Definition: An $n \times n$ matrix A is said to be **Hermitian** if $A = A^H = A^*$. Its conjugate transpose is equal to itself. If A is real, then $A = A^T$.

Theorem: Let A be Hermitian. Then $\langle x, Ax \rangle$ is real for all $x \in \mathbb{C}^n$.

Proof: We will start with properties of inner products.

1- $\langle x, y \rangle = \overline{\langle y, x \rangle}$.

Then we will substitute $A = A^*$ and in the last step we will use,

2- $\langle x, Py \rangle = \langle P^* x, y \rangle$.

Check the properties of inner products here.

$\langle x, Ax \rangle = \overline{\langle Ax, x \rangle} = \overline{\langle A^* x, x \rangle} = \overline{\langle x, Ax \rangle}$ ■

Theorem: Let A be Hermitian. Then all eigenvalues of A are real.

Proof: Let λ be an eigenvalue of A and x be the corresponding eigenvector. Then $Ax = \lambda x$. Then we will use the previous theorem.

$\lambda \langle x, x \rangle = \langle \lambda x, x \rangle = \langle Ax, x \rangle = \overline{\langle x, Ax \rangle} = \overline{\langle x, \lambda x \rangle} = \overline{\lambda \langle x, x \rangle} = \overline{\lambda} \langle x, x \rangle$

Since $\langle x, x \rangle \neq 0$, we can divide both sides by $\langle x, x \rangle$.

$\lambda = \overline{\lambda}$ ■

Theorem: Let A be Hermitian. Then all eigenvectors corresponding to distinct eigenvalues are orthogonal. Let A be Hermitian and $\lambda_i \neq \lambda_j$ be two distinct eigenvalues of A with corresponding eigenvectors e_i and e_j. Then $\langle e_i, e_j \rangle = 0$.

Proof: Let A be Hermitian and $\lambda_i \neq \lambda_j$ be two distinct eigenvalues of A with corresponding eigenvectors e_i and e_j. Then $Ae_i = \lambda_i e_i$ and $Ae_j = \lambda_j e_j$. Then we will use the previous theorem.

$\langle e_i, Ae_j \rangle = \langle e_i, \lambda_j e_j \rangle = \lambda_j \langle e_i, e_j \rangle$
$\langle e_i, Ae_j \rangle = \langle A^* e_i, e_j \rangle = \lambda_i \langle e_i, e_j \rangle$

Then we will subtract the equations.

$(\lambda_j - \lambda_i) \langle e_i, e_j \rangle = 0$

Since $\lambda_j \neq \lambda_i$, we have $\langle e_i, e_j \rangle = 0$ ■

Theorem: Let A be Hermitian. Then its minimal polynomial is

$$m(s) = (s - \lambda_1)(s - \lambda_2) \cdots (s - \lambda_\sigma)$$

where λ_i are the distinct eigenvalues of A.

Proof: Set Equality We will prove that $m(s)$ has no repeated roots. For which we need to use $N((A - \lambda I)) = N((A - \lambda I)^2)$.

In order to show the equality, we will prove that $N((A - \lambda I)) \subseteq N((A - \lambda I)^2)$ and $N((A - \lambda I)^2) \subseteq N((A - \lambda I))$.

1- $N((A - \lambda I)) \subseteq N((A - \lambda I)^2)$

Let $x \in N((A-\lambda I))$. Then $(A-\lambda I)x = 0$. Then we will multiply both sides by $(A-\lambda I)$.

$(A-\lambda I)^2 x = 0$

Then $x \in N((A-\lambda I)^2)$.

2- $N((A-\lambda I)^2) \subseteq N((A-\lambda I))$

Let $x \in N((A-\lambda I)^2)$. Then $(A-\lambda I)^2 x = 0$.

$\langle x, (A-\lambda I)^2 x\rangle = 0$

$\langle x, (A-\lambda I)(A-\lambda I)x\rangle = 0$

$\langle (A-\lambda I)x, (A-\lambda I)x\rangle = 0 = \|(A-\lambda I)x\|^2 \implies (A-\lambda I)x = 0$

Then $x \in N((A-\lambda I))$.

Therefore Hermitian matrices A with distinct eigenvalues have no repeated roots in their minimal polynomials. ■

$$d(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_\sigma)$$
$$m(s) = (s-\lambda_1)(s-\lambda_2)\cdots(s-\lambda_\sigma)$$
$$\mathbb{C}^n = N((A-\lambda_1 I)) \oplus N((A-\lambda_2 I)) \oplus \cdots \oplus N((A-\lambda_\sigma I))$$

Theorem: Let A be Hermitian matrix with characteristic polynomial $d(s) = (s-\lambda_1)^{r_1}(s-\lambda_2)^{r_2}\cdots(s-\lambda_\sigma)^{r_\sigma}$. Then there exist a unitary matrix P such that $P^{-1} = P^*$ and $P^*AP = \Lambda$ where Λ is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \cdots, \lambda_\sigma$.

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_\sigma \end{bmatrix} \text{ and each } \Lambda_i \in \mathbb{C}^{r_i \times r_i}, \Lambda_i = \begin{bmatrix} \lambda_i & 0 & 0 & \cdots & 0 \\ 0 & \lambda_i & 0 & \cdots & 0 \\ 0 & 0 & \lambda_i & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_i \end{bmatrix}$$

where λ_i are the distinct eigenvalues of A.

Proof: See the lecture notes, page 50.

Theorem: Let A be Hermitian matrix with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_\sigma$. Let $\lambda_min = \min_{i=1,2,\cdots,\sigma} \lambda_i$ and $\lambda_max = \max_{i=1,2,\cdots,\sigma} \lambda_i$. Then for all $x \in \mathbb{C}^n$,

$$\lambda_{min}\langle x, x\rangle \leq \langle x, Ax\rangle \leq \lambda_{max}\langle x, x\rangle$$

Proof: See the lecture notes, page 51.

Definition: A Hermitian matrix A is said to be **positive definite** if $\langle x, Ax\rangle > 0$ for all $x \in \mathbb{C}^n$ and $x \neq 0$. A Hermitian matrix A is said to be **positive semidefinite** if $\langle x, Ax\rangle \geq 0$ for all $x \in \mathbb{C}^n$.

Proof: By contradiction. Let A be positive definite Hermitioan, Then $\lambda_i > 0\ \forall i = 1, 2, \cdots, \sigma$.

$\rightarrow$ Suppose not, Let $e \leq 0$ be an eigenvector of A.

From the positive definiteness of A, we have $\langle e, Ae\rangle > 0$.

$\langle e, Ae\rangle = \langle e, \lambda e\rangle = \lambda^*\langle e, e\rangle = \lambda\langle e, e\rangle > 0$

$\lambda\langle e, e\rangle > 0 \implies \lambda > 0$ which is a contradiction. ■

#EE501 - `Linear Systems Theory` at `METU`