CENG 384 - Signals and Systems for Computer Engineers
Spring 2021
Homework 3

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1. (a)

$$x(t) = \frac{1}{2} + cos(\omega_0 t)$$

$$x(t) = \frac{1}{2} + \frac{e^{j\omega_0 t}}{2} + \frac{e^{-j\omega_0 t}}{2}$$

$$a_0 = a_1 = a_{-1} = \frac{1}{2}. \text{ All other } a_k\text{'s } = 0.$$

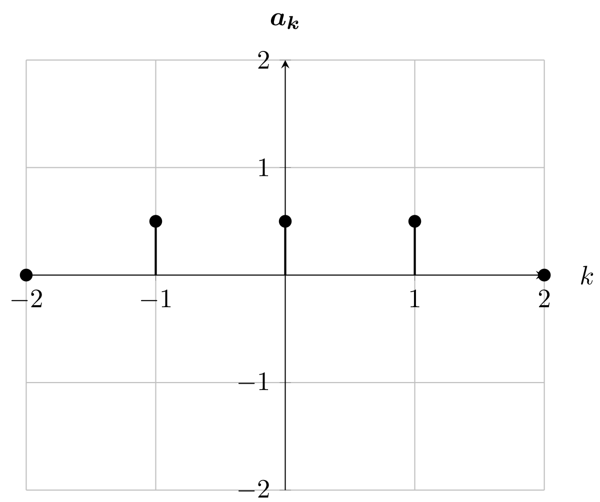

Figure 1: q1a

(b)

$$y(t) = \frac{3}{2} + 2sin(\omega_0 t)$$

$$y(t) = \frac{3}{2} - je^{2j\omega_0 t} + je^{-2j\omega_0 t}$$

$$a_0 = \frac{3}{2}, a_{-2} = -a_2 = j. \text{ All other } a_k\text{'s } = 0.$$

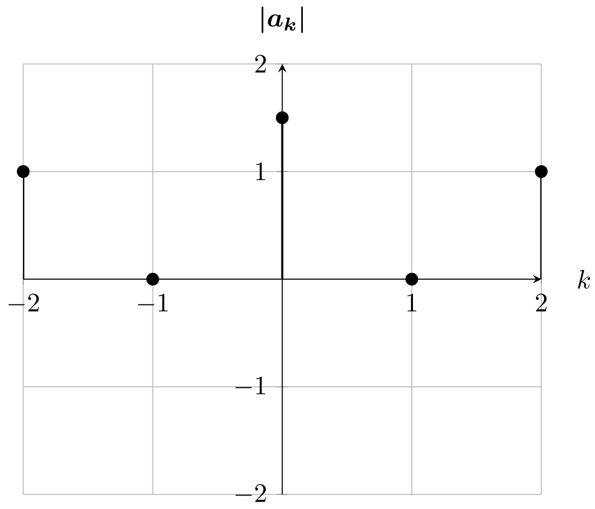

Figure 2: Magnitude of q1b

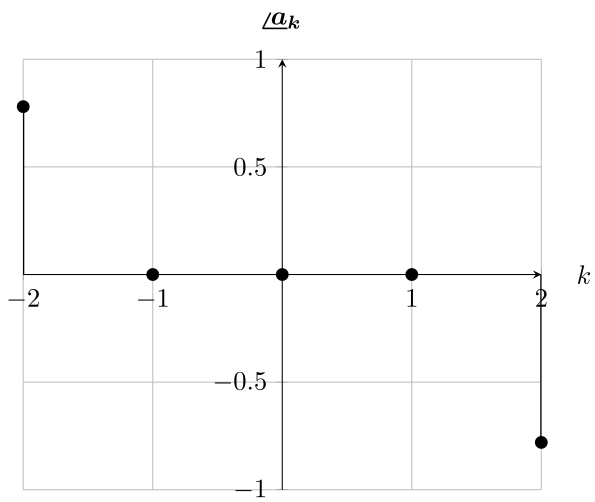

Figure 3: Phase of q1b

(c)

$$z(t) = x(t) + y(t) + cos(2\omega_0 t + \frac{\pi}{4})$$
$$z(t) = x(t) + y(t) + e^{\pi/4} \cdot e^{2j\omega_0 t} + e^{\pi/4} \cdot e^{-2j\omega_0 t}$$

$a_0 = 2, a_1 = a_{-1} = \frac{1}{2}, a_{-2} = e^{\pi/4} + j, a_2 = e^{\pi/4} - j$. All other a_k's $= 0$.

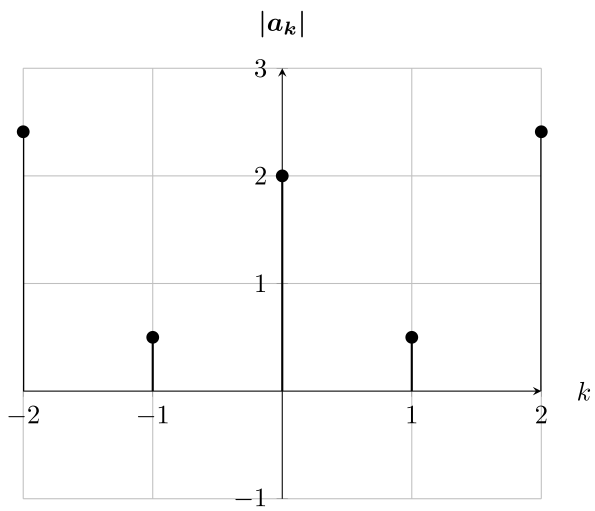

Figure 4: Magnitude of q1c

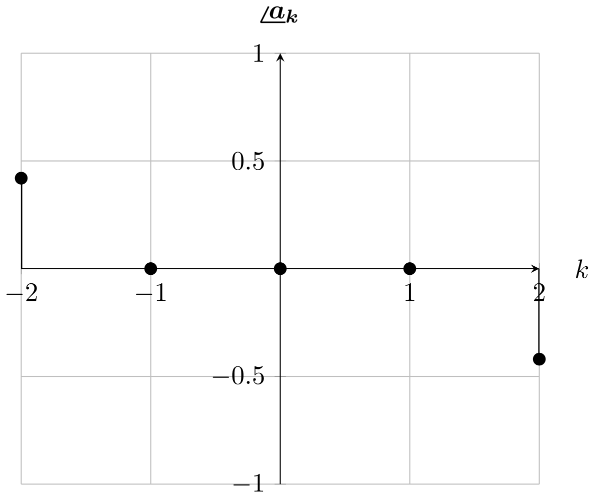

Figure 5: Phase of q1c

2. Note that the period is $T_0 = T$.We take $\omega_0 = 2\pi/T_0 = 2\pi/T$. Choosing the period of integration as 0 to T, we have

$$\frac{A_0}{2} = \frac{1}{T}\int_0^{T_1} x(t)\,dt$$
$$\frac{A_0}{2} = \frac{1}{T}\int_0^{T_1} M\,dt$$
$$\frac{A_0}{2} = \frac{MT_1}{T}$$
$$A_0 = \frac{2MT_1}{T}$$

$$A_k = \frac{2}{T}\int_0^{T_1} x(t)cos(k\omega_0 t)\,dt$$
$$A_k = \frac{2}{T}\int_0^{T_1} Mcos(k\omega_0 t)\,dt$$
$$A_k = \frac{2Msin(k\omega_0 T_1)}{k\omega_0 T}$$

$$B_k = \frac{2}{T}\int_0^{T_1} x(t)sin(k\omega_0 t)\,dt$$
$$B_k = \frac{2}{T}\int_0^{T_1} Msin(k\omega_0 t)\,dt$$
$$B_k = -\frac{2Mcos(k\omega_0 T_1)}{k\omega_0 T}$$

$$x(t) = \frac{1}{2}\frac{2MT_1}{T} \cdot \sum_{k=1}^{\infty} \frac{2Msin(k\omega_0 T_1)}{k\omega_0 T} \cdot cos(k\omega_0 t) - \frac{2Mcos(k\omega_0 T_1)}{k\omega_0 T} \cdot sin(k\omega_0 t)$$

3. (a)

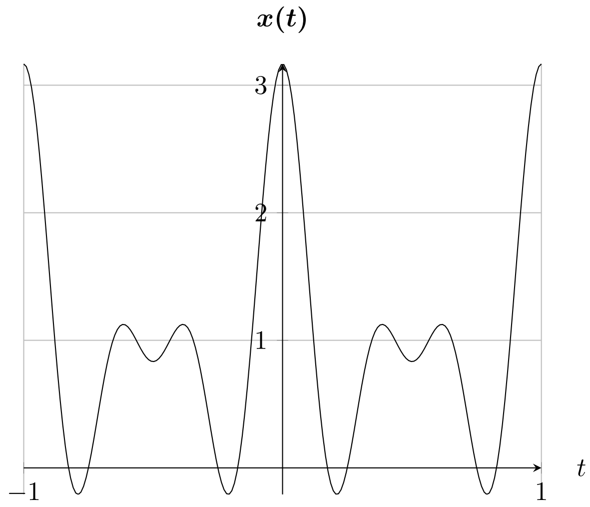

Figure 6: q3a

(b)

$$x(t) = 1 + \frac{1}{2}cos(2\pi t) + cos(4\pi t) + \frac{2}{3}cos(6\pi t)$$

$$x(t) = 1 + \frac{e^{j2\pi t}}{4} + \frac{e^{-j2\pi t}}{4} + \frac{e^{j4\pi t}}{2} + \frac{e^{-j4\pi t}}{2} + \frac{e^{j6\pi t}}{3} + \frac{e^{-j6\pi t}}{3}$$

For $\omega_0 = 2\pi$,

$$a_0 = 1, a_1 = a_{-1} = \frac{1}{4}, a_2 = a_{-2} = \frac{1}{2}, a_3 = a_{-3} = \frac{1}{3}. \text{ All other } a_k\text{'s } = 0.$$

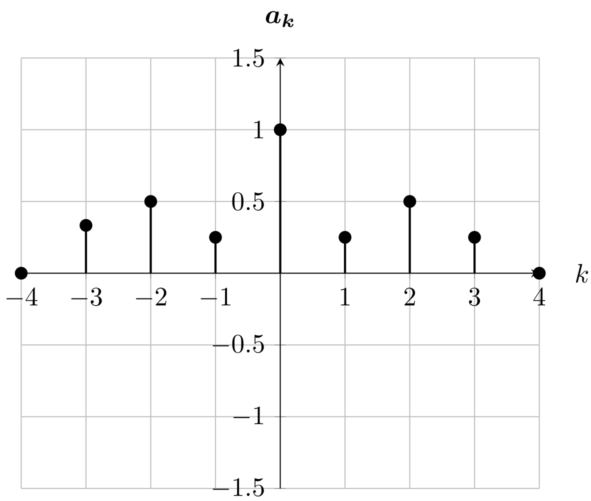

Figure 7: q3b

(c)

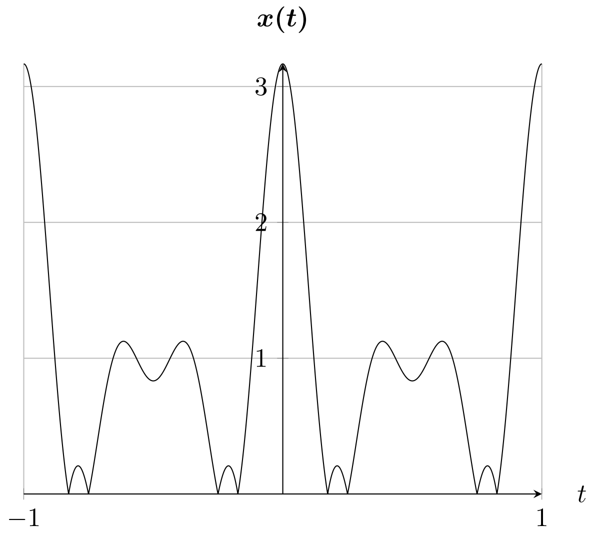

Figure 8: Magnitude of q3c

(d)

4. (a)

1- Calculating a_k' for $x(t-3)$

$$a_k' = \frac{1}{T_0}\int_{T_0} x(t-3)e^{-jk\omega_0 t}\,dt$$

Substituting $\tau = t-3$, we obtain

$$a_k' = a_k e^{-3jk\omega_0}$$

2- Calculating a_k'' for $x(-t)$

$$a_k'' = \frac{1}{T_0}\int_{T_0} x(-t)e^{-jk\omega_0 t}\,dt$$

Substituting $\tau = -t$, we obtain

$$a_k'' = \frac{1}{T_0}\int_{T_0} x(\tau)e^{jk\omega_0 \tau}\,d\tau = -a_k$$

Finally, we combine the results

$$\frac{a_k'}{3} - \frac{2a_k''}{7} = a_k(\frac{e^{-3jk\omega_0}}{3} + \frac{2}{7})$$

(b)

1- Using differentiation property thrice on the function.

$$\frac{dx(t)}{dt} \longleftrightarrow (j\omega_0)a_k$$

$$a_k = (j\omega_0)^3 \cdot a_k$$

5. (a)

$$x[n] = sin(\frac{\pi}{2}n)$$

We choose ω_0 as 2π.Using Euler's relation, we have

$$x[n] = \frac{1}{2j}e^{j(\pi/2)n} - \frac{1}{2j}e^{-j(\pi/2)n}$$

$$a_0 = 0, a_1 = \frac{1}{2j}, a_{-1} = \frac{-1}{2j} \text{ All other } a_k\text{'s } = 0.$$

(b)

$$y[n] = 1 + cos(\frac{\pi}{2}n)$$

We choose ω_0 as 2π.

$$x[n] = 1 + \frac{1}{2}e^{j(\pi/2)n} + \frac{1}{2}e^{-j(\pi/2)n}$$

$$a_0 = 1, a_1 = a_{-1} = \frac{1}{2} \text{ All other } a_k\text{'s } = 0.$$

(c) Multiplication Property is as follows

$$x(t) \longleftrightarrow a_k \text{ and } y(t) \longleftrightarrow b_k$$

then,

$$a_k * b_k \longleftrightarrow \sum_{\forall l} a_l b_{k-l}$$

$$a_k * b_k \to a_1 = \frac{1}{2j}, a_{-1} = \frac{-1}{2j}, a_2 = \frac{1}{4j}, a_{-2} = \frac{-1}{4j} \text{ All other } a_k\text{'s } = 0.$$

(d)

$$z[n] = x[n] * y[n] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot y[k]$$

$$z[n] = sin((\pi/2)n) + sin((\pi/2)n)cos((\pi/2)n)$$

$$z[n] = sin((\pi/2)n) + \frac{1}{2}sin(\pi n)$$

We choose ω_0 as 2π. Using Euler's relation, we have

$$z[n] = \frac{1}{2j}e^{j(\pi/2)n} - \frac{1}{2j}e^{-j(\pi/2)n} + \frac{1}{4j}e^{j(\pi)n} - \frac{1}{4j}e^{-j(\pi)n}$$

$$a_0 = 0, a_1 = \frac{1}{2j}, a_{-1} = \frac{-1}{2j}, a_2 = \frac{1}{4j}, a_{-2} = \frac{-1}{4j} \text{ All other } a_k\text{'s } = 0.$$

Comparing a_k with the a_k from the part (c), we see that both are the same.

6. The Fourier series coefficients of $x[n]$, which is periodic with period N, are given by

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-jk(2\pi/N)n}$$

For $N = 12$,

$$a_k = \frac{1}{12}\sum_{n=0}^{11} x[n]e^{-jk(\pi/6)n}$$

$$a_k = cos(\frac{k\pi}{6}) + sin(\frac{5k\pi}{6})$$

$$a_k = \frac{1}{2}e^{j(\pi k/6)} + \frac{1}{2}e^{-j(\pi k/6)} + \frac{1}{2j}e^{j(5\pi k/6)} - \frac{1}{2j}e^{j(5\pi k/6)}$$

Hence,

$$x[n] = 6\delta[n-1] + 6\delta[n-11] - 6j\delta[n-5] + 6j\delta[n-7], 0 \le n \le 11$$

7. (a)

$$x[n] = \sum_{n=0}^{3} a_k e^{jk(2\pi/4)n}$$

Period is $N = 4$. So the signal $x[n]$ can be expressed as above.

$$x[0] = a_0 + a_1 + a_2 + a_3 = 0$$

$$x[1] = a_0 + a_1 e^{j(\pi/2)} + a_2 e^{j\pi} + a_3 e^{j(3\pi/2)} = 1$$

$$x[2] = a_0 + a_1 e^{j\pi} + a_2 e^{2j\pi} + a_3 e^{j3\pi} = 2$$

$$x[3] = a_0 + a_1 e^{j(3\pi/2)} + a_2 e^{j3\pi} + a_3 e^{j(9\pi/2)} = 1$$

The preceding set of linear equations can be reduced to

$$a_0 + a_1 + a_2 + a_3 = 0$$

$$a_0 + ja_1 - a_2 - ja_3 = 1$$

$$a_0 - a_1 + a_2 - a_3 = 2$$

$$a_0 - ja_1 - a_2 + ja_3 = 1$$

Solving the equations we get

$$a_0 = 1, a_2 = 0, a_1 = a_3 = \frac{-1}{2} \text{ All other } a_k\text{'s } = 0.$$

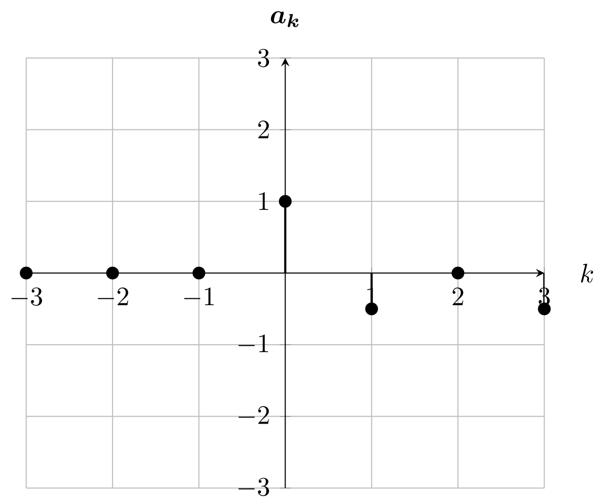

Figure 9: q7a

(b)

$$y[n] = x[n] - \delta[n+5] - \delta[n+1] - \delta[n-3] - \delta[n-7]$$

$$y[n] = \sum_{n=0}^{3} a_k e^{jk(2\pi/4)n}$$

Period is $N = 4$. So the signal $y[n]$ can be expressed as above.

$$y[0] = a_0 + a_1 + a_2 + a_3 = 0$$

$$y[1] = a_0 + a_1 e^{j(\pi/2)} + a_2 e^{j\pi} + a_3 e^{j(3\pi/2)} = 1$$
$$y[2] = a_0 + a_1 e^{j\pi} + a_2 e^{2j\pi} + a_3 e^{j3\pi} = 2$$
$$y[3] = a_0 + a_1 e^{j(3\pi/2)} + a_2 e^{j3\pi} + a_3 e^{j(9\pi/2)} = 0$$

The preceding set of linear equations can be reduced to

$$a_0 + a_1 + a_2 + a_3 = 0$$

$$a_0 + ja_1 - a_2 - ja_3 = 1$$
$$a_0 - a_1 + a_2 - a_3 = 2$$
$$a_0 - ja_1 - a_2 + ja_3 = 0$$

Solving the equations we get

$$a_0 = \frac{3}{4}, a_2 = \frac{1}{4}, a_1 = \frac{1-2j}{2j} a_3 = \frac{1+2j}{2j} \text{ All other } a_k\text{'s } = 0.$$

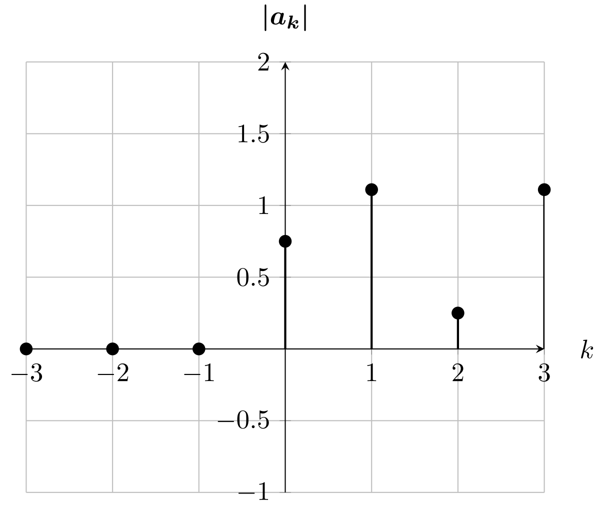

Figure 10: Magnitude of q7b

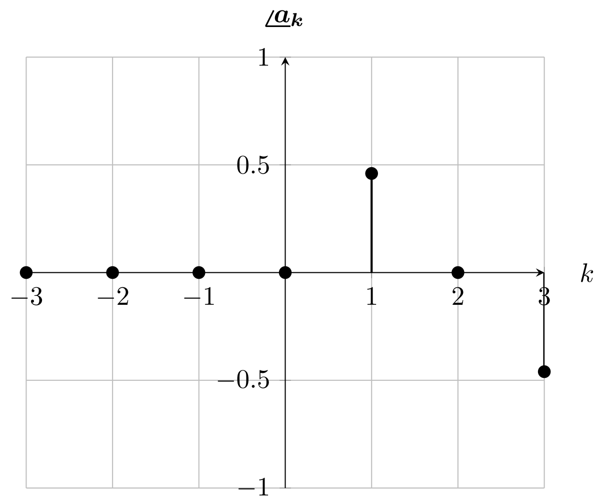

Figure 11: Phase of q7b