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Answer 1

a) If we want every 1 is followed immediately by 0, we can construct bit string length of 9 using the sequences 10 and 0. We can use;

- 4 of 10 and 1 of 0.
- 3 of 10 and 3 of 0.
- 2 of 10 and 5 of 0.
- 1 of 10 and 7 of 0.
- 0 of 10 and 9 of 0. This case is not a valid case since the string including no 1 is not a valid string.

If we calculate all strings per by one rule and sum all of them, we are done.

- $\dfrac{5!}{4! \cdot 1!} = \dfrac{5}{1} = 5$
- $\dfrac{6!}{3! \cdot 3!} = \dfrac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$
- $\dfrac{7!}{2! \cdot 5!} = \dfrac{7 \cdot 6}{2 \cdot 1} = 21$
- $\dfrac{8!}{1! \cdot 7!} = \dfrac{8}{1} = 8$

Hence, $5 + 20 + 21 + 8 = 54$. There are 54 bit strings of length 9 such that every 1 is followed immediately by 0.

b) If we want at least eight 1s in the strings, we can calculate the sum of the number of strings that have eight 1s, nine 1s and ten 1s in them.

- For eight 1s, there are two 0s. $\dfrac{10!}{8! \cdot 2!} = \dfrac{10 \cdot 9}{2 \cdot 1} = 45$
- For nine 1s, there is one 0. $\dfrac{10!}{9! \cdot 1!} = \dfrac{10}{1} = 10$
- For ten 1s, there is zero 0. $\dfrac{10!}{10!} = 1$

Hence, $45 + 10 + 1 = 56$. There are 56 bit strings of length 10 have at least eight 1s in them.

c) In order to a function $f : A \to B$ be a onto function, there must be no remaining element in the set B. In this case, there must be no remaining element in the set with 3 elements.

So lets take our 4 elements from the first set and respectively call them a, b, c, and d.

- For a, there are 4 different elements in the second set, we can select 4 different item.
- For b, there are 3 different elements in the second set, since we do not want any remaining element in the second set, we can select 3 different item.
- Similarly, there are 2 and 1 different elements in the second set, for c and d respectively.

Therefore, the total number of onto functions are $4 \cdot 3 \cdot 2 \cdot 1 = 4! = 24$.

d) We have 5 identical Discrete Mathematics and 7 identical Signals and Systems textbooks. Since there are identical, there are only 3 case that satisfies the condition.

- 1 Discrete Mathematics textbook and 3 Signals and Systems textbooks.
- 2 Discrete Mathematics textbooks and 2 Signals and Systems textbooks.
- 3 Discrete Mathematics textbook and 1 Signals and Systems textbook.

Therefore, there are only 3 different ways that you can make a collection of 4 books.

Answer 2

- Let $S_n = \{1, 2, 3 \cdots n\}$.
- Let's divide S to two different subset such that $A_n \cup B_n = S$, $A_n \cap B_n = \emptyset$, $\forall x \in A_n \to x = 1 \pmod 2$, and $\forall x \in B_n \to x = 0 \pmod 2$. Hypothetically lets call them the main subsets of S.
- Both A_n and B_n have no two consecutive numbers. Therefore any subset of both A_n and B_n does not contain any two consecutive numbers.
- Let's calculate the initial values for the recurrence relation for a_n.

$$a_0 = 1,\ a_1 = 2, \text{ and } a_2 = 3.$$

- Let P be the one of the main subsets, A_n or B_n. If P includes n, then all elements of P except n can make any of the main subsets of S_{n-2}. If P does not include n, then the elements of P must be one of the main subsets of S_{n-1} Therefore,

$$a_n = a_{n-1} + a_{n-2}$$

which is also satisfies the conditions for a_0, a_1, and a_2.

a) Since it's linear homogeneous recurrence, we have to find the characteristic equation.

$$r^2 - r - 1 = 0$$

Roots are $r_1 = \dfrac{1+\sqrt{5}}{2}$ and $r_2 = \dfrac{1-\sqrt{5}}{2}$.

$$a_n = \alpha_1 \cdot (\frac{1+\sqrt{5}}{2})^n + \alpha_2 \cdot (\frac{1-\sqrt{5}}{2})^n$$
$$n = 0 \rightarrow a_0 = \alpha_1 + \alpha_2 = 1$$
$$n = 1 \rightarrow a_1 = \alpha_1 \cdot \frac{1+\sqrt{5}}{2} + \alpha_2 \cdot \frac{1-\sqrt{5}}{2} = 2$$

$$\alpha_1 = \frac{1}{2} + \frac{\sqrt{5}}{5} \text{ and } \alpha_2 = \frac{1}{2} - \frac{\sqrt{5}}{5}$$
$$a_n = (\frac{1}{2} + \frac{\sqrt{5}}{5}) \cdot (\frac{1+\sqrt{5}}{2})^n + (\frac{1}{2} - \frac{\sqrt{5}}{5}) \cdot (\frac{1-\sqrt{5}}{2})^n$$

b) It can be seen that a_n is actually the Fibonacci series, which is $\{1, 1, 2, 3, 5, 8 \cdots\}$

$$F(x) = \sum_{n=0}^{\infty} F_n \cdot x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + 8x^5$$

where F_n is the nth Fibonacci number, such that $F_0 = 1$, $F_1 = 1$, $F_n = F_{n-1} + F_{n-2}$.

$$\begin{aligned}
F(x) = \sum_{n=0}^{\infty} F_n \cdot x^n &= 1 + x + \sum_{n=2}^{\infty} F_n \cdot x^n \\
&= 1 + x + \sum_{n=2}^{\infty} (F_{n-1} + F_{n-2}) \cdot x^n \\
&= 1 + x + x \cdot \sum_{n=2}^{\infty} F_{n-1} \cdot x^{n-1} + x^2 \cdot \sum_{n=2}^{\infty} F_{n-2} \cdot x^{n-2} \\
&= 1 + x + x \cdot (-1 + F_0 \cdot x^0 + \sum_{n=1}^{\infty} F_{n-1} \cdot x^{n-1}) + x^2 \cdot \sum_{n=2}^{\infty} F_{n-2} \cdot x^{n-2} \\
&= 1 + x + x \cdot (-1 + F(x)) + x^2 \cdot F(x) \\
&= 1 + x - x + x \cdot F(x) + x^2 \cdot F(x) \\
F(x) &= 1 + x \cdot F(x) + x^2 \cdot F(x)
\end{aligned}$$

Therefore,

$$F(x) = \frac{1}{1 - x - x^2}$$

Answer 3

- Since it's linear homogeneous recurrence, we have to find the characteristic equation.

$$\begin{aligned} r^3 - 4r^2 - r + 4 &= 0 \\ r(r^2 - 1) - 4(r^2 - 1) &= 0 \\ (r - 4) \cdot (r^2 - 1) &= 0 \\ (r - 4) \cdot (r - 1) \cdot (r + 1) &= 0 \end{aligned}$$

  The roots of the equation are $r_1 = 4$, $r_2 = 1$, $r_3 = -1$.

- $a_n = \alpha_1 \cdot 4^n + \alpha_2 \cdot 1^n - \alpha_3 \cdot (-1)^n$

$$\begin{aligned} n = 0 &\to a_0 = \alpha_1 + \alpha_2 - \alpha_3 = 4 \\ n = 1 &\to a_1 = \alpha_1 \cdot 4 + \alpha_2 + \alpha_3 = 8 \\ n = 2 &\to a_2 = \alpha_1 \cdot 16 + \alpha_2 - \alpha_3 = 34 \end{aligned}$$

- Roots are $\alpha_1 = 2$, $\alpha_2 = 1$, $\alpha_3 = -1$. Therefore,

$$a_n = 2 \cdot 4^n + (-1)^n + 1$$

  which satisfies the initial conditions, $a_0 = 4$, $a_1 = 8$, $a_2 = 34$.

Answer 4

R is an equivalence relation if and only if R is *reflexive*, *symmetric*, and *transitive*.

- *Reflexive*
  Let a (x, y), then if aRa holds, it is reflexive. If $x_1 = x_2 \wedge y_1 = y_2$ is true, then it is reflexive. Since $3x_1 - 2y_1 = 3x_2 - 2y_2$, it is reflexive.

- *Symmetric*
  Let a (x_1, y_1) and b (x_2, y_2), then if $aRb \to bRa$ holds, it is symmetric. Since $3x_1 - 2y_1 = 3x_2 - 2y_2$, it is symmetric.

- *Transitive*
  Let a (x_1, y_1), b (x_2, y_2), and c (x_3, y_3), then if $aRb \wedge bRc \to aRc$ holds, it is transitive. Since $3x_1 - 2y_1 = 3x_2 - 2y_2$, it is transitive.