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A finite element approach to the position problems in open-loop variable geometry trusses

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Abstract

The present paper looks at some kinematic and static-equilibrium problems that arise with variable-geometry trusses (VGTs). The first part of the paper looks at the use of active controls in the correction of static deformations, the second part at the position problems. The separation between deformable- and rigid-body displacements makes it possible to consider separately the corrections in each component of the structure. VGTs are considered as open-loop linkages with redundant rigid-body degrees of freedom. Owing to this redundancy, possible solutions to the inverse problem are in general infinite, for which reason it is necessary to use some optimization criteria. To tackle the problem an optimization procedure with constraints is developed for the purpose of minimizing the displacements of the actuators. Suitable use of the constraints allows us to solve the direct position problem using the same optimization procedure. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Variable geometry trusses (VGTs) are light mechanical systems that can modify their geometry and properties in order to adapt to various operating conditions. As a kind of adaptive structure they should be equipped with actuators that facilitate the controlled variation of its geometry. Some of these actuators may be dedicated to the generation of large rigid-body movements [1,36,39], while others (active links or active controls) serve to reduce small elastic deformations [2–35].

A VGT usually consists of a large number of bi-hinged rods and several actuators, which form a complex truss (Fig. 1). Some of the rods may be active links, i.e., they can vary their length in

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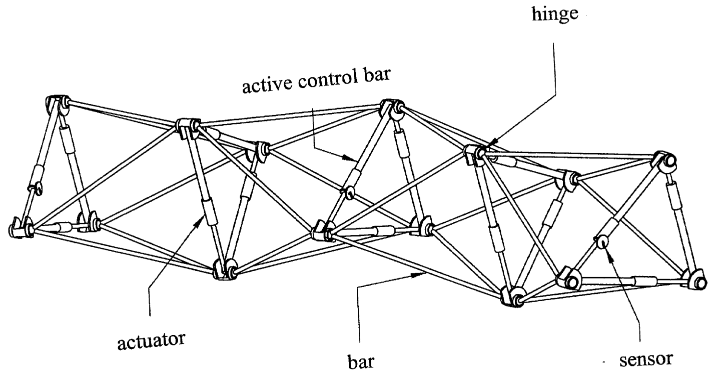


Fig. 1. Modular adaptive structure with actuators, sensors and active links.

a controlled manner for the purpose of correcting small deformations. The cause for these deformations may be thermal phenomena, static and dynamic forces, vibrations in the support, or something else. The simplest kind of VGT is a foldable structure [33,34]. It can remain folded, occupying little space, until called on to function. As one would expect, many structures used in satellite applications are of this type, since they can remain folded during launch and flight, to be deployed only when the satellite has attained orbit. On the other hand, spatial cranes are in general more complex light manipulators used for precise operation in its working space [36].

As mentioned above, some of the rods of the structure can be active links (Fig. 1), which within certain limits can vary in length and produce internal forces in the structure. The task of the active links is to reduce the deformation of the structure by modifying the system's internal force equilibrium. Usually, it is assumed that these deformations are small and that the behavior of the structure is linear. There are other things which may be desired to control, such as certain vibration frequencies, modal dampings, and response to dynamic forces.

From the point of view of the Theory of Mechanisms and Machines, a VGT is a mechanism with redundant degrees of freedom, capable of placing certain precision point at given positions with the minimum possible error [1,37–39]. As mentioned above, a structure of this kind usually comprises a high number of rods, and the applications to which it is usually dedicated call for severe mass limitation, so that the configuration used is most often a very light truss (Fig. 1).

Precisely because of this lightness, the external forces may produce excessive deformations, so that apart from the actuators for large displacements in the system, they may also have a certain number of active links that, by means of variations in length, assist in compensating for deformations. In theory, the actuators themselves, either all or some of them, may be also used as active links, helping in the correction of the position once the rigid-body motion has been performed. Since in general, active links of any kind are present only in a small number, its position in the structure must be chosen carefully if the maximum corrective efficacy is to be ensured. This problem, which arises in the design stage, is a problem of discrete optimization, since the number of possible positions is finite and less than the number of rods in the structure. The Genetic Algorithms have proved highly useful in the solution of this problem, as is described in references [7,11,14,21,23,25,27,28,31,35,36,40], but this aspect will not be focused in the present paper.

The static or kinematic analysis of large displacement problems (e.g., with fold-out structures, spatial cranes, and VGTs in general) and of small deformation in those and other types of structures, adaptive or not, can be undertaken in the manner here described. The present paper will look specifically at several problems relating to mechanical systems of this type. Two of these problems are nonlinear, i.e., the problem of direct position and that of inverse position [1,37,38]. In each case the elements making up the structure are considered to be rigid, except for the actuators that are used to produce the rigid-body movement. The same mathematical model used for the position problems is used to perform the velocity and acceleration analyses. Another problem is that of the small displacements produced by external forces and the compensation of such deformations through active controls.

2. Linear equilibrium conditions on the components of a VGT

Once the desired position has been obtained, for example by means of the procedures described later in this paper, and with the actuators locked, the VGT becomes a structure acted on by external forces. These forces give rise to deformations, which we shall assume to be small, elastic, and linear. They also result in a position error at the precision points. The active links or active controls produce, in each component (Fig. 2), a complementary deformation field of which the purpose is to minimize the resulting deformation and hence also the position error. The action of these active links is equivalent to an adaptive stiffening of the structure.

In consequence of the external forces, which we shall assume here to be of static type, the structure in a given position m is deformed. If a finite-element model of the complete structure at its current position has been devised (including in the model all the rods, actuators and active links), and the external forces reduced to a force vector $\{F_m\}$, are known, then in the linear case we can

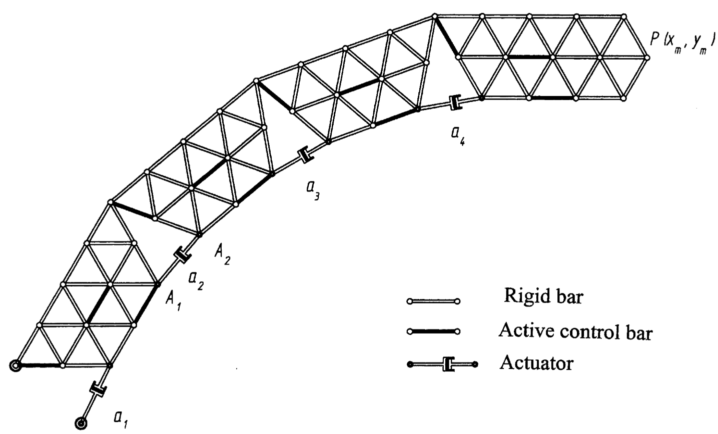


Fig. 2. VGT with four components, ninety-seven rods (of which twelve are active links), forty-eight nodes, and four actuators.

calculate the displacements by means of the following equation:

$$[K_m]\{\delta_m\} = \{F_m\}. \tag{1}$$

Here $[K_m]$ is the stiffness matrix for the finite element model of n degrees of freedom at position m, and $\{\delta_m\}$ are the nodal displacements vector.

With the actuators set at position m, the matrix for the complete structure, $[K_m]$, is positive definite. Each component (i.e., each kinematic element) can be isolated in local coordinates and described in terms of the displacements associated with it. The component-by-component analysis may be useful in configurations like the one shown in Fig. 2.

Let one of these components be e, whose stiffness matrix in local coordinates $[k^e]$ is positive semidefinite and independent of the structure position m. The displacements associated with that component are stored in the vector $\{\delta^e_m\}$, of dimension n_e, extracted from $\{\delta_m\}$ and expressed in local coordinates. The displacements $\{\delta^e_m\}$ contain a part of rigid-body motion, $\{\delta^e_m\}_\mathrm{r}$, and another part of elastic diplacements, $\{\delta^e_m\}_\mathrm{d}$. The following is verified:

$$\{\delta^e_m\} = [R^e][\Gamma^e_m]\{\delta_m\},$$

$$[K_m] = \sum_e [\Gamma^e_m]^\mathrm{T}[\bar{k}^e][\Gamma^e_m]\}. \tag{2}$$

Here $[\bar{k}^e]$ is the matrix of the component e in local coordinates, expanded to dimension n; $[\Gamma^e_m]$ is the matrix for the change of local coordinates of the component e at position m; and $[R^e]$ is a rectangular matrix with n_e rows (number of degrees of freedom of the model of the component e) and n columns. In each row of $[R^e]$, every element is zero except for that which represents the position of one degree of freedom of the component e, in which case the value is unity. The following equation is therefore satisfied:

$$\{\delta^e_m\} = \{\delta^e_m\}_\mathrm{r} + \{\delta^e_m\}_\mathrm{d}. \tag{3}$$

Hence for a vector $\{f^e_m\}$ of external and reaction forces acting on the component e in a position m we can write

$$[k^e]\{\delta^e_m\} = [k^e]\{\delta^e_m\}_\mathrm{d} = \{f^e_m\},$$

$$[k^e]\{\delta^e_m\}_\mathrm{r} = \{0\}. \tag{4}$$

The elastic energy stored in the component is

$$V^e_m = \tfrac{1}{2}\{\delta^e_m\}^\mathrm{T}_\mathrm{d}[k^e]\{\delta^e_m\}_\mathrm{d}. \tag{5}$$

The eigenproblem of the stiffness matrix for the component e is approached in the following form:

$$[k^e]\{\psi^e\}_q = \eta^e_q\{\psi^e\}_q. \tag{6}$$

If a system is semidefinite, as happens when we look at each component separately, then there exist as many zero eigenvalues as rigid-body degrees of freedom, g_e. If from (6) we derive the n_e eigenvalues η^e_q and eigenvectors $\{\psi^e\}_q$ of the component e, we can form the diagonal matrix of eigenvalues $[\eta^e]$, as well as an eigenvector matrix $[\Psi^e]$:

$$[\Psi^e] = [\{\psi^e\}_1\{\psi^e\}_2 \dots \{\psi^e\}_{g_e} \dots \{\psi^e\}_q \dots \{\psi^e\}_{n_e}]. \tag{7}$$

Alternatively we can write (7) as

$$[\Psi^e] = [[\Psi^e]_{\mathrm{r}}[\Psi^e]_{\mathrm{d}}]. \tag{8}$$

The first submatrix in (8) has n_e rows and g_e columns. It groups the eigenvectors associated with the zero eigenvalue of multiplicity order g_e. Hence, it is a case of the null-space of the component's stiffness matrix.

In local coordinates, the eigenvalues and the eigenvectors of the components are independent of position m. Further, the rigid-body eigenvectors, which define the null-space of the stiffness matrix, are the same as would have been derived from a spectral analysis of natural frequencies and modes, with any possible mass matrix (here taken equal to the unit matrix).

The eigenvectors are orthogonal among them and can be normalized in the following form:

$$\begin{aligned} &[\Psi^e]^{\mathrm{T}}[\Psi^e] = [I], \\ &[\Psi^e]^{\mathrm{T}}[k^e][\Psi^e] = [\eta^e], \end{aligned} \tag{9}$$

where $[\eta^e]$ is the diagonal matrix of the ordered eigenvalues (the g_e first diagonal values are zero).

Following a procedure similar to that employed in modal superposition methods in dynamic analysis, we can change over to new modal coordinates $\{x^e\}$ that ensures that the following will be satisfied for any position m:

$$\begin{aligned} &\{\delta^e_m\} = [\Psi^e]\{x^e\}, \\ &\{x^e\} = [\Psi^e]^{\mathrm{T}}\{\delta^e_m\}. \end{aligned} \tag{10}$$

If we substitute into (4) and premultipling by $[\psi^e]^{\mathrm{T}}$ results:

$$[\eta^e]\{x^e\} = [\Psi^e]^{\mathrm{T}}\{f^e_m\} = \{w^e_m\}. \tag{11}$$

The equation in (11) represents a set of n_e uncoupled equations:

$$\begin{aligned} &\eta^e_i x^e_i = w^e_{mi}, \\ &\quad i = 1, 2, \dots n_e. \end{aligned} \tag{12}$$

The rigid-body modes are orthogonal to the forces that act on component e, hence the equation in (11) can also be written in the following form:

$$\begin{bmatrix} [0] & [0] \\ [0] & [\eta^e]_{\mathrm{d}} \end{bmatrix} \begin{Bmatrix} \{x^e\}_{\mathrm{r}} \\ \{x^e\}_{\mathrm{d}} \end{Bmatrix} = \begin{Bmatrix} \{0\} \\ \{w^e_m\}_{\mathrm{d}} \end{Bmatrix}. \tag{13}$$

In view of (10):

$$\begin{aligned} &\{x^e\}_{\mathrm{r}} = [\Psi^e]^{\mathrm{T}}_{\mathrm{r}}\{\delta^e_m\}, \\ &\{x^e\}_{\mathrm{d}} = [\Psi^e]^{\mathrm{T}}_{\mathrm{d}}\{\delta^e_m\}. \end{aligned} \tag{14}$$

The equation in (10) can be rewritten as follows:

$$\{\delta^e_m\} = [[\Psi^e]_r[\Psi^e]_d]\begin{Bmatrix}\{x^e\}_r\\ \{x^e\}_d\end{Bmatrix} = [\Psi^e]_r\{x^e\}_r + [\Psi^e]_d\{x^e\}_d. \tag{15}$$

In accordance with (3) we have

$$\{\delta^e_m\}_r = [\Psi^e]_r[\Psi^e]^T_r\{\delta^e_m\},$$

$$\{\delta^e_m\}_d = [\Psi^e]_d[\Psi^e]^T_d\{\delta^e_m\}. \tag{16}$$

Note that the separation of deformable- and rigid-body movements does not require the calculation of all eigenvectors of the stiffness matrix for each component. We need to calculate only the g_e associated to the zero eigenvalue. The rigid-body movements of an isolated component will be at most six in the three-dimensional case (three translations and three rotations) and three in the two-dimensional case (two translations and one rotation). The vector $\{\delta^e_m\}_r$ is thus derived by means of the first of the equations in (16), which, substituted into (3), provides us with the deformable-solid components $\{\delta^e_m\}_d$:

$$\{\delta^e_m\}_d = [[I] - [\Psi^e]_r[\Psi^e]^T_r]\{\delta^e_m\}. \tag{17}$$

Our task now is to calculate the forces that have to be entered into the active links for the purpose of reducing the displacements by means of a correction vector of displacements $\{\tilde{\delta}^e_m\}$. The residual displacements will be

$$\{\delta^{e*}_m\} = \{\delta^e_m\}_d + \{\tilde{\delta}^e_m\}. \tag{18}$$

Since the number of active controls on a component is reduced, it will not in general be possible to fully compensate the deformations and one must conform with reducing them as much as possible. On the other hand, the forces acting on the active links of a given component are internal and do not result in any rigid-body displacement of the component. Thus the separation of the rigid and deformable displacements is not strictly necessary, nor in a component-by-component approach, but it can help in the selection of the worst positions of the components of a VGT. A component-by-component approach is sometimes recommended, for instance in the case of VGTs formed by a great number of complex components. Obviously, the compensation of deformations can be studied in terms of the entire VGT, with the actuators included as structural components. Such study does not entail any analysis of rigid and deformable displacements.

3. Compensation of small displacements of the precision points

The compensation of deformations can be studied with the aid of a continuous model or with that of a discrete finite element model, like in references [7,8,15–19]. In this paper we refer to the finite element case, in which VGTs are simple to model precisely. Here the relation between the field of corrective displacements and the forces acting on the active links can be written in the well-known form

$$\{\tilde{\delta}^e_m\} = [U^e]\{\tilde{f}^e_m\}. \tag{19}$$

Here $[U^e]$ is a rectangular matrix with u_e columns, being u_e the number of active links in the component e, and n_e rows; this matrix is very simple to evaluate column by column. If we apply a unit force in the active link j, the nodal displacements thus produced are precisely those expressed in the column j of that matrix, as is presented by Haftka et al. in Refs. [7,8,15–19]:

$$[k^e]\{U^e\}_j = \{v^e\}_j. \tag{20}$$

Here $\{v^e\}_j$ is the force vector of the unit force produced by the active link j. We can therefore write

$$\{\delta_m^{e*}\} = \{\delta_m^e\}_\mathrm{d} + [U^e]\{\tilde{f}_m^e\}. \tag{21}$$

A simple and natural measure of the error is the residual elastic deformation energy, where the stiffness matrix of the component is used as the weighting matrix

$$v_m^e = \tfrac{1}{2}\{\delta_m^{e*}\}^\mathrm{T}[k^e]\{\delta_m^{e*}\}. \tag{22}$$

The minimum verifies

$$\frac{\partial v_m^e}{\partial\{\tilde{f}_m^e\}} = [U^e]^\mathrm{T}[k^e](\{\delta_m^e\}_\mathrm{d} + [U^e]\{\tilde{f}_m^e\}) = \{0\}. \tag{23}$$

Which can be developed as

$$[U^e]^\mathrm{T}[k^e][U^e]\{\tilde{f}_m^e\} = -[U^e]^\mathrm{T}([k^e]\{\delta_m^e\}_\mathrm{d}). \tag{24}$$

In the case of the second member, we can, in accordance with the properties described above, make use of $\{\delta_m^e\}$.

The equation in (24) is a system of linear equations, similar to that described in Refs. [17,18], of the following form:

$$[A^e]\{\tilde{f}_m^e\} = \{b_m^e\}, \tag{25}$$

being

$$\begin{aligned}[A^e] &= [U^e]^\mathrm{T}[k^e][U^e],\\ \{b_m^e\} &= -[U^e]^\mathrm{T}[k^e]\{\delta_m^e\}_\mathrm{d}.\end{aligned} \tag{26}$$

Also it is easy to demonstrate that the following relation is verified:

$$\beta_m^e = \frac{v_m^e}{V_m^e} = 1 - \frac{\{\tilde{f}_m^e\}^\mathrm{T}\{b_m^e\}}{\{\tilde{f}_m^e\}^\mathrm{T}\{\delta_m^e\}_\mathrm{d}} = 1 - \frac{\{\tilde{f}_m^e\}^\mathrm{T}\{b_m^e\}}{\{\tilde{f}_m^e\}^\mathrm{T}\{\delta_m^e\}}. \tag{27}$$

This relation may serve as an error function in the discrete optimization process, object of which is to calculate suitable positions for the correction controls, e.g., by using Genetic Algorithms as was described in the Introduction. Note, however, that in the case of a VGT the configuration is variable (positions m), with the result that the question of optimum position for the active controls, and of the corrections entailed, depends on the position m and on the forces acting on the structure. Thus the most unfavorable situation must be considered for the analysis. The effect of the active links, which adapt to external forces, are tantamount to an adaptive tensional stiffening of the component (and of the entire structure).

An interesting case of correction arises when one wishes to control the position of a set of discrete points, or, better, of degrees of freedom, in place of the entire structure. Let us assume that n_c (being $n_c \ll n$) is the number of those degrees of freedom, whose displacements are grouped in a vector $\{\delta_m^*\}_c$:

$$\{\delta_m^*\}_c = [R]_c\{\delta_m^*\}. \tag{28}$$

Here $\{\delta_m^*\}$is the vector of the corrected displacements for the structure, $[R]_c$ is a rectangular matrix of n_c rows and n columns, made up of ones and zeros in such manner that in each row there is only one "1" at the position of the degree of freedom to be controlled. Now the error to be minimized will be of the following form:

$$v_m = \tfrac{1}{2}\{\delta_m^*\}_c^{\mathrm{T}}[\xi]_c\{\delta_m^*\}_c. \tag{29}$$

Here $[\xi]_c$ is a $n_c \times n_c$ diagonal matrix containing the weighting coefficients which ponderate the importance of certain degrees of freedom with respect to the others. It becomes the unit matrix if one has the same interest in the position of the n_c points. Substituting (28) in (29) results in

$$v_m = \tfrac{1}{2}\{\delta_m^*\}^{\mathrm{T}}[R]_c^{\mathrm{T}}[\xi]_c[R]_c\{\delta_m^*\} = \{\delta_m^*\}^{\mathrm{T}}[\Xi]_c\{\delta_m^*\}. \tag{30}$$

The matrix $[\Xi]_c$ is diagonal, of dimensions $n \times n$, in which every element is zero except the ones corresponding to a position on the diagonal of each degree of freedom j to be controlled, in which case the value is that of the corresponding weighting value ξ_j. The minimization of (30) leads to a system of equations similar to the one in (25), whose solution are the forces to be done by the active links. Depending on the number and positions of the active controls and precision points, it is possible to reduce to zero the error in the position of all or some of the precision points. Also, if the noncorrected displacements $\{\delta_m\}$ are considered, then an equation analogous to (30) is obtained for the error V_m. The relation between v_m and V_m gives a value β_m which has now the same utility as the one in (27).

4. Models for the kinematic analyses of VGTs

The models for kinematic analyses, linear and nonlinear, can be sometimes simpler than the ones used in the problems described above, for instance in simple open-loop structures like the one in Fig. 2. The number of degrees of freedom of the finite element models can be now very low, and a convenient choice of elements, nodes (reference points) and constraints leads to a significant reduction in the computational cost of the analyses. Velocities and accelerations, as well as the position problems can be focused from the same reduced finite element models.

The g input parameters known, taken as generalized coordinates of the holonomous system, the direct-position problem is that of determining the position of the whole structure. It can be reduced to an initial-position problem [1,37,41–44] or to a succesive position problem if a sequence of values for those parameters is known. The inverse problem is that of determining the values that these g parameters should have when it is desired that certain points and/or elements in the system occupy the given positions. If the number p of position data is the same as the number g of

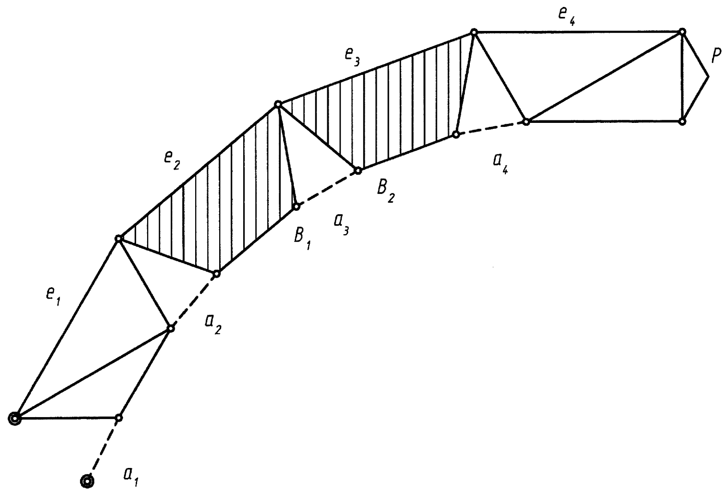


Fig. 3. Model for the system of Fig. 2, consisting of 15 nodes, 30 degrees of freedom, 16 finite rod elements (including four of the active links), and two quadrilateral elements.

generalized coordinates, then the problem reduces again to one of initial position, but if g is greater than p then there are $(g - p)$ redundant Grübler degrees of freedom, and there is no unique solution. To achieve a solution in this case it will be necessary to have an error function, the purpose being to minimize it by means of optimization techniques ([1,37,45]). The problem is therefore more complex than the direct one.

The structure as a whole (in order to simplify the notation, the nodal coordinates in the position m will be considered stored in the vector $\{x\}$) may consist of several components (Fig. 2), each of them comprising a high number of rods. In direct- and inverse-position problems it is assumed that these rods and components are rigid, for which reason it is possible to use a simplified finite element model of each component e. This small model must be able to guarantee that the shape of the components will not vary (Fig. 3). To construct the simplified models of components, one can use finite elements of rod type or other 2D and 3D finite elements, e.g., beams, triangles, and quadrilaterals in two-dimensional models and the many finite elements available for three-dimensional cases (Fig. 3). In the present paper, components will be constructed with rod-type elements in the interests of simplicity [37,41–44].This study can, however, be extended to the case of other finite elements. The reference points are defined in such a manner that they are able to define the shape of the components. These points are taken as the nodes of the reduced finite element model. Once the position problem has been solved on the reduced model, defined by the reference points, the position of the rods which form each component (Fig. 2) can be obtained with a simple transformation of coordinates. These same reference points are also used in the solution of the initial position problem of the components and in the velocity and acceleration analyses (see Refs. [37,42–44]).

The direct problem is simply an initial-position problem in which each actuator has a defined length in the case of translation actuators, or a defined angle in the case of rotation actuators, and hence it can be represented by rod elements of known length defined as the distance between two points (e.g., B1 and B2 in Fig. 3). In the case of the inverse problem, which in general does not have a unique solution, the model is similar to that of the direct problem, but the elements representing the active links (a1–a4 in Fig. 3) are now deformable, e.g., of spring type. Giving different values to the stiffness of those springs, one can simulate operating conditions in which one or another group of actuators suffers greater or smaller displacements [37].

In the kinematic calculation of velocities and accelerations one may use a model similar to any of the foregoing but where it is not necessary to consider elements representing the actuators. In this case the stiffness matrix of the model is singular owing to the possibility of rigid-body movements, but when velocity and acceleration data are fed as kinematic constraints, that singularity vanishes and the problem has a unique solution [37,41–44].

It is well known that the stiffness matrix of a mechanical system modeled by means of finite elements contains all the information necessary for one to predict its linear elastic behavior in reaction to the external forces, but in addition it contains all the kinematic properties of the system. On most occasions the problems analyzed in engineering relate to isostatic or hyperstatic structures, so that kinematic properties reduce simply to the impossibility of rigid-body displacements. The stiffness matrix in these cases is positive definite and does not have zero eigenvalues. On the other hand, in the case of mechanisms there will exist at least one rigid-body movement, and in those mechanisms with open or mixed kinematic chains, that number will be greater, as happens with the adaptive structures described here. In these cases the number of zero eigenvalues belonging to the stiffness matrix is equal to the number of linearly independent rigid-body movements. This latter number is equal to the dimension of the null-space of the stiffness matrix. The eigenvectors associated with these zero eigenvalues constitute a base of the null-space and are those that define any possible rigid-body movement.

A rod element i in two dimensions has a stiffness matrix of the form

$$[k]_i = k_i [g]_i, \tag{31}$$

where k_i depends on the elastic modulus of the rod material, on the area, and on the length, while $[g]_i$ depends only on the position angle of the rod which is dependent of the position m of the structure. We refer to this matrix as the geometric matrix of the element i (see Refs. [37,41–44]). In order to simplify the notation, the subindex m of the position being considered is omitted here.

The kinematic properties of a multibody system depend only on its geometry and on its kinematic pairs, hence they are independent of the stiffness of the elements. It follows that if all stiffness values k_i are made equal to unity, then the assembly of the element matrices (which when expanded to the dimension of the full model in position m we refer to as $[\bar{g}]_i$) provides as follows the geometric matrix $[G]$ of the mechanism comprising b rods:

$$[G] = \sum_{i=1}^{b} [\bar{g}]_i. \tag{32}$$

It is now very simple to demonstrate that the null-space of this geometric matrix $[G]$ is equal to that of the stiffness matrix of the reduced finite element model of the system,$[\hat{K}]$, and also of the null-space of $[K_m]$, [37,41–43].

The elastic energy stored in the reduced model with nodal displacements $\{\delta\}$ is

$$\hat{V} = \tfrac{1}{2}\{\delta\}^{\mathrm{T}}[\hat{K}]\{\delta\}. \tag{33}$$

In the case where this displacement is a rigid-body one, $\{\delta\}_r$, the elastic energy is zero. So also the elastic forces:

$$[G]\{\delta\}_r = \{0\}. \tag{34}$$

Now, the linearity hypothesis calls for small displacements. If we divide (34) by a time increment tending to zero, we see that

$$[G]\{\dot{x}\} = \{0\} \tag{35}$$

is satisfied. Here $\{\dot{x}\}$ is the vector containing the velocities of the model's nodes whose coordinates are stored in the vector $\{x\}$ of the full structure (see the difference with $\{x^e\}$).

To solve the homogeneous system in (35) it is necessary to know the input velocity data into the actuators, as many as the number of rigid-body degrees of freedom. They are introduced as displacement constraints between points in the model [37,43] (i.e., points B_1 and B_2 in Fig. 3). If we differentiate (35) with respect to time we get the following equation for accelerations:

$$[G]\{\ddot{x}\} = -[\dot{G}]\{\dot{x}\}. \tag{36}$$

If we input the known accelerations into the actuators, once the velocities have been calculated, this system of linear equations allows us to derive the nodal acceleration vector. The geometric matrix can be differentiated with respect to time on the basis of the derivatives of the matrices of the elements with respect to their position parameter. For example, in Refs. [37,43] we find this matrix for a two dimensional rod element.

The matrix $[\dot{G}]$ is constructed by means of the sum of the derivatives of the expanded element matrices:

$$[\dot{G}] = \sum_{i=1}^{b} [\dot{\bar{g}}]_i = \sum_{i=1}^{b} \frac{d[\bar{g}]_i}{d\theta_i}\,\dot{\theta}_i, \tag{37}$$

here θ_i is the position angle and $\dot{\theta}_i$ is the angular velocity of rod i, which is easily obtained from the results of the foregoing velocity analysis.

In order to resolve the inverse position problem we shall use as error function the potential function φ of the mechanical system, in the same manner as is presented in Refs. [37,41–44], using a model in which the actuators are replaced by spring elastic elements of known stiffness, as described above:

$$\varphi(\{x\}) = \frac{1}{2}\sum_{i=1}^{r} s_i(d_i(\{x\}) - D_i)^2. \tag{38}$$

Here r is the number of spring elastic elements in the model, elements representing the active links and actuators included; d_i is the length of the elastic element i in any iteration, corresponding to the vector of nodal coordinates $\{x\}$; D_i is the nominal length (without deformation) of this element; and s_i is its stiffness, assumed to be constant.

At the same time we must require that there be no variation in the length of the rigid rods (i.e., all of them except the ones which represent the actuators), i.e., that the following constraints be satisfied:

$$c_i(\{x\}) = l_i(\{x\}) - L_i = 0, \quad i = 1, \ldots, b. \tag{39}$$

Here b is the number of rigid rods, l_i is the length of the rod i in any iteration, when the vector of nodal coordinates is $\{x\}$; and L_i is its undeformed length. Note that not only the function is to be minimized but also that the constraints are nonlinear, since d_i, or l_i, is derived as square root of the sum of the squares of the differences between nodal coordinates. Since in a solution position the rigid components must have no deformation (39), to all rods forming them we can assign a stiffness value equal to unity, while those of the springs representing active links will have different values s_i. It is possible also to have more constraints, as for instance maximum and minimum lenghts of the actuators. These other constraints can be easily included in a similar way as is presented for the ones in (39).

This function can be used also with the direct problem, in which case it is necessary only to modify the significance of the variables. For example, in this case there are no constraints like the ones in (39). Since in any of the possible solution positions each element must have its length free of deformation (since the displacements of the actuators are known), stiffness for each can be assumed equal to unity. Also, in the solution positions, φ becomes zero (null deformation for all elements), and the problem proves easier to solve than the inverse. It can be efficiently solved by means of the Newton–Raphson method [37,41–44,46]. Hence the error function (38), along with the constraints (39), are useful in all position problems.

5. Direct position problem

Consider first the simplest case, in which (38) can be written in the following form:

$$\varphi(\{x\}) = \frac{1}{2}\sum_{i=1}^{b} (l_i(\{x\}) - L_i)^2. \tag{40}$$

Here l has been used to signify that each one of the b elements now behaves in such manner that in the solution position its deformation must be zero.

When we study the relationships between deformations in rod-type elements, these assumed to be elastically deformable and of unity stiffness, we see that if an element i of length L_i suffers a deformation δL_i then there appear elastic forces in the nodes in the direction of the rod [37,41–43]. In a 2D model (x, y coordinates) θ_i is the angle that the rod i forms with the x axis:

$$\{a_2\}_i = -\{a_1\}_i = \delta L_i \begin{Bmatrix} \cos\theta_i \\ \sin\theta_i \end{Bmatrix}. \tag{41}$$

The force vector or action vector at the rod will be

$$\{a\}_i = \begin{Bmatrix} \{a_1\} \\ \{a_2\} \end{Bmatrix}_i = \delta L_i \begin{Bmatrix} -\cos\theta_i \\ -\sin\theta_i \\ \cos\theta_i \\ \sin\theta_i \end{Bmatrix} = \delta L_i \{h\}_i. \tag{42}$$

The vector $\{h\}_i$ has certain interesting properties, for instance:

$$[\bar{g}]_i\{x\} = L_i\{\bar{h}\}_i. \tag{43}$$

Here $\{x\}$ is the coordinate vector of the nodes of the finite element model of dimension N ($2n$ in 2D models and $3n$ in 3D cases). It can also be shown [37,41–43] that the following holds:

$$[\bar{g}]_i = \{\bar{h}\}_i\{\bar{h}\}_i^{\mathrm{T}},$$

$$L_i = \{x\}^{\mathrm{T}}\{\bar{h}\}_i. \tag{44}$$

In view of (41)–(43) we therefore have

$$\{\bar{a}\}_i = \frac{\delta L_i}{L_i}[\bar{g}]_i\{x\}. \tag{45}$$

If we assemble the rod action vectors we get the complete action vector for the system

$$\{A\} = \sum_{i=1}^{b}\{\bar{a}\}_i = \sum_{i=1}^{b}\frac{\delta L_i}{L_i}[\bar{g}]_i\{x\}, \tag{46}$$

from (43) we get

$$\{x\}^{\mathrm{T}}[\bar{g}]_i\{x\} = L_i^2. \tag{47}$$

In Refs. [37,41–43] it is shown that the following holds:

$$\{x\}^{\mathrm{T}}\frac{\partial[\bar{g}]_i}{\partial\{x\}}\{x\} = 0. \tag{48}$$

To look for the minimum we use Newton's second-order method, for which we expand φ in Taylor series around a position $\{x\}_q$, which is the qth approximation to a minimum, and truncation is made by quadratic terms

$$\varphi(\{x\}) \approx \varphi_q(\{x\}) = \varphi(\{x\}_q) + \{\nabla\varphi\}_q^{\mathrm{T}}\{\delta\} + \tfrac{1}{2}\{\delta\}^{\mathrm{T}}[H]_q\{\delta\}. \tag{49}$$

Here $\{\nabla\varphi\}_q$ is the gradient at $\{x\}_q$, $[H]_q$ is the Hessian matrix at the same point, and $\{\delta\}$ is the nodal displacements vector. If we differentiate (49) with respect to $\{x\}$ and equate to the null vector, we get the minimum of φ_q, which is an approximation to the minimum of φ:

$$[H]_q\{\delta\} = -\{\nabla\varphi\}_q. \tag{50}$$

Now, if we resolve the system of equations in (50) we get the vector $\{\delta\}$, which allows us to determine the position corresponding to the following iteration:

$$\{x\}_{q+1} = \{x\}_q + \alpha\{\delta\}. \tag{51}$$

Here α is a correction factor that can in turn be analyzed for purposes of obtaining its optimum value, as is done, for example, in Refs. [44,46]. However, in the direct position problem in which the φ minima take the value zero, good convergence is achieved if that factor is made equal to unity. Once the new position has been determined, the process repeats until convergence is achieved.

We now consider how to derive the gradient vector and the Hessian matrix.

$$\{\nabla\varphi(\{x\})\} = \frac{\partial\varphi(\{x\})}{\partial\{x\}} = \sum_{i=1}^{b} (l_i(\{x\}) - L_i)\frac{\partial l_i(\{x\})}{\partial\{x\}}. \tag{52}$$

This equation can be expanded in the form

$$\{\nabla\varphi(\{x\})\} = \sum_{i=1}^{b} (l_i(\{x\}) - L_i)\frac{[\bar{g}]_i\{x\}}{l_i} = \sum_{i=1}^{b} \delta L_i \frac{[\bar{g}]_i\{x\}}{l_i}. \tag{53}$$

In view of (45), this can be written as follows:

$$\{\nabla\varphi(\{x\})\} = \sum_{i=1}^{b} \{\bar{a}\}_i = \{A\}, \tag{54}$$

where vector $\{A\}$ is concerned, it is necessary only to know the length increment for each link and multiply it by the vector $\{h\}_i$. If the expanded rod vectors are assembled, the result is vector $\{A\}$.

If we again differentiate with respect to $\{x\}$:

$$[H] = \frac{\partial^2\varphi}{\partial\{x\}^2} = \sum_{i=1}^{b}\left(\left\{\frac{\partial l_i}{\partial\{x\}}\right\}\left\{\frac{\partial l_i}{\partial\{x\}}\right\}^{\mathrm{T}} + \delta L_i \frac{\partial^2 l_i}{\partial\{x\}^2}\right). \tag{55}$$

Now if we differentiate in (43) and substitute into (55):

$$[H] = \sum_{i=1}^{b}\left([g]_i + \delta L_i \frac{\partial^2 l_i}{\partial\{x\}^2}\right). \tag{56}$$

Also it can be demonstrated [37,43] that

$$\frac{\partial^2 l_i}{\partial\{x\}^2} = l_i^{-1}([\bar{t}]_i - [\bar{g}]_i). \tag{57}$$

Here $[t]$ is a complementary matrix whose expression according to [37,43] is the following:

$$[t] = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix} \text{2D models,}$$

$$[t] = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \end{bmatrix} \text{3D models.} \tag{58}$$

From this we get

$$[H] = \sum_{i=1}^{b} l_i^{-1}(L_i[\bar{g}]_i + \delta L_i[\bar{t}]_i). \tag{59}$$

Note that the subindex i occurring in the complementary matrix $[t]$ indicates that this matrix should be expanded according to the degrees of freedom of the nodes of the element i. One can also see how, in the case of small displacements, the Hessian matrix becomes simply the geometric matrix $[G]$.

The approach for the direct position problem is very similar to that described in Ref. [43], but the inverse postion problem is focused now using Lagrange Multipliers.

6. Inverse position problem

The Lagrange multiplier method is very well suited for resolving problems of this type [37]. It is based on an expanded error function Λ, sum of the original error function (38) and the product of each of the constraints expressed in (39) by a set of new auxiliary variables, which are the Lagrange multipliers λ_i:

$$\Lambda(\{x\},\{\lambda\}) = \frac{1}{2}\sum_{i=1}^{r} s_i(d_i(\{x\}) - D_i)^2 + \sum_{i=1}^{b} \lambda_i(l_i(\{x\}) - L_i). \tag{60}$$

It is shown that the solution position must satisfy the following:

$$\{\nabla_x \Lambda\} = \left\{\frac{\partial \Lambda}{\partial \{x\}}\right\} = \{0\}. \tag{61}$$

Here we have

$$\left\{\frac{\partial \Lambda}{\partial \{x\}}\right\} = \sum_{i=1}^{r} s_i(d_i(\{x\}) - D_i)\left\{\frac{\partial d_i(\{x\})}{\partial \{x\}}\right\} + \sum_{i=1}^{b} \lambda_i\left\{\frac{\partial l_i(\{x\})}{\partial \{x\}}\right\}. \tag{62}$$

In view of (44) we therefore have

$$\left\{\frac{\partial \Lambda}{\partial \{x\}}\right\} = \sum_{i=1}^{r} s_i(d_i - D_i)\{\bar{h}\}_i + \sum_{i=1}^{b} \lambda_i\{\bar{h}\}_i. \tag{63}$$

If n is the number of nodes of the finite element model, (63) represents a system of N equations plus the b Lagrange multipliers. Hence b additional constraint equations are needed so that (63) may be solved. The problem thus reduces to the system of $(N + b)$ equations, with the same number of unknowns:

$$\left\{\begin{matrix} \left\{\dfrac{\partial \Lambda(\{x\},\{\lambda\})}{\partial \{x\}}\right\} \\ \{c(\{x\})\} \end{matrix}\right\} = \{0\}. \tag{64}$$

Here we have

$$\{c(\{x\})\} = \begin{Bmatrix} c_1(\{x\}) \\ c_2(\{x\}) \\ \vdots \\ c_b(\{x\}) \end{Bmatrix}. \tag{65}$$

Since the equations in (63) are nonlinear, the Newton–Raphson method will once again be used. We begin with

$$\{q\} = \begin{Bmatrix} \{x\} \\ \{\lambda\} \end{Bmatrix}. \tag{66}$$

Accordingly,

$$\left\{\frac{\partial \Lambda}{\partial \{x\}}\right\}_{k+1} \approx \left\{\frac{\partial \Lambda}{\partial \{x\}}\right\}_k + \left[\frac{\partial\{\partial \Lambda/\partial\{x\}\}}{\partial\{q\}}\right]_k (\{q\}_{k+1} - \{q\}_k) \tag{67}$$

and also

$$\{c\}_{k+1} \approx \{c\}_k + \left[\frac{\partial\{c\}}{\partial\{q\}}\right]_k (\{q\}_{k+1} - \{q\}_k). \tag{68}$$

Hence with each iteration k the following linear ones approach the nonlinear equations:

$$\begin{bmatrix} \left[\dfrac{\partial\{\partial \Lambda/\partial\{x\}}{\partial\{q\}}\right] \\ \left[\dfrac{\partial\{c\}}{\partial\{q\}}\right] \end{bmatrix}_k \left(\begin{Bmatrix} \{x\} \\ \{\lambda\} \end{Bmatrix}_{k+1} - \begin{Bmatrix} \{x\} \\ \{\lambda\} \end{Bmatrix}_k\right) = \begin{Bmatrix} -\left\{\dfrac{\partial \Lambda}{\partial\{x\}}\right\} \\ -\{c\} \end{Bmatrix}_k. \tag{69}$$

In greater detail, the matrix for this system, say "[B]", is as follows:

$$[B] = \begin{bmatrix} \left[\dfrac{\partial\{\partial \Lambda/\partial\{x\}\}}{\partial\{q\}}\right] \\ \left[\dfrac{\partial\{c\}}{\partial\{q\}}\right] \end{bmatrix} = \begin{bmatrix} \left[\dfrac{\partial^2 \Lambda}{\partial\{x\}^2}\right]_{(N\times N)} & \left[\dfrac{\partial^2 \Lambda}{\partial\{x\}\partial\{\lambda\}}\right]_{(N\times b)} \\ \left[\dfrac{\partial\{c\}}{\partial\{x\}}\right]_{(b\times N)} & \left[\dfrac{\partial\{c\}}{\partial\{\lambda\}}\right]_{(b\times b)} \end{bmatrix}. \tag{70}$$

We have in consequence:

$$\left[\frac{\partial\{c\}}{\partial\{x\}}\right] = \left[\frac{\partial\{l\}}{\partial\{x\}}\right] = \begin{bmatrix} \{\bar{h}\}_1^{\mathrm{T}} \\ \{\bar{h}\}_2^{\mathrm{T}} \\ \vdots \\ \{\bar{h}\}_b^{\mathrm{T}} \end{bmatrix}. \tag{71}$$

Since the terms c_i do not depend on $\{\lambda\}$, we also have

$$\left[\frac{\partial\{c\}}{\partial\{x\}}\right] = [0]_{(b\times b)}. \tag{72}$$

If we differentiate again we obtain

$$\left[\frac{\partial^2 \Lambda}{\partial\{x\}\partial\{\lambda\}}\right] = [\{\bar{h}\}_1\{\bar{h}\}_2 \cdots \{\bar{h}\}_b] = \left[\frac{\partial\{c\}}{\partial\{x\}}\right]^{\mathrm{T}}. \tag{73}$$

In other words $[B]$ is symmetric:

$$[B] = \begin{bmatrix} \left[\dfrac{\partial^2 \Lambda}{\partial\{x\}^2}\right]_{(N\times N)} & \left[\dfrac{\partial\{c\}}{\partial\{x\}}\right]^{\mathrm{T}}_{(N\times b)} \\ \left[\dfrac{\partial\{c\}}{\partial\{x\}}\right]_{(b\times N)} & [0]_{(b\times b)} \end{bmatrix}. \tag{74}$$

The following therefore holds:

$$\left[\frac{\partial^2 \Lambda}{\partial\{x\}^2}\right] = [\nabla_x^2 \Lambda] = \sum_{i=1}^{r} s_i\left(\left\{\frac{\partial d_i}{\partial\{x\}}\right\}\{\bar{h}\}_i^{\mathrm{T}} + \delta D_i\left[\frac{\partial\{\bar{h}\}_i}{\partial\{x\}}\right]\right) + \sum_{i=1}^{b} \lambda_i\left[\frac{\partial\{\bar{h}\}_i}{\partial\{x\}}\right]. \tag{75}$$

If we operate on this, we derive

$$\left[\frac{\partial^2 \Lambda}{\partial\{x\}^2}\right] = \sum_{i=1}^{r} s_i\left(\{\bar{h}\}_i\{\bar{h}\}_i^{\mathrm{T}} + \delta D_i\left[\frac{\partial^2 d_i}{\partial\{x\}^2}\right]\right) + \sum_{i=1}^{b} \lambda_i\left[\frac{\partial^2 d_i}{\partial\{x\}^2}\right]. \tag{76}$$

Hence the Hessian matrix will take the following form:

$$\left[\frac{\partial^2 \Lambda}{\partial\{x\}^2}\right] = \sum_{i=1}^{r} \frac{s_i}{d_i}(D_i[\bar{g}]_i + \delta D_i[\bar{t}]_i) + \sum_{i=1}^{b} \frac{\lambda_i}{l_i}([\bar{t}]_i - [\bar{g}]_i). \tag{77}$$

For an iteration k, in short, it will be possible to solve the system of linear equations and thus obtain the value of:

$$\{\delta q\}_k = \begin{Bmatrix}\{x\}\\ \{\lambda\}\end{Bmatrix}_{k+1} - \begin{Bmatrix}\{x\}\\ \{\lambda\}\end{Bmatrix}_{k} \Rightarrow \{q\}_{k+1} = \begin{Bmatrix}\{x\}\\ \{\lambda\}\end{Bmatrix}_{k+1} = \begin{Bmatrix}\{x\}\\ \{\lambda\}\end{Bmatrix}_{k} + \{\delta q\}_k. \tag{78}$$

We thus have the new value of $\{q\}$ for the next iteration.

To resolve this problem we can also use the augmented lagrange multiplier (ALM) method, for example as is described in [37]. Since displacements from one point to another in a VGT are not very large, however, the ALM technique does not in general offer advantages in the solving of problems of this particular type.

Three convergence criteria can be used to determine whether a solution position of the position problem has been achieved, ε_i representing in each case a sufficiently small value

$$|\lambda_{i(k)} - \lambda_{i(k-1)}| < \varepsilon_1,$$

$$\sum_{i=1}^{b} (l_i(\{x\}_k) - L_i)^2 < \varepsilon_2,$$

$$|\varphi(\{x\}_k) - \varphi(\{x\}_{k-1})| < \varepsilon_3. \tag{79}$$

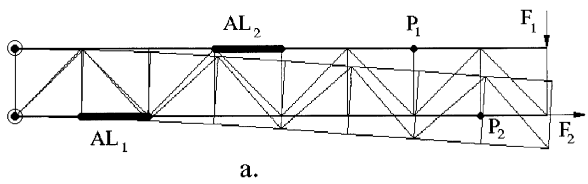
In the inverse position problem it will be required that these three criteria be simultaneously satisfied. In the case of the direct position, it is only necesary to verify the third condition (see also Refs. [37,41–44])

7. Examples

Fig. 4 shows a structure (0.1 m high, 0.8 m long) with two active links (AL_1, AL_2), 33 steel rods with a section of 10^{-5} m^2. The precision points are P_1 and P_2, and the forces acting on the structure are F_1 (10^4 N) and F_2 (5.10^4 N). Fig. 4a presents the structure together with the deformed shape when the rods AL_1 and AL_2 are not active. Fig. 4b presents the situation when these rods are active. The displacements of points P_1 and P_2 are now about 8% of the ones shown in Fig. 4a.

The example of Fig. 5a is a linkage with 16 kinematic elements, 22 rotation pairs. This mechanism has been discretized in a finite element model using five CST triangles (linear triangles), 10 rods and 15 nodes. The mechanism has one rigid-body d.o.f. and the input velocity is the horizontal component of point A. Fig. 5b shows a system like the one of Fig. 2a, with four actuators, its finite element model has only four linear quadrilaterals, each one representing a component. Both systems are shown together with its velocity fields calculated usingEq. (35).

Fig. 6 shows a simple example consisting of two components in the shape of a triangle and two actuators. The object is to step from P_0 to P_1, to which end it is necessary to solve the inverse



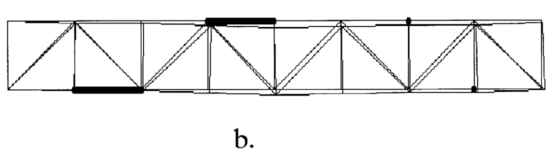


Fig. 4. Structure with two precision points and two active links.

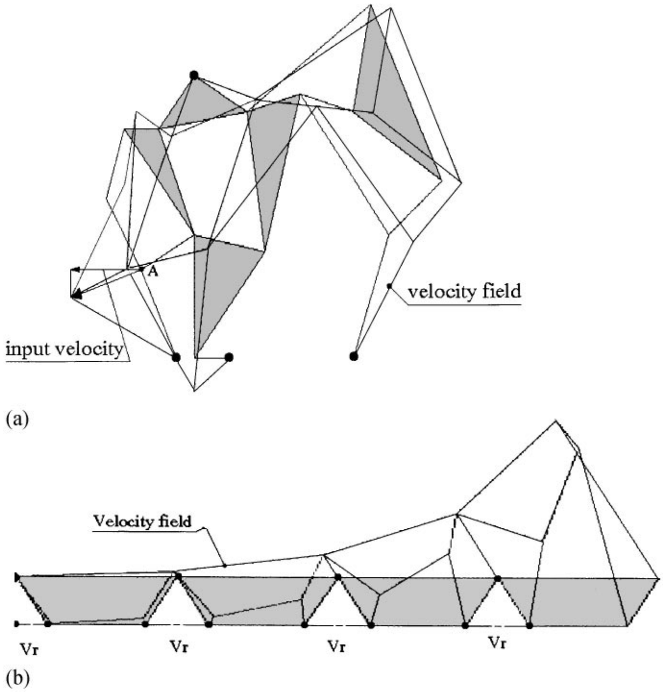


Fig. 5. Finite element models and its velocity fields. (a) Model of a complex closed-loop linkage with one rigid-body degree of freedom. (b) Model of an open-loop mechanism with four rigid-body degree of freedom.

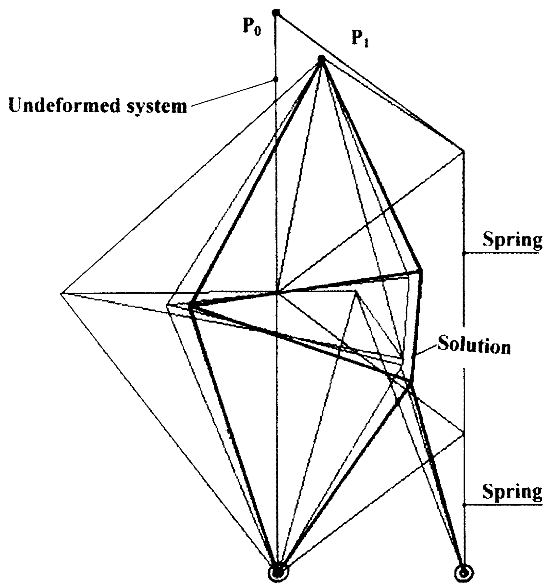
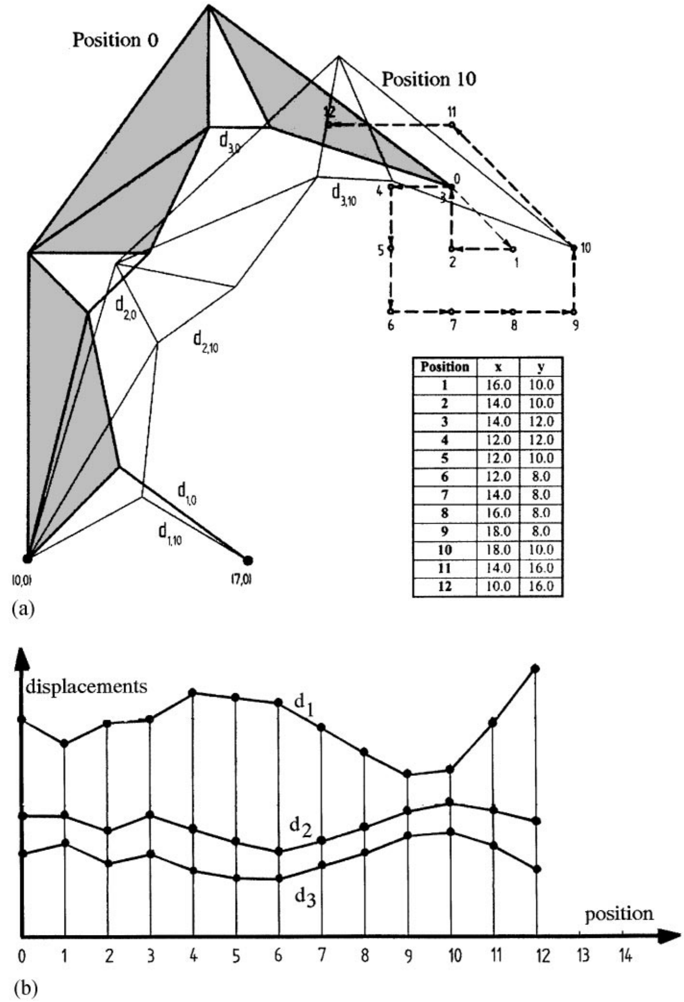


Fig. 6. System with two components and two actuators.



Position	x	y
1	16.0	10.0
2	14.0	10.0
3	14.0	12.0
4	12.0	12.0
5	12.0	10.0
6	12.0	8.0
7	14.0	8.0
8	16.0	8.0
9	18.0	8.0
10	18.0	10.0
11	14.0	16.0
12	10.0	16.0

Fig. 7. Planar system with three actuators and three components. (a) Model with 16 rods, 10 nodes and 20 d.o.f.; (b) Movements of the actuators through 12 points.

position problem, replacing the actuators by springs, in this case of the same stiffness. Because the system has two rigid-body degrees of freedom, the problem is in effect that of the initial position. One of the two possible solutions is achieved after four iterations.

Fig. 7 displays another planar system, also with three actuators and three components. Here the aim is to have point P move through 12 positions within the area to which it has access. When the inverse position problem has been solved for each point, one has the appropriate sequence of movements in the actuators, as indicated in Fig. 7b (a total of 45 iterations).

The convergence is defined using the usual criteria (function error and multipliers) as for instance the ones described in Refs. [37,43].

8. Conclusions

From the point of view of the theory of mechanisms and machines, VGTs can be seen as open-chain mechanisms with redundant degrees of freedom. To solve position problems both direct and inverse ones can use a reduced model in which each of the components is discretized by means of a small number of simple finite elements, e.g., of rod type. Both problems are tackled with the same error function, the difference lying in the constraints used, which refer to the dimensions of the components and of the actuators. The linear kinematic calculation of velocities and accelerations is easily performed according to a procedure based on the properties of the geometric matrix of the reduced model.

These structures have active links, which help in compensating the elastic deformations produced by the loads occurring during operation. The present paper has looked at a simple approach by means of which one can study each component independently, the aim being to correct, one by one, the elastic deformations by using its stiffness matrix as error weighting matrix. The error in this case is the residual elastic energy. If one wishes to correct the positions of only a few precision points in particular, an approach is facilitated that is based on the use of a very simple weighting matrix.

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