Span and Linear Independence

Linear Combination: A `linear combination` of a set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is a vector of the form $c_1v_1 + c_2v_2 + \cdots + c_nv_n$, where $c_1, c_2, \ldots, c_n$ are scalars.

Span: Let $S = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space V. The set of all linear combinations of the vectors in S is called the `span` of S and is denoted by $Span(S)$.

Linear Independence: A set of vectors $S = \{v_1, v_2, \ldots, v_n\}$ in a vector space V is called `linearly independent` iff the only linear combination of the vectors in S that equals the zero vector is the trivial linear combination, that is, $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ implies $c_1 = c_2 = \cdots = c_n = 0$.

> Zero vector cannot be an element of an independent set.

Example: $X = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

- X is linearly independent set since $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0, \forall c_1, c_2 \in \mathbb{R}(F)$.
- $Span(X) = \mathbb{R}^2$

Example: Consider the linear space of polynomials with degree $n \leq 2$. Let subset $S = \{p_1, p_2, p_3\}$ where $p_1(t) = 1$, $p_2(t) = t$, $p_3(t) = t^2$. Is S linearly independent?
Proof: Let $a_1, a_2, a_3 \in \mathbb{R}$,

- $a_1p_1(t) + a_2p_2(t) + a_3p_3(t) = 0$
- $a_1 + a_2t + a_3t^2 = 0$
- $a_1 = a_2 = a_3 = 0$
  - Hence S is linearly independent.

Example: $S = \{cos(t), sin(t), cos(t - \pi/3)\}$
Proof: Let $a_1, a_2, a_3 \in \mathbb{R}$,

- $a_1cos(t) + a_2sin(t) + a_3cos(t - \pi/3) = 0$
- $a_1cos(t) + a_2sin(t) + a_3(cos(t)cos(\pi/3) + sin(t)sin(\pi/3)) = 0$
- $a_1cos(t) + a_2sin(t) + a_3(cos(t)1/2 + sin(t)\sqrt{3}/2) = 0$
- $a_1cos(t) + a_2sin(t) + a_3cos(t)/2 + a_3sin(t)\sqrt{3}/2 = 0$
- $a_1cos(t) + a_3cos(t)/2 + a_2sin(t) + a_3sin(t)\sqrt{3}/2 = 0$
- $(a_1 + a_3/2)cos(t) + (a_2 + a_3\sqrt{3}/2)sin(t) = 0$
- $a_1 + a_3/2 = 0$ and $a_2 + a_3\sqrt{3}/2 = 0$
- $a_1 = -a_3/2$ and $a_2 = -a_3\sqrt{3}/2$
  - Hence S is linearly dependent.

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