# Hilbert Projection Theorem

Let $H$ be a Hilbert space ( inner product space that is complete with respect to the norm induced by the inner product) and $M$ be a finite dimensioal subspace of $H$. Then for any $x \in H$, there exists a unique $y \in M$ such that

$$\min_{m \in M} \|x - m\|$$

has a unique solution $y$. In other words "we can find a unique point in $M$ that is closest to $x$". If $m^*$ is the closest point to $x$ in $M$, then $x - m^* \perp M$.

**Proof:** See lecture notes.

> **Remark:** The proof stated that $m^* = x_1$ is the closest point to $M$. It can also be interpreted as the best approximation of $x$ choosen from the vectors in $M$. The $x_2$ term is the error in the approximation.

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Example: Let $V = \mathbb{R}^2$ and $M = \text{span}\{[1, 1]^T\}$. Find the best approximation of $x = [4, 7]^T$ in $M$.

Solution: We need to find $m^*$ such that $m^* = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\|x - m^*\|$ is minimum.

$$(x - m^*) \perp M \implies < x - m^*, m >= 0 \quad \forall m \in M$$

$$< x - m^*, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 0$$

$$\text{Replace } x \text{ and } m^* \text{ with their values.}$$

$$< \begin{bmatrix} 4 - \alpha \\ 7 - \alpha \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 0$$

$$\text{Recall that } < x, y >= x^T y.$$

$$4 - \alpha + 7 - \alpha = 0$$

$$\alpha = \frac{11}{2}$$

$$m^* = \frac{11}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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Example: Let $x \in V$ and $M = \text{span}\{v_1, v_2\}$. Find the best approximation of $x$ in $M$.

Solution: We need to find $m^*$ such that $m^* = \alpha_1 v_1 + \alpha_2 v_2$ that is in the span of $M$ and $\|x - m^*\|$ is minimum.

$$(x - m^*) \perp M \implies (x - m^*) \perp \text{both } v_1 \text{ and } v_2$$

$$< x - \alpha_1 v_1 - \alpha_2 v_2, v_1 >=< x, v_1 > -\alpha_1 < v_1, v_1 > -\alpha_2 < v_2, v_1 >= 0$$

$$< x - \alpha_1 v_1 - \alpha_2 v_2, v_2 >=< x, v_2 > -\alpha_1 < v_1, v_2 > -\alpha_2 < v_2, v_2 >= 0$$

$$\begin{bmatrix} < v_1, v_1 > & < v_2, v_1 > \\ < v_1, v_2 > & < v_2, v_2 > \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} < x, v_1 > \\ < x, v_2 > \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} < v_1, v_1 > & < v_2, v_1 > \\ < v_1, v_2 > & < v_2, v_2 > \end{bmatrix}^{-1} \begin{bmatrix} < x, v_1 > \\ < x, v_2 > \end{bmatrix}$$

$$m^* = \alpha_1 v_1 + \alpha_2 v_2$$

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Example: Let $V = \mathbb{R}^3$ and $M = \text{span}\{[1, 1, 1]^T, [1, 0, 1]^T\}$. Find the best approximation of $x = [4, 7, 2]^T$ in $M$.

<u>Solution:</u> We need to find $m^*$ such that $m^* = \alpha_1 \begin{bmatrix}1\\1\\1\end{bmatrix} + \alpha_2 \begin{bmatrix}1\\0\\1\end{bmatrix}$ that is in the span of $M$ and $\|x - m^*\|$ is minimum.

$$(x - m^*) \perp M \implies (x - m^*) \perp \text{both } v_1 \text{ and } v_2$$

$$\begin{bmatrix}< v_1, v_1 > & < v_2, v_1 > \\ < v_1, v_2 > & < v_2, v_2 >\end{bmatrix} \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}< x, v_1 > \\ < x, v_2 >\end{bmatrix}$$

$$\text{Recall that } < x, y >= x^T y.$$

$$\begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix} \begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}13\\ 6\end{bmatrix}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}3 & 2\\ 2 & 2\end{bmatrix}^{-1} \begin{bmatrix}13\\ 6\end{bmatrix}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}1 & -1\\ -1 & 3/2\end{bmatrix} \begin{bmatrix}13\\ 6\end{bmatrix}$$

$$\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}7\\ -4\end{bmatrix}$$

$$m^* = \alpha_1 v_1 + \alpha_2 v_2 = 7 \begin{bmatrix}1\\1\\1\end{bmatrix} - 4 \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\7\\3\end{bmatrix}$$

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<u>Example:</u> Let $H$ be the space of square integrable functions on $[-\pi, \pi]$ with inner product $< f, g >= \int_{-\pi}^{\pi} f(t)\bar{g(t)}dt$. Let $M$ be the subspace of $H$, $M = \text{span}\{e^{jkt}/\sqrt{2\pi}\}$, $k \text{ from } -N \text{ to } N$. Note that dimension of $M$ is $2N + 1$.

$$< f_n, f_m >= \int_{-\pi}^{\pi} \frac{e^{jnt}}{\sqrt{2\pi}} \frac{e^{-jmt}}{\sqrt{2\pi}} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(n-m)t} dt$$

$$\text{If } n \neq m, \text{ then } \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(n-m)t} dt = \frac{1}{2\pi} \frac{e^{j(n-m)t}}{j(n-m)} \Big|_{-\pi}^{\pi} = 0$$

$$\text{If } n = m, \text{ then } \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j(n-m)t} dt = \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 dt = 1$$

$$\text{Therefore, } < f_n, f_m >= \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases}$$

Now, let $g \in H$ be an arbitrary function. Then $g = g_1 + g_2$ where $g_1 \in M$ and $g_2 \in M^{\perp}$. We need to find $g_1$ such that $\|g - g_1\|$ is minimum.

$$g_1 = \sum_{k=-N}^{N} \alpha_k \frac{e^{jkt}}{\sqrt{2\pi}} \text{ where } \alpha_k =< g, f_k >= \int_{-\pi}^{\pi} g(t) \frac{e^{-jkt}}{\sqrt{2\pi}} dt$$

$$\alpha_k = \int_{-\pi}^{\pi} g(t) \frac{e^{-jkt}}{\sqrt{2\pi}} dt$$

$$g_1 = \sum_{k=-N}^{N} \int_{-\pi}^{\pi} g(t) \frac{e^{-jkt}}{\sqrt{2\pi}} \frac{e^{jkt}}{\sqrt{2\pi}} dt = \sum_{k=-N}^{N} \int_{-\pi}^{\pi} g(t) \frac{1}{2\pi} dt = \int_{-\pi}^{\pi} g(t) dt$$

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<u>Example:</u> Find the orthagonal projection of the vector $\begin{bmatrix}1\\0\\0\end{bmatrix}$ onto the subspace $M = \text{span}\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\end{bmatrix}\}$.

<u>Solution:</u> We know that the projection matrix is $P = B(B^T B)^{-1} B^T$ where $B$ is the matrix whose columns are the basis vectors of $M$. Therefore,

$$B = \begin{bmatrix}1 & 1\\ 1 & -1\\ 1 & 1\end{bmatrix}$$

$$B^TB = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 3/8 & -1/8 \\ -1/8 & 3/8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \end{bmatrix}$$

$$P \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \\ 1/2 \end{bmatrix}$$

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