Functions of a Matrix

The motivation is to study matrix-valued functions, which stems from the differential equations describing linear systems.

$$\dot{x} = Ax(t)$$

$$x(t) = e^{At}x(0)$$

The power series representation of any function is

$$f(s) = \sum_{i=0}^{\infty} \alpha_i s^i$$

where $s \in \mathbb{C}$.

The power series representation of a matrix-valued function is

$$f(A) = \sum_{i=0}^{\infty} \alpha_i A^i$$

where $A \in \mathbb{C}^{n \times n}$. This solution is another matrix as the same size as A.

The exponential function is defined for matrices as

$$e^t = \sum_{i=0}^{\infty} \frac{t^i}{i!} \implies e^A = \sum_{i=0}^{\infty} \frac{A^i}{i!}$$

By using Cayley-Hamilton Theorem, we can write A^i as a linear combination of $I, A, A^2, \cdots, A^{n-1}$.

$$e^A = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{n-1} A^{n-1}$$

where c_i are scalars.

> Remark: One can use the minimal polynomial of a matrix to express the lth power of a matrix in terms of $I, A, A^2, \cdots, A^{l-1}$. l is the order of the minimal polynomial.
>
> $$e^A = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{l-1} A^{l-1}$$

First Method

Let

$$f(s) = \sum_{i=0}^{\infty} \alpha_i s^i$$

$$f(A) = \sum_{i=0}^{\infty} \alpha_i A^i$$

Define $p(s)$ and $P(A)$ as follows

$$p(s) = c_0 + c_1 s + c_2 s^2 + \cdots + c_{l-1} s^{l-1}$$

$$P(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{l-1} A^{l-1}$$

Then we have the equality

$$f(A) = P(A)$$

where $P(A)$ is a polynomial of A.

<u>Case I: A is diagonalizable</u>

Suppose

$m(s) = (s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_\sigma)$

$l = \sigma$, $m_1 = m_2 = \cdots = m_\sigma = 1$

$f(A) = P(A) = c_0 I + c_1 A + c_2 A^2 + \cdots + c_{l-1} A^{l-1}$

$Ae_i = \lambda_i e_i$

Let e_i be the eigenvector of A corresponding to λ_i, multiply both sides by e_i.

$\sum_{n=0}^{l-1} \alpha_i A^n e_i = \sum_{n=0}^{l-1} c_n \lambda_i^n e_i$

$\sum_{n=0}^{l-1} \alpha_i \lambda_i^n e_i = \sum_{n=0}^{l-1} c_n \lambda_i^n e_i$

$\alpha_i \lambda_i^n = c_n \lambda_i^n$

$\alpha_i = c_n$

$f(A) = P(A) = \sum_{i=0}^{l-1} \alpha_i A^i = \sum_{i=0}^{l-1} c_i A^i$

$f(\lambda_i) = P(\lambda_i)$

<u>Example</u>: $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ find e^A and $log(A)$

<u>Solution</u>:

$d(s) = (s-3)(s-1)$

$\lambda_1 = 3$, $\lambda_2 = 1$

$m(s) = (s-3)(s-1)$, $l = 2$

$p(s) = c_0 + c_1 s$, and $P(A) = c_0 I + c_1 A$

$Ae_1 = 3e_1$, $Ae_2 = e_2$

$f(3) = p(3)$
$f(1) = p(1)$

$f(3) = e^3 = c_0 + 3c_1$
$f(1) = \ e \ = c_0 + c_1$

$c_1 = \frac{e^3 - e}{2}$
$c_0 = \frac{3e - e^3}{2}$

$e^A = \frac{3e - e^3}{2} I + \frac{e^3 - e}{2} A$

$e^A = \frac{3e - e^3}{2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{e^3 - e}{2} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

<u>Case II: A is not diagonalizable</u>

Consider the following example. Let $A \in \mathbb{R}^3$ and $m(s) = (s - \lambda_1)^2(s - \lambda_2)$.

$$J = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix}$$

$$\sum m_i = 2 + 1 = 3$$

$$f(A) = P(A) = c_0 I + c_1 A + c_2 A^2$$

$$f(\lambda_1) = P(\lambda_1) \quad f(\lambda_2) = P(\lambda_2)$$

These two equations are not enough to find c_0, c_1, c_2.

Consider the matrix P that transforms A into its Jordan canonical form J.

We know that $P^{-1} = P^*$ and $P^*AP = J$, and P is in the form of $[e_1, f_1, e_2]$. Where e_1 and e_2 are the eigenvectors of A corresponding to λ_1 and λ_2 respectively. f_1 is the generalized eigenvector of A corresponding to λ_1.

$$P = \begin{bmatrix} \vdots & \vdots & \vdots \\ e_1 & f_1 & e_2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{bmatrix} \vdots & \vdots & \vdots \\ e_1 & f_1 & e_2 \\ \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{bmatrix} = A \begin{bmatrix} \vdots & \vdots & \vdots \\ e_1 & f_1 & e_2 \\ \vdots & \vdots & \vdots \end{bmatrix}$$

$$\begin{aligned} Af_1 &= \lambda_1 f_1 + e_1 \\ A^2 f_1 &= \lambda_1 A f_1 + A e_1 \\ &= \lambda_1^2 f_1 + \lambda_1 e_1 + \lambda_1 e_1 = \lambda_1^2 f_1 + 2\lambda_1 e_1 \\ A^3 f_1 &= \lambda_1^2 A f_1 + 2\lambda_1 A e_1 = \lambda_1^3 f_1 + 3\lambda_1^2 e_1 \\ &\vdots \\ A^k f_1 &= \lambda_1^k f_1 + k\lambda_1^{k-1} e_1 \end{aligned}$$

Return to the equation $f(A) = P(A)$.

$$\sum_{i=0}^{l-1} \alpha_i A^i = \sum_{i=0}^{l-1} c_i A^i$$

Multiply both sides by f_1 from the right.

$$\sum_{i=0}^{l-1} \alpha_i A^i f_1 = \sum_{i=0}^{l-1} c_i A^i f_1$$

$$\sum_{i=0}^{l-1} \alpha_i \lambda_1^i f_1 + \sum_{i=0}^{l-1} \alpha_i i \lambda_1^{i-1} e_1 = \sum_{i=0}^{l-1} c_i \lambda_1^i f_1 + \sum_{i=0}^{l-1} c_i i \lambda_1^{i-1} e_1$$

$$f(\lambda_1) f_1 + f'(\lambda_1) e_1 = P(\lambda_1) f_1 + P'(\lambda_1) e_1$$

$$f(\lambda_1) = P(\lambda_1)$$

$$f'(\lambda_1) = P'(\lambda_1)$$

Which is the additional equation we need to find c_0, c_1, c_2.

<u>General Case</u>

Let $m(s) = (s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_\sigma)^{m_\sigma}$. We have the following set of equations.

$$f^{(t)}(\lambda_i) = P^{(t)}(\lambda_i)$$

where $t = 0, 1, 2, \cdots, m_i - 1$ and $i = 1, 2, \cdots, \sigma$. We have $\sum_{i=1}^{\sigma} m_i = l$ equations.

Example: $A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$ Find $\sin(\pi A)$.

Solution:

$m(s) = s^3(s-1)^2$ and the order of the total minimal polynomial is $l = 5$.

$p(s) = c_0 + c_1 s + c_2 s^2 + c_3 s^3 + c_4 s^4$

$p(A) = c_0 I + c_1 A + c_2 A^2 + c_3 A^3 + c_4 A^4$

We need five equations to find c_0, c_1, c_2, c_3, c_4.

Starting with

$$\begin{aligned} f(s) &= \sin(\pi s) \\ f'(s) &= \pi \cos(\pi s) \\ f''(s) &= -\pi^2 \sin(\pi s) \end{aligned}$$

Then

$$\begin{aligned} p(s) &= c_0 + c_1 s + c_2 s^2 + c_3 s^3 + c_4 s^4 \\ p'(s) &= c_1 + 2c_2 s + 3c_3 s^2 + 4c_4 s^3 \\ p''(s) &= 2c_2 + 6c_3 s + 12 c_4 s^2 \end{aligned}$$

For the first eigenvalue $\lambda_1 = 0$. We have $m_1 = 3$.

$$\begin{aligned} f(\lambda_1) &= p(\lambda_1) \\ f'(\lambda_1) &= p'(\lambda_1) \\ f''(\lambda_1) &= p''(\lambda_1) \end{aligned}$$

$$\begin{aligned} f(0) &= p(0) \\ f'(0) &= p'(0) \\ f''(0) &= p''(0) \end{aligned}$$

$$\begin{aligned} \sin(0) &= 0 = c_0 \\ \pi \cos(0) &= \pi = c_1 \\ -\pi^2 \sin(0) &= 0 = 2c_2 \end{aligned}$$

For the second eigenvalue $\lambda_2 = 1$. We have $m_2 = 2$.

$$\begin{aligned} f(\lambda_2) &= p(\lambda_2) \\ f'(\lambda_2) &= p'(\lambda_2) \end{aligned}$$

$$\begin{aligned} \sin(\pi) &= 0 = c_0 + c_1 + c_2 + c_3 + c_4 \\ \pi \cos(\pi) &= -\pi = c_1 + 2c_2 + 3c_3 + 4c_4 \end{aligned}$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \\ 0 \\ -\pi \\ -\pi \end{bmatrix}$$

$$\begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \pi \\ 0 \\ -2\pi \\ \pi \end{bmatrix}$$

$$p(s) = \pi s - 2\pi s^3 + \pi s^4$$

$$\sin(\pi A) = \pi A - 2\pi A^3 + \pi A^4$$

> Remark: $f(A)$ does not exist when $f^{(t)}(\lambda_i)$ does not exist for some t and i.

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