (14)$$

and c is given by

$$c = [\mathbf{H}v - (\mathbf{I} - \mathbf{R})\mu]/2. \quad (15)$$

Proof: See Appendix. ■

Since (13) is uniquely solvable, the conformal navigation transformation $T(z)$ on U^+ can be solved by (11), as described in Proposition 6.

Proposition 6: For any bounded simply connected workspace U^+ , and any twice continuously differentiable parametric boundary $\gamma(s)$ with $\gamma'(s) \neq 0$, the holomorphic function

$$T(z) = e^{-c}(z - z_c)e^{(z - z_c)f(z)} \quad (16)$$

is the conformal navigation transformation on U^+ .

Proof: According to Lemma 2, the unknown constant $c = -\log(T'(z_c))$ is given by (15). Thus, we have that $T'(z_c) = e^{-c}$. Then, in view of (11), we obtain that: $T(z) = e^{-c}(z - z_c)e^{(z - z_c)f(z)}$. Differentiating $T(z)$ with z , we get that:

$$T'(z) = e^{-c}(e^{(z - z_c)f(z)})$$

$$+ (z - z_c)e^{(z - z_c)f(z)}((z - z_c)f'(z) + f(z)) \quad (17)$$

and $T'(z) \neq 0$ in the interior of U^+ . Since $T(\gamma(s)) = e^{i\theta(s)}$, the function $T(\gamma(s))$ is a homeomorphism on Γ . Hence, $T(z)$ is the conformal navigation transformation on U^+ . ■

Unlike $T(z)$ on U^+ , the conformal navigation transformation on the unbounded simply connected workspace U^- , denoted as $\tilde{T}(z)$, maps U^- to the exterior of the unit circle. The boundary values of $\tilde{T}(z)$ satisfy

$$\tilde{T}(\gamma(s)) = e^{-i\theta(s)}, \quad s \in [0, 2\pi]. \quad (18)$$

With the condition

$$\tilde{T}(\infty) = \infty, \arg(\tilde{T}'(\infty)) = 0, \quad (19)$$

the function $\tilde{T}(z)$ is unique in U^- .

Similar to $T(z)$ on U^+ , the function $\tilde{T}(z)$ on U^- is constructed as

$$\tilde{T}(z) = \tilde{T}'(\infty)(z - z_c)e^{-\tilde{f}(z)}, \quad (20)$$

where $\tilde{f}(z)$ is a holomorphic function on U^- and $\tilde{f}(\infty) = 0$. From (18), we get that:

$$\begin{aligned} \tilde{f}(\gamma(s)) &= \log(|\gamma(s) - z_c|) + \log(\tilde{T}'(\infty)) \\ &\quad + i(\theta(s) + \arg(\gamma(s) - z_c)). \end{aligned} \quad (21)$$

Let $\tilde{\mu}(s) = \log|\gamma(s) - z_c|$, $\tilde{c}(s) = \log(\tilde{T}'(\infty))$, and $\tilde{v}(s) = \theta(s) + \arg(\gamma(s) - z_c)$, then we obtain that:

$$\tilde{f}(\gamma(s)) = \tilde{\mu}(s) + \tilde{c}(s) + i\tilde{v}(s), \quad s \in [0, 2\pi]. \quad (22)$$

Equation (22) has similar solvability as (13), which is described in Lemma 3.

Lemma 3: For the given real-valued function $\tilde{\mu}$, there exists a unique constant \tilde{c} and a unique function \tilde{v} in U^- , such that (22) are boundary values of a holomorphic function \tilde{f} with $\tilde{f}(\infty) = 0$. The function $\tilde{v}(s)$ is the unique solution of the equation

$$\tilde{v} - \mathbf{R}\tilde{v} = -\mathbf{H}\tilde{\mu} \quad (23)$$

and \tilde{c} is given by

$$\tilde{c} = [\mathbf{H}\tilde{v} - (\mathbf{I} - \mathbf{R})\tilde{\mu}]/2. \quad (24)$$

Proof: This lemma can be proved along the same lines as Lemma 2. ■

The solution of $\tilde{T}(z)$ on U^- can be obtained by solving (22), as shown in Proposition 7 below.

Proposition 7: For any unbounded simply connected workspace U^- , and any twice continuously differentiable parametric boundary $\gamma(s)$ with $\gamma'(s) \neq 0$, the holomorphic function

$$\tilde{T}(z) = e^{\tilde{c}}(z - z_c)e^{-\tilde{f}(z)} \quad (25)$$

is the conformal navigation transformation on U^- .

Proof: This proposition can be proved along the same lines as Proposition 6. ■

In order to compute $T(\gamma(s))$ and $\tilde{T}(\gamma(s))$, (14) and (23) are discretized into $n \times n$ linear systems by the Nyström method, where n is the number of points in the discretization of Γ . The singular integral operator \mathbf{H} is discretized using the Fast Fourier transform (FFT), which requires $O(n \log n)$ operations. Then,

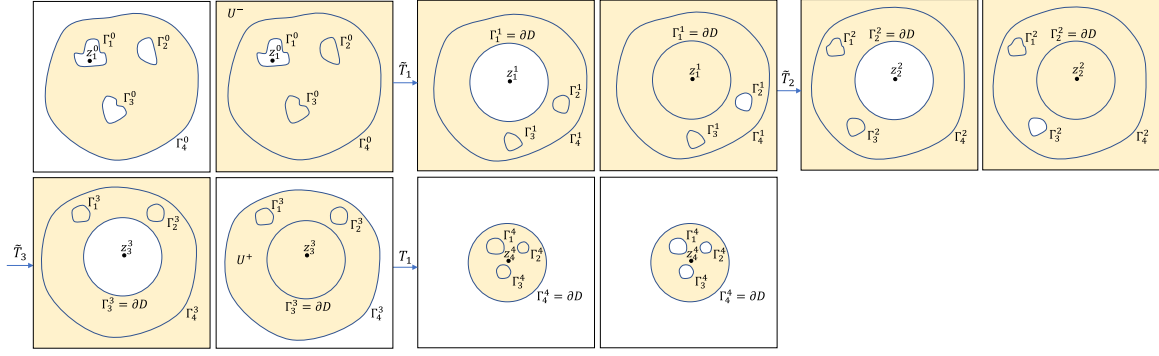


Fig. 1. First iteration in a workspace with three internal obstacles.

the discretized linear system is solved by a combination of the Generalized Minimal Residual Method (GMRES) and the Fast Multipole Method (FMM) in $O(n)$ operations. Consequently, the complexity of computing the function $\theta(s)$ is $O(n \log n)$ operations, which implies that the complexity of computing $T(\gamma(s))$ and $\tilde{T}(\gamma(s))$ is $O(n \log n)$ operations.

B. Conformal Navigation Transformations for Multiply-Connected Workspaces

Let N be a bounded workspace with M internal obstacles in the extended complex plane $\mathbb{C} \cup \{\infty\}$. The boundary $\Gamma = \sqcup_{i=1}^{m+1} \Gamma_i$ is the disjoint union of $M + 1$ closed smooth Jordan curves and Γ_{m+1} is the external boundary. We assume that z_i , $1 \leq i \leq m$, is a given point inside Γ_i , $1 \leq i \leq m$, and z_{m+1} is a given point in the interior of N . The curve Γ_i is parametrized by a 2π -periodic twice continuously differentiable complex function $\gamma_i(s)$ with $\gamma'_i(s) \neq 0$ for $s \in [0, 2\pi]$. The orientation of Γ is such that N is always on the left of Γ .

The construction method of $T(z)$ on N is an efficient and fast iterative method, which is inspired by the conventional Koebe's method on unbounded regions. The building blocks of the method are $T(z)$ on U^+ and $\tilde{T}(z)$ on U^- . Each iteration consists of the following $M + 1$ steps, which are illustrated in Fig. 1 when M is 3.

At the beginning, set $\Gamma_i^0 = \Gamma_i$ and $z_i^0 = z_i$ for $i = 1, 2, \dots, m + 1$. Then, the first step constructs the function \tilde{T}_1 that maps the exterior region of the curve Γ_1^0 onto the exterior of the unit circle and z_1^0 to the origin. Since the curves Γ_i^0 , $i \neq 1$, are in the exterior region of Γ_1^0 , the images of the curves Γ_i^0 , $i \neq 1$, are smooth Jordan curves $\Gamma_i^1 = \tilde{T}_1(\Gamma_i^0)$, $i \neq 1$, which can be computed using (8). Repeat this, and at step $k \leq m$, construct the function \tilde{T}_k , that maps the exterior region of the curve Γ_k^0 onto the exterior region of the unit circle and z_k^{k-1} to the origin. So far, every internal boundary has been mapped to the unit circle once. For the external boundary Γ_{m+1}^m , at the $(m + 1)$ -th step, construct the function T_1 , that maps the interior region of the curve Γ_{m+1}^m onto the open unit disk and z_{m+1}^m to the origin. Then, $\Gamma_{m+1}^{m+1} = T_1(\Gamma_{m+1}^m)$ is the unit circle and $\Gamma_i^{m+1} = T_1(\Gamma_i^m)$, $1 \leq i \leq m$, are smooth Jordan curves, which can be computed using (7).

The above process is repeated for each iteration, and as the number of iterations increases, each boundary becomes gradually more circular. Define $\hat{T}_1 = T_1 \circ \tilde{T}_m \circ \tilde{T}_{m-1} \circ \dots \circ \tilde{T}_1$ as the function after the first iteration and define $\bar{T}_n = \hat{T}_n \circ \hat{T}_{n-1} \circ \dots \circ \hat{T}_1$ as the approximate conformal navigation transformation after n iterations. Then, the above assertion is equivalent to the fact that the function $\bar{T}_n \circ T^{-1}$ tends to the identity function as n increases. The following proposition gives the quantitative convergence estimation.

Proposition 8 [19]: For the workspace N with M internal obstacles, there exist constants $h > 0$, $0 < q < 1$, such that for $n = 1, 2, \dots$ and for all $z \in N$,

$$\|\bar{T}_n \circ T^{-1}(z) - z\| \leq hq^{4n}. \quad (26)$$

For the workspace N with M internal obstacles, each iteration of the presented iterative method requires computing $M + 1$ conformal navigation transformations of simply connected workspaces, e.g., $\tilde{T}_1, \dots, \tilde{T}_m$, and T_1 for the first iteration, which needs $O((M + 1)n \log n)$ operations. For each of these transformations, it is required to use the Cauchy integral formula $M + 1$ times to compute the new boundary components, which needs additional $O((M + 1)^2 n)$ operations. Hence, the computational cost of each iteration is $O((M + 1)^2 n + (M + 1)n \log n)$ which implies that the computational cost of the presented iterative method is $O((M + 1)^2 n + (M + 1)n \log n)$.

The conformal navigation transformation has the advantage of being neither dependent on the initial nor the goal point, which implies that it does not require reconstruction for different initial and goal points. Thus, it only needs to be constructed offline once. Although the online computation of $T(z)$ at the interior point requires computing the Cauchy integral formula, this can be done in $O((M + 1)n)$ operations. Moreover, the rational function can approximate the proposed transformation with high accuracy and provide 10-1000 times faster computation [20]. These significantly increase the possibility of online application of the proposed method.

VI. SIMULATION RESULTS

In order to validate the effectiveness of the proposed methodology, a set of simulation results were presented in this section. Simulations were carried out for both controllers (5) and (6),

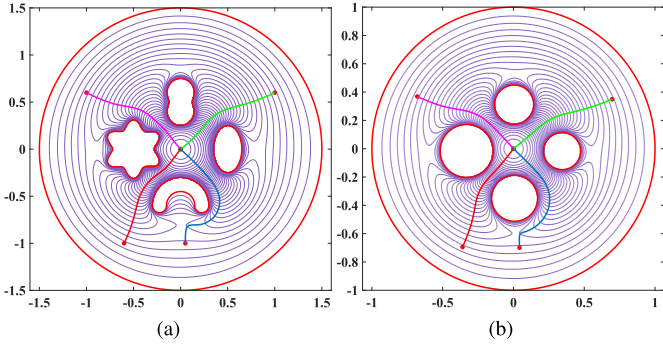


Fig. 2. Trajectories of the kinematic mobile robot for different initial points. (a) 2D workspace. (b) Conformal sphere world.

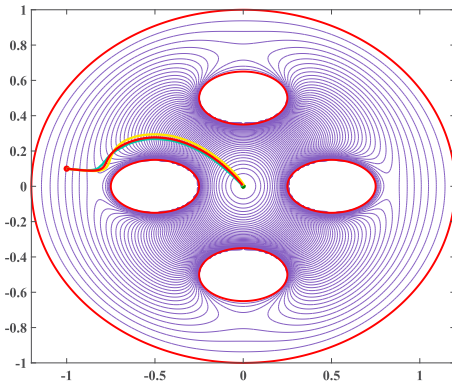


Fig. 3. Trajectories of the dynamics robot (4) under control law (6) for different λ in a two-dimensional workspace with elliptical boundaries.

and the gain K was set as $K = 1$. The parameter k in ϕ_{kr} was set as $k = 6$ in the first two simulations and $k = 30$ in the third simulation.

In the first simulation, the feedback control law (5) provided in Proposition 4 is applied to the kinematic mobile robot (3). The workspace is set up with three star-shaped internal obstacles: an ellipse, an inverse ellipse, a plum; and a non-star-shaped internal obstacle that cannot be handled by the homeomorphism-based transformations: the letter C. The goal point is set as $x_d = (0, 0)$, which the robot aims to converge to from 4 different initial points, namely $x_0 = (1, 0.6)$, $(-1, 0.6)$, $(-0.6, -1)$ and $(0.05, -1)$. Fig. 2 depicts the level sets of the navigation function and robot trajectories in the initial workspace and the transformed sphere world, respectively. As we can observe, the designed feedback control law successfully performs the task of maintaining the robot in the initial workspace, avoiding collisions and stabilizing at the goal point. In this example, the number of iterations is 7 and $\|\bar{T}_7 - \bar{T}_6\| = 5.4675 \times 10^{-14}$.

In the second simulation, the mass of the dynamic mobile robot (4) is set as $m = 1 \text{ kg}$. The dissipative field in the control law (6) is chosen as $d(x, \dot{x}) = -\lambda \dot{x}$. The workspace is bounded by 5 ellipses. The initial and goal points of the robot are $x_0 = (-1, 0.1)$ and $x_0 = (0, 0)$, respectively. Fig. 3 depicts the trajectories of the dynamic robot (4) under the control law (6) for different λ . The dark green trajectory corresponds to the system

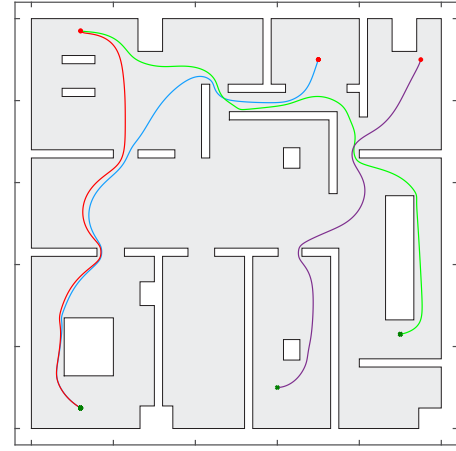


Fig. 4. Trajectories of the kinematic mobile robot (3) under the control law (5) in an obstacle-cluttered office environment.

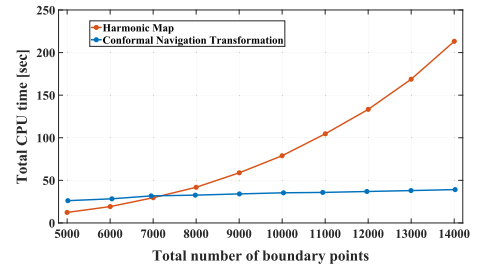


Fig. 5. Total CPU time versus total number of boundary points.

(4) without dynamic, i.e., $\lambda = 0$. The orange trajectory and the red trajectory correspond to a system with $\lambda = 3.2$ and $\lambda = 5$, respectively. As can be seen, all three trajectories successfully arrive and stabilize at the goal point without collision. For different constants λ , we can observe that the larger the constant λ , the closer the trajectory is to that of the system without dynamics. In this example, the number of iterations is 7 and $\|\bar{T}_7 - \bar{T}_6\| = 3.0234 \times 10^{-14}$.

In the third simulation, the feedback control law (5) is validated in an obstacle-cluttered office environment. The corners of the office environment are smoothly parameterized with a graded mesh. Fig. 4 depicts four different trajectories of the kinematic mobile robot (3) in the office workspace (Circles indicate initial points and crosses indicate goal points). As can be seen the proposed controller performs successfully, maintaining the robot in the obstacle-cluttered office environment, avoiding collisions and stabilizing at the goal points.

Last, we compare the computational cost of our construction method with that of the harmonic mapping method [11] in the same examined environment illustrated in Fig. 4. We chose to compare with this method because both can cope with the geometric complexity of the workspace. The total CPU time versus the total number of boundary points is displayed in Fig. 5. The results show that our method becomes more advantageous in terms of computational time as the number of boundary points increases.

Remarks: Our approach has other advantages over the harmonic mapping-based method. First, it is suitable for navigation from the surface of an obstacle, e.g., docking and parts mating. The conformal navigation transformation preserves all the topological properties of the boundary, while the harmonic mapping collapses the boundary to a point and loses the boundary information. Second, it preserves harmonic artificial potential fields between planar environments. The composition of a harmonic function with a conformal navigation transformation in \mathbb{R}^2 is still harmonic.

VII. CONCLUSION

This letter proposes a methodology for navigation problems in complex workspaces based on a novel spatial transformation called the conformal navigation transformation. Unlike the homeomorphism-based transformations, the proposed transformation has the advantages of requiring no tuning, existing uniquely in a given workspace, eliminating angular distortion, and being completely determined by boundary values, which are suitable for navigation problems in complex workspaces. Using the proposed transformation, the navigation function in the geometrically simple sphere world is pulled back to the initial workspace. The negative gradient of the pulled-back navigation function coupled with an appropriate dissipation term can be used as a control law to achieve convergence of the dynamic robot system from almost all initial points to the target point. It is shown that this method provides solutions to both path planning and motion planning subproblems. Furthermore, an iterative method for constructing the proposed transformation is presented, which ensures fast convergence and is applicable to non-star-shaped obstacles. Finally, simulation results support the theoretical results.

Future research directions include extending the solution to account for conformal geometric structures of workspaces, constrained stabilization in complex workspaces, dynamic obstacles, multi-agent navigation problems, and more realistic scenarios with local sensing.

APPENDIX

Define the function $C(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{\mu+iv}{G} \frac{d\gamma}{\gamma-z}$ for $z \notin \Gamma$. Let $C^{\pm}(\gamma(s)) = \lim_{z \rightarrow \gamma^{\pm}(s)} C(z)$. According to Sokhotski-Plemelj theorem [21], we have that

$$G(s)C^{\pm}(\gamma(s)) = \pm \frac{1}{2} \left(\mu(s) + iv(s) + \int_0^{2\pi} \frac{1}{i\pi} \frac{G(s)}{G(t)} \frac{\gamma'(t)}{\gamma(t) - \gamma(s)} (\mu(t) + iv(t)) dt \right). \quad (27)$$

Define $\mathbf{H}\mu = \int_0^{2\pi} H\mu(t)dt$, $\mathbf{R}v = \int_0^{2\pi} Rv(t)dt$, where $H = \text{Re} \left(\frac{1}{\pi} \frac{G(s)}{G(t)} \frac{\gamma'(t)}{\gamma(t) - \gamma(s)} \right)$, $R = \text{Im} \left(\frac{1}{\pi} \frac{G(s)}{G(t)} \frac{\gamma'(t)}{\gamma(t) - \gamma(s)} \right)$. From (27), we get $\text{Re}(G(s)C^{\pm}(\gamma(s))) = \frac{1}{2}(\pm v + \mathbf{R}v + \mathbf{H}\mu)$, $\text{Im}(G(s)C^{\pm}(\gamma(s))) = \frac{1}{2}(\pm \mu + \mathbf{R}\mu + \mathbf{H}v)$, and the jump relation $G(s)C^{+}(\gamma(s)) - G(s)C^{-}(\gamma(s)) = \mu(s) + iv(s)$. Let

$f(z) = C(z)$. According to the jump relation of boundary values, $f(z)$ has the boundary values $G(s)f^{+}(\gamma(s)) = G(s)C^{+}(\gamma(s)) = G(s)C^{-}(\gamma(s)) + \mu(s) + iv(s)$, where $f^{+}(\gamma(s)) = \lim_{z \rightarrow \gamma^{+}(s)} f(z)$. Since $f(z)$ is a holomorphic function on U^{+} , we obtain that $f(z) = C(z) = 0$ for $z \in U^{-}$. This implies $\text{Im}(G(s)C^{-}(\gamma(s))) = 0$, which is equivalent to (14). Since $\text{Im}(G(s)C^{-}(\gamma(s))) = 0$, we have $G(s)C^{-}(\gamma(s)) = \text{Re}(G(s)C^{-}(\gamma(s))) = \frac{1}{2}(-v + \mathbf{R}v + \mathbf{H}\mu)$. Hence, the boundary values $G(s)f^{+}(\gamma(s)) = \frac{1}{2}(-v + \mathbf{R}v + \mathbf{H}\mu) + \mu(s) + iv(s)$. This implies $c = [\mathbf{H}\mu - (\mathbf{I} - \mathbf{R})v]/2$. Since $\frac{1}{2\pi} \arg(G(s))|_0^{2\pi} = \frac{1}{2\pi} \arg(\gamma(s) - z_c)|_0^{2\pi} > 0$, then according to the Theorem 9 in [19], (13) has a unique solution, which completes the proof. ■

REFERENCES

- [1] O. Khatib, "Real-time obstacle avoidance for manipulators and mobile robots," in *Proc. IEEE Int. Conf. Robot. Automat.*, 1985, pp. 500–505.
- [2] E. Rimon and D. E. Koditschek, "Exact robot navigation using artificial potential functions," *IEEE Trans. Robot. Automat.*, vol. 8, no. 5, pp. 501–518, Oct. 1992.
- [3] E. Rimon and D. E. Koditschek, "The construction of analytic diffeomorphisms for exact robot navigation on star worlds," *Trans. Amer. Math. Soc.*, vol. 327, no. 1, pp. 71–116, 1991.
- [4] S. G. Loizou and K. J. Kyriakopoulos, "Navigation of multiple kinematically constrained robots," *IEEE Trans. Robot.*, vol. 24, no. 1, pp. 221–231, Feb. 2008.
- [5] H. G. Tanner and A. Boddu, "Multiagent navigation functions revisited," *IEEE Trans. Robot.*, vol. 28, no. 6, pp. 1346–1359, Dec. 2012.
- [6] C. Li and H. G. Tanner, "Navigation functions with time-varying destination manifolds in star worlds," *IEEE Trans. Robot.*, vol. 35, no. 1, pp. 35–48, Feb. 2019.
- [7] S. G. Loizou and E. D. Rimon, "Mobile robot navigation functions tuned by sensor readings in partially known environments," *IEEE Robot. Automat. Lett.*, vol. 7, no. 2, pp. 3803–3810, Apr. 2022.
- [8] C. K. Verginis and D. V. Dimarogonas, "Adaptive robot navigation with collision avoidance subject to 2nd-order uncertain dynamics," *Automatica*, vol. 123, Jan. 2021, Art. no. 109303.
- [9] S. Berkane and D. V. Dimarogonas, "Constrained stabilization on the n-sphere," *Automatica*, vol. 125, Mar. 2021, Art. no. 109416.
- [10] S. Paternain, D. E. Koditschek, and A. Ribeiro, "Navigation functions for convex potentials in a space with convex obstacles," *IEEE Trans. Autom. Control*, vol. 63, no. 9, pp. 2944–2959, Sep. 2018.
- [11] P. Vlantis, C. Vrohidis, C. P. Bechlioulis, and K. J. Kyriakopoulos, "Robot navigation in complex workspaces using harmonic maps," in *Proc. IEEE Int. Conf. Robot. Automat.*, 2018, pp. 1726–1731.
- [12] S. G. Loizou, "The navigation transformation," *IEEE Trans. Robot.*, vol. 33, no. 6, pp. 1516–1523, Dec. 2017.
- [13] N. Constantinou and S. G. Loizou, "Robot navigation on star worlds using a single-step navigation transformation," in *Proc. IEEE Conf. Decis. Control*, 2020, pp. 1537–1542.
- [14] D. E. Blair, *Inversion Theory and Conformal Mapping*. Providence, RI, USA: Amer. Math. Soc., 2000.
- [15] J. H. Gallier and D. Xu, *A Guide to the Classification Theorem for Compact Surfaces*. Berlin, Germany: Springer, 2013.
- [16] Z.-X. He and O. Schramm, "Koebe uniformization for "almost circle domains"," *Amer. J. Math.*, vol. 117, no. 3, pp. 653–667, 1995.
- [17] L. Fan, J. Liu, W. Zhang, and P. Xu, "Conformal navigation transformations with application to robot navigation in complex workspaces," 2022, *arXiv:2208.06876*. [Online]. Available: <https://arxiv.org/abs/2208.06876>
- [18] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, vol. 299. Berlin, Germany: Springer, 1992.
- [19] R. Wegmann, A. H. M. Murid, and M. M. S. Nasser, "The Riemann-Hilbert problem and the generalized Neumann kernel," *J. Comput. Appl. Math.*, vol. 182, no. 2, pp. 388–415, Jan. 2005.
- [20] Y. Nakatsukasa, O. Sète, and L. N. Trefethen, "The AAA algorithm for rational approximation," *SIAM J. Sci. Comput.*, vol. 40, no. 3, pp. A1494–A1522, 2018.
- [21] F. D. Gakhov, *Boundary Value Problems*. Mineola, NY, USA: Dover, 1990.