Middle East Technical University   Department of Computer Engineering

CENG 223

Discrete Computational Structures

Fall '2019-2020

Take Home Exam 4 - ANSWERS SHEET

Question 1

Solve the following and explain your answers:

a) How many bit strings of length 9 are there such that every 1 is followed immediately by a 0?

Solution:
We consider four different cases here. In all cases we will use permutations with repetition.

- If we have one 10, then we will have seven 0s. So in this case we will have $\frac{8!}{7!1!} = 8$ many strings.
- If we have two 10s, then we will have five 0s. So in this case we will have $\frac{7!}{5!2!} = 21$ many strings.
- If we have three 10s, then we will have three 0s. So in this case we will have $\frac{6!}{3!3!} = 20$ many strings.
- If we have four 10s, then we will have one 0. So in this case we will have $\frac{5!}{4!1!} = 5$ many strings.

When we add up all the results we get $8 + 21 + 20 + 5 = \mathbf{54}$ many strings.

b) How many bit strings of length 10 have at least eight 1s in them.

Solution:
We consider three different cases here. The number of 1s can be eight, nine or ten.

- If we have ten 1s, all the bit positions will have 1 so there will be 1 string in total (which is equal to $\binom{10}{10}$).
- If we have nine 1s, then we have one 0. So this 0 can be inserted into $\binom{10}{1} = 10$ different places (which is equal to $\binom{10}{9}$).
- If we have eight 1s, then we have two 0s. So these 0s can be inserted into the string in $\binom{10}{2} = 45$ many ways (which is equal to $\binom{10}{8}$).

Finally, we add them up and get the result:

$$\binom{10}{10} + \binom{10}{9} + \binom{10}{8}$$

$$= 1 + 10 + 45$$

$$= \mathbf{56}$$

c) How many *onto* functions are there from a set with 4 elements to a set with 3 elements?

Solution:

We use inclusion-exclusion principle here. Number of onto functions will be equal to

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}(n-k)^m$$

where $m = 4$ and $n = 3$

So plug them into the equation and get the following:

$$\binom{3}{0}3^4 - \binom{3}{1}2^4 + \binom{3}{2}1^4 - \binom{3}{3}0^4$$

$$= 81 - 48 + 3 - 0$$

$$= \mathbf{36}$$

d) We have 5 Discrete Mathematics textbooks and 7 Signals and Systems textbooks at hand. In how many ways can you make a collection of 4 books from these 12 textbooks with the condition that at least one Discrete Mathematics textbook and at least one Signals and Systems textbook must be in the collection.

Solution:

We use combination here because the order in which these books are chosen does not matter. Number of ways = $\binom{12}{4}-\binom{7}{4}-\binom{5}{4}$ = **455**. Because, we subtract the number of different collections with 4 books which don't include any DM book and the number of different collections with 4 books which don't include any SS book from the number of different collections with 4 books. So we get the number of different collections with 4 books which include at least one DM book and at least one SS book.

Note: The number of different collections with 4 books which don't include any DM book is $\binom{7}{4}$. Because we choose all the books from SS books to satisfy this condition. And also the number of different collections with 4 books which don't include any SS book is $\binom{5}{4}$. Because we choose all the books from DM books to satisfy this condition.

Question 2

Let a_n be the number of subsets of the set $\{1, 2, 3 \cdots n\}$ that do not contain two consecutive numbers.

a) **Determine the recurrence relation for a_n.**

Let's first determine the number of subsets for the sets up to length 5 in Table-1.

We can generate the recurrence relation for a_n by using a_{n-1} and a_{n-2}. The subsets generated in length n always includes the elements that included in the set of length $n-1$. Then by adding the new number (n) to the list we can include the subsets that include n. The subsets generated by including n to the subsets of the length $n-1$ causes consecutive numbers in subsets, hence the new subsets are generated by adding n to $n-2$ length subsets. **Therefore a_n can be evaluated by** $a_n = a_{n-1} + a_{n-2}$.

Table 1: The subsets that do not contain two consecutive numbers and the a_n values of the set $\{1, 2, 3 \cdots n\}$ where $n \leq 5$.

n	Set	Subsets	a_n
1	{1}	{}, {1}	2
2	{1,2}	{}, {1}, {2}	3
3	{1,2,3}	{}, {1}, {2}, {3}, {1,3}	5
4	{1,2,3,4}	{}, {1}, {2}, {3}, {1,3}, {4}, {1,4}, {2,4}	8
5	{1,2,3,4,5}	{}, {1}, {2}, {3}, {1,3}, {4}, {1,4}, {2,4}, {5}, {1,5}, {2,5}, {3,5}, {1,3,5}	13

To trace our solution in Table-1, lets examine the case for $n = 3$. $a_3 = 5$ and the elements are $(\{\}, \{1\}, \{2\}, \{3\}, \{1, 3\})$. $(\{\}, \{1\}, \{2\})$ are the exact same members for the set when $n = 2$. The second part of the subsets are generated by adding 3(which is n) to the subsets generated when $n = 1$ $(\{\}, \{1\})$. We get $(\{3\}, \{1, 3\})$ and add these to the subset list for $n = 3$. At the end we generate the subset of the set with length 3, and a_3 can be evaluated by $a_3 = a_1 + a_2$ which is $a_3 = 3 + 2$.

b) **Solve it by using generating functions.**

In this question we will solve the recurrence relation by first generating it with a generating function and then try to find the value of that generating function. Let's call the generating function we will generate $F(x)$.

Recall that generating function of a series $a_1, a_2, a_3, \cdots, a_n$ can be represented as the following: $G(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 \cdots$

STEP-1: So we can actually write our series as $2, 3, 5, 8, 13, 21 \cdots$ which is $F(x) = 2 + 3x + 5x^2 + 8x^3 + 13x^4 \cdots$

For the sake of simplicity we will now write $F(x)$ as $F(x) = a_1 + a_2x + a_3x^2 + a_4x^3 + a_5x^4 \cdots$

STEP-2: We know that our recurrence relation is $a_n = a_{n-1} + a_{n-2}$. So, let's write all the elements of the sequence $F(x)$ in terms of a_1 and a_2:
$F(x) = a_1 + a_2x + (a_1 + a_2)x^2 + (a_1 + 2a_2)x^3 + (2a_2 + 3a_3)x^4 \cdots$

STEP-3: Let's take our sequence into a_1 and a_2 parenthesis:
$F(x) = a_1(1 + x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 \cdots) + a_2(x + x^2 + 2x^3 + 3x^4 + 5x^5 \cdots)$

STEP-4: Let's take the 1 inside the a_1 parenthesis out:
$F(x) = a_1 + a_1(x^2 + x^3 + 2x^4 + 3x^5 + 5x^6 \cdots) + a_2(x + x^2 + 2x^3 + 3x^4 + 5x^5 \cdots)$

STEP-5: Let's take a_1's sequence into x^2 parenthesis and a_2's sequence into x parenthesis:
$F(x) = a_1 + a_1x^2(1 + x + 2x^2 + 3x^3 + 5x^4 \cdots) + a_2x(1 + x + 2x^2 + 3x^2 + 5x^4 \cdots)$

STEP-6: Let's instantiate 2 into a_1 and 3 into a_2:
$F(x) = 2 + 2x^2(1 + x + 2x^2 + 3x^3 + 5x^4 \cdots) + 3x(1 + x + 2x^2 + 3x^2 + 5x^4 \cdots)$

STEP-7: Let's take $(1 + x)$'s inside parenthesis out:
$F(x) = 2 + 2x^2 + 2x^3 + 2x^2(2x^2 + 3x^3 + 5x^4 \cdots) + 3x + 3x^2 + 3x(2x^2 + 3x^2 + 5x^4 \cdots)$

STEP-8: If we take x^2's out we can recognize that the sequence in the parenthesis exactly the same sequence of the $F(x)$ provided in **STEP-1**.
$F(x) = 2 + 2x^2 + 2x^3 + 2x^4(2 + 3x + 5x^2 \cdots) + 3x + 3x^2 + 3x^3(2 + 3x + 5x^2 \cdots)$ which is equal to:
$F(x) = 2 + 2x^2 + 2x^3 + 2x^4(F(x)) + 3x + 3x^2 + 3x^3(F(x))$

STEP-9: Let's collect all of the $F(x)$'s into left hand side:
$F(x) - 2x^4(F(x)) - 3x^3(F(x)) = 2 + 2x^2 + 2x^3 + 3x + 3x^2$
Take the expression into $F(x)$ parenthesis: $F(x)(1 - 2x^4 - 3x^3) = 2 + 3x + 5x^2 + 2x^3$
As a result $F(x)$ can be calculated as, $F(x) = \dfrac{2 + 3x + 5x^2 + 2x^3}{1 - 2x^4 - 3x^3}$

Question 3

Solve the following recurrence relation with the given initial conditions:

$$a_n = 4a_{n-1} + a_{n-2} - 4a_{n-3}$$

with $a_0 = 4$, $a_1 = 8$, $a_2 = 34$.

Solution:
Reorder the equation:

$$a_n - 4a_{n-1} - a_{n-2} + 4a_{n-3} = 0$$

Write down the characteristic equation:

$$\lambda^3 - 4\lambda^2 - \lambda + 4 = 0$$

Factorize the characteristic equation and find the λ values:

$$\begin{gathered} \lambda^2(\lambda - 4) - (\lambda - 4) = 0 \\ (\lambda^2 - 1)(\lambda - 4) = 0 \\ (\lambda - 1)(\lambda + 1)(\lambda - 4) = 0 \\ \lambda_1 = -1,\ \lambda_2 = 1,\ \lambda_3 = 4 \end{gathered}$$

Write down a_n and find the unknown coefficients:

$$a_n = A(-1)^n + B(1)^n + C(4)^n$$

$$\begin{gathered} a_0 = A + B + C = 4 \\ a_1 = -A + B + 4C = 8 \\ a_2 = A + B + 16C = 34 \\ a_2 - a_0 = A + B + 16C - (A + B + C) = 15C = 30 \\ C = 2 \end{gathered}$$

$$\begin{gathered} a_0 = A + B + 2 = 4 \rightarrow A + B = 2 \\ a_1 = -A + B + 8 = 8 \rightarrow -A + B = 0 \\ A = B = 1 \end{gathered}$$

Write down a_n with the known coefficients:

$$\begin{gathered} a_n = (-1)^n + 2(4)^n + 1 \\ a_n = (-1)^n + 2^{2n+1} + 1 \end{gathered}$$

Question 4

Let R be a binary relation on real numbers defined by $(x_1, y_1)\, R\, (x_2, y_2)$ iff $3x_1 - 2y_1 = 3x_2 - 2y_2$. Prove that R is an equivalence relation. Give a graphical representation of $[(2, 3)]$ and $[(2, -3)]$ in the Cartesian coordinate system, where $[(x, y)]$ denotes the equivalence class of (x, y) with respect to R.

Solution

For the given relation to be an equivalence class it should be

1. reflexive,
2. symmetric,
3. transitive.

Reflexivity. Let $x_1 \in \mathbb{R}$, $y_1 \in \mathbb{R}$. Then clearly $3x_1 - 2y_1 = 3x_1 - 2y_1$, that is, $(x_1, y_1)\, R\, (x_1, y_1)$. Hence the given relation is reflexive.

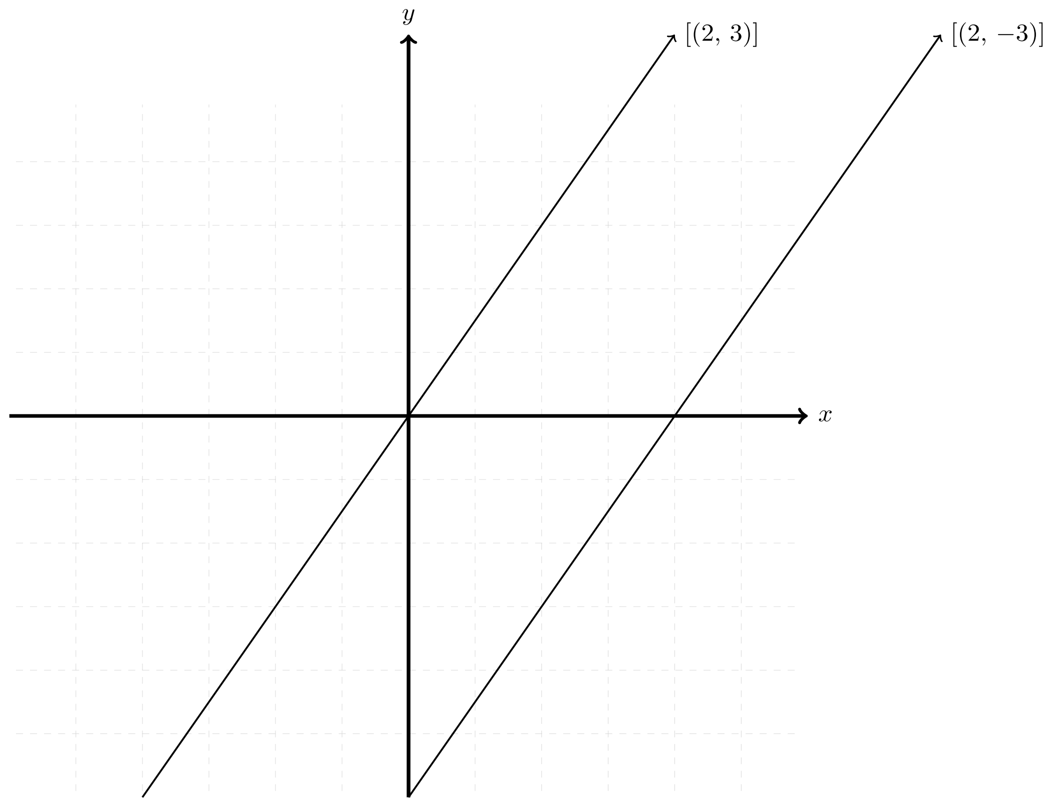
Symmetry. Let $(x_1, y_1)\, R\, (x_2, y_2)$ where $x_1, x_2, y_1, y_2 \in \mathbb{R}$. Then by the definition of the relation R we have $3x_1 - 2y_1 = 3x_2 - 2y_2$, whereby we can use the symmetric property of the algebraic symbol of equivalence, $=$, to acquire $3x_2 - 2y_2 = 3x_1 - 2y_1$. This means that $(x_2, y_2)\, R\, (x_1, y_1)$ which implies (and is implied by) $(x_2, y_2)\, R\, (x_1, y_1)$. Since our choice of x_1, x_2, y_1, y_2 was arbitrary, this holds for all real pairs, related by R. Hence R is symmetric.

Transitivity. Let $(x_1, y_1)\, R\, (x_2, y_2)$, and Let $(x_2, y_2)\, R\, (x_3, y_3)$. Then $3x_1 - 2y_1 = 3x_2 - 2y_2$ and $3x_2 - 2y_2 = 3x_3 - 2y_3$. It follows from the transitivity of $=$ that $3x_1 - 2y_1 = 3x_3 - 2y_3$, *i.e.* $(x_1, y_1)\, R\, (x_3, y_3)$. Hence we have shown that R transitive.

Since R satisfies all three of the criteria, it is an equivalence relation.

Equivalence class of $[(2, 3)]$. This class consists of all real pairs (x, y) satisfying $(x, y)\, R\, (2, 3)$. Then the elements are described exactly by $3x - 2y = 0$, which describes a line.

Equivalence class of $[(2, -3)]$. Similar arguments. The elements in this class are described by $3x - 2y = 12$, again a line.



y
$[(2,\ 3)]$
$[(2,\ -3)]$
x