THE 3 Solutions

Answer 1

Assume $p-1 \equiv k \pmod{y}$. It follows that $p-1 = by+k$. Also, $x^{p-1} = (x^y)^b x^k \equiv x^k \pmod{p}$ since $x^y \equiv 1 \pmod{p}$ given in the question. Fermat's Little Theorem states that $x^{p-1} \equiv 1$ (mod p), so $x^k \equiv 1 \pmod{p}$.
We have $x^y \equiv 1 \pmod{p}$ and $x^k \equiv 1 \pmod{p}$ where y is the smallest positive integer satisfying the condition and $0 \leq k < y$, so k should be 0. It follows that, $p-1 = by+k = by+0 = by$. Hence $y \mid (p-1)$.

Answer 2

See that

$$2n^2+10n-7 = 2n^2+10n-72+65 = (2n-8)(n+9)+65 = 2(n-4)(n+9)+65.$$

$13 \mid (2n^2+10n-7)$ if and only if $13 \mid 2(n-4)(n+9)$. Hence, if $13 \nmid 2(n-4)(n+9)$ then 13 and 169 (since $169 = 13 \times 13$) don't divide $2n^2+10n-7$.
Assume that $13 \mid 2(n-4)(n+9)$. Then it follows that $13 \mid (n-4)$ or $13 \mid (n+9)$. For $2(n-4)(n+9)$ to be divisible by 169, 13 must divide both $(n+9)$ and $(n-4)$. Since $n+9 \equiv n-4 \pmod{13}$, it follows that $169 \mid 2(n-4)(n+9)$ for some positive integer n. However, $169 \nmid 65$. Thus, $169 \nmid (2n^2+10n-7) = 2(n-4)(n+9)+65$.

Answer 3

$a \equiv b (\text{mod } m)$ implies $m \mid (a-b)$. Therefore, there exists a positive integer x such that $mx = a-b$. Similarly, there exists another positive integer y such that $ny = a-b$. So, $a-b = mx = ny$. Since $gcd(m,n) = 1$, $m \nmid n$. However $m \mid mx = ny$. Thus, we get $m \mid y$ and $y = mz$ for some positive integer z. By replacing y, we get $nmz = (nm)z = a-b$. Hence $a \equiv b (\text{mod } m \times n)$.

Answer 4

$$P(n,k) : \sum_{j=1}^{n} j(j+1)(j+2)....(j+k-1) = \frac{n(n+1)(n+2)....(n+k)}{k+1}$$

To show that this proposition is true,
a) We should show that $P(1,k)$ is true for all positive integers k.

$$P(1,k) : 1(1+1)(1+2)....(1+k-1) = \frac{1(1+1)(1+2)....(1+k)}{k+1}$$

$$1.2.3.....k = \frac{1.2.3.....k.\cancel{(k+1)}}{\cancel{(k+1)}} \qquad (k+1>0)$$

$$1.2.3....k = 1.2.3....k$$

So $P(1,k)$ is true.
b) We should show that $P(n,k) \rightarrow P(n+1,k)$ is true for all positive integers n and k. We assume the following is true :

$$P(n,k) : \sum_{j=1}^{n} j(j+1)(j+2)....(j+k-1) = \frac{n(n+1)(n+2)....(n+k)}{k+1} \qquad (*)$$

Now we should show that $P(n+1,k)$ is also true.

$$P(n+1,k) : \sum_{j=1}^{n+1} j(j+1)(j+2)...(j+k-1) \stackrel{?}{=} \frac{(n+1)(n+2)(n+3)..(n+1+k)}{k+1}$$

To get $P(n+1,k)$, we should add $(n+1)(n+2)(n+3)...(n+1+k-1)$ to the both sides of the equation $(*)$ denoted above, that is :

$$\sum_{j=1}^{n} [j(j+1)...(j+k-1)] + (n+1)(n+2)..(n+1+k-1) = \frac{n(n+1)....(n+k)}{k+1} + (n+1)(n+2)..(n+1+k-1)$$

Then the left hand side becomes as we desired $(P(n+1,k))$:

$$P(n+1,k) : \sum_{j=1}^{n+1} j(j+1)(j+2)...(j+k-1) = \frac{n(n+1)....(n+k) + (k+1)[(n+1)(n+2)..(n+1+k-1)]}{k+1}$$

$$= \frac{[(n+1)(n+2)..(n+k)](n+1+k)}{k+1}$$

$$= \frac{(n+1)(n+2)(n+3)...(n+k)(n+1+k)}{k+1}$$

So $P(n,k) \rightarrow P(n+1,k)$ is true for all positive integers n and k.
As a result, from part a and part b we say:
$P(n,k) : \sum_{j=1}^{n} j(j+1)(j+2)....(j+k-1) = \frac{n(n+1)(n+2)....(n+k)}{k+1}$ is true for all positive integers k and n.

Answer 5

Basis STEP : We must show the statement is true for n = 0 which is the smallest value of n for this inequality.
$H_n \leq 7^n$ must be satisfied for n=0.
$H_0 \leq 7^0$ is true since H_0 is given as 1. Same as H_1 and H_2 :
$H_1 = 3 \leq 7^1$

$H_2 = 5 \leq 7^2$
This finishes the basis step of induction.
Inductive STEP : Assume that $H_j \leq 7^j$ for $0 \leq j \leq k$ where $k \geq 2$. We must show that $H_{k+1} \leq 7^{k+1}$ to prove it is true for all values of n. We will use the defined equation as follows:

$$H_{k+1} = 5H_k + 5H_{k-1} + 63H_{k-2} \quad (*)$$

Since we assumed that it is true for all values of $H_j(where \quad 0 \leq j \leq k)$, $H_k \leq 7^k$, $H_{k-1} \leq 7^{k-1}$ and $H_{k-2} \leq 7^{k-2}$. Now to show $H_{k+1} \leq 7^{k+1}$ we will substitute H_{k+1} with the equation $(*)$ defined above. Thus if we prove $5H_k + 5H_{k-1} + 63H_{k-2} \leq 7^{k+1}$, we can finish the induction.

$$5H_k + 5H_{k-1} + 63H_{k-2} \leq 5 \times 7^k + 5 \times 7^{k-1} + 63 \times 7^{k-2}$$

$$5H_k + 5H_{k-1} + 63H_{k-2} \leq 7^{k-2}(5 \times 7^2 + 5 \times 7 + 63)$$

$$5H_k + 5H_{k-1} + 63H_{k-2} \leq 7^{k-2}(343)$$

Since $343 = 7^3$:

$$5H_k + 5H_{k-1} + 63H_{k-2} \leq 7^{k+1}$$

Now using the equation (*) again :

$$H_{k+1} \leq 7^{k+1}$$

So $H_n \leq 7^n$ for any value of $n > 0$. Thus this finishes the induction.