Cayley-Hamilton Theorem

Every square matrix satisfies its own characteristic equation.

$$\mathbf{A}^n + d_1\mathbf{A}^{n-1} + \cdots + d_{n-1}\mathbf{A} + d_n\mathbf{I} = 0$$

> Remark: Cayley-Hamilton Theorem basically states that nth and higher powers of a matrix can be expressed in terms of lower powers of the matrix. This is a very useful property in solving linear systems.

$A^n = -d_1A^{n-1} - \cdots - d_{n-1}A - d_nI$

For previous example, we have:

$A^2 - 4A + 3I = 0$

$A^2 = 4A - 3I$

$A^3 = 4A^2 - 3A = 4(4A - 3I) - 3A = 13A - 12I$

$A^4 = 4A^3 - 3A^2 = 4(13A - 12I) - 3(4A - 3I) = 49A - 48I$

$exp(s) = \sum_{n=0}^{\infty} \frac{s^n}{n!} = I + s + \frac{s^2}{2!} + \frac{s^3}{3!} + \cdots$

$exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{A^2}{2!} + \frac{A^3}{3!} + \cdots$

$A^{-1} =?$

$A^2 - 4A + 3I = 0$

$A - 4I + 3A^{-1} = 0$

$A^{-1} = \frac{(4I-A)}{3}$

Example: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ $A^{100} =?$

Solution:

$A^{100} = \alpha A + \beta I$

$\lambda_1 = 3, \lambda_2 = 1, \text{and } e_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, e_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$Ae_1 = \lambda_1 e_1 \Rightarrow A^2e_1 = \lambda_1^2 e_1 \Rightarrow A^3e_1 = \lambda_1^3 e_1 \Rightarrow \cdots \Rightarrow A^{100}e_1 = \lambda_1^{100}e_1$

$Ae_2 = \lambda_2 e_2 \Rightarrow A^2e_2 = \lambda_2^2 e_2 \Rightarrow A^3e_2 = \lambda_2^3 e_2 \Rightarrow \cdots \Rightarrow A^{100}e_2 = \lambda_2^{100}e_2$

$A^{100} = \alpha A + \beta I \Rightarrow A^{100}e_1 = \alpha Ae_1 + \beta e_1 \Rightarrow \lambda_1^{100}e_1 = \alpha\lambda_1 e_1 + \beta e_1 \Rightarrow \lambda_1^{100} = \alpha\lambda_1 + \beta \implies 3^{100} = 3\alpha + \beta$

$A^{100} = \alpha A + \beta I \Rightarrow A^{100}e_2 = \alpha Ae_2 + \beta e_2 \Rightarrow \lambda_2^{100}e_2 = \alpha\lambda_2 e_2 + \beta e_2 \Rightarrow \lambda_2^{100} = \alpha\lambda_2 + \beta \implies 1^{100} = \alpha + \beta$

$\begin{cases} 3^{100} = 3\alpha + \beta \\ 1 = \alpha + \beta \end{cases} \Rightarrow \begin{matrix} \alpha = \frac{3^{100}-1}{2} \\ \beta = \frac{-3^{100}+3}{2} \end{matrix}$

$A^{100} = \frac{3^{100}-1}{2}A + \frac{-3^{100}+3}{2}I$

Minimal Polynomial

Definition: The monic polynomial is the polynomial with the highest degree coefficient equal to 1.

$s^n + a_1 s^{n-1} + \cdots \rightarrow$ monic

$2s^n + a_1 s^{n-1} + \cdots \rightarrow$ not monic

Definition: The minimal polynomial of a matrix is the monic polynomial of the lowest degree that has the matrix as a root.

$$m(A) = 0_{n\times n}$$

Theorem: Given a matrix $A = \mathbb{C}^{n\ timesn}$, let m(s) be its minimal polynomial. Then, the minimal polynomial is the smallest degree polynomial that satisfies the following:

i) m(s) is unique.
ii) m(s) divides d(s) with no remainder.

$$\exists q(s) \text{ s.t. } d(s) = q(s)m(s)$$

iii) Every root of m(s) is a root of d(s).

Example: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{matrix} d(s){=}(s{-}1)(s{-}2)(s{-}3) \\ m(s){=}(s{-}1)(s{-}2)(s{-}3) \end{matrix}$

> Remark: When A has distinct eigenvalues (which implies diagonalizability but converse is not true), the minimal polynomial is the same as the characteristic polynomial.

Example: $A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} d(s){=}(s{-}1)(s{-}2)^2 \\ m(s){=}(s{-}1)(s{-}2) \end{matrix} \qquad A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} \begin{matrix} d(s){=}(s{-}1)(s{-}2)^2 \\ m(s){=}(s{-}1)(s{-}2)^2 \end{matrix}$

A way to check if a minimal polynomial is correct is to check if the characteristic polynomial is zero when the minimal polynomial is substituted for s.

Let $m_1(s) = (s-1)(s-2)$ then $m_1(A) = (A - 1I)(A - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$

Let $m_2(s) = (s-1)(s-2)$ then $m_2(A) = (A - 1I)(A - 2I) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$

Then, $m_2(s) = (s-1)(s-2)^2 \implies m_2(A) = (A - 1I)(A - 2I)^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \checkmark$

$\mathbb{R}^3 = N(A - 1I) \oplus N((A - 2I)^2)$ where,

$$N(A - 1I) = N\left(\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \ \dim\Big(N(A - 1I) \Big) = 1$$

$$N((A - 2I)^2) = N\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right) = \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \ \dim\Big(N((A - 2I)^2) \Big) = 2$$

> Remark: There is a relationship between decomposition form and minimal polynomial in terms of power.
>
> $$m(s) = (s - \lambda_1)^{d_1} \cdots (s - \lambda_k)^{d_k}$$
>
> $$\mathbb{R}^n = N(A - \lambda_1 I)^{d_1} \oplus \cdots \oplus N(A - \lambda_k I)^{d_k}$$

Theorem:

$$\mathbb{C}^n = N(A - \lambda_1 I)^{m_1} \oplus N(A - \lambda_2 I)^{m_2} \oplus \cdots \oplus N(A - \lambda_k I)^{m_k}$$

$$d(s) = (s-\lambda_1)^{r_1}\cdots(s-\lambda_k)^{r_k}$$

$$r_1 + r_2 + \cdots + r_k = n$$

$$m(s) = (s-\lambda_1)^{m_1}\cdots(s-\lambda_k)^{m_k}$$

$$1 \le m_i \le r_i$$

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & 0 & \cdots & 0 \\ 0 & \bar{A}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \bar{A}_k \end{bmatrix} \begin{array}{l} \bar{A}{=}B^{-1}AB, \text{B is composed of the basis vectors} \\ B{=}\begin{bmatrix} e_1 & e_2 & \cdots & e_n \end{bmatrix} \text{ for the } N(A{-}\lambda_i I)^{m_i} \end{array}$$

Size of $\bar{A}_i$ is $\dim\left(N(A-\lambda_i I)^{m_i}\right)$

Example: $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} d(s){=}(s{-}1)^3(s{-}2) \\ m(s){=}(s{-}1)^2(s{-}2) \end{array}$

Solution: Let Σ_1 be $A - \lambda_1 I$ and Σ_2 be $A - \lambda_2 I$.

$$\Sigma_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\left(N(\Sigma_1)\right) = 2 \neq 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1) = 4 - 2 = 2$$

Then we need to check the dimension of $N(\Sigma_1^2)$.

$$\Sigma_1^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\left(N(\Sigma_1^2)\right) = 1 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^2) = 4 - 1 = 3 = r_1$$

Lets briefly check the dimension of $N(\Sigma_1^3)$.

$$\Sigma_1^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\left(N(\Sigma_1^3)\right) = 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^3) = 4 - 1 = 3 = r_1$$

$$\vdots$$

$$\Sigma_1^4 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\left(N(\Sigma_1^4)\right) = 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^4) = 4 - 1 = 3 = r_1$$

Lets check the second eigenvalue.

$$\Sigma_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dim\left(N(\Sigma_2)\right) = 1 = r_2 \rightarrow \dim V - \dim R(\Sigma_2) = 4 - 3 = 1$$

Then the minimal polynomial is $m(s) = (s-1)^2(s-2)$.

Example: $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{array}{l} d(s){=}(s{-}1)^3(s{-}2) \\ m(s){=}(s{-}1)^3(s{-}2) \end{array}$

Solution: Let Σ_1 be $A - \lambda_1 I$ and Σ_2 be $A - \lambda_2 I$.

$$\Sigma_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\Big(N(\Sigma_1)\Big) = 1 \neq r_1 \rightarrow \dim V - \dim R(\Sigma_1) = 4 - 3 = 1$$

Then we need to check the dimension of $N(\Sigma_1^2)$.

$$\Sigma_1^2 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\Big(N(\Sigma_1^2)\Big) = 2 \neq r_1 \rightarrow \dim V - \dim R(\Sigma_1^2) = 4 - 2 = 2$$

Check the dimension of $N(\Sigma_1^3)$.

$$\Sigma_1^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \dim\Big(N(\Sigma_1^3)\Big) = 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^3) = 4 - 1 = 3$$

Now we can check the second eigenvalue.

$$\Sigma_2 = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \dim\Big(N(\Sigma_2)\Big) = 1 = r_2 \rightarrow \dim V - \dim R(\Sigma_2) = 4 - 3 = 1$$

Then the minimal polynomial is $m(s) = (s-1)^3(s-2)$.

<u>Example:</u> $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \begin{matrix} d(s) = (s-2)^7(s-3)^3 \\ m(s) = ? \end{matrix}$

<u>Solution:</u> For each jordan block, we need to check the dimension of $N(\Sigma_1^i)$.

The largest jordan block will have the largest dimension of $N(\Sigma_1^i)$. The rest of the jordan blocks will have dimension of $N(\Sigma_1^i)$ equal to the size of the jordan block. Hence the geometric multiplicity wont increase.

Let Σ_1 be $A - \lambda_1 I$ and Σ_2 be $A - \lambda_2 I$. Check the largest jordan block.

$$\Sigma_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \dim\Big(N(\Sigma_1)\Big) = 1 \neq r_1 \rightarrow \dim V - \dim R(\Sigma_1) = 3 - 2 = 1$$

Then we need to check the dimension of $N(\Sigma_1^2)$.

$$\Sigma_1^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dim\Big(N(\Sigma_1^2)\Big) = 2 \neq r_1 \rightarrow \dim V - \dim R(\Sigma_1^2) = 3 - 1 = 2$$

Check the dimension of $N(\Sigma_1^3)$.

$$\Sigma_1^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dim\Big(N(\Sigma_1^3)\Big) = 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^3) = 3 - 0 = 3$$

We need to continue to see that further powers of Σ_1 will not increase the dimension of $N(\Sigma_1^i)$.

$$\Sigma_1^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \dim\Big(N(\Sigma_1^4)\Big) = 3 = r_1 \rightarrow \dim V - \dim R(\Sigma_1^4) = 3 - 0 = 3$$

This implies that the further jordan blocks will not increase the dimension of $N(\Sigma_1^i)$. Hence they are not needed to be checked for minimal polynomial.

Now we can check the second eigenvalue.

$$\Sigma_2 = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} \dim\left(N(\Sigma_2)\right) = 1 \neq r_2 \rightarrow \dim V - \dim R(\Sigma_2) = 2 - 1 = 1$$

$$\Sigma_2^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \dim\left(N(\Sigma_2^2)\right) = 2 = r_2 \rightarrow \dim V - \dim R(\Sigma_2^2) = 2 - 0 = 2$$

Then the minimal polynomial is $m(s) = (s-2)^7(s-3)^2$.

> Remark:
>
> $$\dim\left(N(A-\lambda_i I)\right) = \#\text{ of jordan blocks corresponding to }\lambda_i\text{ with size }\geq 1$$
>
> $$\dim\left(N(A-\lambda_i I)^2\right) - \dim\left(N(A-\lambda_i I)\right) = \#\text{ of jordan blocks corresponding to }\lambda_i\text{ with size }\geq 2$$
>
> $$\dim\left(N(A-\lambda_i I)^3\right) - \dim\left(N(A-\lambda_i I)^2\right) = \#\text{ of jordan blocks corresponding to }\lambda_i\text{ with size }\geq 3$$
>
> $$\dim\left(N(A-\lambda_i I)^k\right) - \dim\left(N(A-\lambda_i I)^{k-1}\right) = \#\text{ of jordan blocks corresponding to }\lambda_i\text{ with size }\geq \text{k}$$

Example: $A \in \mathbb{R}^4$ Which has a single eigenvalue $\lambda_1 = 7$ and the geometric multiplicity is 2. Find all possible jordan forms.

Solution: The characteristic polynomial is $d(s) = (s-7)^4$ and the minimal polynomial is $m(s) = (s-7)^2$. This suggests that,

$$\dim\left(N(A-7I)^2\right) = 4 = r_1 \rightarrow \dim(V) = \dim\left(N(A-7I)^2\right)$$

Then for the $\dim\left(N(A-7I)\right)$ we have 3 possibilities, 1, 2, or 3.

a) Let $\dim N(A-7I) = 3$

$$\#\text{ of jordan blocks } = 3$$

$$\#\text{ of jordan blocks w/ size } \geq 2 = 4 - 3 = 1$$

$$\#\text{ of jordan blocks w/ size } \geq 3 = 3 - 3 = 0$$

$$A = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

a) Let $\dim N(A-7I) = 2$

$$\#\text{ of jordan blocks } = 2$$

$$\#\text{ of jordan blocks w/ size } \geq 2 = 4 - 2 = 2$$

$$\#\text{ of jordan blocks w/ size } \geq 3 = \dim\left(N(A-7I)^3\right) - \dim\left(N(A-7I)^2\right) = 4 - 4 = 0$$

$$A = \begin{bmatrix} 7 & 1 & 0 & 0 \\ 0 & 7 & 0 & 0 \\ 0 & 0 & 7 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}$$

a) Let $\dim N(A-7I) = 1$

$$\# \text{ of jordan blocks } = 1$$

$$\# \text{ of jordan blocks w/ size } \geq 2 = 4 - 1 = 3$$

However, this is not possible since there cannot be 3 jordan blocks with size greater than or equal to 2.

<u>Example:</u> Suppose a matrix $A \in \mathbb{R}^{8\times 8}$ has the following subspace dimensions:

$$\dim\left(N(A-3I)\right) = 5$$
$$\dim\left(N(A-3I)^2\right) = 7$$
$$\dim\left(N(A-3I)^3\right) = 8$$

a) Find the characteristic polynomial of A.

b) Find the minimal polynomial of A.

c) Find the possible Jordan forms of A.

<u>Solution:</u>

a) We know that the characteristic polynomial is $d(s) = (s-3)^8$.

b) We know that the minimal polynomial is $m(s) = (s-3)^3$ since the dimension of $N(A-3I)^3$ is same as the dimension of the vector space. This implies that the geometric multiplicity is 3.

c) Starting from the largest jordan block, having geometric multiplicity 3 suggest that the largest jordan block has size 3.

$$\dim\left(N(A-3I)^3\right) - \dim\left(N(A-3I)^2\right) = \# \text{ of jordan blocks corresponding to } \lambda_1 \text{ with size } \geq 3 = 8 - 7 = 1$$

$$\dim\left(N(A-3I)^2\right) - \dim\left(N(A-3I)\right) = \# \text{ of jordan blocks corresponding to } \lambda_1 \text{ with size } \geq 2 = 7 - 5 = 2$$

We have 2 that are greater than or equal to 2. This implies that we have 1 jordan block with size 3 and 1 jordan blocks with size 2.

$$\dim\left(N(A-3I)\right) = \# \text{ of jordan blocks corresponding to } \lambda_1 \text{ with size } \geq 1 = 5$$

This implies that we have 3 jordan blocks with size 1.

$$A = \begin{bmatrix} 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$
