Direct Sum

Definition: Let V be a vector space and let $M_1, M_2, \ldots, M_k$ be subspaces of V. The **sum** of the subspaces M is defined as

$$M = \{m = m_1 + m_2 + \ldots + m_k \mid m_i \in M_i, i = 1, 2, \ldots, k\}$$

Theorem: The sum of subspaces M is a subspace of V.

Proof:
Let x, y $\in M$ Then,
x = $m_1 + m_2 + \ldots + m_k$
y = $\bar{m_1} + \bar{m_2} + \ldots + \bar{m_k}$

$\alpha x + \beta y = \alpha(m_1 + m_2 + \ldots + m_k) + \beta(\bar{m_1} + \bar{m_2} + \ldots + \bar{m_k})$
$\alpha x + \beta y = (\alpha m_1 + \beta \bar{m_1}) + (\alpha m_2 + \beta \bar{m_2}) + \ldots + (\alpha m_k + \beta \bar{m_k})$

Since $\alpha m_i + \beta \bar{m_i} \in M_i$ for $i = 1, 2, \ldots, k$, we have $\alpha x + \beta y \in M$.

> **Remark**: Let $V = M_1 + M_2 + \ldots + M_k$ and let $M_1, M_2 \ldots, M_k$ are linearly independent. Let $x \in V$
>
> $$x = m_1 + m_2 + \ldots + m_k$$
>
> $$\text{where } m_i \in M_i \text{ for } i = 1, 2, \ldots, k$$
>
> $$m_1 + m_2 + \ldots + m_k = 0 \implies m_1 = m_2 = \ldots = m_k = 0$$
>
> $$\text{since } M_1, M_2 \ldots, M_k \text{ are linearly independent}$$

Definition: Let $M_1, M_2, \ldots, M_k$ be subspaces of V.

$$\begin{aligned} &\text{i) } M = M_1 + M_2 + \ldots + M_k \\ &\text{ii) } M_1, M_2, \ldots, M_k \text{ are linearly independent} \end{aligned}$$

Then, M is called the **direct sum** of $M_1, M_2, \ldots, M_k$ and we write $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$.

> When we have a direct sum, summation of dimensions of subspaces is equal to the dimension of the direct sum.

Example: $V = \mathbb{R}^4 \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4$

$M_1 = \{x \in \mathbb{R}^4 \mid x_3 = x_4 = 0\}$ dimension of M_1 is 2
$M_2 = \{x \in \mathbb{R}^4 \mid x_1 = x_2 = 0\}$ dimension of M_2 is 2
$M_3 = \{x \in \mathbb{R}^4 \mid x_1 = 0\}$ dimension of M_3 is 3

- $M_1 + M_2 = \mathbb{R}^4$
- $M_1 \oplus M_2 = \mathbb{R}^4$

Definition: If $M = V$, then $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ is called a **direct sum decomposition** of V.

> **Remark**: Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ be a direct sum decomposition of V. Then, the decomposition is unique. **Proof**: Let $M = M_1 \oplus M_2 \oplus \ldots \oplus M_k$ and $M = \bar{M_1} \oplus \bar{M_2} \oplus \ldots \oplus \bar{M_k}$ be two direct sum decompositions of V. Then,
>
> $$\begin{aligned} &\text{i) } M = M_1 + M_2 + \ldots + M_k \\ &\text{ii) } M_1, M_2, \ldots, M_k \text{ are linearly independent} \\ &\text{iii) } M = \bar{M_1} + \bar{M_2} + \ldots + \bar{M_k} \\ &\text{iv) } \bar{M_1}, \bar{M_2}, \ldots, \bar{M_k} \text{ are linearly independent} \end{aligned}$$

> Let $x \in M_1 \cap \bar{M}_1$, then $x \in M_1$ and $x \in \bar{M}_1$. Since $M_1 \oplus \bar{M}_1$, we have $x = 0$.
> Let $x \in M_2 \cap \bar{M}_2$, then $x \in M_2$ and $x \in \bar{M}_2$. Since $M_2 \oplus \bar{M}_2$, we have $x = 0$.
> $\vdots$
> Let $x \in M_k \cap \bar{M}_k$, then $x \in M_k$ and $x \in \bar{M}_k$. Since $M_k \oplus \bar{M}_k$, we have $x = 0$.
> Since $x = 0$ for all $x \in M_1 \cap \bar{M}_1, M_2 \cap \bar{M}_2, \dots, M_k \cap \bar{M}_k$, we have $M_1 = \bar{M}_1, M_2 = \bar{M}_2, \dots, M_k = \bar{M}_k$.

Definition: Let V be an inner product space and let $M_1, M_2, \dots, M_k$ be subspaces of V. The **orthogonal sum** of the subspaces M is defined as

$$< m_1, m_2 >= 0 \quad \forall\, m_1 \in M_1 \text{ and } m_2 \in M_2$$

Orthogonality is denoted by $M_1 \perp M_2$.

Definition: Let $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$ and let $M_1 \perp M_2 \perp \dots \perp M_k$. Then, M is called the **orthogonal direct sum** of $M_1, M_2, \dots, M_k$ and we write $M = M_1 \overset{\perp}{\oplus} M_2 \overset{\perp}{\oplus} \dots \overset{\perp}{\oplus} M_k$.

Definition: Let $M = M_1 \oplus M_2 \oplus \dots \oplus M_k$ be a direct sum decomposition of V. Then, the **orthogonal complement** of M_i is defined as

$$M_i^{\perp} = \{x \in V \mid < x, m_i >= 0 \quad \forall\, m_i \in M_i\}$$

> **Theorem**: $M_i^{\perp}$ is a subspace of V.
> **Proof**: $x, y \in M^{\perp}$ then $\alpha x + \beta y \in M^{\perp}$ should be shown.
>
> - $0 \in M_i^{\perp}$ since $< 0, m_i >= 0$ for all $m_i \in M_i$.
> - Let $x, y \in M_i^{\perp}$ and let $\alpha, \beta \in \mathbb{R}$. Then,
>
> $$< \alpha x + \beta y, m_i >= \alpha < x, m_i > + \beta < y, m_i >= 0$$
>
> for all $m_i \in M_i$. Therefore, $\alpha x + \beta y \in M_i^{\perp}$.

Example: $V = \mathbb{R}^3 \qquad M = \operatorname{span}\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \right\} \qquad M^{\perp} = ?$

Solution: Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in M^{\perp}$

$< x, m_1 >= 0 \qquad < x, m_2 >= 0$

$< x, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} >= 0$

$x_2 - x_3 = 0$

$x_2 = x_3$

$< x, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} >= 0$

$-x_1 + x_3 = 0$

$x_1 = x_3$

$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$M^{\perp} = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$

Theorem: Let V be an inner product space and let M be a subspace of V. Then, we can always write $M \oplus M^\perp = V$. That is V can always be written as the direct sum of a subspace and its orthogonal complement.

Proof: We need to show two things:

1. M and $M^\perp$ are linearly independent.
2. Any $x \in V$ can be written as $x = m + m^\perp$ where $m \in M$ and $m^\perp \in M^\perp$.

See lecture notes for the full proof.

#EE501 - `Linear Systems Theory` at `METU`