Student Information

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Answer 1

- $x > 0$, p is a prime and $p \nmid x$.
- From Fermat's Little Theorem, $x^{p-1} \equiv 1 \pmod{p}$.
- To prove that $y \mid (p-1)$, we can prove that $(p-1) = yk + t$, $0 \le t < k$ when $t = 0$.

  $$x^{(p-1)} \equiv 1 \pmod{p}$$
  $$x^{(yk+t)} \equiv 1 \pmod{p}$$
  $$(x^y)^k \cdot x^t \equiv 1 \pmod{p}$$
  $$(x^y) \equiv 1 \pmod{p} \text{ so } x^t \equiv 1 \pmod{p}$$

  Since $t < k$, as a result t must be 0 to hold the equation.
- Since $t = 0$ and $(p-1) = yk$, $y \mid (p-1)$.

Answer 2

- Let $n = 3k + r$ such that $k \in \mathbb{Z}^+$ and $0 \le r < 169$.
- We have to prove that $169 \nmid 2n^2 + 10n - 7, \forall n \in \mathbb{Z}^+$, thus we have to show that $2n^2 + 10n - 7 \equiv 0 \pmod{169}$ is wrong.
- $2(3k + r)^2 + 10(3k + r) - 7 \equiv 0 \pmod{169}$.
- $2r^2 + 10r - 7 \equiv 0 \pmod{169}$.
- So, if we can show that for every value of r, the statement is wrong, then we can show that the main statement is wrong.
- $r = 0 \rightarrow 2r^2 + 10r - 7 \equiv 162 \pmod{169}$
  $r = 1 \rightarrow 2r^2 + 10r - 7 \equiv 5 \pmod{169}$
  $r = 2 \rightarrow 2r^2 + 10r - 7 \equiv 21 \pmod{169}$
  $r = 3 \rightarrow 2r^2 + 10r - 7 \equiv 41 \pmod{169}$
  $r = 4 \rightarrow 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
  $r = 5 \rightarrow 2r^2 + 10r - 7 \equiv 93 \pmod{169}$
  $r = 6 \rightarrow 2r^2 + 10r - 7 \equiv 125 \pmod{169}$
  $r = 7 \rightarrow 2r^2 + 10r - 7 \equiv 161 \pmod{169}$
  $r = 8 \rightarrow 2r^2 + 10r - 7 \equiv 32 \pmod{169}$
  $r = 9 \rightarrow 2r^2 + 10r - 7 \equiv 76 \pmod{169}$
  $r = 10 \rightarrow 2r^2 + 10r - 7 \equiv 124 \pmod{169}$
  $r = 11 \rightarrow 2r^2 + 10r - 7 \equiv 7 \pmod{169}$
  $r = 12 \rightarrow 2r^2 + 10r - 7 \equiv 63 \pmod{169}$
  $r = 13 \rightarrow 2r^2 + 10r - 7 \equiv 123 \pmod{169}$
  $r = 14 \rightarrow 2r^2 + 10r - 7 \equiv 18 \pmod{169}$
  $r = 15 \rightarrow 2r^2 + 10r - 7 \equiv 86 \pmod{169}$

$r = 16 \to 2r^2 + 10r - 7 \equiv 158 \pmod{169}$
$r = 17 \to 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
$r = 18 \to 2r^2 + 10r - 7 \equiv 145 \pmod{169}$
$r = 19 \to 2r^2 + 10r - 7 \equiv 60 \pmod{169}$
$r = 20 \to 2r^2 + 10r - 7 \equiv 148 \pmod{169}$
$r = 21 \to 2r^2 + 10r - 7 \equiv 71 \pmod{169}$
$r = 22 \to 2r^2 + 10r - 7 \equiv 167 \pmod{169}$
$r = 23 \to 2r^2 + 10r - 7 \equiv 98 \pmod{169}$
$r = 24 \to 2r^2 + 10r - 7 \equiv 33 \pmod{169}$
$r = 25 \to 2r^2 + 10r - 7 \equiv 141 \pmod{169}$
$r = 26 \to 2r^2 + 10r - 7 \equiv 84 \pmod{169}$
$r = 27 \to 2r^2 + 10r - 7 \equiv 31 \pmod{169}$
$r = 28 \to 2r^2 + 10r - 7 \equiv 151 \pmod{169}$
$r = 29 \to 2r^2 + 10r - 7 \equiv 106 \pmod{169}$
$r = 30 \to 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
$r = 31 \to 2r^2 + 10r - 7 \equiv 28 \pmod{169}$
$r = 32 \to 2r^2 + 10r - 7 \equiv 164 \pmod{169}$
$r = 33 \to 2r^2 + 10r - 7 \equiv 135 \pmod{169}$
$r = 34 \to 2r^2 + 10r - 7 \equiv 110 \pmod{169}$
$r = 35 \to 2r^2 + 10r - 7 \equiv 89 \pmod{169}$
$r = 36 \to 2r^2 + 10r - 7 \equiv 72 \pmod{169}$
$r = 37 \to 2r^2 + 10r - 7 \equiv 59 \pmod{169}$
$r = 38 \to 2r^2 + 10r - 7 \equiv 50 \pmod{169}$
$r = 39 \to 2r^2 + 10r - 7 \equiv 45 \pmod{169}$
$r = 40 \to 2r^2 + 10r - 7 \equiv 44 \pmod{169}$
$r = 41 \to 2r^2 + 10r - 7 \equiv 47 \pmod{169}$
$r = 42 \to 2r^2 + 10r - 7 \equiv 54 \pmod{169}$
$r = 43 \to 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
$r = 44 \to 2r^2 + 10r - 7 \equiv 80 \pmod{169}$
$r = 45 \to 2r^2 + 10r - 7 \equiv 99 \pmod{169}$
$r = 46 \to 2r^2 + 10r - 7 \equiv 122 \pmod{169}$
$r = 47 \to 2r^2 + 10r - 7 \equiv 149 \pmod{169}$
$r = 48 \to 2r^2 + 10r - 7 \equiv 11 \pmod{169}$
$r = 49 \to 2r^2 + 10r - 7 \equiv 46 \pmod{169}$
$r = 50 \to 2r^2 + 10r - 7 \equiv 85 \pmod{169}$
$r = 51 \to 2r^2 + 10r - 7 \equiv 128 \pmod{169}$
$r = 52 \to 2r^2 + 10r - 7 \equiv 6 \pmod{169}$
$r = 53 \to 2r^2 + 10r - 7 \equiv 57 \pmod{169}$
$r = 54 \to 2r^2 + 10r - 7 \equiv 112 \pmod{169}$
$r = 55 \to 2r^2 + 10r - 7 \equiv 2 \pmod{169}$
$r = 56 \to 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
$r = 57 \to 2r^2 + 10r - 7 \equiv 132 \pmod{169}$
$r = 58 \to 2r^2 + 10r - 7 \equiv 34 \pmod{169}$
$r = 59 \to 2r^2 + 10r - 7 \equiv 109 \pmod{169}$
$r = 60 \to 2r^2 + 10r - 7 \equiv 19 \pmod{169}$
$r = 61 \to 2r^2 + 10r - 7 \equiv 102 \pmod{169}$
$r = 62 \to 2r^2 + 10r - 7 \equiv 20 \pmod{169}$
$r = 63 \to 2r^2 + 10r - 7 \equiv 111 \pmod{169}$
$r = 64 \to 2r^2 + 10r - 7 \equiv 37 \pmod{169}$
$r = 65 \to 2r^2 + 10r - 7 \equiv 136 \pmod{169}$
$r = 66 \to 2r^2 + 10r - 7 \equiv 70 \pmod{169}$
$r = 67 \to 2r^2 + 10r - 7 \equiv 8 \pmod{169}$
$r = 68 \to 2r^2 + 10r - 7 \equiv 119 \pmod{169}$
$r = 69 \to 2r^2 + 10r - 7 \equiv 65 \pmod{169}$
$r = 70 \to 2r^2 + 10r - 7 \equiv 15 \pmod{169}$
$r = 71 \to 2r^2 + 10r - 7 \equiv 138 \pmod{169}$
$r = 72 \to 2r^2 + 10r - 7 \equiv 96 \pmod{169}$
$r = 73 \to 2r^2 + 10r - 7 \equiv 58 \pmod{169}$
$r = 74 \to 2r^2 + 10r - 7 \equiv 24 \pmod{169}$
$r = 75 \to 2r^2 + 10r - 7 \equiv 163 \pmod{169}$
$r = 76 \to 2r^2 + 10r - 7 \equiv 137 \pmod{169}$
$r = 77 \to 2r^2 + 10r - 7 \equiv 115 \pmod{169}$
$r = 78 \to 2r^2 + 10r - 7 \equiv 97 \pmod{169}$
$r = 79 \to 2r^2 + 10r - 7 \equiv 83 \pmod{169}$
$r = 80 \to 2r^2 + 10r - 7 \equiv 73 \pmod{169}$
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$r = 163 \to 2r^2 + 10r - 7 \equiv 5 \pmod{169}$
$r = 164 \to 2r^2 + 10r - 7 \equiv 162 \pmod{169}$
$r = 165 \to 2r^2 + 10r - 7 \equiv 154 \pmod{169}$
$r = 166 \to 2r^2 + 10r - 7 \equiv 150 \pmod{169}$
$r = 167 \to 2r^2 + 10r - 7 \equiv 150 \pmod{169}$
$r = 168 \to 2r^2 + 10r - 7 \equiv 154 \pmod{169}$

- So, $2n^2 + 10n - 7 \not\equiv 0 \pmod{169}$.
- Hence, $169 \nmid (2n^2 + 10n - 7)$, $\forall n \in \mathbb{Z}^+$.

Answer 3

- $a - b = mp$, $p \in \mathbb{Z}$ from $a \equiv b \pmod{m}$.
- $n \mid a - b$ from $a \equiv b \pmod{n}$.
- Hence, $n \mid mp$.
- From $gcd(m, n) = 1$, we can say that $n \mid p$.
- Let $p = nt$.
- $a - b = mp = mnt$, so $mn \mid a - b$.
- Hence, $a \equiv b \pmod{m \times n}$.

Answer 4

Since $j \geq 1$ and $k, n \geq 0$, we can define function f such that

$$f(n,k) = \sum_{j=1}^{n} \frac{(j+k-1)!}{(j-1)!} = \frac{(n+k)!}{(k+1)\cdot(n-1)!}$$

1) **Base Case**

   for $n = 1$

$$\begin{aligned} \sum_{j=1}^{n} \frac{(j+k-1)!}{(j-1)!} &= \frac{(n+k)!}{(k+1)\cdot(n-1)!} \\ \frac{k!}{1} &= \frac{(k+1)!}{(k+1)\cdot 0!} \\ k! &= k! \end{aligned}$$

2) **Inductive Step**

   Assume that it is true for n.

$$f(n,k) = \sum_{j=1}^{n} \frac{(j+k-1)!}{(j-1)!} = \frac{(n+k)!}{(k+1)\cdot(n-1)!}$$

We have to show that it is also true that $n+1$.

$$f(n+1,k) = \sum_{j=1}^{n+1} \frac{(j+k-1)!}{(j-1)!} = \frac{(n+k+1)!}{(k+1)\cdot n!}$$

$$\begin{aligned}
\frac{(n+k)!}{(k+1)\cdot(n-1)!} + \frac{(n+k)!}{n!} &= \frac{(n+k+1)!}{(k+1)\cdot n!} \\
\frac{(n+k)!\cdot n}{(k+1)\cdot n!} + \frac{(n+k)!}{n!} &= \frac{(n+k+1)\cdot(n+k)!}{(k+1)\cdot n!} \\
\frac{(n+k)!}{n!} &= \frac{(k+1)\cdot(n+k)!}{(k+1)\cdot n!} \\
\frac{(n+k)!}{n!} &= \frac{(n+k)!}{n!}
\end{aligned}$$

Hence, by induction the statement is true.

Answer 5

1) **Base Case**

$H_0 = 1 \leq 7^0 = 1$
$H_1 = 3 \leq 7^1 = 7$
$H_2 = 5 \leq 7^2 = 49$

2) **Inductive Step**

Assume that it is true for $n \geq 3$,

$$H_{n-1} \leq 7^{n-1},\ \ H_{n-2} \leq 7^{n-2}, \text{and}\ \ H_{n-3} \leq 7^{n-3}$$

We have to show that it is also true for H_n.

$$\begin{aligned}
H_n &= 5H_{n-1} + 5H_{n-2} + 63H_{n-3} \\
H_n &\leq 5\cdot 7^{n-1} + 5\cdot 7^{n-2} + 63\cdot 7^{n-3} \\
H_n &\leq 35\cdot 7^{n-2} + 5\cdot 7^{n-2} + 9\cdot 7^{n-2} \\
H_n &\leq 49\cdot 7^{n-2} \\
H_n &\leq 7^n
\end{aligned}$$

Hence, $H_n \leq 7^n$, for all $n \geq 0$ by induction.