

**Definition:** Let  $V$  and  $W$  be linear spaces over the same field  $F$ . A mapping  $\mathcal{T} : V \rightarrow W$  is called a linear mapping satisfying

$$\mathcal{T}(ax + by) = a\mathcal{T}(x) + b\mathcal{T}(y) \quad \forall x, y \in V \quad \forall a, b \in F$$

here  $V$  is called the domain of  $T$  and  $W$  is called the codomain of  $\mathcal{T}$ .

**Example:** Let  $V = W$  polynomials of degree less than  $n$  in  $S$ ;  $\mathcal{T} = \frac{d}{ds}$

**Solution:** Let  $p, q \in V$  and  $\alpha_1, \alpha_2 \in F$  then ,

$$p(s) = \sum_{i=0}^{n-1} a_i s^i \text{ and } q(s) = \sum_{i=0}^{n-1} b_i s^i$$

$$\alpha_1 p(s) + \alpha_2 q(s) = \sum_{i=0}^{n-1} (\alpha_1 a_i + \alpha_2 b_i) s^i$$

$$\frac{d}{ds}(\alpha_1 p(s) + \alpha_2 q(s)) = \sum_{i=0}^{n-1} (\alpha_1 a_i + \alpha_2 b_i) i s^{i-1} = \alpha_1 \sum_{i=0}^{n-1} a_i i s^{i-1} + \alpha_2 \sum_{i=0}^{n-1} b_i i s^{i-1} = \alpha_1 \frac{dp}{ds} + \alpha_2 \frac{dq}{ds}$$

$$\mathcal{T}(\alpha_1 p + \alpha_2 q) = \frac{d}{ds}(\alpha_1 p + \alpha_2 q) = \alpha_1 \frac{dp}{ds} + \alpha_2 \frac{dq}{ds} = \alpha_1 \mathcal{T}(p) + \alpha_2 \mathcal{T}(q) \blacksquare$$

**Example:** Let  $V = W = \mathbb{R}^2$ . Let  $\mathcal{A}$  be defined as,

$$\mathcal{A} = \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} \text{ where } x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$$

**Solution:** Let  $a, b \in F$  and  $x_1, x_2 \in X$  with  $x_1 = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$  and  $x_2 = \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}$  then,

$$\begin{aligned} \mathcal{A}(ax_1 + bx_2) &= \mathcal{A}\left(a \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} + b \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix}\right) = \mathcal{A}\left(\begin{bmatrix} a\alpha_1 \\ a\alpha_2 \end{bmatrix} + \begin{bmatrix} b\beta_1 \\ b\beta_2 \end{bmatrix}\right) = \mathcal{A}\left(\begin{bmatrix} a\alpha_1 + b\beta_1 \\ a\alpha_2 + b\beta_2 \end{bmatrix}\right) = \\ &= \begin{bmatrix} a\alpha_1 + b\beta_1 \\ a\alpha_1 + a\alpha_2 + b\beta_1 + b\beta_2 \end{bmatrix} = a \begin{bmatrix} \alpha_1 \\ \alpha_1 + \alpha_2 \end{bmatrix} + b \begin{bmatrix} \beta_1 \\ \beta_1 + \beta_2 \end{bmatrix} = a\mathcal{A}(x_1) + b\mathcal{A}(x_2) \blacksquare \end{aligned}$$

**Example:** Let  $V = W = \mathbb{R}$ . is  $\mathcal{A}x = (1 - x)$  linear or not ?

**Solution:** Let  $a, b \in F$  and  $x_1, x_2 \in X$  then,

$$\begin{aligned} \mathcal{A}(ax_1 + bx_2) &\stackrel{?}{=} a\mathcal{A}(x_1) + b\mathcal{A}(x_2) \\ 1 - (ax_1 + bx_2) &\stackrel{?}{=} a(1 - x_1) + b(1 - x_2) \\ 1 - ax_1 - bx_2 &\stackrel{?}{=} a - ax_1 + b - bx_2 \\ 1 &\neq a + b \quad \forall a, b \in F \blacksquare \\ &\text{hence not linear} \end{aligned}$$

Rotation transformations in  $\mathbb{R}^2$  are linear transformations.

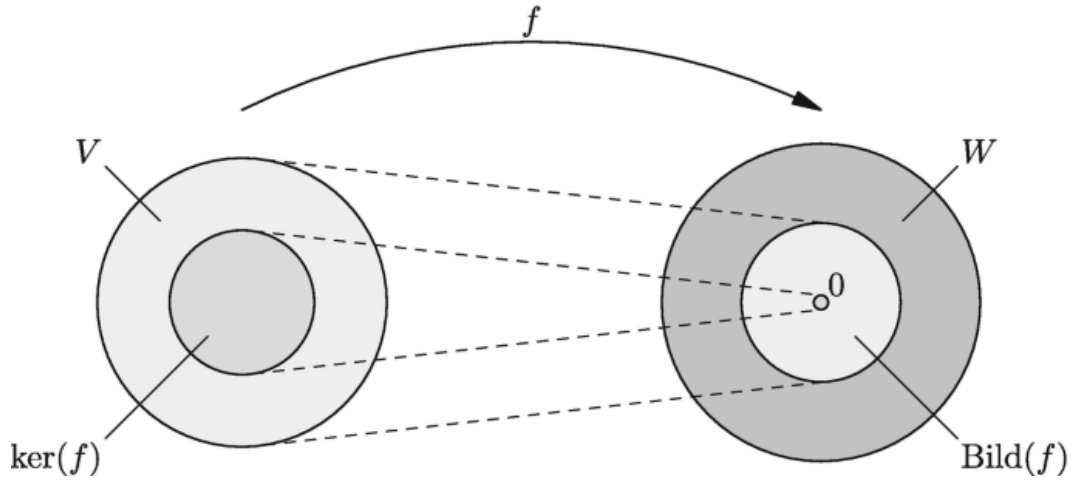
Integration and differentiation are linear transformations.

**Definition:** Given a linear mapping  $\mathcal{T} : V \rightarrow W$ , the set of all vectors  $x \in V$  such that  $\mathcal{T}(x) = 0_W$  is called the null space of  $\mathcal{T}$  and is denoted by  $N(\mathcal{T})$ . That is,

$$N(\mathcal{T}) := \{x \in V : \mathcal{T}(x) = 0_W\}$$

**Definition:** Given a linear mapping  $\mathcal{T} : V \rightarrow W$ , the set of all vectors  $w \in W$  such that  $w = \mathcal{T}(v)$  for some  $v \in V$  is called the range of  $\mathcal{T}$  and is denoted by  $R(\mathcal{T})$ . That is,

$$R(\mathcal{T}) := \{w \in W : w = \mathcal{T}(v) \text{ for some } v \in V\}$$




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Claim: For a given linear mapping  $\mathcal{T} : V \rightarrow W$ ,  $N(\mathcal{T})$  is a linear subspace of  $V$ .

**Proof:** Let  $x_1, x_2 \in N(\mathcal{T})$  and  $a \in F$  show,

(S1).  $x_1 + x_2 \in N(\mathcal{T})$

(S2).  $ax_1 \in N(\mathcal{T})$

1-  $\mathcal{T}(x_1 + x_2) = \mathcal{T}(x_1) + \mathcal{T}(x_2) = 0_W + 0_W = 0_W \implies x_1 + x_2 \in N(\mathcal{T})$

2-  $\mathcal{T}(ax_1) = a\mathcal{T}(x_1) = a0_W = 0_W \implies ax_1 \in N(\mathcal{T})$  ■

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Claim: For a given linear mapping  $\mathcal{T} : V \rightarrow W$ ,  $R(\mathcal{T})$  is a subspace of  $W$ .

**Proof:** Let  $x_1, x_2 \in R(\mathcal{T})$  and  $a \in F$  show,

(S1).  $x_1 + x_2 \in R(\mathcal{T})$

(S2).  $ax_1 \in R(\mathcal{T})$

**Definition:** A linear transformation  $\mathcal{T} : V \rightarrow W$  is called one-to-one if  $x_1 \neq x_2$  implies  $\mathcal{T}(x_1) \neq \mathcal{T}(x_2)$  for all  $x_1, x_2 \in V$ .

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Theorem: Let  $\mathcal{T} : V \rightarrow W$  be a linear transformation. Then mapping  $\mathcal{T}$  is one-to-one if and only if  $N(\mathcal{T}) = \{0_V\}$ .

**Proof:** We will prove the statement by contrapositive. Since it is an if and only if statement, we will prove both directions.

(Backward direction) Assume that  $N(\mathcal{T}) = \{0_V\}$  and  $\mathcal{T}(x_1) = \mathcal{T}(x_2)$  for some  $x_1, x_2 \in V$ . Then,

$$\mathcal{T}(x_1) - \mathcal{T}(x_2) = 0_W$$

$$\mathcal{T}(x_1 - x_2) = 0_W$$

$$x_1 - x_2 \in N(\mathcal{T})$$

$$x_1 - x_2 = 0_V$$

$$x_1 = x_2$$

$\mathcal{T}$  is one-to-one.

(Forward direction) Assume that  $\mathcal{T}$  is one-to-one and  $x \in N(\mathcal{T})$ . Then,

$$\mathcal{T}(x) = 0_W$$

$$\mathcal{T}(0_V) = 0_W$$

$$x = 0_V$$

$$N(\mathcal{T}) = \{0_V\} \blacksquare$$

**Definition:** A linear transformation  $\mathcal{T} : V \rightarrow W$  is called onto if  $R(\mathcal{T}) = W$ , otherwise if  $R(\mathcal{T}) \subset W$  then  $\mathcal{T}$  is called into.

**Example:** Let  $V := \{f : [0, 1] \rightarrow \mathbb{R} \text{ and } f \text{ is integrable}\}$ . A transformation  $\mathcal{A} : V \rightarrow \mathbb{R}$  is defined as,

$$\mathcal{A}(f(s)) = \int_0^1 f(s) ds$$

is  $\mathcal{A}$  one-to-one ?

**Solution:** Integration operation resulting in one-to-one transformation probably not true. Hence we can exploit the fact that the integration might result in zero.

Let  $f(s) = 2s - 1$  then,

$$\mathcal{A}(f(s)) = \int_0^1 (2s - 1) ds = [s^2 - s]_0^1 = 0$$

$$\mathcal{A}(f(s)) = 0$$

Then  $\mathcal{A}(0) = 0_w$  and  $\mathcal{A}(f(s)) = 0_w$  for some  $f(s) \neq 0$ .

$\mathcal{A}$  is not one-to-one.

moreover,

Let  $f(s) = a$  then,

$$\mathcal{A}(f(s)) = \int_0^1 a ds = [as]_0^1 = a$$

Shows that  $\mathcal{A}$  is onto.

## Matrix Representations

### Linear Transformations

**Definition:** Let  $\mathcal{T} : V \rightarrow W$  be a linear transformation with  $\dim(V) = n$  and  $\dim(W) = m$ . Let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  be a basis for  $V$  and  $\mathcal{C} = \{w_1, w_2, \dots, w_m\}$  be a basis for  $W$ . Then, the matrix representation of  $\mathcal{T}$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$  is the  $m \times n$  matrix  $A$  such that,

$$[w]_{\mathcal{C}} = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_m \end{bmatrix} \quad [v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$[\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

$$\mathcal{T}(v_j) = \sum_{i=1}^m a_{ij} w_i \text{ for } j = 1, 2, \dots, n$$

**Remark:** The matrix representation of  $\mathcal{T}$  with respect to  $\mathcal{B}$  and  $\mathcal{C}$  is denoted by  $[\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}}$ .

Now we have a transformation represented as,

$$[w]_c = [\mathcal{T}]_{\mathcal{B}}^{\mathcal{C}} [v]_b$$

## A formal procedure to obtain the matrix representation of a linear transformation

1. Take each basis vector  $v_j$  in  $\mathcal{B}$
2. Apply  $\mathcal{A}$  to  $v_j$  :  $\mathcal{A}(v_j)$
3. Express the result in terms of the basis vectors in  $\mathcal{C}$  :  $\mathcal{A}(v_j) = \sum_{i=1}^m a_{ij} w_i$
4. The  $j$ th column of  $[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}}$  is the vector  $\begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$

Example:  $V = \{\text{Polynomials of degree less than } 3\}$  and  $W = \{\text{Polynomials of degree less than } 2\}$   
 Let  $\mathcal{A} : V \rightarrow W$  be defined as,

$$\mathcal{A}(p(s)) = \frac{dp(s)}{ds}$$

Find the matrix representation of  $\mathcal{A}$  with respect to the bases  $\mathcal{B} = \{1, 1+s, 1+s+s^2, 1+s+s^2+s^3\}$  and  $\mathcal{C} = \{1, 1+s, 1+s+s^2\}$ .

**Solution:**  $[w]_c = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}$  and  $[v]_b = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$$\mathcal{A}(v_1) = \frac{d}{ds}(1) = 0 = 0w_1 + 0w_2 + 0w_3$$

$$\mathcal{A}(v_2) = \frac{d}{ds}(1+s) = 1 = 1w_1 + 0w_2 + 0w_3$$

$$\mathcal{A}(v_3) = \frac{d}{ds}(1+s+s^2) = 1+2s = -1w_1 + 2w_2 + 0w_3$$

$$\mathcal{A}(v_4) = \frac{d}{ds}(1+s+s^2+s^3) = 1+2s+3s^2 = -1w_1 - 1w_2 + 3w_3$$

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The full matrix representation of  $\mathcal{A}$  is,

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

**Example:** Let  $V = \mathbb{R}^2$  and  $\mathcal{A} : V \rightarrow V$  be defined as,

$$\mathcal{A}(x) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + x \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

Find the matrix representation of  $\mathcal{A}$  with respect to the bases

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}. \text{ and}$$

$$\mathcal{C} = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}.$$

**Solution:**  $[w]_{\mathcal{C}} = \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}$  and  $[v]_{\mathcal{B}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$

$$\mathcal{A}(v_1) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = 1w_1 + 0w_2 - 1w_3 + 0w_4$$

$$\mathcal{A}(v_2) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1w_1 + 0w_2 + 1w_3 - 1w_4$$

$$\mathcal{A}(v_3) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = 1w_1 + 0w_2 + 1w_3 - 1w_4$$

$$\mathcal{A}(v_4) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -1w_1 + 0w_2 + 1w_3 + 0w_4$$

$$[\mathcal{A}]_{\mathcal{B}}^{\mathcal{C}} = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

## Change of Basis

## Linear Transformations

**Definition:** Let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  and  $\mathcal{C} = \{w_1, w_2, \dots, w_n\}$  be two bases for a linear space  $V$ . The change of basis matrix from  $\mathcal{B}$  to  $\mathcal{C}$  is the  $n \times n$  matrix  $P$  such that,

$$[w]_{\mathcal{C}} = A[v]_{\mathcal{B}}$$

$$[w]_{\mathcal{C}} = \bar{A}[v]_{\mathcal{B}}$$

We know that a change of basis is a linear transformation. Hence,

$$[v]_B = P[v]_{\bar{B}}$$

$$[w]_C = AP[v]_{\bar{B}}$$

in codomain perspective,

$$[w]_C = Q[w]_{\bar{C}}$$

$$[w]_{\bar{C}} = Q^{-1}A[v]_B$$

$$[w]_{\bar{C}} = Q^{-1}AP[v]_{\bar{B}}$$

Example:

$$V = \{\text{Polynomials with degree less than 3}\}$$

$$W = \{\text{Polynomials with degree less than 2}\}$$

$$\mathcal{B} = \{1, 1 + s, 1 + s + s^2, 1 + s + s^2 + s^3\}$$

$$\mathcal{C} = \{1, 1 + s, 1 + s + s^2\}$$

$$A = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\bar{\mathcal{B}} = \{1, s, s^2, s^3\}$$

**Solution:** First we will find the change of basis in the domain matrix from  $\mathcal{B}$  to  $\bar{\mathcal{B}}$ . That is more clearly stated as,

$$[w]_C = \bar{A}[v]_{\bar{B}}$$

and given  $[v]_B = P[v]_{\bar{B}}$ ,  $\bar{A}$  is equal to,

$$[w]_C = AP[v]_{\bar{B}}$$

In order to find  $P$  we need to write the basis vectors in  $\bar{\mathcal{B}}$  in terms of  $\mathcal{B}$ .

$$1 = 1(1) + 0(1 + s) + 0(1 + s + s^2) + 0(1 + s + s^2 + s^3)$$

$$s = -1(1) + 1(1 + s) + 0(1 + s + s^2) + 0(1 + s + s^2 + s^3)$$

$$s^2 = 0(1) + -1(1 + s) + 1(1 + s + s^2) + 0(1 + s + s^2 + s^3)$$

$$s^3 = 0(1) + 0(1 + s) + -1(1 + s + s^2) + 1(1 + s + s^2 + s^3)$$

$$P = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

now  $\bar{A}$  is equal to,

$$\bar{A} = AP = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Now we will change the basis in the codomain to canonical basis while keeping the basis of the domain as also in canonical form. That is,

$$[w]_{\mathcal{C}} = Q^{-1}AP[v]_{\mathcal{B}}$$

$$[w]_{\mathcal{C}} = Q[w]_{\bar{\mathcal{C}}}$$

$$[w]_{\bar{\mathcal{C}}} = Q^{-1}[w]_{\mathcal{C}}$$

In order to find  $Q^{-1}$  in a single step, we can write the basis vectors in  $\mathcal{C}$  in terms of  $\bar{\mathcal{C}}$ .

$$1 = 1(1) + 0(s) + 0(s^2)$$

$$1 + s = 1(1) + 1(s) + 0(s^2)$$

$$1 + s + s^2 = 1(1) + 1(s) + 1(s^2)$$

$$Q^{-1} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

as the final steps,

$$[w]_{\bar{\mathcal{C}}} = Q^{-1}\bar{A}[v]_{\mathcal{B}}$$

$$Q^{-1}\bar{A} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & -2 & 0 \\ 0 & 0 & 2 & -3 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$[w]_{\bar{\mathcal{C}}} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix} [v]_{\mathcal{B}}$$

Given the matrix representation of a linear transformation  $\mathcal{A} : V \rightarrow W$  with respect to bases  $\mathcal{B}$  and  $\mathcal{C}$ , one can draw the following diagram.

