Range and Null Space

Definition: Let $T : V \to W$ be a linear transformation. The set of all vectors $y \in W$ such that $y = T(x)$ for some $x \in V$ is called the **range** of T and denoted by $\mathcal{R}(T)$.

Definition: Let $T : V \to W$ be a linear transformation. The set of all vectors $x \in V$ such that $T(x) = 0$ is called the **null space** of T and denoted by $\mathcal{N}(T)$.

Let $T : V \to W$ be a linear transformation. Let $dim(V) = n$ and $dim(W) = m$. Recall that,

- $\mathcal{R}(T) : \text{Range space of } T, \mathcal{R}(T) \subseteq W, dim(\mathcal{R}(T)) = dim(col(T))$
- $\mathcal{N}(T) : \text{Null space of } T, \mathcal{N}(T) \subseteq V, dim(\mathcal{N}(T)) = dim(null(T))$

Theorem: $dim(\mathcal{R}(T)) + dim(\mathcal{N}(T)) = dim(V)$

Proof: Let $B = \{v_1, v_2, \ldots, v_k\}$ be a basis for $\mathcal{N}(T)$. Extend B to a basis $B' = \{v_1, v_2, \ldots, v_k, v_{k+1}, \ldots, v_n\}$ for V. Then B' is a basis for V and $B'' = \{T(v_{k+1}), T(v_{k+2}), \ldots, T(v_n)\}$ is a basis for $\mathcal{R}(T)$. Hence $dim(\mathcal{R}(T)) = n - k = dim(V) - dim(\mathcal{N}(T))$ ■

Definition: Let $T : V \to W$ be a linear transformation. The **rank** of T is defined as $rank(T) = dim(\mathcal{R}(T))$.

> **Remark**: $rank(T)$ is equal to the number of linearly independent columns of T.

> **Remark**: In general, a basis for $\mathcal{R}(T)$ and $\mathcal{N}(T)$ can be found by finding a basis for the column space and null space of the matrix representation of T with respect to some basis for V and W.

Eample: $V = \mathbb{R}^{2x2}$, $W = V$, Find a basis for $\mathcal{R}(\mathcal{A})$

$$B = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$
$$C = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$
$$\mathcal{A}(x) = Sx + xS^\top = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x + x \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}^\top$$
$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix}$$

Solution: Start with finding a basis for $\mathcal{R}(\mathcal{A})$

$$\mathcal{R}(\mathcal{A}) := \{w \in W \mid \exists v \in V,\ \mathcal{A}v = w\}$$
$$\mathcal{R}(A) := \{y \in \mathbb{R}^4 \mid \exists x \in \mathbb{R}^4,\ Ax = y\}$$

- Start with finding a basis for $\mathcal{R}(A)$
- Convert the vectors in the basis into matrices using the basis C.

Finding a basis for $\mathcal{R}(A)$ is equivalent to finding a basis for the column space of A. The basis for the column space of A can be found by reducing A to independent columns by elementary column operations and then taking the non-zero columns of the reduced matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\text{basis for } \mathcal{R}(A) = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 2 \\ -1 \end{bmatrix} \right\}$$

$$\begin{bmatrix}1\\0\\-1\\0\end{bmatrix} = [w_1]_C \implies w_1 = 1c_1 + 0c_2 + (-1)c_3 + 0c_4 = c_1 - c_3$$

$$= \begin{bmatrix}1&0\\0&0\end{bmatrix} - \begin{bmatrix}1&1\\1&0\end{bmatrix} = \begin{bmatrix}0&-1\\-1&0\end{bmatrix}$$

$$\begin{bmatrix}0\\0\\2\\-1\end{bmatrix} = [w_2]_C \implies w_2 = 0c_1 + 0c_2 + 2c_3 + (-1)c_4 = 2c_3 - c_4$$

$$= \begin{bmatrix}2&2\\2&0\end{bmatrix} - \begin{bmatrix}1&1\\1&1\end{bmatrix} = \begin{bmatrix}1&1\\1&-1\end{bmatrix}$$

$$\text{basis for } \mathcal{R}(\mathcal{A}) = \left\{ \begin{bmatrix}0&-1\\-1&0\end{bmatrix}, \begin{bmatrix}1&1\\1&-1\end{bmatrix} \right\}$$
$$dim(\mathcal{R}(A)) = 2 = \text{rank}(A)$$

1. A is a $4x4$ matrix, $dim(\mathcal{R}(A)) = 4 - dim(\mathcal{N}(A))$
2. $dim(\mathcal{N}(A)) = 2$ since $rank(A) = 2$
3. $dim(\mathcal{R}(A)) = 4 - 2 = 2$
4. $dim(\mathcal{R}(\mathcal{A})) = 2$

Now, we need to find a basis for $\mathcal{N}(\mathcal{A})$.

$$\mathcal{N}(\mathcal{A}) := \{v \in V \mid \ \mathcal{A}v = 0\}$$
$$\mathcal{N}(A) := \{x \in \mathbb{R}^4 \mid \ Ax = 0_W\}$$

- Start with finding a basis for $\mathcal{N}(A)$
- Convert the vectors in the basis into matrices using the basis B.

Finding a basis for $\mathcal{N}(A)$ is equivalent to finding a basis for the null space of A. The basis for the null space of A can be found by reducing A by elementary row operations and then equating the $\bar{A}x = 0$.

$$A = \begin{bmatrix}1&1&1&-1\\0&0&0&0\\-1&1&1&1\\0&-1&-1&0\end{bmatrix} \sim \begin{bmatrix}1&0&0&-1\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} = \bar{A}$$

$$\bar{A}x = 0$$
$$\begin{bmatrix}1&0&0&-1\\0&1&1&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$$
$$x_1 - x_4 = 0 \implies x_1 = x_4$$
$$x_2 + x_3 = 0 \implies x_2 = -x_3$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}x_1\\x_2\\-x_2\\x_1\end{bmatrix} = x_1\begin{bmatrix}1\\0\\0\\1\end{bmatrix} + x_2\begin{bmatrix}0\\1\\-1\\0\end{bmatrix}$$

$$\text{basis for } \mathcal{N}(A) = \left\{ \begin{bmatrix}1\\0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\-1\\0\end{bmatrix} \right\}$$

$$\begin{bmatrix}1\\0\\0\\1\end{bmatrix} = [v_1]_B \implies v_1 = 1b_1 + 0b_2 + 0b_3 + 1b_4 = b_1 + b_4$$

$$= \begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}0&0\\0&1\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} = [v_2]_B \implies v_2 = 0b_1 + 1b_2 + (-1)b_3 + 0b_4 = b_2 - b_3$$

$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\text{basis for } \mathcal{N}(\mathcal{A}) = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

$$dim(\mathcal{N}(A)) = 2 = \text{nullity}(A)$$

Definition: Let $T : V \to W$ be a linear transformation. The **nullity** of T is defined as $nullity(T) = dim(\mathcal{N}(T))$.

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