Basis

Definition: Let V be a vector space. A (finite) set of vectors $S = \{v_1, v_2, \dots, v_n\}$ is called a `basis set` for V iff

- $Span(S) = V$
- S is `Linearly Independent`

A (finite dimensional) linear space V has many bases. All bases of a linear space have the same number of elements. This number is called the `dimension` of the linear space.

Example: $V = \mathbb{R}^2$, consider the two `base`s:

$S_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

1. S_1 is linearly independent set since $c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0$, $\forall c_1, c_2 \in \mathbb{R}(F)$.
2. $Span(S_1) = V = \mathbb{R}^2$ Hence S_1 is a basis for V.

$S_2 = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \end{bmatrix} \right\}$

1. S_2 is linearly independent set since $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0$, $\forall c_1, c_2 \in \mathbb{R}(F)$.
2. $Span(S_2) = V = \mathbb{R}^2$

Example: $V = \mathbb{R}^2$, and $F = \mathbb{R}$, consider the base:
$B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ $y = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ $[y]_B$=?

Solution:

1. B is linearly independent set since $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ implies $c_1 = c_2 = 0$, $\forall c_1, c_2 \in \mathbb{R}(F)$.
2. $Span(B) = V = \mathbb{R}^2$
3. $[y]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
4. $y = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ c_2 \end{bmatrix}$
5. $c_1 + c_2 = 2$ and $c_2 = 3$
6. $c_1 = -1$ and $c_2 = 3$
7. $[y]_B = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$

Example: $V = Span\{cos(t), sin(t)\}$, and $F = \mathbb{R}$, consider the base:
$B = \{cos(t), sin(t)\}$ $y = cos(t - \frac{\pi}{3})$ $[y]_B$=?

Solution:

1. B is linearly independent set since $c_1 cos(t) + c_2 sin(t) = 0$ implies $c_1 = c_2 = 0$, $\forall c_1, c_2 \in \mathbb{R}(F)$.
2. $Span(B) = V = Span\{cos(t), sin(t)\}$
3. $[y]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$
4. $y = c_1 cos(t) + c_2 sin(t) = cos(t - \frac{\pi}{3}) = cos(t)cos(\frac{\pi}{3}) + sin(t)sin(\frac{\pi}{3}) = \frac{1}{2}cos(t) + \frac{\sqrt{3}}{2}sin(t)$
5. $c_1 = \frac{1}{2}$ and $c_2 = \frac{\sqrt{3}}{2}$
6. $[y]_B = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$

Claim: For a given basis B, the representation $[y]_B$ of a vector y is unique.
Proof: By contradiction.
Assume $[y]_B = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ and $[y]_B = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$

Then $y = c_1b_1 + c_2b_2 = d_1b_1 + d_2b_2$
$c_1b_1 + c_2b_2 - d_1b_1 - d_2b_2 = 0$
$(c_1 - d_1)b_1 + (c_2 - d_2)b_2 = 0$
Since B is linearly independent, $c_1 - d_1 = 0$ and $c_2 - d_2 = 0$
Hence $c_1 = d_1$ and $c_2 = d_2$ ■

> **Remark**: The representation of a vector y in a basis B is unique. The representation of a vector y in a basis B is called the `coordinate vector` of y with respect to B.

Ordered Basis

Definition: Let V be a vector space. An ordered set of basis vectors $S = \{v_1, v_2, \ldots, v_n\}$ is called an `ordered basis` for V. If $y = (x_1, x_2, ..., x_n)$ is an ordered basis for V, then every vector $x \in V$ can be written as a linear combination of the basis vectors as follows:

Theorem: Let V be an n-dimensional vector space over $\mathbb{R}$. Let B_1 and B_2 be two bases for V. Then there exists a unique $n \times n$ real invertible matrix P such that $[x]_{B_2} = P[x]_{B_1}$ for all $x \in V$.

Proof: By construction.

Example: Consider $V =$ polynomials of degree ≤ 2 with coefficients in $\mathbb{R}$ and the bases:
$B_1 = \left\{1, t-1, (t-1)^2\right\}$
$B_2 = \left\{1, t, t^2\right\}$
Find the matrix P such that $[x]_{B_1} = P[x]_{B_2}$ for all $x \in V$.

Solution:

1. $[x]_{B_1} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$ and $[x]_{B_2} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
2. $x = c_1 + c_2(t-1) + c_3(t-1)^2 = c_1 + c_2t - c_2 + c_3t^2 - 2c_3t + c_3$
3. $x = (c_1 - c_2 + c_3) + (c_2 - 2c_3)t + c_3t^2$
4. $d_1 = c_1 - c_2 + c_3$
5. $d_2 = c_2 - 2c_3$
6. $d_3 = c_3$
7. $c_1 = d_1 + d_2 + d_3$
8. $c_2 = d_2 + 2d_3$
9. $c_3 = d_3$
10. $[x]_{B_1} = \begin{bmatrix} d_1 + d_2 + d_3 \\ d_2 + 2d_3 \\ d_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix} = P[x]_{B_2}$
11. $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

Example: Show that Matrix P is invertible.
Proof: Let V be an n-dimensional vector space over $\mathbb{R}^{n \times n}$. B_1 and B_2 are bases for V. A vector v holds,

- $[v]_{B_1} = P[v]_{B_2}$
- $\exists\, Q \in \mathbb{R}^{n \times n}$ s.t.
  - $[v]_{B_2} = Q[v]_{B_1}$
  - $[v]_{B_1} = P[v]_{B_2} = P(Q[v]_{B_1}) = (PQ)[v]_{B_1}$ implies $PQ = I$
  - similarly $QP = I$
- P is invertible and $P^{-1} = Q$ ■
