Subspaces

Definition: Let V be a vector space. A subset W of V is called a `subspace` of V iff W is a vector space with respect to the operations of V, that is,

- (S1) $w_1 + w_2 \in W \ \forall w_1, w_2 \in W$ (closure under addition)
- (S2) $cw \in W \ \forall c \in F$ and $\forall w \in W$ (closure under scalar multiplication)

> **Remark**: W is a subspace of V iff W is nonempty and W is closed under addition and scalar multiplication. All other axioms are inherited from the original vector space V.

Example: linear space $V = \mathbb{R}^2$, subspace $W = [\alpha \ 0]^T \in \mathbb{R}^2 : \alpha \in \mathbb{R}$
Solution: Let $w_1 = [\alpha_1 \ 0]^T$ and $w_2 = [\alpha_2 \ 0]^T$ and $c \in \mathbb{R}$,

- (S1) $w_1 = \begin{bmatrix} \alpha_1 \\ 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} \alpha_2 \\ 0 \end{bmatrix}$ be two arbitrary elements of W. Then $w_1 + w_2 = \begin{bmatrix} \alpha_1 + \alpha_2 \\ 0 \end{bmatrix} \in W$
- (S2) $cw1 = \begin{bmatrix} c\alpha_1 \\ 0 \end{bmatrix} \in W$ for all $c \in \mathbb{R}$. Hence W is a subspace of V.

Example: linear space $V = \mathbb{R}^2$, subspace $W = [\alpha \ 1]^T \in \mathbb{R}^2 : \alpha \in \mathbb{R}$
Solution: Let $w_1 = [\alpha_1 \ 1]^T$ and $w_2 = [\alpha_2 \ 1]^T$

- (S1) $w_1 = \begin{bmatrix} \alpha_1 \\ 1 \end{bmatrix}$ and $w_2 = \begin{bmatrix} \alpha_2 \\ 1 \end{bmatrix}$ be two arbitrary elements of W. Then $w_1 + w_2 = \begin{bmatrix} \alpha_1 + \alpha_2 \\ 2 \end{bmatrix} \notin W$

> **Remark**: In $\mathbb{R}^2$, a subspace is a line through the origin. Any line through the origin is a subspace of $\mathbb{R}^2$.

In function spaces, an example can be given as follows:
Example: linear space $V =$ set of all real-valued functions of a real variable $t \to f(t)$;
subspace $W_1 =$ set of all continuous functions [+]
subspace $W_2 =$ set of all constant functions [+]
subspace $W_3 =$ set of all functions periodic with π [+] subspace $W_4 =$ set of all functions which are discontinuous at $t = 1$ [-]

> **Remark**: 0 vector is a subspace of any vector space, even subspace of itself, and it is the smallest subspace.

[+] W_1, W_2, W_3 are subspaces of V
- [ ] W_4 is not a subspace of V

Example: Show that $Y + Z$ is a linear subspace of X, if Y and Z are also linear subspaces of X.
Proof: Let $w_1 + w_2 \in W$, with
$w_1 = y_1 + z_1$ where $y_1 \in Y, \ z_1 \in Z$
$w_2 = y_2 + z_2$ where $y_2 \in Y, \ z_2 \in Z$
then

$$w_1 + w_2 = y_1 + z_1 + y_2 + z_2 = (y_1 + y_2) + (z_1 + z_2)$$
$$y_1 + y_2 \in Y, z_1 + z_2 \in Z \Rightarrow w_1 + w_2 \in Y + Z$$

Shows that $Y + Z$ is closed under addition. (lemma 1)

Let $cw_1 \in W, \forall c \in F$

$$cw_1 = c(y_1 + z_1) = (cy_1) + (cz_1)$$
$$cy_1 \in Y, cz_1 \in Z \Rightarrow cw_1 \in Y + Z$$

Shows that $Y + Z$ is closed under scalar multiplication. (lemma 2) Hence $Y + Z$ is a linear subspace of X.

Example: If Y and Z are subspaces of X, then $Y \cap Z$ is a subspace of X.
Proof:

- $0 \in Y$, $0 \in Z$ then by definition $0 \in Y \cap Z$
- for $u, w \in Y$ and $u, w \in Z \implies u, w \in Y \cap Z$

now we need to show that $u + w \in Y \cap Z$ (closure under addition)

- starting $u \in Y, w \in Y \implies u + w \in Y$
- similarly $u \in Z, w \in Z \implies u + w \in Z$
  - hence $u + w \in Y \cap Z$

now we need to show that $cu \in Y \cap Z$ (closure under scalar multiplication)

- starting $u \in Y \implies cu \in Y$, $\forall c \in F$
- similarly $u \in Z \implies cu \in Z$, $\forall c \in F$
  - hence $cu \in Y \cap Z$

Hence $Y \cap Z$ is a subspace of X.

Example: For Y and Z are subspaces of X, show that whether $Y \cup Z$ is a subspace of X or not.
Proof: Prove by contradiction.

- Assume $Y \cup Z$ is a subspace of X. Then $Y \cup Z$ is closed under addition and scalar multiplication.
- Let $Y = \{(y, 0) : y \in \mathbb{R}\}$ and $Z = \{(0, z) : z \in \mathbb{R}\}$.
- Then $u_1 = (1, 0) \in Y$ and $u_2 = (0, 1) \in Z$.
- $u_1 + u_2 = (1, 1) \notin Y \cup Z$.

Example: Is $\mathbb{R}^2$ a subspace of complex vector space $\mathbb{C}^2$?
Proof: Note that we consider complex vector space, so if $x \in \mathbb{R}^2$ then $x \in \mathbb{C}^2$.

- $i \in \mathbb{C}$ and $x \in \mathbb{R}^2$ then $ix \in \mathbb{R}^2$.
  - $i(1, 1) = (i, i) \notin \mathbb{R}^2$

Hence $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$.

Sums of Subspaces

Subspaces

Definition: Suppose $W_1, ..., W_m$ are subspaces of a vector space V. The sum of $W_1, ..., W_m$ is the set of all possible sums of elements of $W_1, ..., W_m$:

$$W_1 + ... + W_m = \{w_1 + ... + w_m : w_i \in W_i, i = 1, ..., m\}$$

> **Remark**: $W_1 + ... + W_m$ is the smallest subspace of V containing $W_1, ..., W_m$.

Direct Sums

Definition: Suppose $W_1, ..., W_m$ are subspaces of a vector space V. The sum $W_1 + ... + W_m$ is called a direct sum if each element of $W_1 + ... + W_m$ can be written in one and only one way as a sum $w_1 + ... + w_m$ with $w_i \in W_i$.

> **Remark**: if $W_1 + ... + W_m$ is a direct sum, then $W_1 \oplus ... \oplus W_m$ is used to denote the direct sum.

Example: Suppose U_j is a subspace of F^n of those vectors whose jth component is zero. As $U_2 = \{(0, x, ..., 0) \in F^n : x \in F\}$, Then.

$$U_1 \oplus ... \oplus U_n = F^n$$

Condition for Direct Sum: Suppose $W_1, ..., W_m$ are subspaces of a vector space V. Then $W_1 + ... + W_m$ is a direct sum iff the only way to write 0 as a sum $w_1 + ... + w_m$ with $w_i \in W_i$ is by taking $w_1 = ... = w_m = 0$. In other words, Suppose

W and U are subspaces of a vector space V. Then $W \oplus U$ is a direct sum iff $W \cap U = \{0\}$.

Proof: Proving the statements with iff is equivalent to proving the statements separately, in both directions.

$$W \oplus U \text{ is a direct sum} \iff W \cap U = \{0\}$$