Hilbert Projection Theorem

Let H be a Hilbert space (inner product space that is complete with respect to the norm induced by the inner product) and M be a finite dimensioal subspace of H. Then for any $x \in H$, there exists a unique $y \in M$ such that

$$\min_{m \in M} \|x - m\|$$

has a unique solution y. In other words "we can find a unique point in M that is closest to x". If m^* is the closest point to x in M, then $x - m^* \perp M$.

Proof: See lecture notes.

> **Remark:** The proof stated that $m^* = x_1$ is the closest point to M. It can also be interpreted as the best approximation of x choosen from the vectors in M. The x_2 term is the error in the approximation.

Example: Let $V = \mathbb{R}^2$ and $M = \text{span}\{[1, 1]^T\}$. Find the best approximation of $x = [4, 7]^T$ in M.

Solution: We need to find m^* such that $m^* = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\|x - m^*\|$ is minimum.

$$(x - m^*) \perp M \implies < x - m^*, m >= 0 \quad \forall m \in M$$

$$< x - m^*, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 0$$

$$\text{Replace } x \text{ and } m^* \text{ with their values.}$$

$$< \begin{bmatrix} 4 - \alpha \\ 7 - \alpha \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} >= 0$$

$$\text{Recall that } < x, y >= x^T y.$$

$$4 - \alpha + 7 - \alpha = 0$$

$$\alpha = \frac{11}{2}$$

$$m^* = \frac{11}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Example: Let $x \in V$ and $M = \text{span}\{v_1, v_2\}$. Find the best approximation of x in M.

Solution: We need to find m^* such that $m^* = \alpha_1 v_1 + \alpha_2 v_2$ that is in the span of M and $\|x - m^*\|$ is minimum.

$$(x - m^*) \perp M \implies (x - m^*) \perp \text{both } v_1 \text{ and } v_2$$

$$< x - \alpha_1 v_1 - \alpha_2 v_2, v_1 >=< x, v_1 > -\alpha_1 < v_1, v_1 > -\alpha_2 < v_2, v_1 >= 0$$

$$< x - \alpha_1 v_1 - \alpha_2 v_2, v_2 >=< x, v_2 > -\alpha_1 < v_1, v_2 > -\alpha_2 < v_2, v_2 >= 0$$

$$\begin{bmatrix} < v_1, v_1 > & < v_2, v_1 > \\ < v_1, v_2 > & < v_2, v_2 > \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} < x, v_1 > \\ < x, v_2 > \end{bmatrix}$$

$$\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} < v_1, v_1 > & < v_2, v_1 > \\ < v_1, v_2 > & < v_2, v_2 > \end{bmatrix}^{-1} \begin{bmatrix} < x, v_1 > \\ < x, v_2 > \end{bmatrix}$$

$$m^* = \alpha_1 v_1 + \alpha_2 v_2$$

Example: Let $V = \mathbb{R}^3$ and $M = \text{span}\{[1, 1, 1]^T, [1, 0, 1]^T\}$. Find the best approximation of $x = [4, 7, 2]^T$ in M.

Solution of Linear Equations

Consider the linear equation as,

$$Ax = B$$

$$\text{where } A \in \mathbb{C}^{m \times n},\ B \in \mathbb{C}^{m \times 1} \text{ and } x \in \mathbb{C}^{n \times 1} \text{ is unknown.}$$

a) **solution exists**: when $b \in \mathrm{R}(A)$
b) **solution is unique**: when $N(A) = \{0\}$

#EE501 - `Linear Systems Theory` at `METU`