Linear (Vector) Spaces

Definition: A linear space (or vector space) over a `field` F is a set V with two operations, addition and scalar multiplication such that the following axioms hold:

Vector Addition $(+) : V \times V \to V$ such that $x + y \in V$ for all $x, y \in V$.

- (A1) $x + y = y + x \ \forall x, y \in V$ (commutativity)
- (A2) $x + (y + z) = (x + y) + z \ \forall x, y, z \in V$ (associativity)
- (A3) There exists an element $0 \in V$ such that $x + 0 = x \ \forall x \in V$ (existence of additive identity)
- (A4) For every $x \in V$ there exist an element $-x \in V$ such that $x + (-x) = 0 \ \forall x \in V$ (existence of additive inverse)

Scalar Multiplication $(\cdot) : F \times V \to V$ such that $\alpha x \in V$ for all $\alpha \in F$ and $x \in V$.

- (M1) $\alpha(bx) = (\alpha b)x \ \forall \alpha, \beta \in F$ and $\forall x \in V$ (associativity)
- (M2) $\alpha(x + y) = \alpha x + \alpha y \ \forall \alpha \in F$ and $\forall x, y \in V$ (distributivity)
- (M3) $(\alpha + \beta)x = \alpha x + \beta x \ \forall \alpha, \beta \in F$ and $\forall x \in V$ (distributivity)
- (M4) $1_F x = x \ \forall x \in V$ (existence of multiplicative identity)

Elements of the vector space are called vectors, or points.

Scalar multiplication in a vector space depends on the field F. Thus when we say $x \in V$, we mean that x is a vector in the vector space V over the field F or (V, F).
For example, $\mathbb{R}^n$ is a vector space over the field $\mathbb{R}$, and $\mathbb{C}^n$ is a vector space over the field $\mathbb{C}$.

> Sharing the same field is a necessary condition for two vector spaces to be comparable. For example, $\mathbb{R}^n$ and $\mathbb{C}^n$ are not comparable.

> The simplest vector space contains only one point. In other words, $\{0\}$ is a vector space.

Example: Show that $x0_F = 0_V$ for all $x \in V$
Proof:

1. $y = x0_F = 0_V$ is assumed to be true.
2. $x + y = x + x0_F = x1_F + x0_F$ (M4)
3. $x + y = x(1_F + 0_F) = x1_F$ (M3)
4. $x + y + (-x) = x1_F + (-x1_F)$ (A4)
5. $y = x(1_F + (-1_F)) = x0_V = 0_V$ (A2) ■

Example: Show that $x0_F = 0_v$
Proof:

1. Assume $y = x0_F$ and show $y = 0_V$.
2. $x + y = x + x0_F = x1_F + x0_F$ (M4)
3. $x + y = x1_F + x0_F = x(1_F + 0_F)$ (M3)
4. $x + y = x(1_F + 0_F) = x1_F$ (A3)
5. $x + y = x1_F = x$ (M4)
6. $x + y + (-x) = x + (-x)$ (A4)
7. $y = x + (-x) = 0_V$ (A4) ■

> **Remark**: A vector space has a unique additive identity.

Function Space

Definition: Let S be a set and F be a field. The set of all functions from S to F is denoted by F^S.

For $f, g \in F^S$, the sum $f + g \in F^S$ is the function defined by

$$(f+g)(x) = f(x) + g(x),\ \forall x \in S$$

For $f \in F^S$ and $\alpha \in F$, the product $\alpha f \in F^S$ is the function defined by

$$(\alpha f)(x) = \alpha f(x),\ \forall x \in S$$

Example: Set of all `polynomials` with degree n with coefficients in F is denoted by $F[x]_n$.

> What essentially function spaces are, is that they are the set of all functions from a set to a field. For example, the set of all polynomials with degree n with coefficients in F is denoted by $F[x]_n$.

#EE501 - `Linear Systems Theory` at `METU`