You should print out preliminary work separately from other parts.

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6 Preliminary Work

6.1 Assignment #1: Equations of Motion

A schematic of the Linear Flexible Joint Cart (LFJC-E) mounted on an IP02 linear-cart-and track system, excluding the motor and gearbox models is represented in Figure 2, below. The model of the motor and gearbox components has been derived in the preliminary work for Experiment #2. The LFJC-plus-IP02 system's nomenclature is provided in Appendix A. As illustrated in Figure 2, the positive direction of linear displacement is to the right when facing the cart.

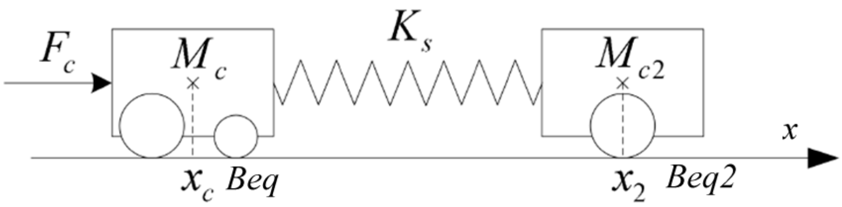


Figure 2 – Schematic of the SLFJ System

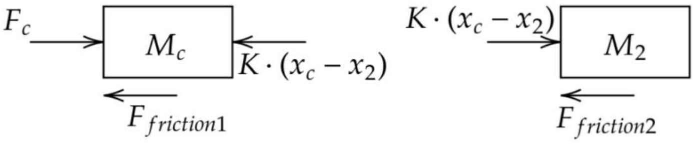
Using the terminology given in the appendix and Figure 2, derive the equations of motion for the system. The resulting EOM should have the following format:

$$\frac{\partial^2}{\partial t^2}x_c = \left(\frac{\partial^2}{\partial t^2}x_c\right)(x_c, x_2, F_c) \quad \text{and} \quad \frac{\partial^2}{\partial t^2}x_2 = \left(\frac{\partial^2}{\partial t^2}x_2\right)(x_c, x_2, F_c) \qquad [1]$$

Hint #1:
By neglecting the Coulomb (a.k.a. static) friction of the SLFJ system, the two EOM should be linear. They represent a pure spring-mass system with viscous friction causing the damping.

Hint #2:
You can use the method of your choice to model the system's dynamics. In EE302, we mostly used the free-body diagrams to model mechanical systems. An alternative is the energy based Lagrangian approach. In this case, since the system has two Degrees-Of-Freedom (DOF), there should be two Lagrangian coordinates (a.k.a. generalized coordinates). The chosen two coordinates are namely: x_c and x_2. Also, the input to the system is defined to be F_c, the linear force applied by the motorized cart.

Hint #3:
Ignore the rotational inertia of motor and simply consider the system as seen in Figure 2. The rotational inertia of motor will be considered while calculating the mass of the cart.



$$F_{net1} = M_c \cdot \ddot{x}_c = F_c - K_s \cdot (x_c - x_2) - \beta_{eq1} \cdot \dot{x}_c \quad \Rightarrow \ddot{x}_c = \frac{F_c}{M_c} - \frac{K_s(x_c - x2)}{M_c} - \frac{\beta_{eq1}\dot{x}_c}{M_c}$$

$$F_{net2} = M_2 \cdot \ddot{x}_2 = K_s(x_c - x_2) - \beta_{eq2} x_c \;\Rightarrow\; \ddot{x}_2 = \frac{K_s(x_c - x_2)}{M_2} - \frac{\beta_{eq2} x_c}{M_2}$$

$$\begin{bmatrix} \dot{X}_c \\ \dot{X}_2 \\ \ddot{X}_c \\ \ddot{X}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_s}{M_c} & \frac{K_s}{M_c} & -\frac{\beta_{eq1}}{M_c} & 0 \\ \frac{K_s}{M_2} & -\frac{K_s}{M_2} & 0 & -\frac{\beta_{eq2}}{M_2} \end{bmatrix} X + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_c} \\ 0 \end{bmatrix} U$$

6.2 Assignment #2: Linear State-Space Representation

In order to design and implement a LQ Regulator for our system, a linear state-space representation of that system needs to be derived. It is reminded that state-space matrices, by definition, represent a set of linear differential equations that describe the system's dynamics. Since the two EOM of the SLFJ (a pure spring-mass-damper system), as found in Assignment #1, should already be linear, they can be transformed into matrix notation.

The state vector of the mechanical system X is often chosen to include the generalized coordinates as well as their first-order time derivatives. Taking the state vector X as

$$X^T = \left[x_c(t), x_2(t), \frac{d}{dt} x_c(t), \frac{d}{dt} x_2(t) \right] \qquad [2]$$

1. Determine the linear state-space representation of the system. That is, from your EOM determined above, determine the **A** and **B** matrices in

$$\frac{\partial}{\partial t} X = AX + BU \qquad [3]$$

where we have the input to the mechanical system U initially given by $U = F_c$.

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_s}{M_c} & \frac{K_s}{M_c} & -\frac{\beta_{eq1}}{M_c} & 0 \\ \frac{K_s}{M_2} & -\frac{K_s}{M_2} & 0 & -\frac{\beta_{eq2}}{M_2} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{M_c} \\ 0 \end{bmatrix}$$

2. From the system's state-space representation previously found, transform the matrices A and B for the case where the system's input U is equal to the IP02 cart's DC motor voltage V_m, instead of the linear force F_c. The system's input U can now be expressed by $U = V_m$.

   Hint: In order to convert the previously found force equation state-space representation for the motor voltage input, it is reminded that the driving force, F_c, generated by the DC motor and acting on the cart through the motor pinion has already been determined in previous laboratories. You should be able to verify that F_c can be expressed by:

$$F_c(t) = -\frac{\eta_g K_g^2 \eta_m K_t K_m \left(\frac{d}{dt} x_c(t)\right)}{R_m r_{mp}^2} + \frac{\eta_g K_g \eta_m K_t V_m(t)}{R_m r_{mp}} \qquad [4]$$

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{K_s}{M_c} & \frac{K_s}{M_c} & -\frac{\beta_{eq1}}{M_c} - \frac{\eta_g K_g^2 \eta_m K_t K_m}{M_c R_m r_{mp}^2} & 0 \\ \frac{K_s}{M_2} & -\frac{K_s}{M_2} & 0 & -\frac{\beta_{eq2}}{M_2} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 \\ 0 \\ \frac{\eta_g K_g \eta_m K_t}{M_c R_m r_{mp}} \\ 0 \end{bmatrix}$$

3. Now, numerically evaluate the matrices A and B as found in Question 2, that is to say in case the system's input U is $U = V_m$.

   Hint1:
   Evaluate matrices A and B by using the model parameter values given in References [2] and [3]. The system configuration you are going to use in your in-lab session is composed of a LFJC-E module connected to an IP02 motorized cart.

Hint2:

Calculate masses of carts as follows:

$$M_c = M_{cart\_IP02} + M_{extra\_weight\_IP02} + \frac{\eta_g K_g^2 J_m}{r_{mp}^2} + M_{spring\_connecting\_piece} + \left(M_{spring}/2\right)$$

$$M_{c2} = M_{cart\_LFJ} + 2 * M_{extra\_weight\_LFJ} + M_{spring\_connecting\_piece} + \left(M_{spring}/2\right)$$

Hint3:

Beq of a IP02 cart with extra mass is not given in the user manual of IP02. Take Beq with extra load as 5.4.

$M_c =$ $1.1456\ kg$

$M_{c2} =$ $0.5425\ kg$

$$\mathbf{A} = \begin{bmatrix} 0.000 & 0.000 & 1.000 & 0.000 \\ 0.000 & 0.000 & 0.000 & 1.000 \\ -139.665 & 139.665 & -11.456 & 0.000 \\ 294.931 & -294.931 & 0.000 & -2.028 \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0.000 \\ 0.000 \\ 1.505 \\ 0.000 \end{bmatrix}$$

4. Calculate the open-loop poles from the system's state-space representation, as previously evaluated in Question 3. Is the system stable? Justify your answer.

$$\lambda_{1,2} = -2.347 \pm j20.275,\ \lambda_3 = 0 \text{ and } \lambda_4 = -8.790$$

The system is not stable since there exist a pole on the origin.

What is the type of the system? Justify your answer.

The system is type one, since there is only one pole laying on the origin (one integrator).

Considering the output matrices $C_1 = [1\ 0\ 0\ 0]$ and $C_2 = [0\ 0\ 1\ 0]$, comment on the system's observability.

$$\mathrm{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix}$$

is defined as the observability matrix. If it is full rank then the system is observable.

$$\text{for } C_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix},\ \mathrm{O}_1 \textit{ is rank } 4\ (\textit{full rank}) \Rightarrow \textit{Fully Observable}$$
$$\text{for } C_2 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix},\ \mathrm{O}_2 \textit{ is rank } 3\ (\textit{not full rank}) \Rightarrow \textit{Not Fully Observable}$$

What can you infer regarding the system's dynamic behavior? Do you see the need for a closed-loop controller? Explain.

When we look at the poles of the system, infact two of them have imaginary components thus the result will have decaying oscillations. Another pole is at the origin and it brings unstability to the system.

A controller is needed in order for the system to behave as intended.

6.3 *Assignment #3: Free Oscillation Differential Equation*

The theoretical results derived in this assignment will be used in your in-lab session to finely estimate your system parameter values K_s and B_{eq2} from your own experimental data. This process of estimating parameters of a model from experimentation is known as "System Identification". Answer the following questions:

1. Let us consider a linear (i.e., one-dimensional) spring-damper-mass system, of parameters K_s, B_{eq2}, and M_{c2}, respectively. Also let us call $x_2(t)$ the output displacement of the mass centre of gravity as a function of time. Determine the ordinary differential equation (ODE) describing the free-oscillatory motion of the mass.

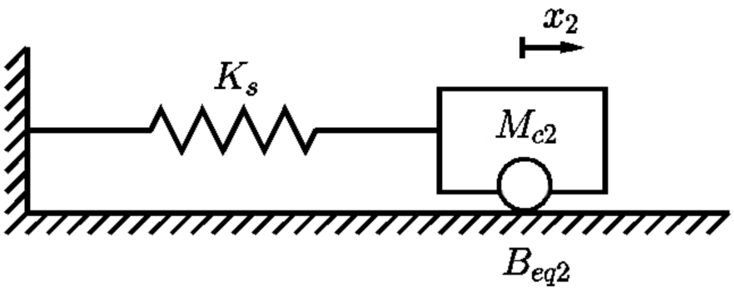


Figure 3 – Mass-Spring-Damper System corresponding to load-cart

$$\ddot{x}_2 = -\frac{\beta_{eq2}}{M_{c2}}\dot{x}_2 - \frac{K_s}{M_{c2}}x_2$$

2. As you probably found in Question 1, the ordinary differential equation (ODE) describing the free-oscillatory motion of the mass is a second-order differential equation. Show (by determining ζ and ω_n as a function of the given three system parameters K_s, B_{eq2}, and M_{c2}) that this ODE, leads to the following characteristic equation:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \qquad [5]$$

$$\ddot{x}_2 = -\frac{\beta_{eq2}}{M_{c2}}\dot{x}_2 - \frac{K_s}{M_{c2}}x_2$$

$$\Downarrow \ \textit{Laplace domain} \ \Downarrow$$

$$s^2 = -\frac{\beta_{eq2}}{M_{c2}}s - \frac{K_s}{M_{c2}}$$

Together with the final functions, also provide any relevant details in the box below.

$$s^2 + \frac{\beta_{eq2}}{M_{c2}}s + \frac{K_s}{M_{c2}} = 0, \quad 2\zeta\omega_n = \frac{\beta_{eq2}}{M_{c2}} \; and \; \omega_n^2 = \frac{K_s}{M_{c2}}$$

$$\boxed{\omega_n = \sqrt{\frac{K_s}{M_{c2}}}, \; \zeta = \frac{\beta_{eq2}}{2\sqrt{M_{c2}K_s}}}$$

Note:
Remember from your EE302 lectures that for an underdamped system ($\zeta < 1$), the roots of the quadratic Equation [4] are a complex conjugate pair given as

$$s_1 = -\zeta\,\omega_n + j\,\omega_n\sqrt{1-\zeta^2} \quad \text{and} \quad s_2 = -\zeta\,\omega_n - j\,\omega_n\sqrt{1-\zeta^2} \qquad [6]$$

3. Let us assume now that ω_n and ζ can be determined from experimental measurements. M_{c2} can also be easily measured experimentally (but will be given for this experiment). Therefore, determine from your findings in question 2 the two relationships giving K_s and B_{eq2} as functions of the experimentally measured system parameters ζ, ω_n, and M_{c2}.Also provide your steps in the box provided below.

$$\boxed{\begin{aligned} K_s &= M_{c2}\omega_n^2 \\ \beta_{eq2} &= 2\zeta\sqrt{M_{c2}K_s} = 2\zeta\omega_n M_{c2} \end{aligned}}$$

4. Now, we will proceed by defining how to determine ω_n and ζ from experimental measurements. Suppose now that the spring-mass-damper system depicted in Figure 3 is subjected to a force impulse input. This input can be emulated in the form of starting the system with a compressed spring (corresponding to a non-zero energy stored in the system). The effect of an impulse on such a system is similar, namely it instantaneously sets the system to non-zero initial conditions. Figure 4 illustrates the emulation of this

force impulse response by starting the system from a compressed spring state. The x-axis position of the load-cart is zero when the spring is at rest. This is also the steady-state value of the load-cart position response.

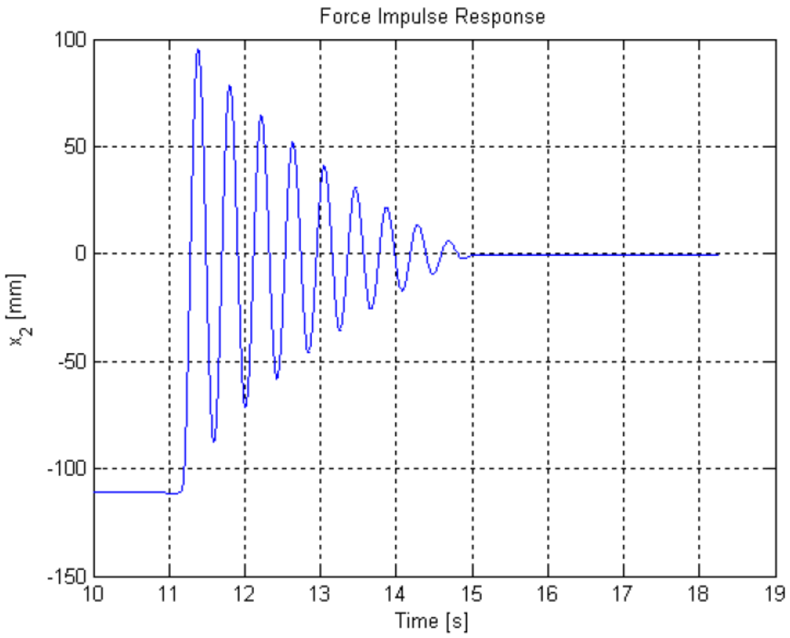


Figure 4 – Typical force impulse response of the system given in Figure 3

5. Let us start by estimating ζ. Based on measurements from the response curve given in Figure 4. In your EE302 lectures, review the section where you have computed the “peak-time” for a second order, under-damped system. There, the approach was to compute the derivative of the time response of the system and equate to zero. This would give you and expression for all the positive and negative peaks of the decaying sinusoidal response. In EE302, you have found that successive peaks (both positive and negative) occur at

$$\omega_d t = k\pi, \qquad [7]$$

where, $k = 1,2,\ldots$ Also remember that the “envelope” of the decaying sinusoid is given by an exponential function of the form

$$|x_2(t)| \leq A exp(-\zeta\omega_n t). \qquad [8]$$

From these results, together with the relation between ω_n and ω_d, compute, in closed form, the ratio of the amplitudes of the first two positive peaks (R_p) in the damped sinusoidal response for the system as a function of ζ. You will use this expression to compute ζ once you measure the magnitudes of these two peaks from the experimental response.

$$\omega_d t = k\pi,\ \ for\ t_1\ and\ t_2;$$
$$\omega_d t_1 = \pi$$
$$\omega_d t_2 = 3\pi$$

$$R_p = \frac{|x_2(t_1)|}{|x_2(t_2)|} = \frac{Ae^{-\zeta\omega_n \frac{\pi}{\omega_d}}}{Ae^{-\zeta\omega_n \frac{3\pi}{\omega_d}}} = e^{\zeta\omega_n \frac{2\pi}{\omega_d}} \Rightarrow \boxed{R_p = e^{\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}}}$$

$R_p =$ $\boxed{R_p = e^{\frac{2\zeta\pi}{\sqrt{1-\zeta^2}}}}$

Now, from your above result, derive the expression for ζ as a function of the ratio of the first two positive peaks R_p.

$$\frac{\log_e R_p}{\pi} = \frac{2\zeta\pi}{\pi\sqrt{1-\zeta^2}} = \frac{2\zeta}{\sqrt{1-\zeta^2}} \Rightarrow \boxed{\zeta = \frac{|\log_e R_p|}{\sqrt{4\pi^2 + \log_e^2 R_p}}}$$

7. At this point, you are ready to determine the damped natural frequency w_d when you can experimentally measure the oscillation period of the damped sinusoidal response. Additionally, knowing ζ, you can also determine the undamped natural frequency ω_n.

from $t_2 - t_1$ *we can find* ω_d *such that ;* $\quad \omega_d = \dfrac{2\pi}{t_2 - t_1}$ *using* R_p *such that;* $\quad \zeta = \dfrac{\lvert\log_e R_p\rvert}{\sqrt{4\pi^2 + \log_e^2 R_p}}$ *then,* $\omega_n = \dfrac{\omega_d}{\sqrt{1-\zeta^2}}$

8. Once you have the values for ζ and ω_n, you can proceed to determine the system parameters K_s and B_{eq2} hence completing the system identification problem. As part of this step, consolidate all your above results in the form of a Matlab function: `[Ks, Beq2] = Identify(p1, p2, Period)` where `p1` and `p2` are the amplitudes of the first two positive peaks in the experimental response and `Period` is the period of the damped sinusoidal signal. You will use this Matlab function during the actual experimental procedure. Save it into a USB drive and keep it with you at all times.