Ceng222 Homework1 Solutions

Answer 1

Let R, G, B denote the events that the red, green, blue ball is picked, respectively. Let X, Y be the events that the selected box is X, Y, respectively.

a)

Since the selection of the box X is given, all the balls in the box have equal probability of being picked, and 2 out of 6 balls are green in box X, $P(G|X) = \frac{2}{6} = \frac{1}{3}$.

b)

Notice that X and Y are mutually exclusive and also exhaustive. Therefore, we have $P(Y) = 1 - P(X) = 0.6$. From the law of total probability, we also have

$$\begin{aligned} P(R) &= P(R|X)P(X) + P(R|Y)P(Y), \\ &= \frac{2}{6}\frac{4}{10} + \frac{1}{5}\frac{6}{10}, \\ &= \frac{19}{75}. \end{aligned}$$

c)

From the Bayes' Rule and the law of total probability, we have

$$\begin{aligned} P(Y|B) &= \frac{P(B|Y)P(Y)}{P(B)}, \\ &= \frac{P(B|Y)P(Y)}{P(B|Y)P(Y) + P(B|X)P(X)}, \\ &= \frac{\frac{2}{5}\frac{6}{10}}{\frac{2}{5}\frac{6}{10} + \frac{2}{6}\frac{4}{10}}, \\ &= \frac{9}{14}. \end{aligned}$$

Answer 2

a)

We are going to prove it in two parts.

Part 1: If events A and B are mutually exclusive, then $\overline{A}$ and $\overline{B}$ are exhaustive.

Proof: Assume that A and B are mutually exclusive. Then, we know that $A \cap B = \emptyset$. Observe that $\overline{A} \cup \overline{B} = \overline{A \cap B} = \overline{\emptyset} = \Omega$, where the first equality comes from De Morgan's laws. Since their union is the sample space, $\overline{A}$ and $\overline{B}$ are exhaustive.

Part 2: If events $\overline{A}$ and $\overline{B}$ are exhaustive, then A and B are mutually exclusive.

Proof: Assume that $\overline{A}$ and $\overline{B}$ are exhaustive. Then we know that $\overline{A} \cup \overline{B} = \Omega$. Observe that $A \cap B = \overline{\overline{A} \cup \overline{B}} = \overline{\Omega} = \emptyset$. Therefore, A and B are mutually exclusive.

From Part 1 and Part 2, it can be easily seen that A and B are mutually exclusive if and only if $\overline{A}$ and $\overline{B}$ are exhaustive.

b)

We are going to disprove it by giving a counter example. Let A, B, C are events such that $A = B \neq \emptyset$ and $C = \overline{A}$. Then, observe that $\overline{A} \cup \overline{B} \cup \overline{C} = \overline{A} \cup \overline{C} = \overline{A} \cup \overline{\overline{A}} = \overline{A} \cup A = \Omega$, where the first equality comes from the fact that $A = B$, the second equality is because $C = \overline{A}$. As one can see $\overline{A}$, $\overline{B}$, and $\overline{C}$ are exhaustive.

However, since $A = B \neq \emptyset$, $A \cap B = A \neq \emptyset$. Therefore, A, B, and C are not mutually exclusive, even though $\overline{A}$, $\overline{B}$, and $\overline{C}$ are exhaustive.

Answer 3

a) (w/o Binomial)

The number of possible (equally likely) outcomes of throwing 5 dice is 6^5 since each die has 6 possible outcomes: 1, 2, 3, 4, 5, 6. If we find the number of outcomes that has exactly 2 successes then their ratio will give us the probability of having exactly 2 successful dice.

There are $\binom{5}{2}$ ways of selecting the 2 successful dice among 5. Those successful dice can take 2 values each (5, 6) and therefore, there are 2^2 possible assignments. The unsuccessful ones can take 4 values each (1, 2, 3, 4) and therefore, there are 4^3 possible assignments.

Therefore, the answer is

$$\frac{\text{number of outcomes where there are exactly 2 successful dice}}{\text{number of all possible outcomes}} = \frac{\binom{5}{2}2^2 4^3}{6^5} \approx 0.329.$$

a) (w/ Binomial)

We can see it as 5 Bernoulli trials where the probability of success is $p = \dfrac{2}{6} = \dfrac{1}{3}$. Then, (a) asks for the probability of two success in Binomial distribution where 5 trials are done. Let X denote the number of successes in that Binomial distribution. Then, the answer can be found using the pmf of Binomial distribution:

$$\begin{aligned}
&P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \\
&P(X = 2) = \binom{5}{2} (\frac{1}{3})^2 (\frac{2}{3})^3, \\
&P(X = 2) \approx 0.329.
\end{aligned}$$

b)

Similarly, it asks for $P(X \geq 2) = 1 - P(X \leq 1)$. To calculate it, we can use the Binomial distribution table or we can sum individual pmf's for all $x \leq 1$. If we take the second option, we get

$$\begin{aligned}
P(X \geq 2) &= 1 - (P(X = 0) + P(X = 1)), \\
&= 1 - \left[\binom{5}{0} (\frac{1}{3})^0 (\frac{2}{3})^5 + \binom{5}{1} (\frac{1}{3})^1 (\frac{2}{3})^4 \right], \\
&\approx 0.539.
\end{aligned}$$

Answer 4

a)

From the law of total probability,

$$\begin{aligned} P(A=1, C=0) &= \sum_b P(A=1, B=b, C=0), \\ &= P(A=1, B=0, C=0) + P(A=1, B=1, C=0), \\ &= 0.06 + 0.09, \\ &= 0.15. \end{aligned}$$

b)

Similarly,

$$\begin{aligned} P(B=1) =& \sum_{a,c} P(A=a, B=1, C=c), \\ =& P(A=0, B=1, C=0) + P(A=0, B=1, C=1) \\ &+ P(A=1, B=1, C=0) + P(A=1, B=1, C=1), \\ =& 0.21 + 0.02 + 0.09 + 0.08, \\ =& 0.4. \end{aligned}$$

c)

If A and B are independent, then $P(A=a, B=b) = P(A=a)P(B=b)$ **for all** a**,** b.

We are going to show that A and B are **not** independent. Therefore, it is sufficient to show that $P(A=a, B=b) \neq P(A=a)P(B=b)$ for some a, b.

Observe that $P(A=1) = 0.55$, $P(B=1) = 0.4$, $P(A=1)P(B=1) = 0.22$. However, $P(A=1, B=1) = 0.17 \neq P(A=1)P(B=1)$.

d)

If A and B are conditionally independent given $C=1$, then

$$P(A=a, B=b|C=1) = P(A=a|C=1)P(B=b|C=1)$$

for all a, b.

We are going to show that A and B are independent random variables given $C = 1$. Therefore, we should show the equality for all a and b.

If we calculate the probabilities using the law of total probabilities, we get

$$
\begin{aligned}
P(C = 1) &= 0.5, \\
P(A = 0|C = 1) &= \frac{P(A = 0, C = 1)}{P(C = 1)} = \frac{0.1}{0.5} = 0.2, \\
P(A = 1|C = 1) &= \frac{P(A = 1, C = 1)}{P(C = 1)} = \frac{0.4}{0.5} = 0.8, \\
P(B = 0|C = 1) &= \frac{P(B = 0, C = 1)}{P(C = 1)} = \frac{0.4}{0.5} = 0.8, \\
P(B = 1|C = 1) &= \frac{P(B = 1, C = 1)}{P(C = 1)} = \frac{0.1}{0.5} = 0.2, \\
P(A = 0, B = 0|C = 1) &= \frac{P(A = 0, B = 0, C = 1)}{P(C = 1)} = \frac{0.08}{0.5} = 0.16, \\
P(A = 0, B = 1|C = 1) &= \frac{P(A = 0, B = 1, C = 1)}{P(C = 1)} = \frac{0.02}{0.5} = 0.04, \\
P(A = 1, B = 0|C = 1) &= \frac{P(A = 1, B = 0, C = 1)}{P(C = 1)} = \frac{0.32}{0.5} = 0.64, \\
P(A = 1, B = 1|C = 1) &= \frac{P(A = 1, B = 1, C = 1)}{P(C = 1)} = \frac{0.08}{0.5} = 0.16.
\end{aligned}
$$

Now observe that

$$
\begin{aligned}
P(A = 0|C = 1)P(B = 0|C = 1) &= (0.2)(0.8) = 0.16 = P(A = 0, B = 0|C = 1), \\
P(A = 0|C = 1)P(B = 1|C = 1) &= (0.2)(0.2) = 0.04 = P(A = 0, B = 1|C = 1), \\
P(A = 1|C = 1)P(B = 0|C = 1) &= (0.8)(0.8) = 0.64 = P(A = 1, B = 0|C = 1), \\
P(A = 1|C = 1)P(B = 1|C = 1) &= (0.8)(0.2) = 0.16 = P(A = 1, B = 1|C = 1).
\end{aligned}
$$

Therefore, given $C = 1$, A and B are conditionally independent.