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Answer 1

a) (i) $A \cap (B \cup C)$

(ii) $(A \cap B) \cup C$

(iii) $A - ((A \cap B) - C)$

b) (i) $(A \times B) \times C = A \times (B \times C)$

- Let A, B, C different sets given as;
  $A = \{a, b\}$
  $B = \{c, d\}$
  $C = \{e\}$
- $A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$
  $B \times C = \{(c, e), (c, e)\}$
- $(A \times B) \times C = \{((a, c), e), ((a, d), e), ((b, c), e), ((b, d), e)\}$
  $A \times (B \times C) = \{(a, (c, e)), (a, (d, e)), (b, (c, e)), (b, (d, e))\}$

As a result, $(A \times B) \times C \neq A \times (B \times C)$

(ii)

$$(A \cap B) \cap C = A \cap (B \cap C)$$

A	B	C	$A \cap B$	$B \cap C$	$(A \cap B) \cap C$	$A \cap (B \cap C)$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	0	0	0
1	0	0	0	0	0	0
0	1	1	0	1	0	0
0	1	0	0	0	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

As a result, $(A \cap B) \cap C = A \cap (B \cap C)$.

(iii)

$$(A \oplus B) \oplus C = A \oplus (B \oplus C)$$

A	B	C	$A \oplus B$	$B \oplus C$	$(A \oplus B) \oplus C$	$A \oplus (B \oplus C)$
1	1	1	0	0	1	1
1	1	0	0	1	0	0
1	0	1	1	1	0	0
1	0	0	1	0	1	1
0	1	1	1	0	0	0
0	1	0	1	1	1	1
0	0	1	0	1	1	1
0	0	0	0	0	0	0

As a result, $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

Answer 2

a) $A_0 \subseteq f^{-1}(f(A_0))$

- To show $A_0 \subseteq f^{-1}(f(A_0))$, we have to prove that $\forall x(x \in A_0 \to x \in f^{-1}(f(A_0))$.
- Since $x \in A_0$ and $A_0 \subseteq A$, there must be $f(x) \in B$.
- Since f is injective, $f(a) = f(b) \to a = b$, there must be $C_0 \subseteq B$ such that $f(x) \in C_0$.
- As a result, there must be $x \in f^{-1}(C_0)$.
- Since also $f(A_0) = C_0$, $A_0 = f^{-1}(f(A_0))$
- This also implies $A_0 \subseteq f^{-1}(f(A_0))$

b) $f(f^{-1}(B_0)) \subseteq B_0$

- To show $f(f^{-1}(B_0)) \subseteq B_0$, we have to show that $\forall x(x \in f(f^{-1}(B_0)) \to x \in B_0)$.
- Let $f^{-1}(B_0) \in C_0$, then $f(C_0) = B_0$.
- $f(C_0) \subseteq B_0$ if and only if f is surjective, since there can be elements that does not belong to $f(f^{-1}(B_0))$.
- As a result, $f(f^{-1}(B_0)) \subseteq B_0$.

Answer 3

Let A be a nonempty set. Show that the following are equivalent

(i) A is countable

(ii) There is a surjective function $f : \mathbb{Z}^+ \to A$

(iii) There is a injective function $f : A \to \mathbb{Z}^+$

1) If A is countable, that means either A is a finite set, or A has same cardinality with the set Z^+ (defn. 3 from textbook p. 171).

2) The sets A and $\mathbb{Z}^+$ have the same cardinality if and only if there is one to one correspondence from A to $\mathbb{Z}^+$ (defn. 1 from textbook p. 170).

3) If there is one to one correspondence from A to $\mathbb{Z}^+$, then they the cardinality of A is less then or same as the cardinality of $\mathbb{Z}^+$ (defn. 2 from textbook p. 170).

4) Let $f : \mathbb{Z}^+ \to A$. If A is countable and $|A| \leq |\mathbb{Z}^+|$, then using (3) we can define f such that f is a surjective function. $(i) \to (ii)$

5) Let $g : A \to \mathbb{Z}^+$. As a result from f and (3) we can define g as injective, since there is one to one correspondence from A to $\mathbb{Z}^+$. $(ii) \to (iii)$

6) Since $|A| \leq |\mathbb{Z}^+|$, we can say that A is countable using (1). $(iii) \to (i)$

Answer 4

a) Show that the set of finite binary strings is countable.

- We can define a function f from binary strings which is a sequence of $0s$ and $1s$ to positive integers such as $f(\text{"0001"}) = 1$, $f(\text{"0101"}) = 5$ etc.
- Since we can enumerate positive integers, $\mathbb{Z}^+$ is countable, the set of finite binary strings must be countable.

b) Show that the set of infinite binary strings is uncountable.

- If the consider infinite binary strings, then we cannot define a function f from binary strings which is a sequence of $0s$ and $1s$ to positive integers since there will be infinite positive integers that can be represented by a binary string.
- Since we cannot define the function f, we cannot represent binary strings as positive integers.
- Since we cannot enumerate the infinite binary strings as they are uncountable, also the set of infinite binary string is uncountable.

Answer 5

a) Determine whether $\log n!$ is $\Theta(n \log n)$.

1) Consider that $\log(a \cdot b) = \log(a) + \log(b)$.
2) In order to prove $\log n!$ is $\Theta(n \log n)$, we have to prove both $\log n!$ is $O(n \log n)$ and $n \log n$ is $\Omega(\log n!)$.
3) $\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \dots 1)$
   $\log(n!) = \log(n) + \log(n-1) + \log(n-2) + \cdots + \log(1)$
   $\log(n!) \leq \log n + \log n + \log n + \cdots + \log n$ (n times)
   $\log(n!) \leq n \log n$
4) So, $\log n!$ is $O(n \log n)$.
5) $\log(n!) = \log(n \cdot (n-1) \cdot (n-2) \cdot (n-3) \dots 1)$
   $\log(n!) = \log(n) + \log(n-1) + \log(n-2) + \cdots + \log(1)$
   $\log(n!) \geq \log(n) + \cdots + \log(\frac{n}{2} + 1) + \log(\frac{n}{2})$ (delete second half)
   $\log(n!) \geq \log(\frac{n}{2}) + \cdots + \log(\frac{n}{2}) + \log(\frac{n}{2})$ (replace all by $\frac{n}{2}$, which is much smaller)
   $\log(n!) \geq \frac{n}{2} \cdot \log(\frac{n}{2})$
6) So, $\log n!$ is $\Omega(n \log n)$.
7) As a result from both (4) and (6), $\log n!$ is $\Theta(n \log n)$.

b) Which function grows faster, $n!$ or 2^n?

- Let $f(n) = \dfrac{n!}{2^n}$
- $L = \lim_{n\to\infty} |\dfrac{f(n+1)}{f(n)}| = \lim_{n\to\infty} |\dfrac{(n+1)!}{2^{n+1}} \cdot \dfrac{2^n}{n!}| = \lim_{n\to\infty} |\dfrac{(n+1) \cdot n!}{2 \cdot 2^n} \cdot \dfrac{2^n}{n!}| = \dfrac{n+1}{2}$
- For $n \in \mathbb{Z}^+, L \geq 1$.
- As a result, from ratio test $n!$ grows faster than 2^n.