CMP 694 Graph Theory
Hacettepe University

Lecture 9: Edge-coloring of graphs and Hamiltonian Cycles

Lecturer:
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Resources:
"Introduction to Graph Theory" by Douglas B. West

Definitions

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Example: Edge-coloring of K_{2n} is a modeling of scheduling problem.

Bipartite Graphs, Petersen Graph

Theorem (König, 1916)

If G is bipartite, then $\chi'(G) = \Delta(G)$.

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Thus, there are two types of graphs: the ones that have edge-chromatic number $\Delta(G)$ or $\Delta(G) + 1$.

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Proposition (A necessary condition)

If G has a Hamilton cycle, then for each nonempty set $S \subset V$, the graph $G - S$ has at most $|S|$ components.

See Example 7.2.5 in West.

Example: Two cliques or order $\lceil (n+1)/2 \rceil$ and $\lfloor (n+1)/2 \rfloor$ merged at one vertex. This graph has a very high minimum degree, but it is not hamiltonian.

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- By the maximality of G, adding any other edge to G would create a Hamiltonian cycle. So, let $uv \notin E(G)$. There is a Ham. path $v_1, v_2, \ldots, v_n$ with ends $u = v_1$ and $v = v_n$.

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Since $n \notin S \cup T$, $|S \cup T| \leq n-1$, done.

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Theorem (Ore, 1960)

Let G be a simple graph. If u and v are distinct non-adjacent vertices such that $\deg(u) + \deg(v) \geq n(G)$, then G is Hamiltonian iff $G + uv$ is Hamiltonian.

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The closure fo a graph G, denoted by $C(G)$, is the graph with the same vertex set as G that is obtained by iteratively adding the edges to G whose endvertices are a non-adjacent pair with degree sum at least n.

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Theorem (Chvatal's condition, 1972)

Let G be a simple graph with vertex degrees $d_1 \leq \ldots d_n$, where $n \geq 3$. If for each $i < n/2$, $d_i > i$ or $d_{n-i} \geq n - i$, then G is Hamiltonian.

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- **Claim:** $C(G) = K_n$.
  To prove this, again assume on the contrary that $C(G) \neq K_n$. We will show that there is an i for which BCC does not hold, i.e.
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Example: The graph $K_i \vee (\bar{K}_i + K_{n-2i})$ is an example where Chvátal's condition is not satisfied, but still the degrees are high.