Norm

Definition: Let (V, F) be a vector space. A `norm` on V is a function

$\| \cdot \| : V \to \mathbb{R} \geq 0$

such that for all $x, y \in V$ and $c \in F$ the following axioms hold:

- (P1) $\|x\| \geq 0$ and $\|x\| = 0$ iff $x = 0_v$ (positive definiteness)
- (P2) $\|cx\| = |c|\|x\| \quad \forall c \in \mathbb{R}$ (homogeneity)
- (P3) $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

The triplet $(V, F, \| \cdot \|)$ is called a `normed linear space`.

> **Remark**: Perhaps the most important property of a norm is that it induces a metric on the vector space. Hence we can quantify the `distance` between two vectors in a vector space. Namely, the distance between two vectors x and $y \in V$ is the norm of the vector $x - y$ or $y - x$: $\|(y - x)\|$. Since the norm is always positive, the distance is also positive. The distance between two vectors is zero iff the vectors are the same. $x = x - 0$, the norm of x is the distance of x to the origin. With a proper distance definition (norm), one can begin studying the geometry of the space.

> **Remark**: We can define a "sphere" in V using the norm concept. The sphere is the set of all vectors in V with a fixed norm. For example, in $\mathbb{R}^2$, the sphere with radius r is the set of all vectors with norm r.

$S = v \in V | \|v - v_0\| \leq r$

<u>Example</u>: Let $V = \mathbb{R}^2$, $F = R$, and let $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

i. $\|x\|_1 := |\alpha_1| + |\alpha_2|$, is $\|x\|_1$ a norm?

- (P1) Let $x = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$,
  - then $\|x\|_1 = |c_1| + |c_2| \geq 0$ and $\|x\|_1 = 0$ iff $c_1 = c_2 = 0$ (forwards direction)
  - $\|x\|_1 = 0$ iff $c_1 = c_2 = 0$ implies $\|x\|_1 = |c_1| + |c_2| \geq 0$ (backwards direction)
- (P2) Let $c \in \mathbb{R}$, then $\|cx\|_1 = |c_1 c| + |c_2 c| = |c|(|c_1| + |c_2|) = |c|\|x\|_1$
- (P3) Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$,
  then $\|x + y\|_1 = |x_1 + y_1| + |x_2 + y_2| \leq |x_1| + |y_1| + |x_2| + |y_2| = \|x\|_1 + \|y\|_1$

ii. $\|x\|_2 := \sqrt{x_1^2 + x_2^2}$, is $\|x\|_2$ a norm?

- (P1) $\|x\|_2 = \sqrt{x_1^2 + x_2^2} \geq 0$ and $\|x\|_2 = 0$ iff $x_1 = x_2 = 0$ (forwards direction) $\|x\|_2 = 0$ iff $x_1 = x_2 = 0$ implies $\|x\|_2 = \sqrt{x_1^2 + x_2^2} \geq 0$ (backwards direction)
- (P2) Let $c \in \mathbb{R}$, then $\|cx\|_2 = \sqrt{(cx_1)^2 + (cx_2)^2} = |c|\sqrt{x_1^2 + x_2^2} = |c|\|x\|_2$
- (P3) Let $y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$,
  then $\|x + y\|_2 = \sqrt{(x_1 + y_1)^2 + (x_2 + y_2)^2} \leq \sqrt{x_1^2 + y_1^2} + \sqrt{x_2^2 + y_2^2} = \|x\|_2 + \|y\|_2$

iii.

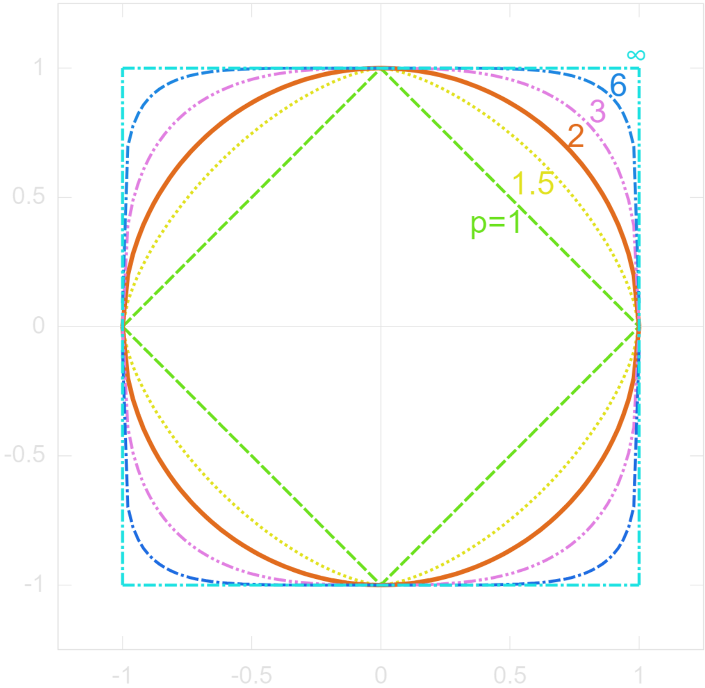
All these norms can be generalized into what is called the p-norm.

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$$

where $p \geq 1$ and $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

> **Remark**: Note that $lim_{p\to\infty}\|x\|_p = \|x\|_\infty = max_i|x_i|$

Geometric visualization of the p-norms in $\mathbb{R}^2$ is given below.



<u>Example</u>: Let $\|x\|_{\frac{1}{2}} = \left(|\alpha_1|^{1/2} + |\alpha_2|^{1/2}\right)^2$ where, $x = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$ is $\|x\|_{\frac{1}{2}}$ a norm ?

- Show a counter example for (P3),
  Let $x = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $y = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  then $\|x+y\|_{\frac{1}{2}} = \left(|1|^{1/2} + |1|^{1/2}\right)^2 = 4$
  but $\|x\|_{\frac{1}{2}} + \|y\|_{\frac{1}{2}} = \left(|1|^{1/2} + |0|^{1/2}\right)^2 + \left(|0|^{1/2} + |1|^{1/2}\right)^2 = 2$
  hence $\|x+y\|_{\frac{1}{2}} \nleq \|x\|_{\frac{1}{2}} + \|y\|_{\frac{1}{2}}$

<u>Example</u>: Norms on function spaces. $V : \{f(.)|f[0,1] \to \mathbb{R} \text{ s.t. } \int_0^1 f(t)^p dt < \infty, 1 \leq p < \infty\}$

- $\|f\|_p = (\int_0^1 |f(t)|^p dt)^{\frac{1}{p}}$ (p-norm)
- $\|f\|_\infty = max_{t\in[0,1]}|f(t)|$ (infinity-norm)

Matrix Norms

Definition: Let $A \in \mathbb{R}^{m\times n}$ be a matrix. A `matrix norm` is a function that maps matrices to non-negative real numbers. A matrix norm must satisfy the norm axioms. A norm on matrices can be defined as,

$$\|A\| = \max_{ij} |a_{ij}|$$

where a_{ij} is the element of A at the i th row and j th column.

<u>**Example**</u>: Let $V = \mathbb{R}^{n\times m}$ and $A = [a_{ij}]$

$$\|A\|_1 = \max_{1\leq j\leq m} \sum_{i=1}^{n} |a_{ij}| \quad \text{(absolute sum of rows)}$$

Another norm definition is Frobenuis norm:

$$\|A\|_F = (\sum_{i=1}^{n}\sum_{j=1}^{m}|a_{ij}|^p)^{\frac{1}{p}}$$

$$\text{where } 1 \leq p \leq \infty$$

Induced Norm

Definition: $A : \mathbb{R}^n \to \mathbb{R}^m$ be an $m \times n$ matrix. Let $\|\cdot\|_{\mathbb{R}^n}$ be a norm on $\mathbb{R}^n$ and $\|\cdot\|_{\mathbb{R}^m}$ be a norm on $\mathbb{R}^m$. The norm of A induced by these norms is defined as,

$$\|A\| := \max_{x \in \mathbb{R}^n} \frac{\|Ax\|_{\mathbb{R}^m}}{\|x\|_{\mathbb{R}^n}}$$

> **Remark**: The induced `matrix` norm is defined in terms of vector norms. An equivalent definition is given below:

$$\|A\| := \max_{\|x\|_{\mathbb{R}^n}=1} \|Ax\|_{\mathbb{R}^m}$$

> **Remark**: The induced norm of a matrix is the maximum amplification of the norm of a vector under the action of the matrix. In other words, the induced norm of a matrix is the maximum amount by which the matrix can stretch a vector.

A nice visualization of the induced norm is given below.