Ceng222 Homework 2 Solutions

Answer 1

Let's calculate the marginal pmfs (probability mass functions) from joint pmfs.

$$P(X=0) = \sum_y P(X=0, Y=y) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = \frac{1}{4}.$$
$$P(X=1) = \sum_y P(X=1, Y=y) = \frac{4}{12} + \frac{2}{12} = \frac{6}{12} = \frac{1}{2}.$$
$$P(X=2) = \sum_y P(X=2, Y=y) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12} = \frac{1}{4}.$$
$$P(Y=0) = \sum_x P(X=x, Y=0) = \frac{1}{12} + \frac{4}{12} + \frac{1}{12} = \frac{6}{12} = \frac{1}{2}.$$
$$P(Y=2) = \sum_x P(X=x, Y=2) = \frac{2}{12} + \frac{2}{12} + \frac{2}{12} = \frac{6}{12} = \frac{1}{2}.$$

We may use $P(X=x)$ and $P_X(x)$ interchangeably and both will be pmfs.

a)

From the definition of expected value (for the discrete case),

$$\begin{aligned}
E(X) &= \sum_x x P_X(x), \\
&= 0P_X(0) + 1P_X(1) + 2P_X(2), \\
&= 0\frac{1}{4} + 1\frac{1}{2} + 2\frac{1}{4}, \\
&= 1.
\end{aligned}$$

From the definition of variance,

$$\begin{aligned} Var(X) &= E(X - EX)^2, \\ &= \sum_x (x-\mu)^2 P_X(x) \qquad\qquad \text{where } \mu = E(X), \\ &= (0-1)^2 P_X(0) + (1-1)^2 P_X(1) + (2-1)^2 P_X(2), \\ &= 1\frac{1}{4} + 0\frac{1}{2} + 1\frac{1}{4}, \\ &= \frac{1}{2}. \end{aligned}$$

b)

Let $Z = X + Y$. So the question asks for the pmf of Z.

$$\begin{aligned} P(Z=0) &= P(X=0, Y=0) = \frac{1}{12}. \\ P(Z=1) &= P(X=1, Y=0) = \frac{4}{12}. \\ P(Z=2) &= P(X=2, Y=0) + P(X=0, Y=2) = \frac{1}{12} + \frac{2}{12} = \frac{3}{12}. \\ P(Z=3) &= P(X=1, Y=2) = \frac{2}{12}. \\ P(Z=4) &= P(X=2, Y=2) = \frac{2}{12}. \end{aligned}$$

c)

To calculate $Cov(X,Y)$ we can use both $Cov(X,Y) = E((X - EX)(Y - EY))$, $Cov(X,Y) = E(XY) - E(X)E(Y)$. Let's use the second one.

We need to calculate

$$
\begin{aligned}
E(Y) &= \sum_y y P_Y(y), \\
&= 0P_Y(0) + 2P_Y(2), \\
&= 0\frac{1}{2} + 2\frac{1}{2}, \\
&= 1,
\end{aligned}
$$

and also

$$
\begin{aligned}
E(XY) &= \sum_{x,y} xyP(X = x, Y = y), \\
&= 0(\frac{1}{12} + \frac{4}{12} + \frac{1}{12} + \frac{2}{12}) + 2\frac{2}{12} + 4\frac{2}{12}, \\
&= 1.
\end{aligned}
$$

We already know $E(X) = 1$. Thus,

$$
\begin{aligned}
Cov(X,Y) &= E(XY) - E(X)E(Y), \\
&= 1 - (1)(1), \\
&= 0.
\end{aligned}
$$

d)

Let's show for the case that A and B are discrete random variables. Continuous case can be shown in a similar manner.

Assume that A and B are independent discrete random variables. Then, we know that $E(AB) = E(A)E(B)$ (proof of this can be seen in the proof of the "Properties of expectations", 3.5 in the book).

Then, from the definition of covariance,

$$\begin{aligned} Cov(A, B) &= E(AB) - E(A)E(B), \\ &= E(A)E(B) - E(A)E(B), \\ &= 0. \end{aligned}$$

Thus, we have shown that if A and B are independent discrete random variables, then $Cov(A, B) = 0$.

e)

We are going to give a counter example to show that X and Y are not independent. If X and Y were independent, then $P(X = x, Y = y)$ would be equal to $P(X = x)P(Y = y)$ for all x, y. However, notice that $P(X = 0, Y = 0) = \dfrac{1}{12}$, but $P(X = 0)P(Y = 0) = \left(\dfrac{1}{4}\right)\left(\dfrac{1}{2}\right) = \dfrac{1}{8}$. Therefore, they are not independent.

Note: We cannot use "independence implies Cov=0" since the other way is not true.

Answer 2

We can see each pen inspection as an independent Bernoulli trial where success is a pen being broken with probability $p = 0.2$.

a)

There are 12 independent Bernoulli trials and this describes Binomial distribution. Let X be a random variable described as the number of successes in this 12 independent Bernoulli trials with probability $p = 0.2$. Then, the pmf of this variable is

$$P(X = x) = \binom{12}{x}(0.2)^x(0.8)^x.$$

The question asks $P(X \geq 3) = 1 - P(X \leq 2) = 1 - F(2)$. This can be obtained by calculating $1 - \sum_{x=0}^{2} P(X = x)$ or by obtaining the value of $F(2)$ from the Binomial distribution table A2 in our book.

The values to use in the table are $n = 2$, $p = 0.2$, $x = 2$. So we obtain $F(2) = 0.558$. Thus, $P(X \geq 3) = 1 - F(2) = 0.442$.

b)

This description fits the Negative Binomial distribution. Let X be a random variable described as the number of trials needed to obtain 2 successes. The pmf of X is,

$$P(X = x) = \binom{x-1}{1}(0.2)^2(0.8)^{x-2}.$$

The question asks $P(X = 5) = \binom{4}{1}(0.2)^2(0.8)^3 \approx 0.08192$.

This question could also be though as getting exactly one broken pen in the first 4 trials (Binomial distribution, $\binom{4}{1}(0.2)(0.8)^3$) and the fifth one as a success (multiplying with probability of success $p = 0.2$).

c)

Similarly, this question also can be solved using Negative Binomial distribution. Let X be a random variable described as the number of trials to get 4 successes. Then, the question basically asks $E(X) = \dfrac{k}{p} = \dfrac{4}{0.2} = 20$.

Another approach can be using 4 independent Geometric random variables representing each success and summing them up, which actually leads to the expected value of Negative Binomial distribution.

Note: Approaching this problem using the expected value formula of Binomial distribution would give the same result but is **wrong** in general. It is dangerous to calculate expected value of a variable using its occurrence in another expectation formula. Because occurrence of n in the expected value formula of Binomial distribution is not a random variable, it is just its parameter.

Answer 3

a)

Let T be the random variable that describes the time until the first phone call. Since Bob gets 4 calls on average, from the expected value formula of Exponential distribution $E(T) = \dfrac{1}{\lambda} = 4$ and therefore $\lambda = \dfrac{1}{4}$ calls per hour.

The question asks the probability of Bob not getting phone call for 2 hours. This means that Bob gets his first phone call after 2 hours. Therefore, the answer is

$$
\begin{aligned}
P(T > 2) &= 1 - P(T <= 2), \\
&= 1 - (1 - e^{-2\lambda}) && \text{cdf of T}, \\
&= e^{-\frac{1}{2}}, \\
&\approx 0.6065.
\end{aligned}
$$

This question could also be solved using Poisson distribution with $\lambda_p = \lambda t = \dfrac{1}{2}$ and the answer would be P(0) We are going to take a similar approach in (3b).

b)

α independent steps that takes Exponential(λ) amount of time can be described using Gamma distribution.

The question asks the probability of Bob getting at most 3 phone calls in the first 10 hours. This can be restated as Bob getting his **fourth** phone call after 10 hours. Let T be the random variable that describes the total time of 4 phone calls ($\alpha = 4$) with $\lambda = \dfrac{1}{4}$.

Therefore, question asks $P(T > 10)$. This could be obtained using gamma distribution, however, converting it into Poisson distribution would be much easier. If we apply the Gamma-Poisson formula we get

$$
\begin{aligned}
P(T > 10) &= P(X < \alpha), \\
&= P(X < 4), \\
&= P(X \leq 3),
\end{aligned}
$$

where X is a Poisson random variable with $\lambda_p = \lambda t = 2.5$. So the question basically asks us to find F(3) for this Poisson random variable and we can do it by using the table A3 in our book (with $\lambda_p = 2.5$, $x = 3$). So the answer is 0.758.

c)

Let's define our Gamma random variable T as the time of the fourth phone call ($\alpha = 4$) with $\lambda = \frac{1}{4}$. Then, if we restate the question as we did in (3b), it basically asks $P(T > 16|T > 10)$.

From the definition of conditional probability, we get

$$P(T > 16|T > 10) = \frac{P(T > 16 \cap T > 10)}{P(T > 10)}.$$

Notice that $(T > 16) \subset (T > 10)$ since if it happened after 16 hours, then it definitely happened after 10 hours. Therefore, $P(T > 16 \cap T > 10) = P(T > 16)$ and $P(T > 16|T > 10) = \frac{P(T > 16)}{P(T > 10)}$.

To calculate the probability values $P(T > 16)$ and $P(T > 10)$, we can convert them to Poisson distribution using Gamma-Poisson formula

$$\begin{aligned}
P(T > 16) &= P(X_1 < \alpha), \\
&= P(X_1 < 4), \\
&= P(X_1 \leq 3), and \\
P(T > 10) &= P(X_2 < \alpha), \\
&= P(X_2 < 4), \\
&= P(X_2 \leq 3),
\end{aligned}$$

where X_1 and X_2 are Poisson random variables with $\lambda_{p_1} = 16\lambda = 4$ and $\lambda_{p_2} = 10\lambda = 2.5$, respectively.

These values can be obtained using the Poisson distribution table A3 in our book. So the answer is

$$\begin{aligned}
P(T > 16|T > 10) &= \frac{P(X_1 \leq 3)}{P(X_2 \leq 3)}, \\
&\approx \frac{0.433}{0.758}, \\
&= 0.571.
\end{aligned}$$