CMP 694 Graph Theory
Hacettepe University

Lecture 7: Vertex Coloring and Upper Bounds

Lecturer:
Lale Özkahya

Resources:
“Introduction to Graph Theory” by Douglas B. West

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$.
The vertices of the same color form a color class.

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$. The vertices of the same color form a *color class*.

proper coloring: A coloring, where any two neighboring vertices have different colors

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$. The vertices of the same color form a color class.

proper coloring: A coloring, where any two neighboring vertices have different colors

k-colorable: A graph is k-colorable if it has a proper k-coloring.

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$. The vertices of the same color form a color class.

proper coloring: A coloring, where any two neighboring vertices have different colors
k-colorable: A graph is k-colorable if it has a proper k-coloring.

chromatic number of a graph G, $\chi(G)$: The least k such that G is k-colorable.

Definitions

k-coloring of a graph G: A labeling $f : V(G) \implies S$, where $|S| = k$. The vertices of the same color form a color class.

proper coloring: A coloring, where any two neighboring vertices have different colors
k-colorable: A graph is k-colorable if it has a proper k-coloring.

chromatic number of a graph G, $\chi(G)$: The least k such that G is k-colorable.
Examples: bipartite graphs have chromatic number 2, odd cycles, Petersen graph have chromatic number 3. Why? What is the chromatic number of Q_n?

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Proposition

For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Proposition

For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Remark: Can you find examples, for which equalities do not hold in the above inequalities?

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Proposition

For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Remark: **Can you find examples, for which equalities do not hold in the above inequalities?** When $G = C_{2r+1} \vee K_s$. $\omega(G) = s + 2$ and $\chi(G) \geq s + 3$.

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Proposition

For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Remark: Can you find examples, for which equalities do not hold in the above inequalities? When $G = C_{2r+1} \vee K_s$. $\omega(G) = s + 2$ and $\chi(G) \geq s + 3$.

The chromatic number of the *disjoint union* of two graphs:

$$\chi(G + H) = \max(\chi(G), \chi(H)\}.$$

Relation of $\chi(G)$ to other graph parameters

Clique number, $\omega(G)$: maximum order of a clique (complete subgraph) in G.

Proposition

For every graph G, $\chi(G) \geq \omega(G)$ and $\chi(G) \geq \frac{n(G)}{\alpha(G)}$.

Remark: **Can you find examples, for which equalities do not hold in the above inequalities?** When $G = C_{2r+1} \vee K_s$. $\omega(G) = s + 2$ and $\chi(G) \geq s + 3$.

The chromatic number of the *disjoint union* of two graphs:

$$\chi(G + H) = \max(\chi(G), \chi(H)\}.$$

The chromatic number of the *join* of two graphs:

$$\chi(G \vee H) = \chi(G) + \chi(H).$$

Another Product of Graphs: *Cartesian product*

The cartesian product of G and H, $G\Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting an edge between the vertices uv and $u'v'$ iff

1. $u = u'$ and $vv' \in E(H)$, or
2. $v = v'$ and $uu' \in E(G)$.

Another Product of Graphs: *Cartesian product*

The cartesian product of G and H, $G \square H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting an edge between the vertices uv and $u'v'$ iff

1. $u = u'$ and $vv' \in E(H)$, or
2. $v = v'$ and $uu' \in E(G)$.

Can you draw the cartesian product of two paths, say $P_3 \square P_4$?

Another Product of Graphs: *Cartesian product*

The cartesian product of G and H, $G \Box H$, is the graph with vertex set $V(G) \times V(H)$ specified by putting an edge between the vertices uv and $u'v'$ iff

1. $u = u'$ and $vv' \in E(H)$, or
2. $v = v'$ and $uu' \in E(G)$.

Can you draw the cartesian product of two paths, say $P_3 \Box P_4$?

The chromatic number of the *cartesian product* of two graphs (Vizing, 1963, Aberth, 1964):

$$\chi(G \Box H) = \max\{\chi(G), \chi(H)\}.$$

Upper Bounds

Proposition

$\chi(G) \leq \Delta(G) + 1$.

Upper Bounds

Proposition

$\chi(G) \leq \Delta(G) + 1.$

Proposition (Welsh-Powell, 1967)

If a graph G has a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then

$$\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}.$$

Proof idea: Apply greedy coloring to the vertices ordered with nonincreasing degrees.

Upper Bounds

Proposition

$\chi(G) \leq \Delta(G) + 1.$

Proposition (Welsh-Powell, 1967)

If a graph G has a degree sequence $d_1 \geq d_2 \geq \cdots \geq d_n$, then

$$\chi(G) \leq 1 + \max_i \min\{d_i, i-1\}.$$

Proof idea: Apply greedy coloring to the vertices ordered with nonincreasing degrees.

Note: Every graph has some vertex ordering for which greedy coloring uses exactly $\chi(G)$ colors. (Exercise 33)

Color-critical (or k-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then G is called k-critical (or color-critical).
Example: Every odd cycle is a 2-critical graph, any K_n is n-critical.

Lemma

If H is a k-critical graph, then $\delta(H) \geq k - 1$.

Color-critical (or k-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then G is called k-critical (or color-critical).
Example: Every odd cycle is a 2-critical graph, any K_n is n-critical.

Lemma

If H is a k-critical graph, then $\delta(H) \geq k - 1$.

Proof idea: Assume, there is a vertex with degree $k - 2$ or less, find a contradiction.

Color-critical (or k-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then G is called k-critical (or color-critical).

Example: Every odd cycle is a 2-critical graph, any K_n is n-critical.

Lemma

If H is a k-critical graph, then $\delta(H) \geq k - 1$.

Proof idea: Assume, there is a vertex with degree $k - 2$ or less, find a contradiction.

Theorem (Szekeres-Wilf, 1968)

For any graph G,

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H).$$

Color-critical (or k-critical) graphs

If $\chi(H) < \chi(G) = k$ for every proper subgraph $H \subset G$, then G is called k-critical (or color-critical).

Example: Every odd cycle is a 2-critical graph, any K_n is n-critical.

Lemma

If H is a k-critical graph, then $\delta(H) \geq k - 1$.

Proof idea: Assume, there is a vertex with degree $k - 2$ or less, find a contradiction.

Theorem (Szekeres-Wilf, 1968)

For any graph G,

$$\chi(G) \leq 1 + \max_{H \subseteq G} \delta(H).$$

Proof idea: Let H' be a k-critical subgraph of G.

$$\chi(G) - 1 = \chi(H') - 1 \leq \delta(H') \leq \max_{H \subseteq G} \delta(H).$$

Brook's Theorem

Theorem (Brooks, 1941)

If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Sketch of the proof: Let $k = \Delta(G)$. For $k \geq 3$, trivial for $k = 1, 2$.

- **Case 1:** G **is not** k**-regular.** Let $\deg(v_n) < k$, construct a spanning tree of G using BFS starting at v_n, label the vertices v_i with decreasing index i as they are added to the tree. Greedy algorithm uses at most k colors.

Brook's Theorem

Theorem (Brooks, 1941)

If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Sketch of the proof: Let $k = \Delta(G)$. For $k \geq 3$, trivial for $k = 1, 2$.

- **Case 1: G is not k-regular.** Let $\deg(v_n) < k$, construct a spanning tree of G using BFS starting at v_n, label the vertices v_i with decreasing index i as they are added to the tree. Greedy algorithm uses at most k colors.
- **Case 2: G is k-regular and has a cut-vertex:** Say x is a cut-vertex and H_1 is a component of $G - x$ and $H_2 = G - \{x\} - H_1$. Color $H_1 \cup \{x\}$ and $H_2 cup \{x\}$ separately. Permute colors in both colorings such that x has the same color in both. Done.

Brook's Theorem

- **Case 3:** G **is** k**-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)

Brook's Theorem

- **Case 3: G is k-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)
- Use either BFS or DFS to find a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n such that vertex indices increase along the paths to the root.

Brook's Theorem

- **Case 3: G is k-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)
- Use either BFS or DFS to find a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n such that vertex indices increase along the paths to the root.
- Color greedily $v_1, v_2, \ldots, v_n$ by coloring v_1 and v_2 the same. Done.

Brook's Theorem

- **Case 3: G is k-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)
- Use either BFS or DFS to find a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n such that vertex indices increase along the paths to the root.
- Color greedily $v_1, v_2, \ldots, v_n$ by coloring v_1 and v_2 the same. Done.

Claim

Every k-regular 2-connected graph has a triple as v_1, v_2, v_n.

Brook's Theorem

- **Case 3: G is k-regular and 2-connected:** Assume some vertex v_n has neighbors v_1 and v_2, that are not adjacent, and $G - \{v_1, v_2\}$ is connected. (We show later, that this is always true.)
- Use either BFS or DFS to find a spanning tree of $G - \{v_1, v_2\}$ rooted at v_n such that vertex indices increase along the paths to the root.
- Color greedily $v_1, v_2, \ldots, v_n$ by coloring v_1 and v_2 the same. Done.

Claim

Every k-regular 2-connected graph has a triple as v_1, v_2, v_n.

Proof: Since G is not complete, there are two vertices of distance 2, say v_1 and v_2. We let the common neighbor of them be v_n.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Remark: This construction obtains a $k+1$-chromatic graph, when the input graph is k-chromatic. Examples: $G = K_2$ and $G = C_5$.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Remark: This construction obtains a $k+1$-chromatic graph, when the input graph is k-chromatic. Examples: $G = K_2$ and $G = C_5$.

Theorem (Mycielski, 1955)

From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.

- U is an independent set. So, triangles could be induced by some u_i and neighbors in $N(v_i)$, contradiction, because G has no triangle.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Remark: This construction obtains a $k+1$-chromatic graph, when the input graph is k-chromatic. Examples: $G = K_2$ and $G = C_5$.

Theorem (Mycielski, 1955)

From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.

- U is an independent set. So, triangles could be induced by some u_i and neighbors in $N(v_i)$, contradiction, because G has no triangle.
- We can easily extend a k-coloring of G to color U. Then, color w with an extra color. So, at most $k+1$ colors are sufficient.

Graphs with large chromatic number

Construction (Mycielski's construction)

For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Remark: This construction obtains a $k+1$-chromatic graph, when the input graph is k-chromatic. Examples: $G = K_2$ and $G = C_5$.

Theorem (Mycielski, 1955)

From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.

- U is an independent set. So, triangles could be induced by some u_i and neighbors in $N(v_i)$, contradiction, because G has no triangle.
- We can easily extend a k-coloring of G to color U. Then, color w with an extra color. So, at most $k+1$ colors are sufficient.
- Also, at least $k+1$ colors are needed. To show that start with a proper coloring of G' and obtain a proper coloring of G using less colors.