[Inner Product Spaces]

Definition: Let (V, F) be a vector space. An `inner product` is a map of the form $\langle \cdot, \cdot \rangle : V \times V \to F$ such that for all $x, y, z \in V$ and $c \in F$ the following axioms hold:

(P1) $\langle x, y \rangle = \overline{\langle y, x \rangle}$ (conjugate symmetry)

(P2) a. $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$ (linearity in the first argument) b. $\langle cx, y \rangle = c\langle x, y \rangle$ (homogeneity in the first argument)

(P3) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0_v$ (positive definiteness)

fill missing parts

Theorem: *"Cauchy-Schwarz Inequality"* Let (V, F) be a vector space with an inner product $\langle \cdot, \cdot \rangle$. Then for all $x, y \in V$ the following inequality holds:

$$|\langle x, y \rangle|^2 \leq \langle x, x \rangle \langle y, y \rangle$$

Proof: Let $x, y \in V$ and $c \in F$.

1. $$\begin{aligned} 0 &\leq \langle x - cy, x - cy \rangle \\ &= \langle x, x \rangle - \langle x, cy \rangle - \langle cy, x \rangle + \langle cy, cy \rangle \\ &= \langle x, x \rangle - \overline{c}\langle x, y \rangle - c\langle y, x \rangle + |c|^2\langle y, y \rangle \\ &= \langle x, x \rangle - \overline{c}\langle x, y \rangle - c\overline{\langle x, y \rangle} + |c|^2\langle y, y \rangle \\ &= \langle x, x \rangle - 2Re(c\langle x, y \rangle) + |c|^2\langle y, y \rangle \end{aligned}$$

#EE501 - `Linear Systems Theory` at `METU`