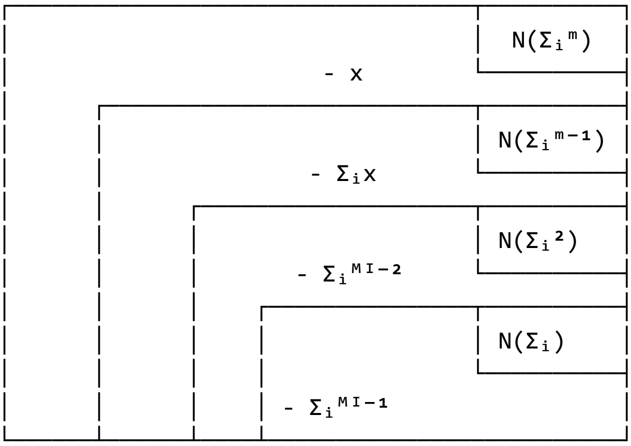
> **Let A be an nxn matrix and Ā be its Jordan canonical form.**
>
> - $\bar{A} = B^{-1}AB$, where B is invertible and composed of the basis vectors for the $N(A-\lambda_i I)^{m_i}$
> - $\text{rank}(A) = \text{rank}(BA) = \text{rank}(AB) = \text{rank}(\bar{A})$
> - $\dim(N(A-\lambda_i I)) = \dim(N(\bar{A}-\lambda_i I))$

$A \in \mathbb{C}^{n\times n}$
$\bar{A} = J = B^{-1}AB$
$\mathbb{C}^n = N(A-\lambda_1 I)^{m_1} \oplus \cdots \oplus N(A-\lambda_k I)^{m_k}$
    We look for basis vectors for $N((A-\lambda_i I)^{m_i})$
$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \end{bmatrix}$
$B_i = \begin{bmatrix} b_i^2 & b_i^3 & \cdots & b_i^{r_i} \end{bmatrix}$ where $b_i, \cdots, b_i^{r_i}$ have to be basis vectors for $N((A-\lambda_i I)^{m_i})$

Let $M_i := A - \lambda_i I$, and let's choose a vector x such that $x \in N(M_i^j)$ or $M_i^j x = 0$, but $x \notin N(M_i^{j-1})$

Now consider the chain of vectors $x, M_i x, M_i^2 x, \cdots, M_i^{j-1}x$, which is linearly independent. We can see this by contradiction. Suppose that the chain is linearly dependent. Then there exists a vector $M_i^k x$ such that $M_i^k x = \sum_{l=0}^{k-1} c_l M_i^l x$, where $c_l \in \mathbb{C}$. But then $M_i^k x = 0$, which is a contradiction. Thus, the chain is linearly independent.

$$\{M_i^{m_i-1}x, M_i^{m_i-2}x, \cdots, M_i x, x\} \text{ is linearly independent}$$



<u>Example</u>: Find the Jordan canonical form of $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

<u>Solution</u>:

$d(s) = \det(sI - A) = (s-1)[(s-3)(s-1)+1] = (s-2)^2(s-1)$

$\lambda_1 = 2, \lambda_2 = 1$

$\Sigma_1 \;=\; A - 2I \;=\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\; \dim N(\Sigma_1) = 3 - 2 = 1 \neq r_1 = 2$

$\Sigma_1^2 = (A-2I)^2 = \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}\; \dim N(\Sigma_1^2) = 3 - 1 = 2 = r_1$

No need to check Σ_2 because $r_2 = 1$

$m(s) = (s-2)^2(s-1) = d(s)$

From the minimal polynomial, we can see that the Jordan canonical form will have a 2x2 block for $\lambda_1 = 2$ and a 1x1 block for $\lambda_2 = 1$

$$\bar{A} = J = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to PJ = AP$$

Now we will construct the chain of vectors, which will be used to construct the basis vectors.

$x \in N(\Sigma_1^2)$ but $x \notin N(\Sigma_1)$

$$\Sigma_1^2 x = 0 \to \begin{bmatrix} 0 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to x_3 = 0$$

$$\Sigma_1 x \neq 0 \to \begin{bmatrix} -1 & 1 & -1 \\ -1 & 1 & 2 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to x_1 \neq x_2$$

Select $x = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ such that $x \in N(\Sigma_1^2)$ but $x \notin N(\Sigma_1)$

The chain of vectors for $\lambda_1 = 2$ is

$$\left\{ \Sigma_1 x \ , \ x \right\} = \left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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┌─────────────────────────────────────┬─────────┐
│                                     │ N(Σ₁²)  │
│                         - x         └─────────┤
│                   ┌─────────────────┬─────────┤
│                   │                 │ N(Σ₁)   │
│                   │                 └─────────┤
│                   │  - Σ₁x                    │
└───────────────────┴───────────────────────────┘
```

Now we will construct the chain of vectors for $\lambda_2 = 1$

$$\Sigma_2 = A - I = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \quad \dim N(\Sigma_2) = 3 - 2 = 1 = r_2$$

$y \in N(\Sigma_2)$

$$\Sigma_2 y = 0 \to \begin{bmatrix} 0 & 1 & -1 \\ -1 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \to y = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

The chain of vectors for $\lambda_2 = 1$ is

$$\left\{ y \right\} = \left\{ \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} \right\}$$

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┌─────────────────────────────────────┬─────────┐
│                                     │ N(Σ₂)   │
│                                     └─────────┤
│                                      - y      │
└───────────────────────────────────────────────┘
```

Now we will construct the change of basis matrix P, or B in the general case.

$$P = \left[\begin{array}{c|c|c} \Sigma_1 x & x & y \end{array}\right] = \begin{bmatrix} -1 & 1 & 4 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$J = P^{-1}AP \ = \begin{bmatrix} -1 & 1 & 4 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 4 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$AP = P\bar{A} \implies \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 4 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 & 4 \\ -1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Special Case I

*A **has a single eigenvalue** λ_i **and** $m_i = r_i$*

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┌───────────────────────────────────┬─────────┬─────────┬────────┐
│ N(Σᵢ)          - Σᵢᵐ⁻¹x           │         │ .....   │        │
├───────────────────────────────────┘         │         │        │
│ N(Σᵢ²)             - Σᵢᵐ⁻²x                 │         │        │
├─────────────────────────────────────────────┘ ..      │        │
│ .                                                 ..  │        │
├───────────────────────────────────────────────────────┘        │
│ N(Σᵢᵐᴵ)                           - x                          │
└────────────────────────────────────────────────────────────────┘
```

For this case, we can see that $m(s) = d(s)$, which means that the Jordan canonical form will have a single block for λ , and the size of the block will be $m_i = r_i$. This is because the minimal polynomial is the product of the linear factors of the characteristic polynomial, and the characteristic polynomial is the product of the eigenvalues. Since there is only one eigenvalue, the minimal polynomial will be the same as the characteristic polynomial.

$$\bar{A} = J = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

<u>consider the case:</u> $\left\{\Sigma_i^{m_i-1}x, \Sigma_i^{m_i-2}x, \cdots, \Sigma_i x, x\right\}$ where it is sufficient to span $N(\Sigma_i^{m_i})$, which is the same as $N(\Sigma_i^{m_i-1})$, which is the same as $N(\Sigma_i^{m_i-2})$, and so on. There are r_i vectors in this chain, which is the same as the dimension of $N(\Sigma_i^{m_i})$.

Special Case II

*A **has a single eigenvalue** λ_i **and** $m_i = 1$*

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┌────────────────────────────────────────────────────────────────┐
│ N(Σᵢ)                 - x₃                                     │
│                                 - x₂                           │
│                                          - x₁                  │
└────────────────────────────────────────────────────────────────┘
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For this case, we can see that $m(s) = (s - \lambda_i)$, which means that the Jordan canonical form will have a single block for λ , and the size of the block will be $m_i = 1$. This is because the minimal polynomial is the product of the linear factors of the characteristic polynomial, and the characteristic polynomial is the product of the eigenvalues. Since there is only one eigenvalue, the minimal polynomial will be the same as the characteristic polynomial. The number of chains will be equal to the dimension of $N(\Sigma_i)$, which is r_i.

$\dim N(\Sigma_i) = r_i$

$$\text{chain } 1 = \left\{x_1\right\}$$
$$\text{chain } 2 = \left\{x_2\right\}$$
$$\vdots$$
$$\text{chain } r_i = \left\{x_{r_i}\right\}$$

$$J = \begin{bmatrix} \lambda & 0 & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}$$

Example: Given $m_i = 6$ and $r_i = 8$, Find the largest K value such that $y \in N(\Sigma_i^K)$ but $y \notin N(\Sigma_i^{K-1})$

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┌──────────────────────────────┬────────┬─────────┬────────┬────────┬─────────┐
│ N(Σᵢ)    -Σᵢy    - Σᵢ⁵        │        │         │        │        │         │
├──────────────────────────────┘        │         │        │        │         │
│ N(Σᵢ²)      -y           - Σᵢ⁴x       │         │        │        │         │
├───────────────────────────────────────┘         │        │        │         │
│ N(Σᵢ³)                            - Σᵢ³x         │        │        │         │
├─────────────────────────────────────────────────┘        │        │         │
│ N(Σᵢ⁴)                                   - Σᵢ²x           │        │         │
├──────────────────────────────────────────────────────────┘        │         │
│ N(Σᵢ⁵)                                           - Σᵢx            │         │
├───────────────────────────────────────────────────────────────────┘         │
│ N(Σᵢ⁶)                                                   - x                │
└─────────────────────────────────────────────────────────────────────────────┘
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First chain $\left\{\Sigma_i^5 x, \Sigma_i^4 x, \Sigma_i^3 x, \Sigma_i^2 x, \Sigma_i x, x\right\}$ has r_i vectors, which is the same as the dimension of $N(\Sigma_i^6)$

Second chain $\left\{\Sigma_i y, y\right\}$ has 2 vectors, which is the same as the dimension of $N(\Sigma_i^1)$

$$B = \begin{bmatrix} \Sigma_i^5 x & \Sigma_i^4 x & \Sigma_i^3 x & \Sigma_i^2 x & \Sigma_i x & x & \Sigma_i y & y \end{bmatrix}$$

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

Example: Let $A \in \mathbb{C}^{6\times 6}$ with a single eigenvalue. $m(s) = (s - \lambda_i)^3$, $\dim N(\Sigma_i) = 2$, Find the Jordan canonical form of A and construct the change of basis matrix B.

Solution:

$d(s) = (s - \lambda_i)^6 \rightarrow \dim\,(v) = 6 \text{ single eigenvalue } \lambda_i$

$\dim N(\Sigma_i) = 2 \implies \text{There are 2 Jordan blocks or chains}$

$m(s) = (s - \lambda_i)^3 \implies \text{The size of largest Jordan block is } 3 \times 3$

Then we would have 2 Jordan blocks of size 3x3. The change of basis matrix B would be composed of the basis vectors.

$$\bar{A} = J = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{bmatrix}$$

Jordan chains for λ_i are $\left\{\Sigma_i^2 x, \Sigma_i x, x\right\}$ and $\left\{\Sigma_i^2 y, \Sigma_i y, y\right\}$

$B = \left[\Sigma_i^2 x \quad \Sigma_i x \quad x \quad \Sigma_i^2 y \quad \Sigma_i y \quad y\right]$

Try to find x and y such that $x \in N(\Sigma_i^2)$ but $x \notin N(\Sigma_i)$ and $y \in N(\Sigma_i)$

ASK FOR HELP FOR NEXT

<u>Example</u>: Let $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 \end{bmatrix}$ Find the characteristic polynomial, minimal polynomial, Jordan canonical form and the basis for A.

<u>Solution</u>:

$d(s) = \det(sI - A) = (s-1)^3(s-2)$

For the minimal polynomial, we need to check the null space of $A - \lambda_i I$ for each eigenvalue.

$$\Sigma_1 = \quad A - I \quad = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \dim N(\Sigma_1) = 4 - 2 = 2 \neq r_1$$

$$\Sigma_1^2 = (A - I)^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \dim N(\Sigma_1^2) = 4 - 1 = 3 = r_1$$

$$\Sigma_2 = \quad A - 2I \quad = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \end{bmatrix} \dim N(\Sigma_2) = 4 - 1 = 3 = r_2$$

$m(s) = (s-1)^2(s-2)$ Largest Jordan block is 2x2

$$\bar{A} = J = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Jordan chains for $\lambda_1 = 1$ are $\left\{\Sigma_1 x, x\right\}$

choose $x = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ Then $\left\{\Sigma_1 x, x\right\} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

choose $y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ Then $\left\{y\right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$

choose $z = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ Then $\left\{z\right\} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$B = \begin{bmatrix} \Sigma_1 x & x & y & z \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$$

#EE501 - `Linear Systems Theory` at `METU`