CMP 694 Graph Theory
Hacettepe University

Lecture 7: Vertex Coloring and Upper Bounds

Lecturer:
Lale Özkahya

Resources:
“Introduction to Graph Theory” by Douglas B. West

Outline

Definitions

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Examples: bipartite graphs have chromatic number 2, odd cycles, Petersen graph have chromatic number 3. Why? What is the chromatic number of Q_n?

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Construction (Mycielski's construction)

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From a k-chromatic triangle-free graph G, Mycielski's construction produces a $k+1$-chromatic triangle-free graph.

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- Also, at least $k+1$ colors are needed. To show that start with a proper coloring of G' and obtain a proper coloring of G using less colors.

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For an input graph G with vertices $\{v_1, \ldots, v_n\}$, a new graph G' is obtained by adding vertices $U = \{u_1, \ldots, u_n\}$ and another vertex w. The edge set og G' contains $E(G)$, the edges between u_i and $N_G(v_i)$ for all i. Moreover, let $N(w) = U$.

Remark: This construction obtains a $k+1$-chromatic graph, when the input graph is k-chromatic. Examples: $G = K_2$ and $G = C_5$.

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- Also, at least $k+1$ colors are needed. To show that start with a proper coloring of G' and obtain a proper coloring of G using less colors.

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Example: Edge-coloring of K_{2n} is a modeling of scheduling problem.

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Thus, there are two types of graphs: the ones that have edge-chromatic number $\Delta(G)$ or $\Delta(G) + 1$.