Linear Mapping

Definition: Let V and W be linear spaces over the same field F. A mapping $\mathcal{T}: V \to W$ is called a linear mapping satisfying

$$\mathcal{T}(ax+by) = a\mathcal{T}(x) + b\mathcal{T}(y) \qquad \forall x, y \in V \qquad \forall a, b \in F$$

here V is called the domain of T and W is called the codomain of $\mathcal{T}$.

Example: Let $V = W$ polynomials of degree less than n in S; $\mathcal{T} = \frac{d}{ds}$

Solution: Let $p, q \in V$ and $\alpha_1, \alpha_2 \in F$ then ,
p(s) = $\sum_{i=0}^{n-1} a_i s^i$ and q(s) = $\sum_{i=0}^{n-1} b_i s^i$
$\alpha_1 p(s) + \alpha_2 q(s) = \sum_{i=0}^{n-1} (\alpha_1 a_i + \alpha_2 b_i) s^i$
$\frac{d}{ds}(\alpha_1 p(s) + \alpha_2 q(s)) = \sum_{i=0}^{n-1} (\alpha_1 a_i + \alpha_2 b_i) i s^{i-1} = \alpha_1 \sum_{i=0}^{n-1} a_i s^{i-1} + \alpha_2 \sum_{i=0}^{n-1} b_i s^{i-1} = \alpha_1 \frac{dp}{ds} + \alpha_2 \frac{dq}{ds}$
$\mathcal{T}(\alpha_1 p + \alpha_2 q) = \frac{d}{ds}(\alpha_1 p + \alpha_2 q) = \alpha_1 \frac{dp}{ds} + \alpha_2 \frac{dq}{ds} = \alpha_1 \mathcal{T}(p) + \alpha_2 \mathcal{T}(q)$ ■

Example: Let $V = W = \mathbb{R}^2$. Let $\mathcal{A}$ be defined as,

$$\mathcal{A} = \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} \text{ where } x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

Solution: Let $a, b \in F$ and $x_1, x_2 \in X$ with $x_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ and $x_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$ then,
$\mathcal{A}(ax_1 + bx_2) = \mathcal{A}(a \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + b \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}) = \mathcal{A}(\begin{bmatrix} a\alpha_1 \\ a\alpha_2 \end{bmatrix} + \begin{bmatrix} b\beta_1 \\ b\beta_2 \end{bmatrix}) = \mathcal{A}(\begin{bmatrix} a\alpha_1 + b\beta_1 \\ a\alpha_2 + b\beta_2 \end{bmatrix}) =$
$\begin{bmatrix} a\alpha_1 + b\beta_1 \\ a\alpha_1 + a\alpha_2 + b\beta_1 + b\beta_2 \end{bmatrix} = a \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} + b \begin{bmatrix} \beta_1 \\ \beta_1 + \beta_2 \end{bmatrix} = a\mathcal{A}(x_1) + b\mathcal{A}(x_2)$ ■

Example: Let $V = W = \mathbb{R}$. is $\mathcal{A}x = (1 - x)$ linear or not ?

Solution: Let $a, b \in F$ and $x_1, x_2 \in X$ then,

$$\begin{aligned} \mathcal{A}(ax_1 + bx_2) &\stackrel{?}{=} a\mathcal{A}(x_1) + b\mathcal{A}(x_2) \\ 1 - (ax_1 + bx_2) &\stackrel{?}{=} a(1 - x_1) + b(1 - x_2) \\ 1 - ax_1 - bx_2 &\stackrel{?}{=} a - ax_1 + b - bx_2 \\ 1 &\neq a + b \ \forall a, b \in F \ \blacksquare \\ &\text{hence not linear} \end{aligned}$$

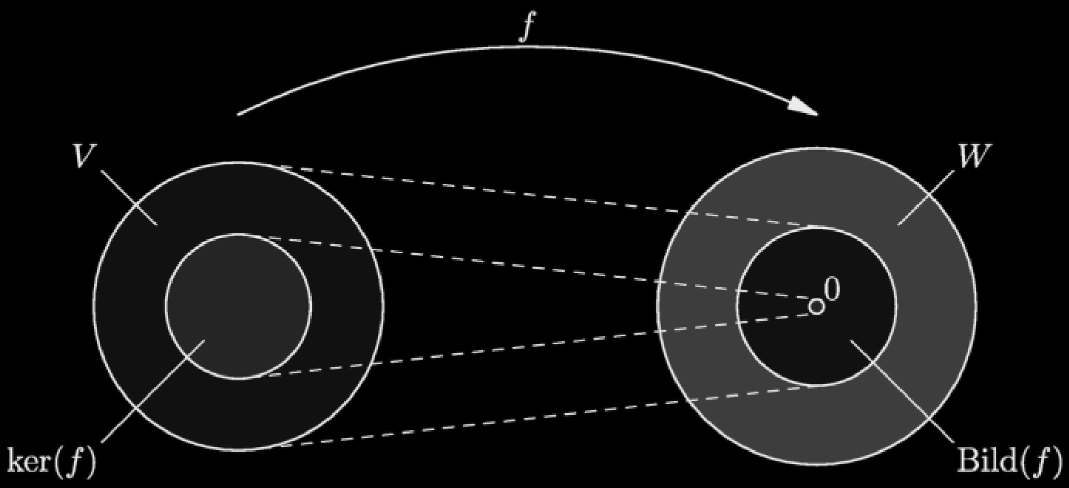
> Rotation transformations in $\mathbb{R}^2$ are linear transformations.
> Integration and differentiation are linear transformations.

Definition: Given a linear mapping $\mathcal{T}: V \to W$, the set of all vectors $x \in V$ such that $\mathcal{T}(x) = 0_W$ is called the null space of $\mathcal{T}$ and is denoted by $N(\mathcal{T})$. That is,

$$N(\mathcal{T}) := \{x \in V \ : \ \mathcal{T}(x) = 0_W\}$$

Definition: Given a linear mapping $\mathcal{T} : V \to W$, the set of all vectors $w \in W$ such that $w = \mathcal{T}(v)$ for some $v \in V$ is called the range of $\mathcal{T}$ and is denoted by $R(\mathcal{T})$. That is,

$$R(\mathcal{T}) := \{w \in W \ : \ w = \mathcal{T}(v) \text{ for some } v \in V\}$$



<u>Claim</u>: For a given linear mapping $\mathcal{T} : V \to W$, $N(\mathcal{T})$ is a linear subspace of V.

<u>**Proof**</u>: Let $x_1, x_2 \in N(\mathcal{T})$ and $a \in F$ show,

(S1). $x_1 + x_2 \in N(\mathcal{T})$
(S2). $ax_1 \in N(\mathcal{T})$

1- $\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1) + \mathcal{T}(x_2) = 0_W + 0_W = 0_W \implies x_1 + x_2 \in N(\mathcal{T})$
2- $\mathcal{T}(ax_1) = a\mathcal{T}(x_1) = a0_W = 0_W \implies ax_1 \in N(\mathcal{T})$ ■

<u>Claim</u>: For a given linear mapping $\mathcal{T} : V \to W$, $R(\mathcal{T})$ is a subspace of W.

<u>**Proof**</u>: Let $x_1, x_2 \in R(\mathcal{T})$ and $a \in F$ show,

(S1). $x_1 + x_2 \in R(\mathcal{T})$
(S2). $ax_1 \in R(\mathcal{T})$

> **Definition**: A linear transformation $\mathcal{T} : V \to W$ is called one-to-one if $x_1 \neq x_2$ implies $\mathcal{T}(x_1) \neq \mathcal{T}(x_2)$ for all $x_1, x_2 \in V$.

<u>Theorem</u>: Let $\mathcal{T} : V \to W$ be a linear transformation. Then mapping $\mathcal{T}$ is one-to-one if and only if $N(\mathcal{T}) = \{0_V\}$.

<u>**Proof**</u>: We will prove the statement by contrapositive. Since it is an if and only if statement, we will prove both directions.

(Bacward direction) Assume that $N(\mathcal{T}) = \{0_V\}$ and $\mathcal{T}(x_1) = \mathcal{T}(x_2)$ for some $x_1, x_2 \in V$. Then,
$\mathcal{T}(x_1) - \mathcal{T}(x_2) = 0_W$
$\mathcal{T}(x_1 - x_2) = 0_W$

$x_1 - x_2 \in N(\mathcal{T})$
$x_1 - x_2 = 0_V$
$x_1 = x_2$
$\mathcal{T}$ is one-to-one.

(Forward direction) Assume that $\mathcal{T}$ is one-to-one and $x \in N(\mathcal{T})$. Then,
$\mathcal{T}(x) = 0_W$
$\mathcal{T}(0_V) = 0_W$
$x = 0_V$
$N(\mathcal{T}) = \{0_V\}$ ■

Definition: A linear transformation $\mathcal{T}: V \rightarrow W$ is called onto if $R(\mathcal{T}) = W$, otherwise if $R(\mathcal{T}) \subset W$ then $\mathcal{T}$ is called into.

Example: Let $V := \{f : [0,1] \rightarrow \mathbb{R} \text{ and f is integrable}\}$. A transformation $\mathcal{A}: V \rightarrow \mathbb{R}$ is defined as,

$$\mathcal{A}(f(s)) = \int_0^1 f(s)ds$$
$$\text{is } \mathcal{A} \text{ one-to-one ?}$$

Solution: Integration operation resulting in one-to-one transformation probably not true. Hence we can exploit the fact that the integration might result in zero.

Let $f(s) = 2s-1 then,

$\mathcal{A}(f(s)) = \int_0^1 (2s-1)ds = \left[s^2 - s\right]_0^1 = 0$
$\mathcal{A}(f(s)) = 0$
Then $\mathcal{A}(0) = 0_w$ and $\mathcal{A}(f(s)) = 0_w$ for some $f(s) \neq 0$.
$\mathcal{A}$ is not one-to-one.

moreover,

Let $f(s) = a then,

$\mathcal{A}(f(s)) = \int_0^1 ads = [as]_0^1 = a$
Shows that $\mathcal{A}$ is onto.

Matrix Representations

Definition: Let $\mathcal{T}: V \rightarrow W$ be a linear transformation with $dim(V) = n$ and $dim(W) = m$. Let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ be a basis for V and $\mathcal{C} = \{w_1, w_2, ..., w_m\}$ be a basis for W. Then, the matrix representation of $\mathcal{T}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ is the $m \times n$ matrix A such that,

$$[w]_c = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} [v]_b = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$[\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathcal{T}(v_j) = \sum_{i=1}^{m} a_{ij} w_i \text{ for } j = 1, 2, ..., n$$

> **Remark**: The matrix representation of $\mathcal{T}$ with respect to $\mathcal{B}$ and $\mathcal{C}$ is denoted by $[\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}}$.

Now we have a transformation represented as,

$$[w]_c = [\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}} [v]_b$$

A formal procedure to obtain the matrix representation of a linear transformation

1. Take each basis vector v_j in $\mathcal{B}$
2. Apply $\mathcal{A}$ to v_j : $\mathcal{A}(v_j)$
3. Express the result in terms of the basis vectors in $\mathcal{C}$: $\mathcal{A}(v_j) = \sum_{i=1}^{m} a_{ij} w_i$
4. The jth column of $[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}}$ is the vector $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

<u>Example</u>: $V = \{Polynomials\ of\ degree\ less\ than\ 3\}$ and $W = \{Polynomials\ of\ degree\ less\ than\ 2\}$
Let $\mathcal{A} : V \to W$ be defined as,

$$\mathcal{A}(p(s)) = \frac{dp(s)}{ds}$$

Find the matrix representation of $\mathcal{A}$ with respect to the bases $\mathcal{B} = \{1, 1+s, 1+s+s^2, 1+s+s^2+s^3\}$ and $\mathcal{C} = \{1, 1+s, 1+s+s^2\}$.

<u>Solution</u>: $[w]_c = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$ and $[v]_b = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$$\mathcal{A}(v_1) = \frac{d}{ds}(1) = 0 = 0w_1 + 0w_2 + 0w_3$$
$$\mathcal{A}(v_2) = \frac{d}{ds}(1+s) = 1 = 1w_1 + 0w_2 + 0w_3$$
$$\mathcal{A}(v_3) = \frac{d}{ds}(1+s+s^2) = 1 + 2s = -1w_1 + 2w_2 + 0w_3$$
$$\mathcal{A}(v_4) = \frac{d}{ds}(1+s+s^2+s^3) = 1 + 2s + 3s^2 = -1w_1 - 1w_2 + 3w_3$$

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The full matrix representation of $\mathcal{A}$ is,

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

<u>Example</u>: Let $V = \mathbb{R}^2$ and $\mathcal{A} : V \to V$ be defined as,

$$\mathcal{A}(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + x \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Find the matrix representation of $\mathcal{A}$ with respect to the bases

$\mathcal{B} = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\}$. and

$\mathcal{C} = \{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\}$.

<u>**Solution**</u>: $[w]_c = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$ and $[v]_b = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$$\mathcal{A}(v_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = 1w_1 + 0w_2 - 1w_3 + 0w_4$$

$$\mathcal{A}(v_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1w_1 + 0w_2 + 1w_3 - 1w_4$$

$$\mathcal{A}(v_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1w_1 + 0w_2 + 1w_3 - 1w_4$$

$$\mathcal{A}(v_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1w_1 + 0w_2 + 1w_3 + 0w_4$$

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Change of Basis

Definition: Let $\mathcal{B} = \{v_1, v_2, ..., v_n\}$ and $\mathcal{C} = \{w_1, w_2, ..., w_n\}$ be two bases for a linear space V. The change of basis matrix from $\mathcal{B}$ to $\mathcal{C}$ is the $n \times n$ matrix P such that,

$$[w]_C = A[v]_B$$

$$[w]_C = \bar{A}[v]_{\bar{B}}$$

We know that a change of basis is a linear transformation. Hence,

$$[v]_B = P[v]_{\bar{B}}$$

$$[w]_C = AP[v]_{\bar{B}}$$

in codomain perspective,

$$[w]_C = Q[w]_{\bar{C}}$$

$$[w]_{\bar{C}} = Q^{-1}A[v]_B$$

$$[w]_{\bar{C}} = Q^{-1}AP[v]_{\bar{B}}$$

Example:

$$\begin{aligned} V &= \{\text{Polynomials with degree less than 3}\} \\ W &= \{\text{Polynomials with degree less than 2}\} \\ \mathcal{B} &= \{1, 1+s, 1+s+s^2, 1+s+s^2+s^3\} \\ \mathcal{C} &= \{1, 1+s, 1+s+s^2\} \\ A &= \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \\ \bar{B} &= \{1, s, s^2, s^3\} \end{aligned}$$

Solution: First we will find the change of basis in the domain matrix from $\mathcal{B}$ to $\bar{\mathcal{B}}$. That is more clearly stated as,

$$[w]_C = \bar{A}[v]_{\bar{B}}$$

and given $[v]_B = P[v]_{\bar{B}}$, $\bar{A}$ is equal to,

$$[w]_C = AP[v]_{\bar{B}}$$

In order to find P we need to write the basis vectors in $\bar{\mathcal{B}}$ in terms of $\mathcal{B}$.

$$\begin{aligned} 1 &= 1(1) + 0(1+s) + 0(1+s+s^2) + 0(1+s+s^2+s^3) \\ s &= -1(1) + 1(1+s) + 0(1+s+s^2) + 0(1+s+s^2+s^3) \\ s^2 &= 0(1) + -1(1+s) + 1(1+s+s^2) + 0(1+s+s^2+s^3) \\ s^3 &= 0(1) + 0(1+s) + -1(1+s+s^2) + 1(1+s+s^2+s^3) \end{aligned}$$

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

now $\bar{A}$ is equal to,

$$\bar{A} = AP = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Now we will change the basis in the codomain to canonical basis while keeping the basis of the domain as also in canonical form. That is,

$$[w]_{\bar{C}} = Q^{-1}AP[v]_{\bar{B}}$$

$$[w]_C = Q[w]_{\bar{C}}$$

$$[w]_{\bar{C}} = Q^{-1}[w]_C$$

In order to find Q^{-1} in a single step, we can write the basis vectors in $\mathcal{C}$ in terms of $\bar{\mathcal{C}}$.

$$\begin{aligned} 1 &= 1(1) + 0(s) + 0(s^2) \\ 1 + s &= 1(1) + 1(s) + 0(s^2) \\ 1 + s + s^2 &= 1(1) + 1(s) + 1(s^2) \end{aligned}$$

$$Q^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

as the final steps,

$$[w]_{\bar{C}} = Q^{-1}\bar{A}[v]_{\bar{B}}$$

$$Q^{-1}\bar{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$[w]_{\bar{C}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} [v]_{\bar{B}}$$

Given the matrix representation of a linear transformation $\mathcal{A} : V \to W$ with respect to bases $\mathcal{B}$ and $\mathcal{C}$, one can draw the following diagram.

Matrix Representation → Change of Basis → Matrix Representation → Change of Basis → Matrix Representation

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#EE501 - `Linear Systems Theory` at `METU`