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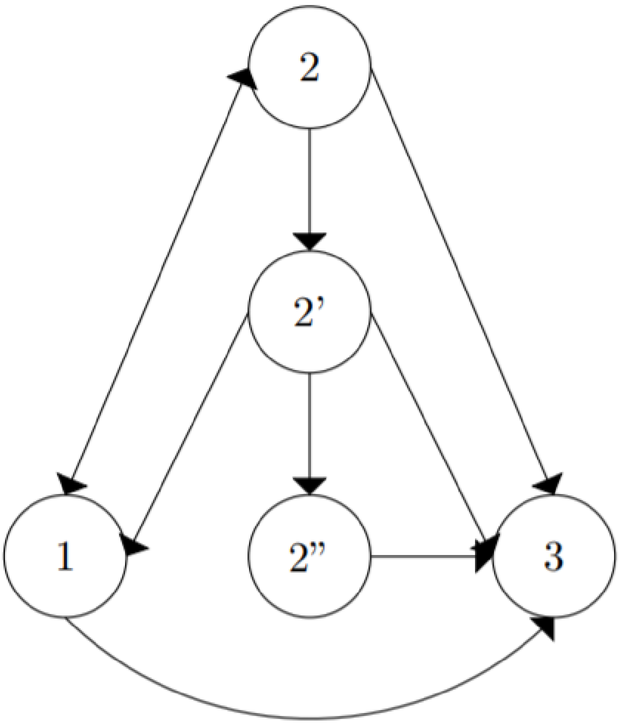
Monte Carlo simulation of an insurance

June 13, 2019

1. Description of a problem

Consider a Markov Chain given by graph 1. State 1 is interpreted as active, various states 2 represent different stages of some sickness and state 3 represents death. In any given year there is also a chance to remain in your current state.
The probability of transitions are given by fig. 2 where $T_1(t) = \frac{l_t}{l_0}$, $T_2(t) = min(10\frac{l_t}{l_0}, 1)$ $T_{2'} = min(50\frac{l_t}{l_0}, 1)$ and $T_{2''} = min(250\frac{l_t}{l_0}, 1)$, provided that l_t is a respective lifetable. Assumed interest rate is 0.02.

Figure 1. State transition graph



* Tomasz Jurczyk - coding
** Denis Grenda - coding
*** Michał Smolinski - documentation

Figure 2. State transition probabilities

$$\mathbb{P}(t) = \begin{array}{c|ccccc} & 1 & 2 & 2' & 2'' & 3 \\ \hline 1 & \frac{1}{10}\max(9-9T_1, 9.9-10T_1) & \frac{1}{10}\min(1-T_1, 0.1) & 0 & 0 & T_1 \\ 2 & 0.3-\frac{3}{7}\max(T_2-0.3,0) & \frac{5}{7}\max(T_2+0.4,0.7)-T_2 & 0.2-\frac{2}{7}\max(T_2-0.3,0) & 0 & T_2 \\ 2' & 0.2-\frac{2}{5}\max(T_{2'}-0.5,0) & 0 & \frac{3}{5}\max(T_{2'}+\frac{2}{3},\frac{7}{6})-T_{2'} & 0.1-\frac{1}{5}\max(T_{2'}-0.5,0) & T_{2'} \\ 2'' & 0 & 0 & 0 & 1-T_{2''} & T_{2''} \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array}$$

2. State transition probabilities

Figures 3 - 10 contain transition probabilities for states listed in previous part of the document. Figs 3 and 4 tell us how likely is transition from state 1. The line listed in red is basically the lifetable data. The line in blue is similar to lifetable's reciprocal, and it presents the probability of deceasing from other reasons than modeled disease. On the other hand, probability of getting sick (in green) is roughly constant over time

In figures 5 and 6 we see transitions from state 2. Probability of getting healthy is constant over time (up to age 77), just like the prob. of staying in this state. Getting deceased outright from state 2 becomes dramatically more likely after age 60

Figures 7 and 10 show us transitions from state 2'. They looks similar to state 2 plots (probabilities of getting active 2 and of progressing to 2" are constant), but rise in $p_{2',3}$ and fall in $p_{2',2'}$ are more sharp. After age 62 it is unlikely to survive a year in state 2'.

State 2" is terminal one. Hence figures 9 and 9 show only probabilities of staying alive and getting dead, which are reciprocals of each other. After age 45 it is unlikely to survive a year in that state.

Figure 3. Transition probabilities from state 1 - male

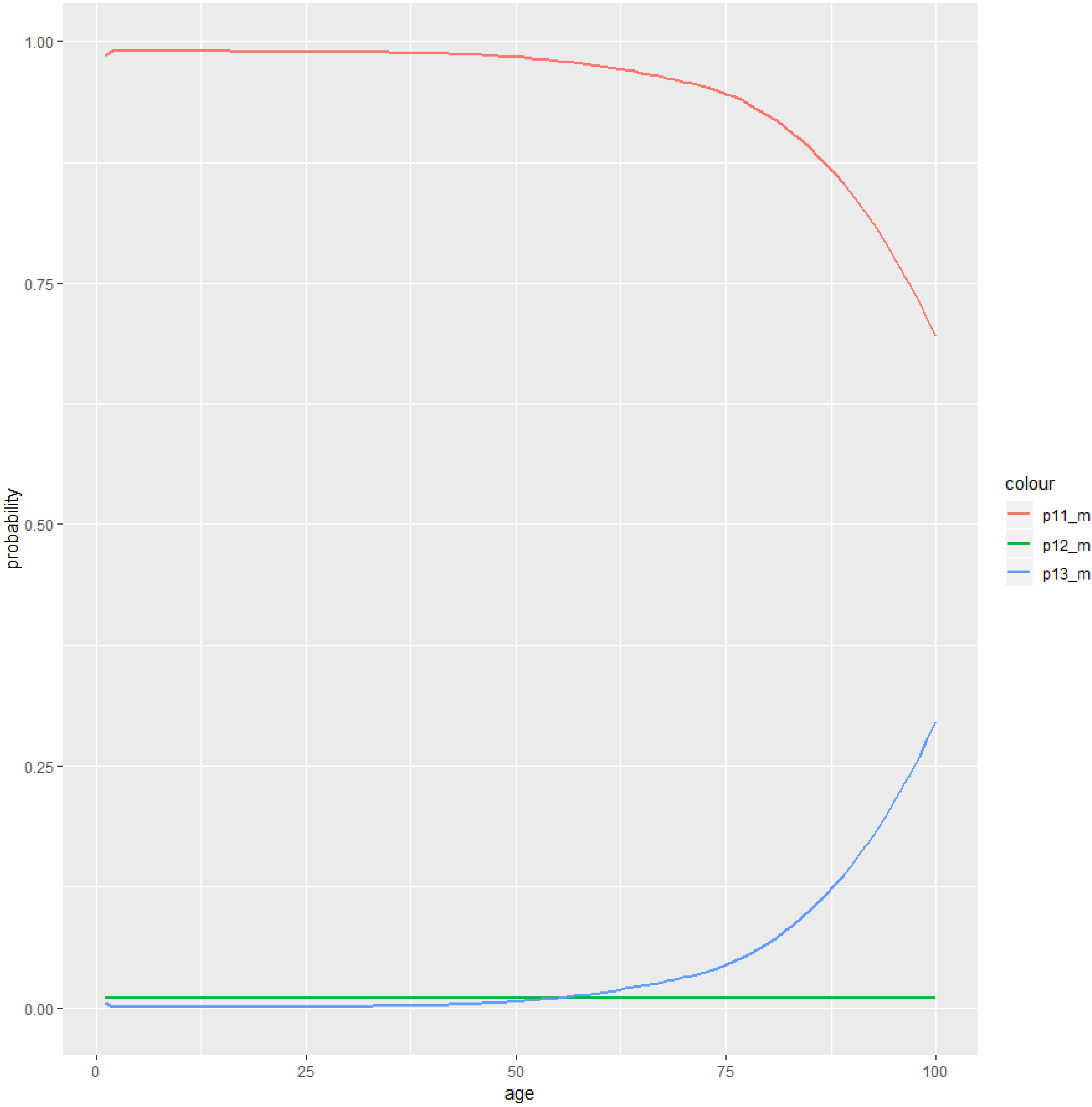


Figure 4. Transition probabilities from state 1 - female

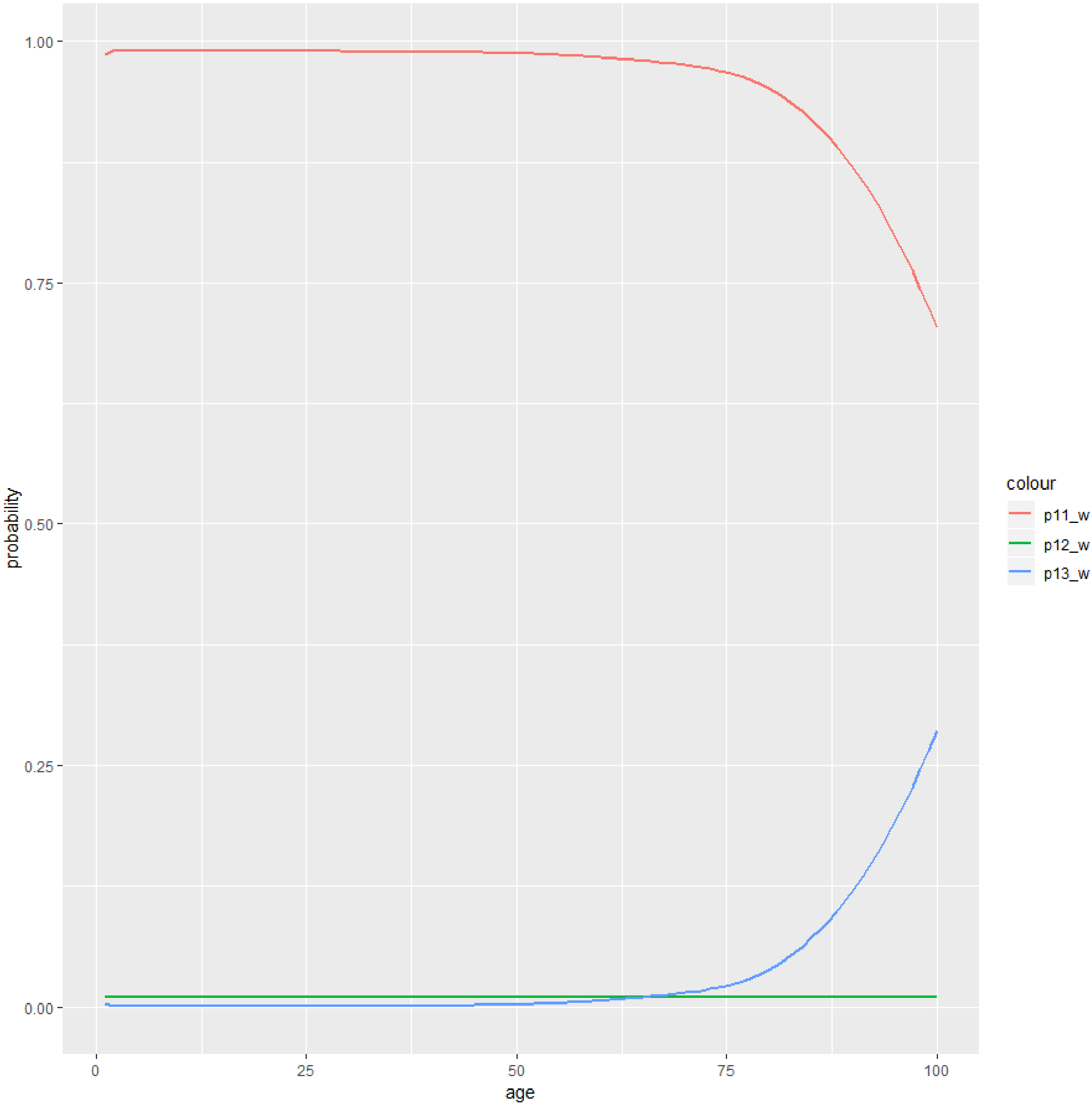


Figure 5. Transition probabilities from state 2 - male

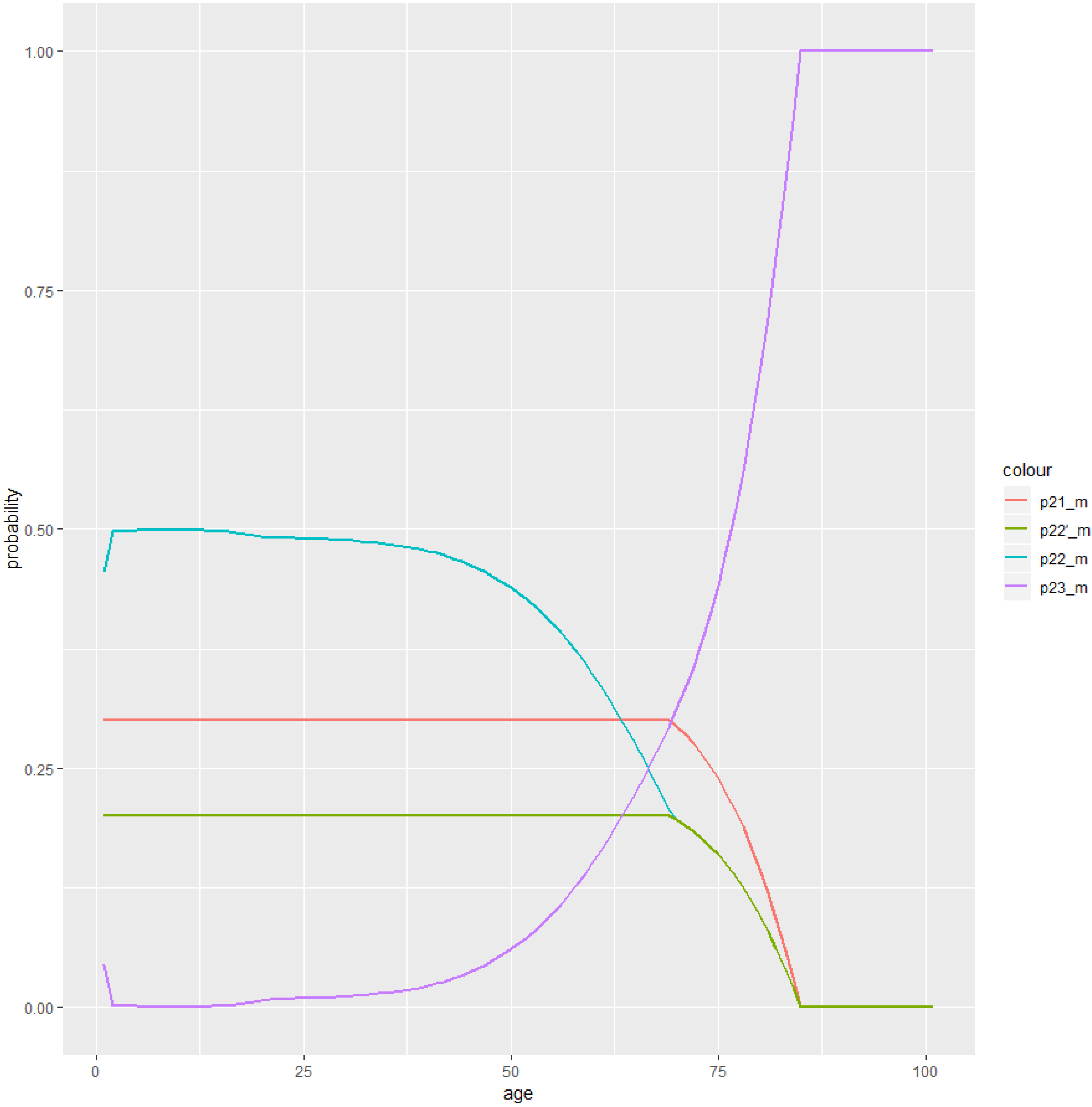


Figure 6. Transition probabilities from state 2 - female

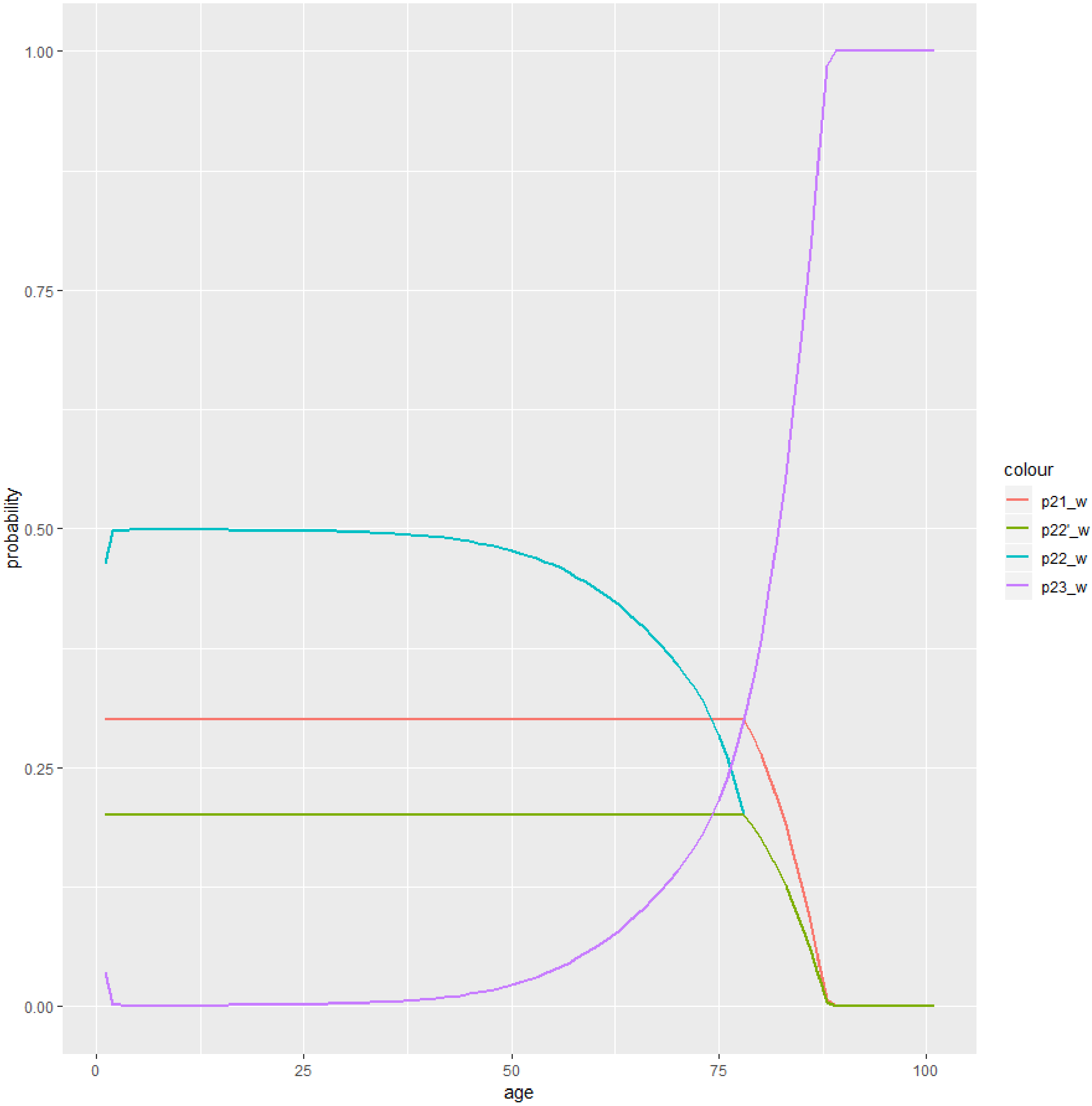


Figure 7. Transition probabilities from state 2' - male

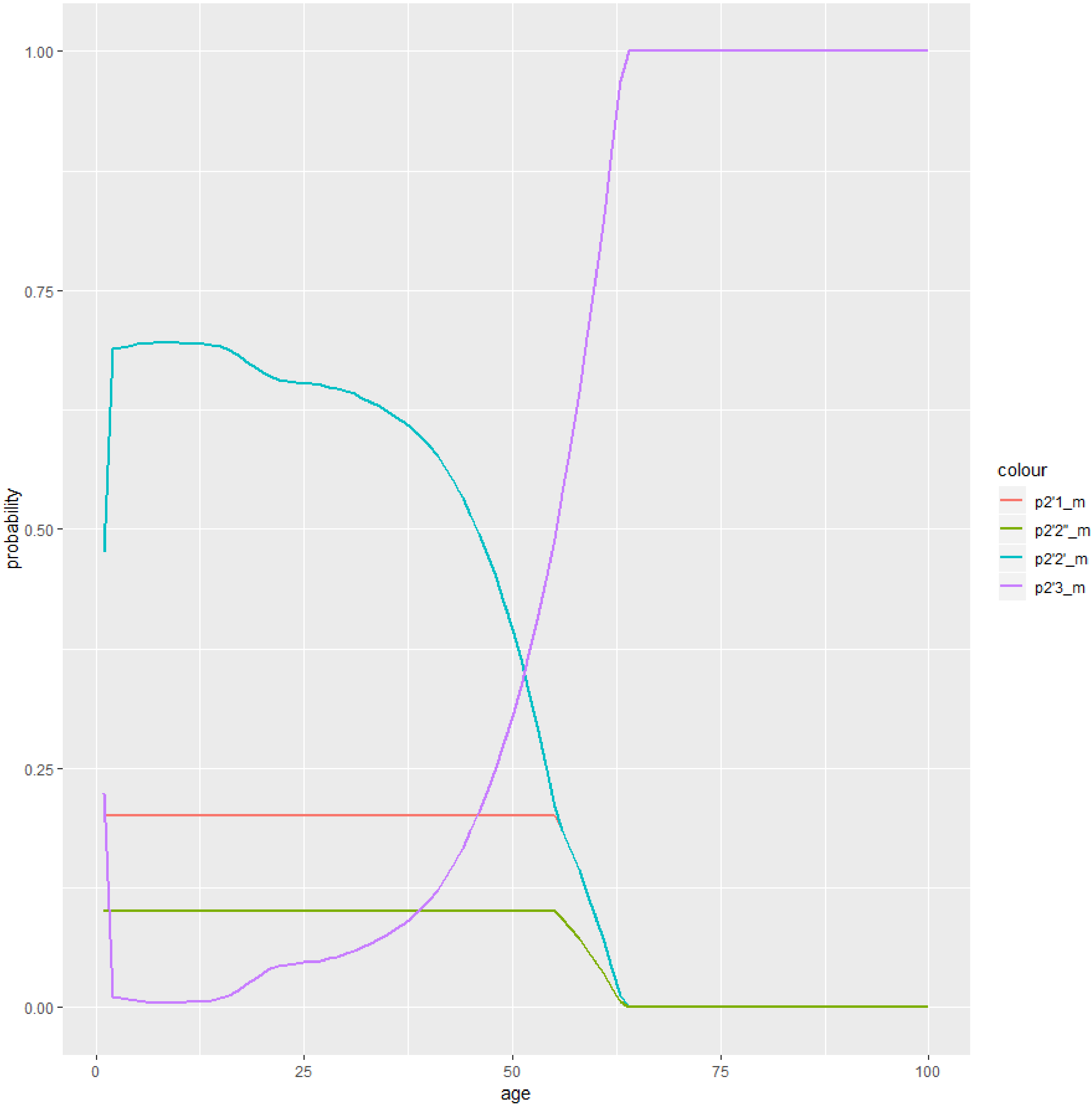


Figure 8. Transition probabilities from state 2' - female

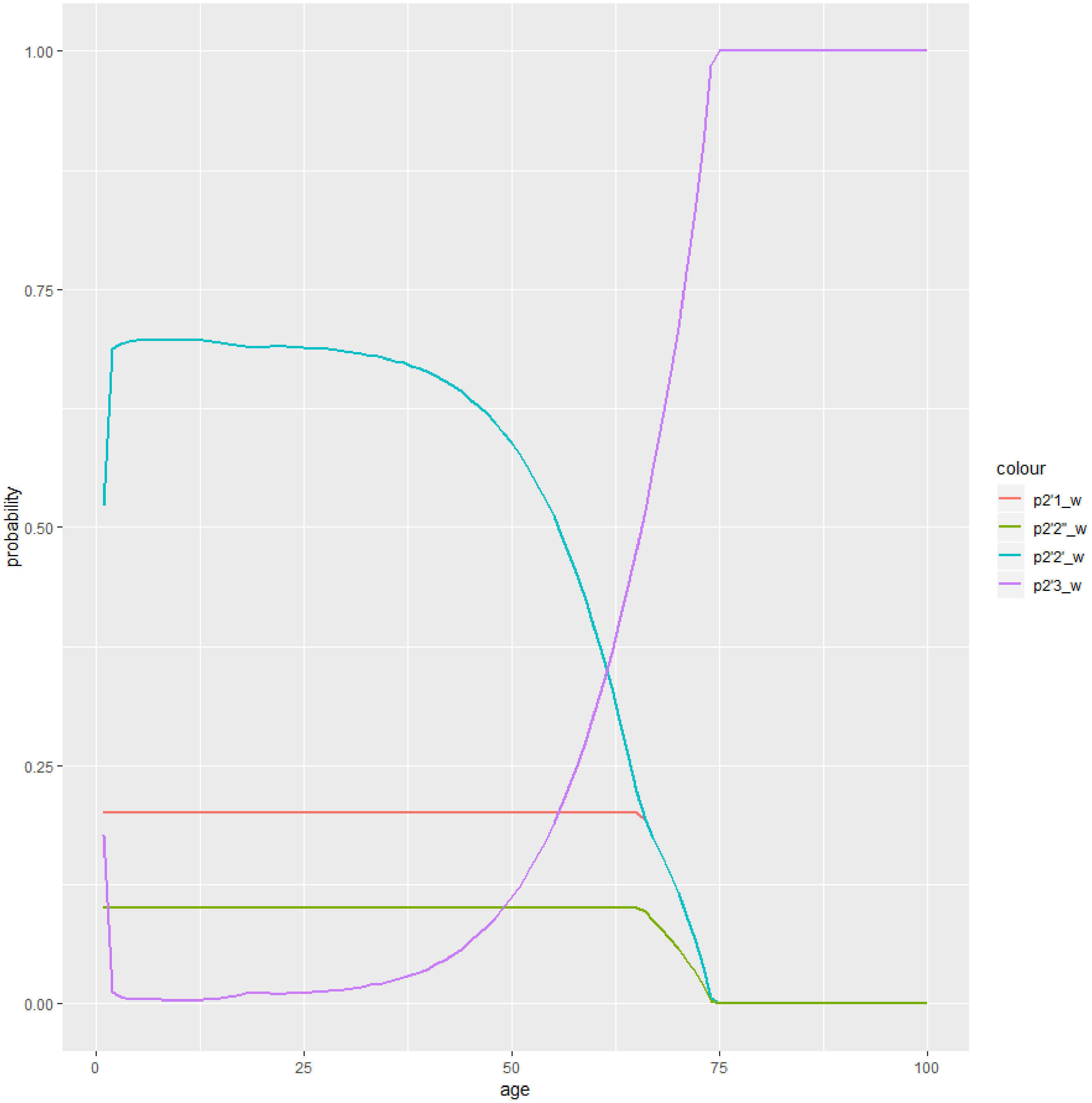


Figure 9. Transition probabilities from state 2" - male

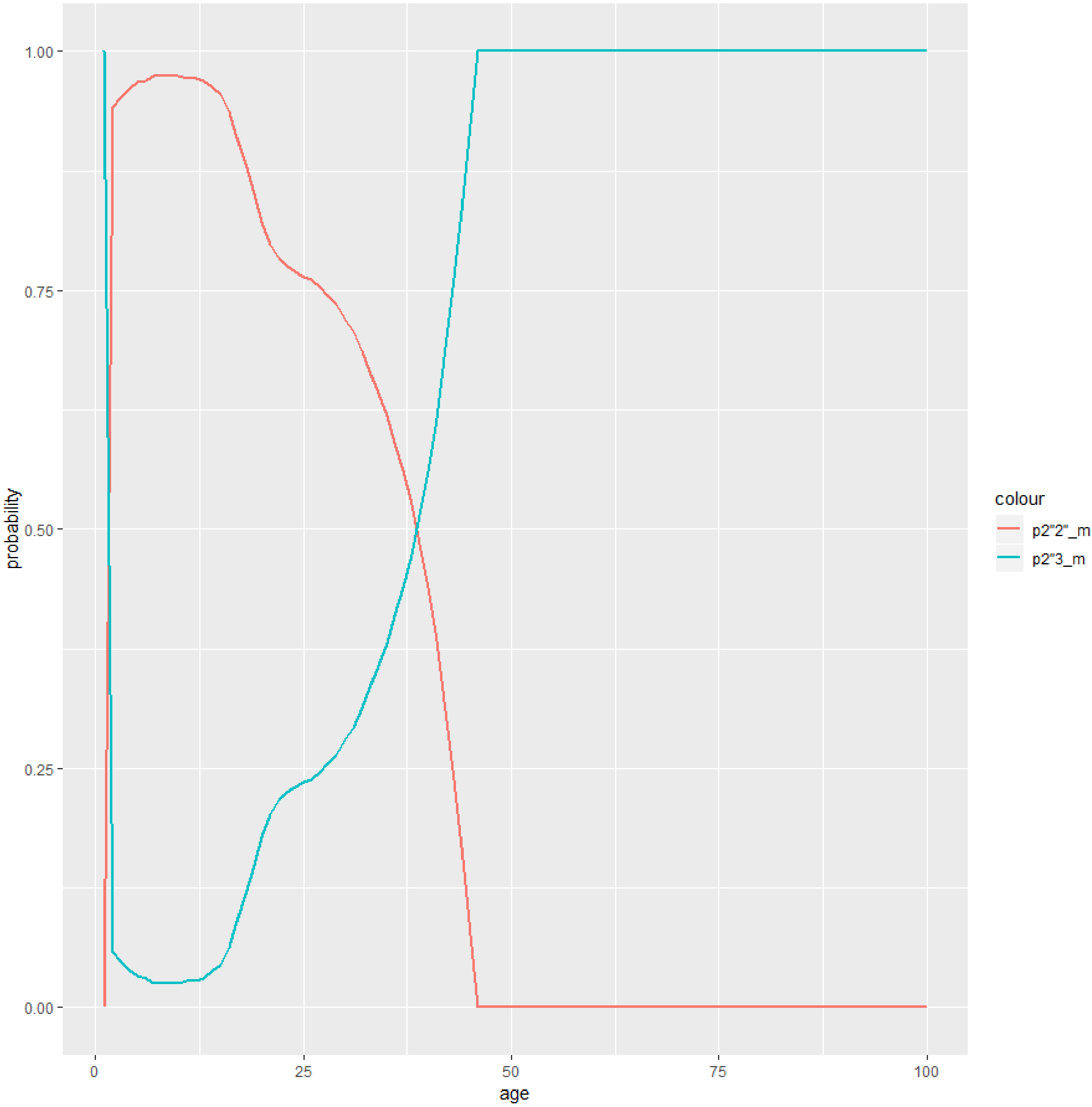
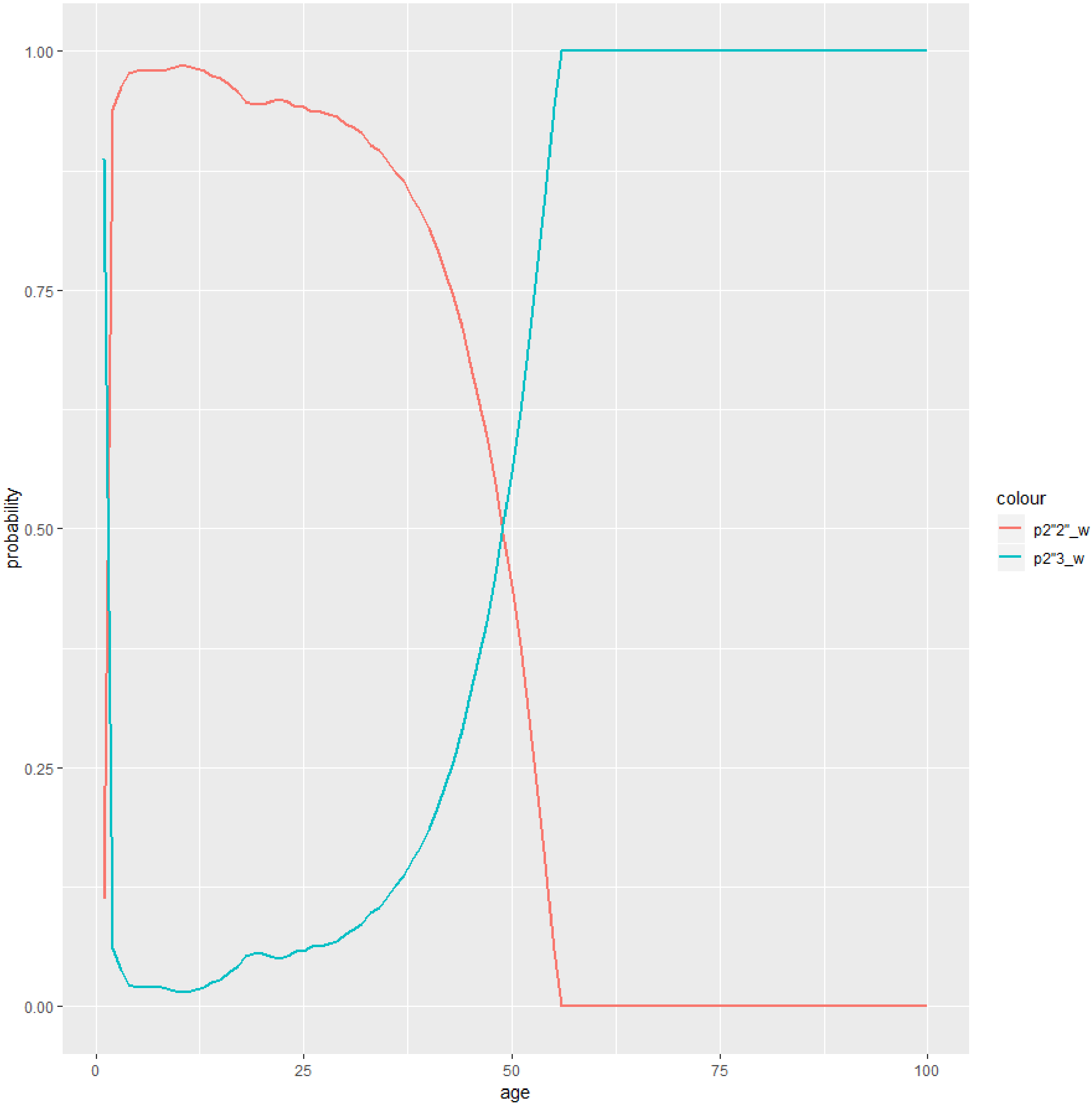


Figure 10. Transition probabilities from state 2" - female



3. Task 1 - state occupancy periods

Figures 11, 12, 13 and 14 show us, how much does average person spend in respective state. The results were achieved by performing 500 simulations, with 300 random lives each, then taking the average for each simulation. Every live is simulated using the distribution implied by lifetables.

Figure 11 reads that people generally stay active for at least few decades (on average 66.97 years for men and 74.46 for women). Figures 12, 13, 14 describe the course of disease. The illness progresses very rapidly, causing that average affected lives less than three years.

Mean lengths of each "2" stage are:
— state 2: 1.18 years for males and 1.35 years for females,
— state 2': 0.53 years for males and 0.66 for females,
— state 2": 0.227 years for males and 0.5 years for females.

Figure 11. Time spent in state 1

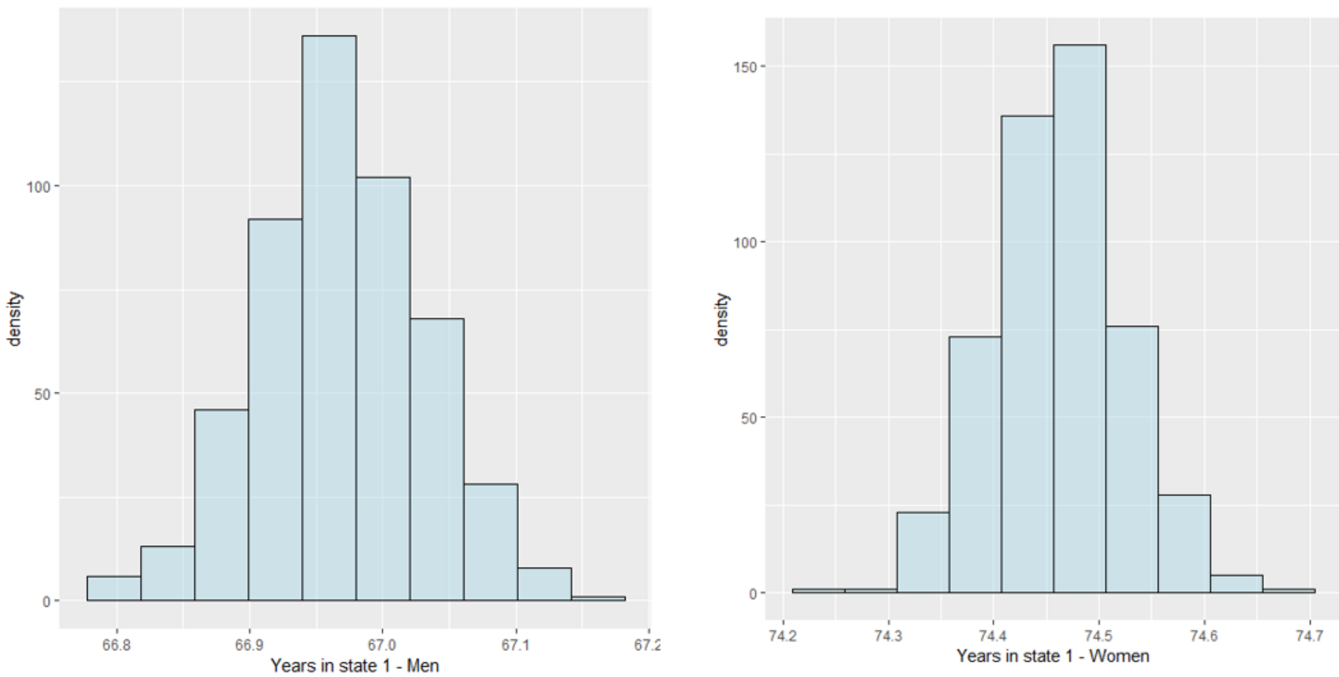


Figure 12. Time spent in state 2

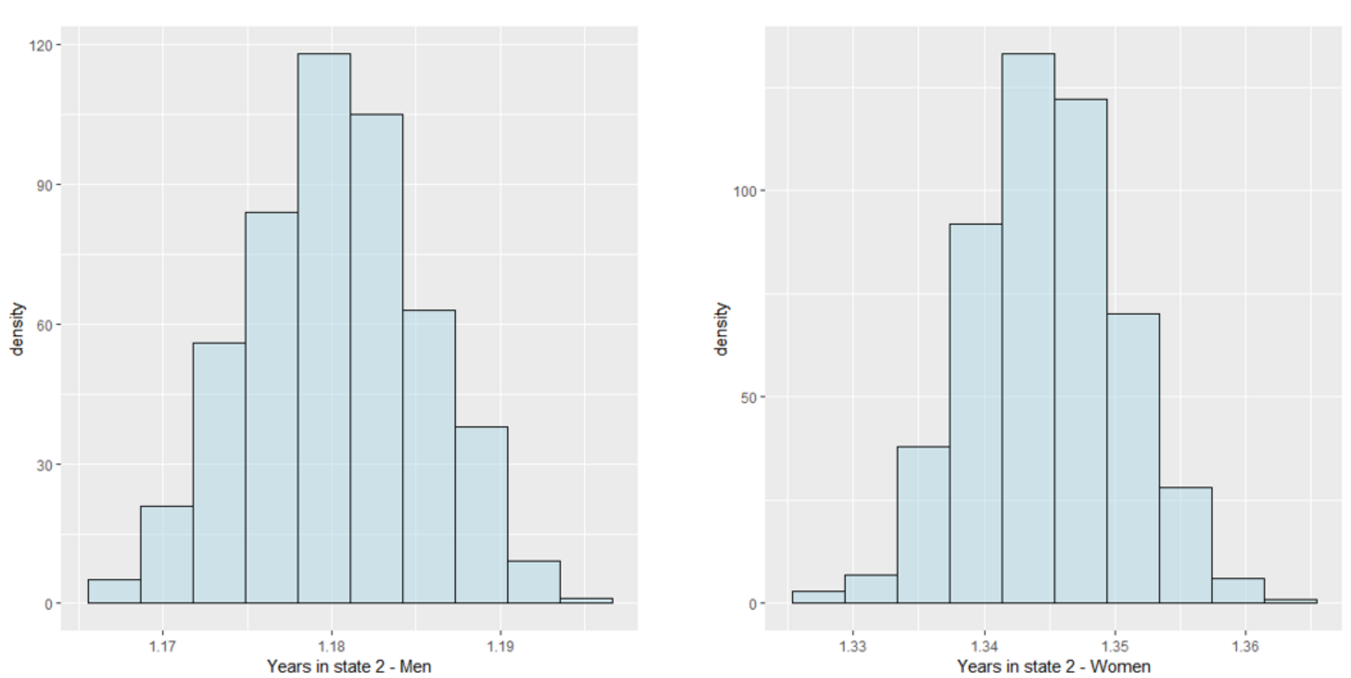


Figure 13. Time spent in state 2'

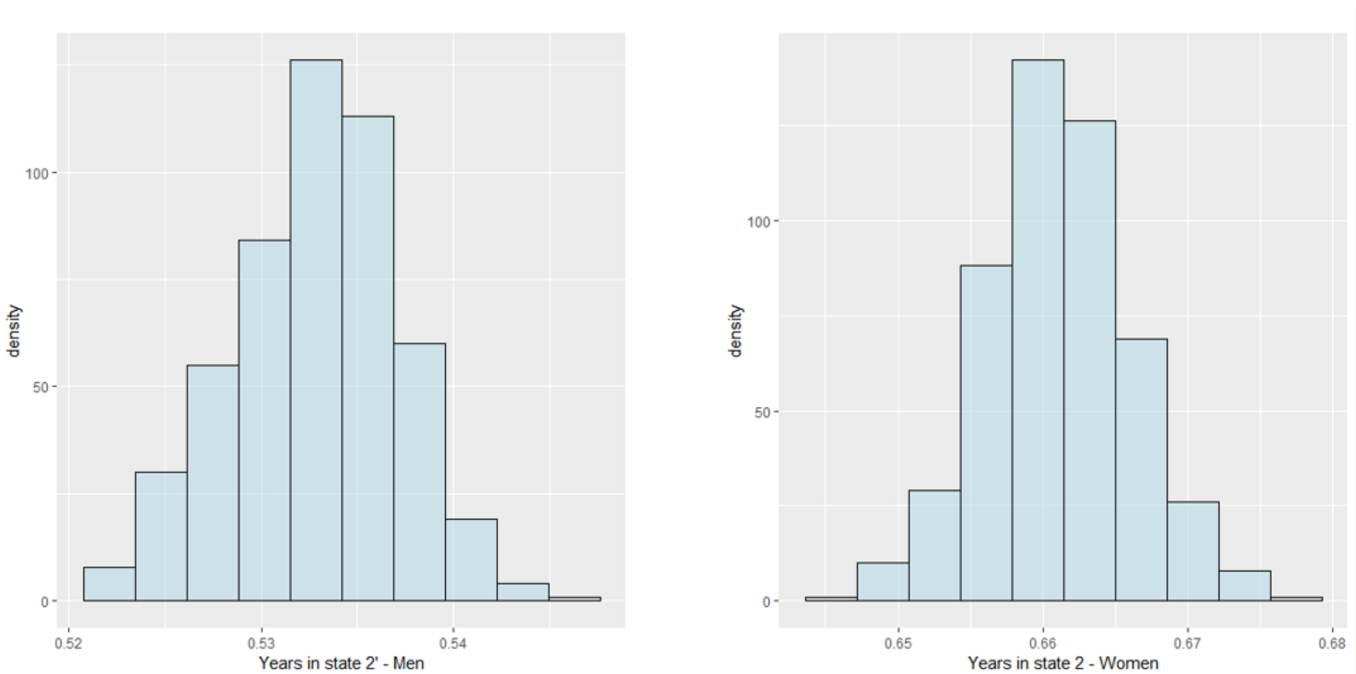
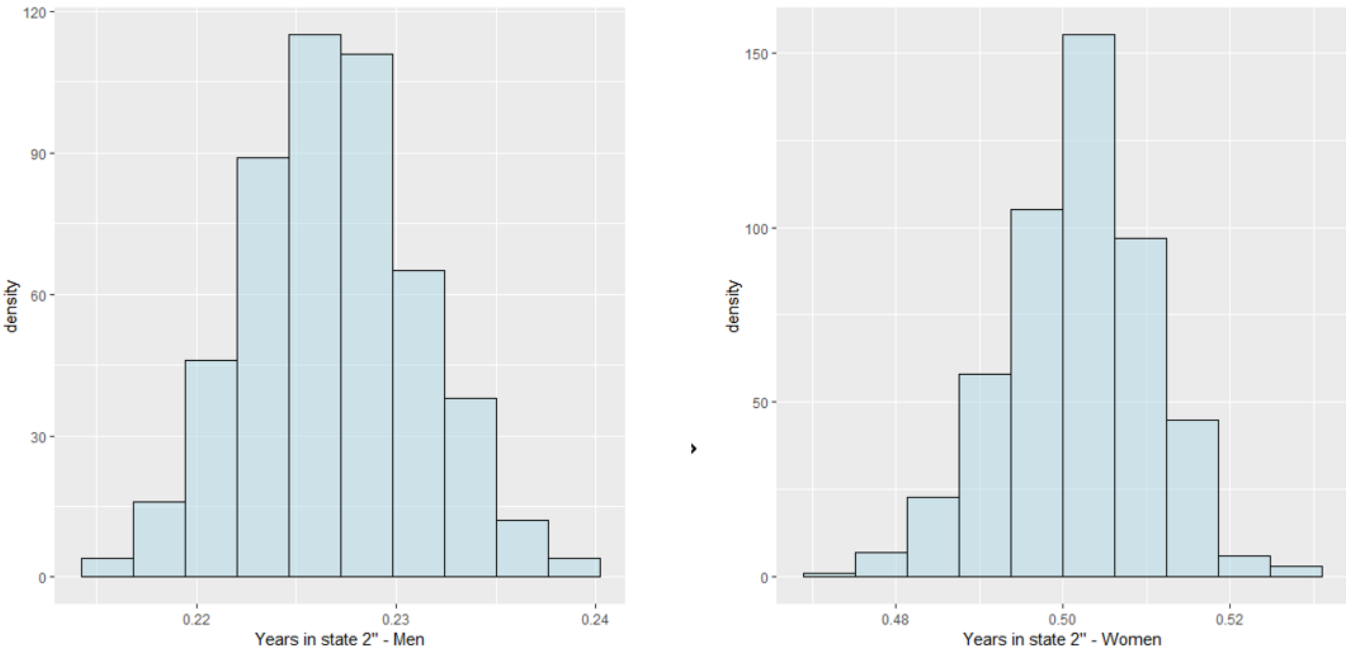


Figure 14. Time spent in state 2”



4. Task 2 - benefit calculation

Consider following payment policy:

— State 2 - 15000 PLN at the end of each year which person began in this state
— State 2’ - 25000 PLN at the end of each year which person began in this state
— State 2” - 100000 PLN lump sum at the end of each year which person entered this state + 50000 PLN at the end of each year which person began in this state
— State 3 - 100000 PLN lump sum at the end of each year which person entered this state

Sum of total benefits are plotted in fig. 15. Average males’ benefit is 55 189.71 and females’ benefit is 60 512.67
Figures 16 - 19 show average benefits in respeectective non-active states. Benefits at state 2’ are the smallest, averaging under 9000. The largest are benefits at state 3, which reach over 20000

Figure 15. Total benefits

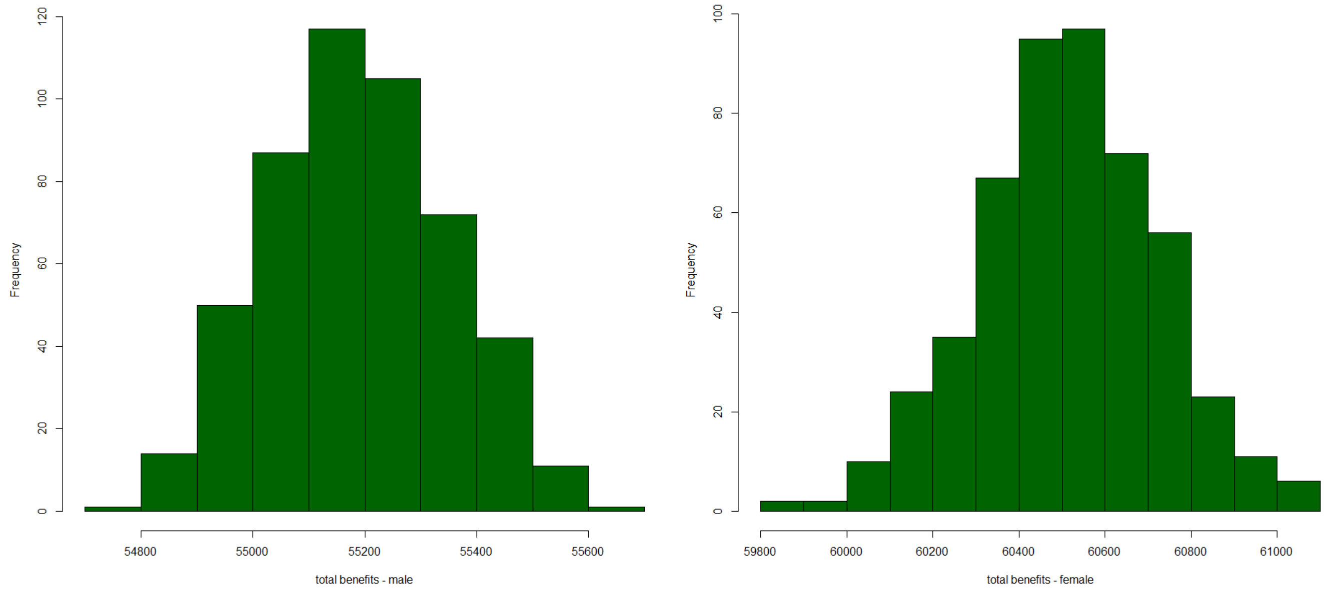


Figure 16. Benefits at state 2

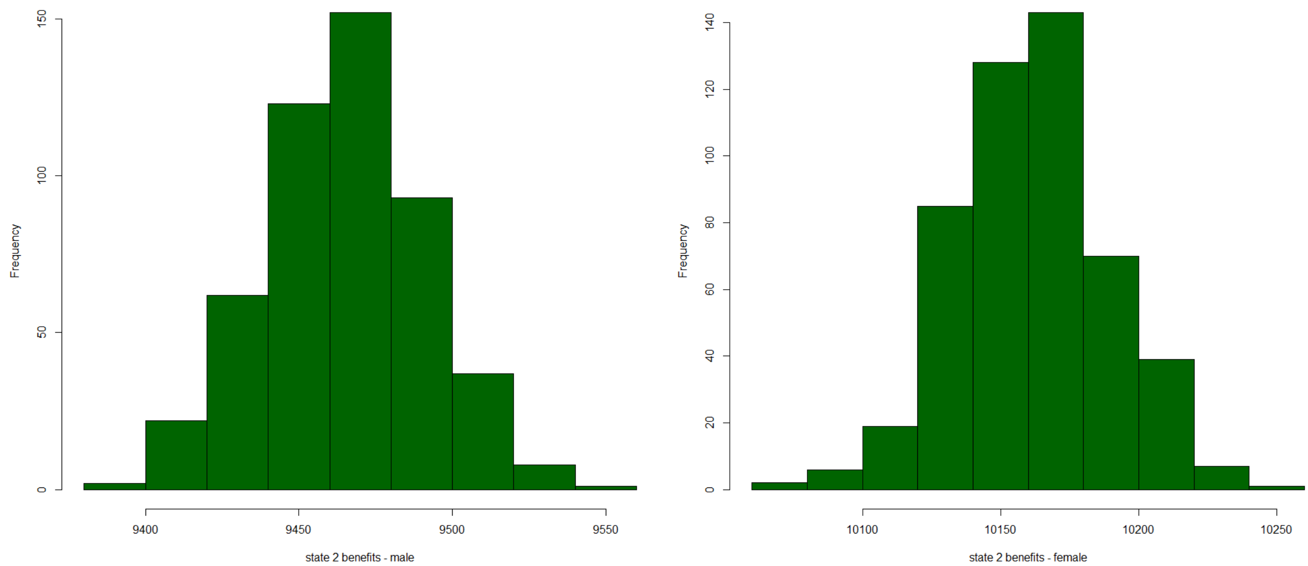


Figure 17. Benefits at state 2'

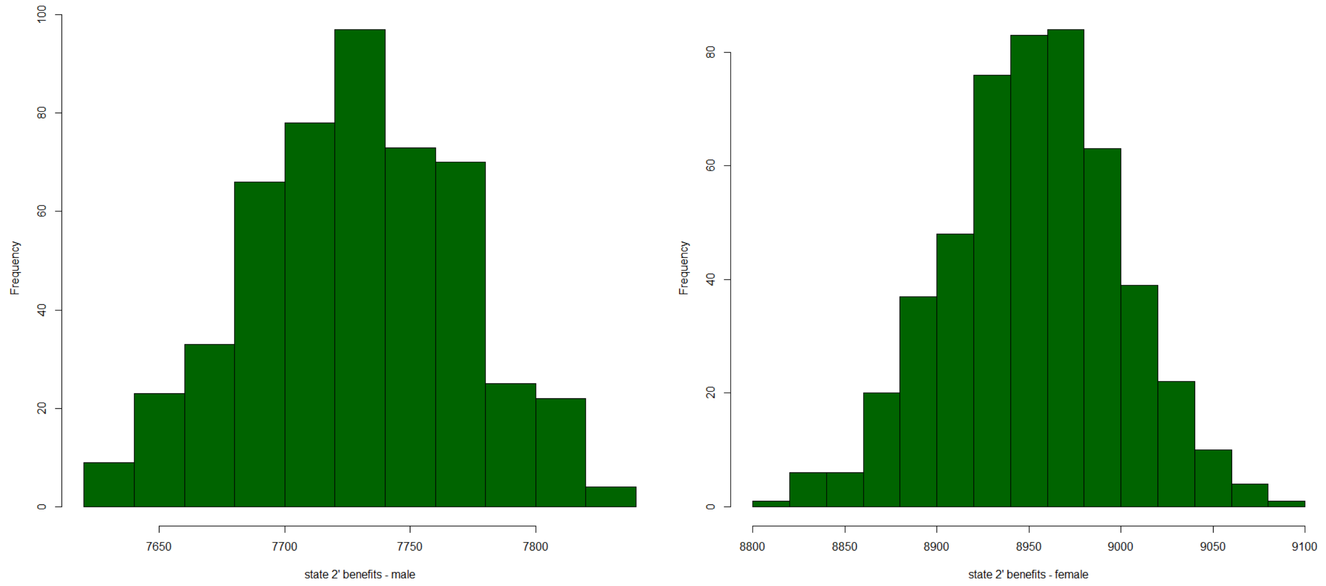


Figure 18. Benefits at state 2"

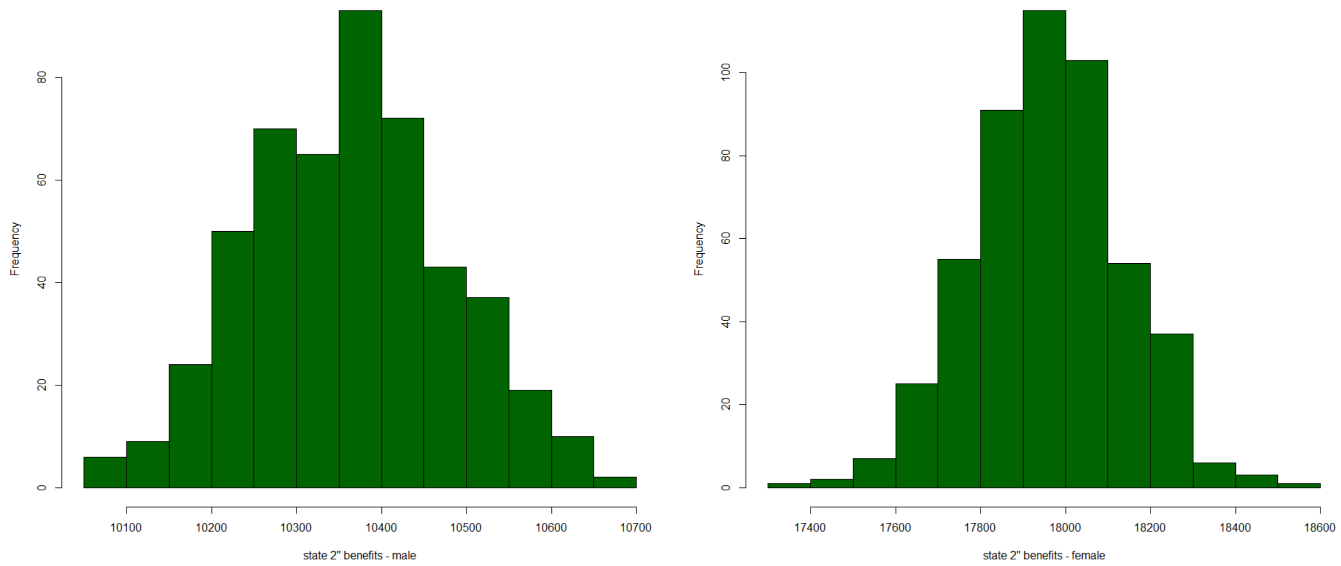
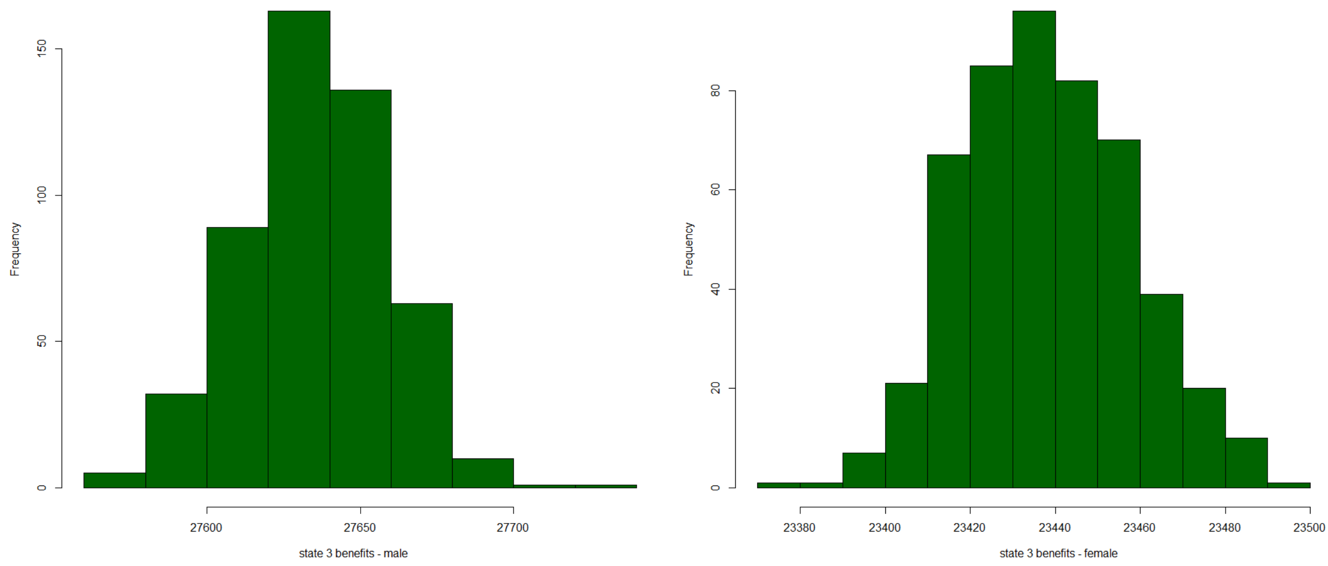


Figure 19. Benefits at state 3

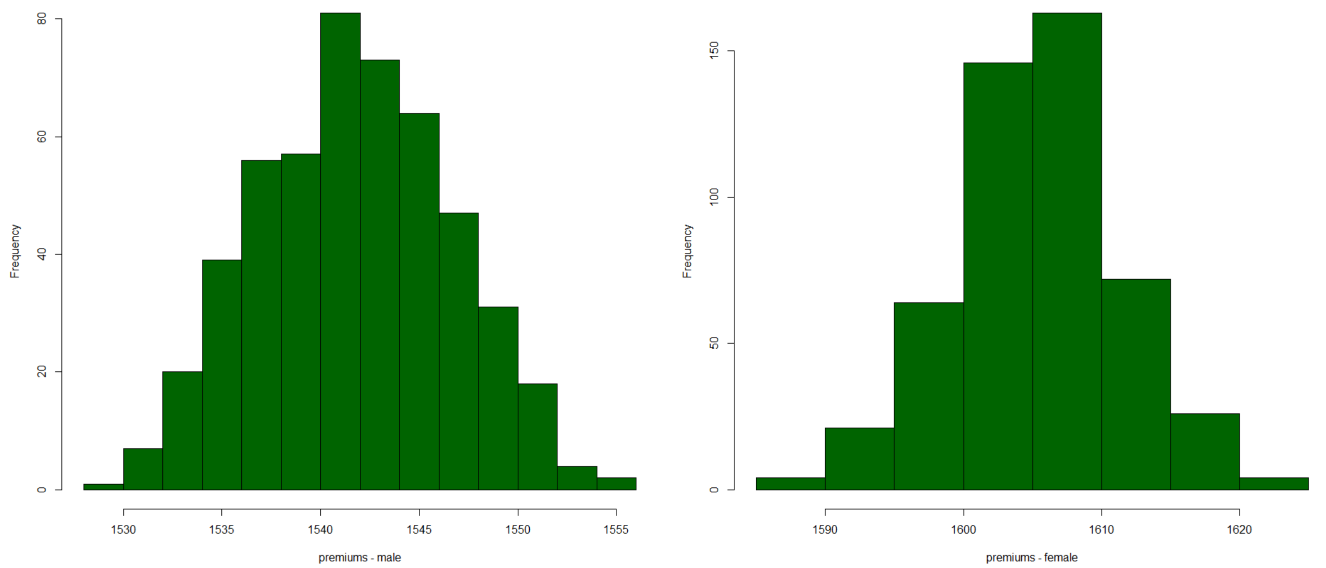


5. Task 3 - net premiums

Consider the payment policy, detailed in previous section. Male and female net premiums are shown in figure 20. Male premiums average at 1 541.74 and female ones at 1 605.29.
The difference is around 4 percent. Because net premiums are closely related to actuarial present value of the benefits, we may conclude that present value of the benefits are similar for both sexes, despite vast difference in the sum of benefits.
Differences in calculated values are very low between simulations. The ninety-five percent confidence intervals are:

— $[1541.31, 1542.18]$ for male premiums
— $[1604.76, 1605.82]$ for female premiums

Figure 20. Yearly net premiums



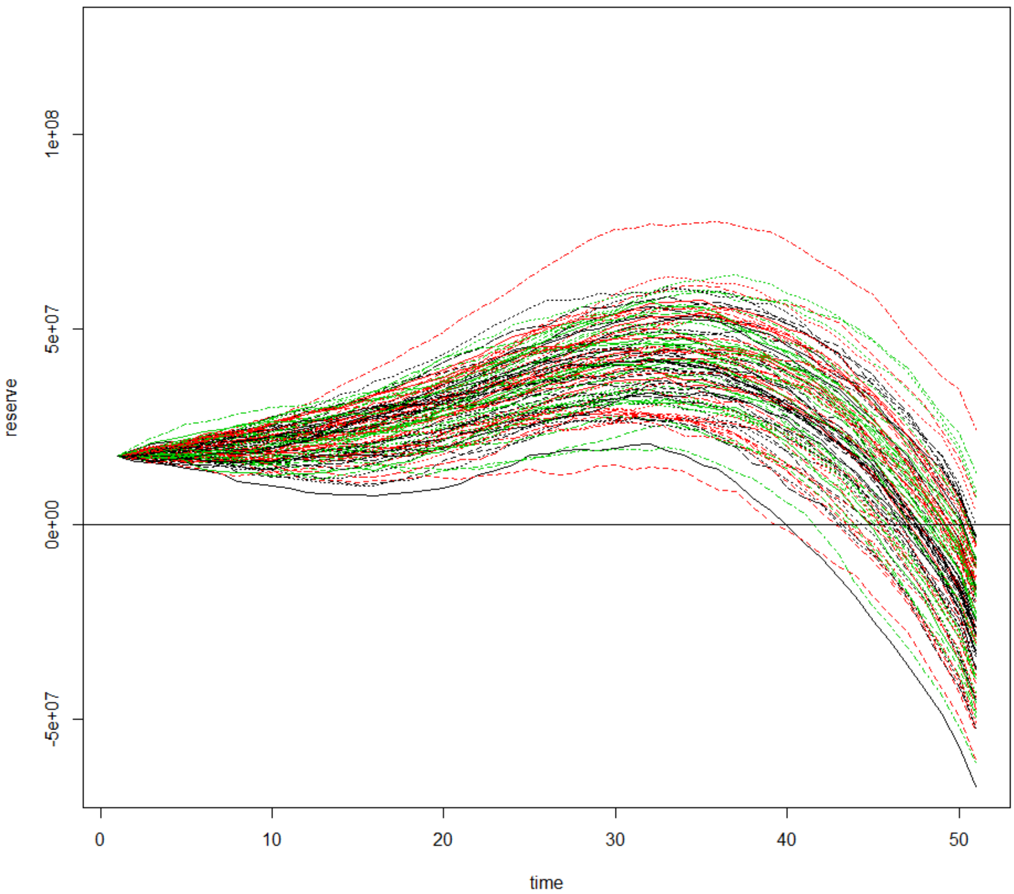
6. Task 4 - company's profit

We want to calculate the long-term profitablilty of the insurance for its issuer. To achieve this, we simulate the surplus process, given by:

$$S_t = C + P_t - B_t$$

where C is the starting capital, P_t is the sum of premiums up to time t and B_t is the sum of benefits up to time t.
We perform the simulation by calculating 100 random paths of S_t process, assuming starting capital of 10 million, and 5000 clients, each aged 25, but of random health status.
The results are shown in fig. 21. Almost all the paths hit zero between 40th and 50th year, what means that after 40 years the insurance brings net loss, and its term should be restricted.

Figure 21. Company's surplus on considered insurance

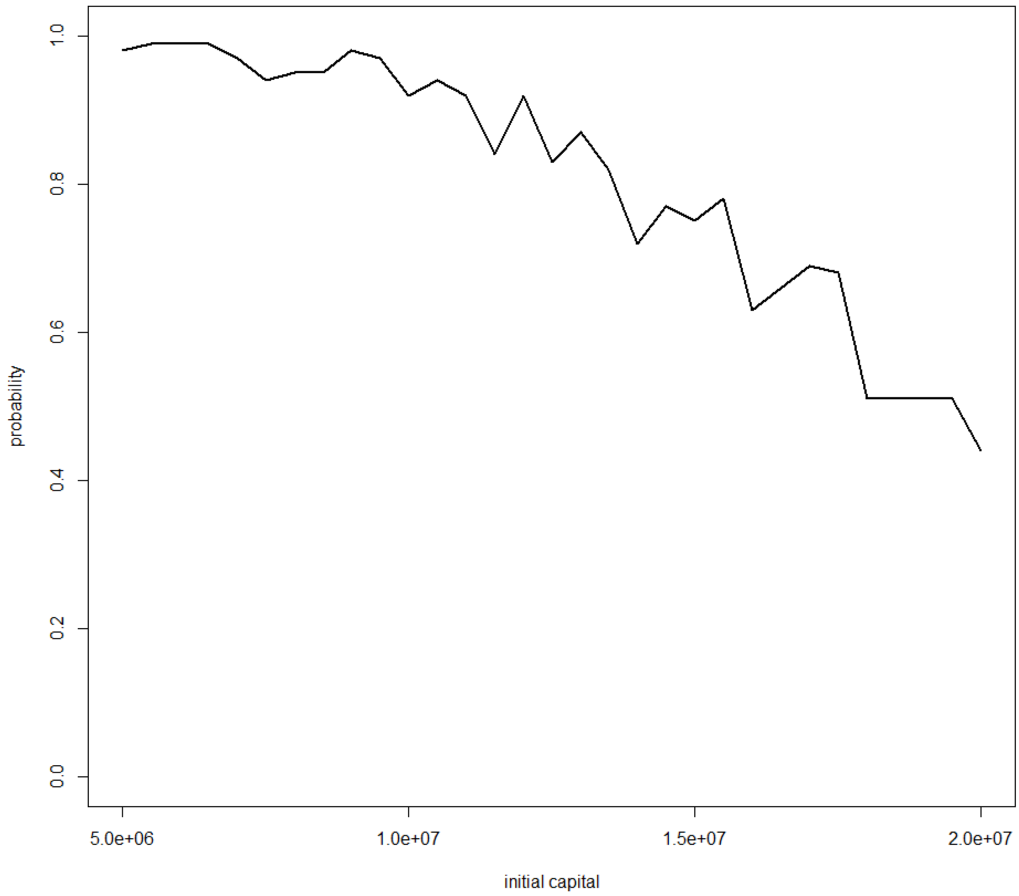


7. Task 5 - default probability under net premium

Using Monte Carlo simulation, we estimated the fifty-year probability of default as 92%, if we take assumptions from task 4. It turned out that maintaining that insurance product is highly inprofitable long-term.

Figure 22 contains the probability of ruin for net premium. The plot tells us that even with reserves equal to 20 mil, company goes bankrupt in nearly half of cases.

Figure 22. Fifty-year probability of default for $r = 0$ and different starting reserves



8. Task 6 - linear premium

In this task we assume that insured people pay linear premium $P = (1+r)N$ where N is the net premium and r is some fixed coefficient ($r = 0$ means that insured pay net premium).
We want to examine how the choice of r influence the probability of default.
The results are visible in figures **??** and 24. In figure 23 we see the influence of parameter r when $C = 5'000'000$. Setting $r = 0.12$ suffices to avoid default.
Fig. 24 shows such probability when $C = 10'000'000$ (assumptions from task 4). In order to stay solvent for fifty years, one must set $r > 0.1$.
Fig. 25 show how the probability is dependent on C and r. Conclusion from it is that one should set C larger that 16 mil or C at least 10 mil and r at least 0.1. Setting starting reserves under 10 mil is very risky.

Figure 23. Probability of default, assuming starting capital of 5 mil.

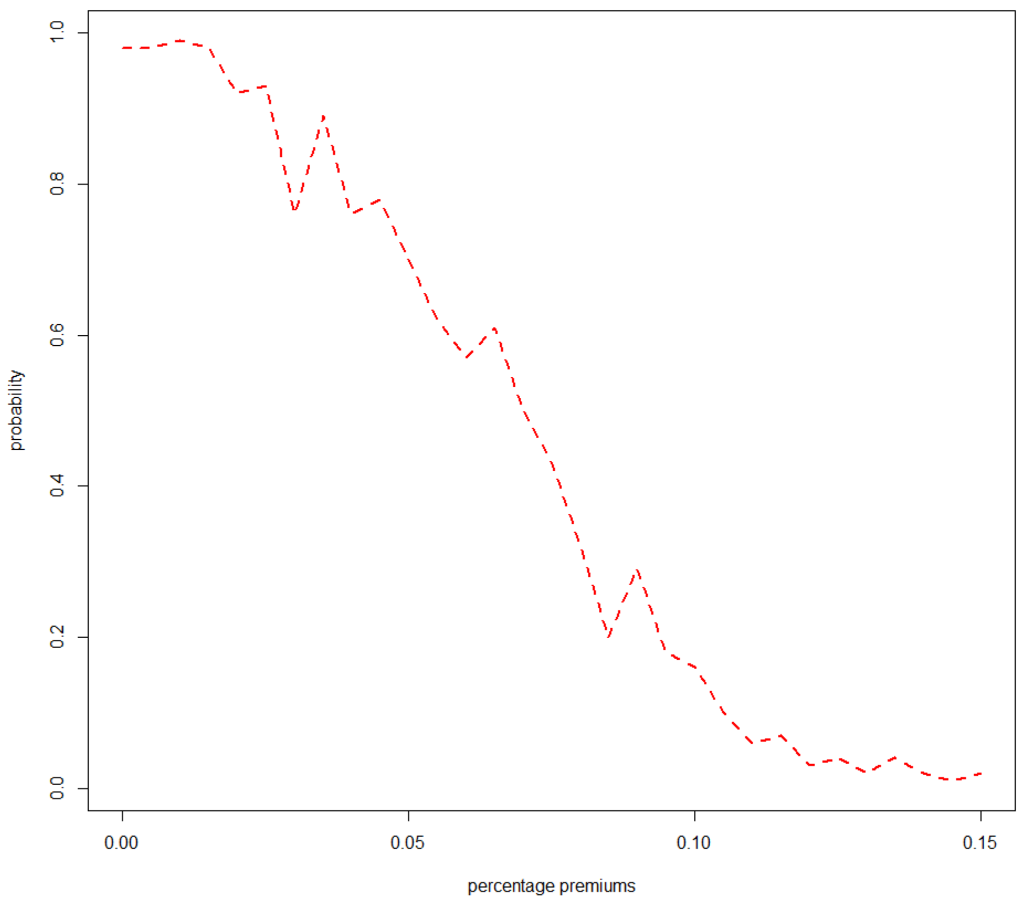


Figure 24. Probability of default, assuming starting capital of 10 mil.

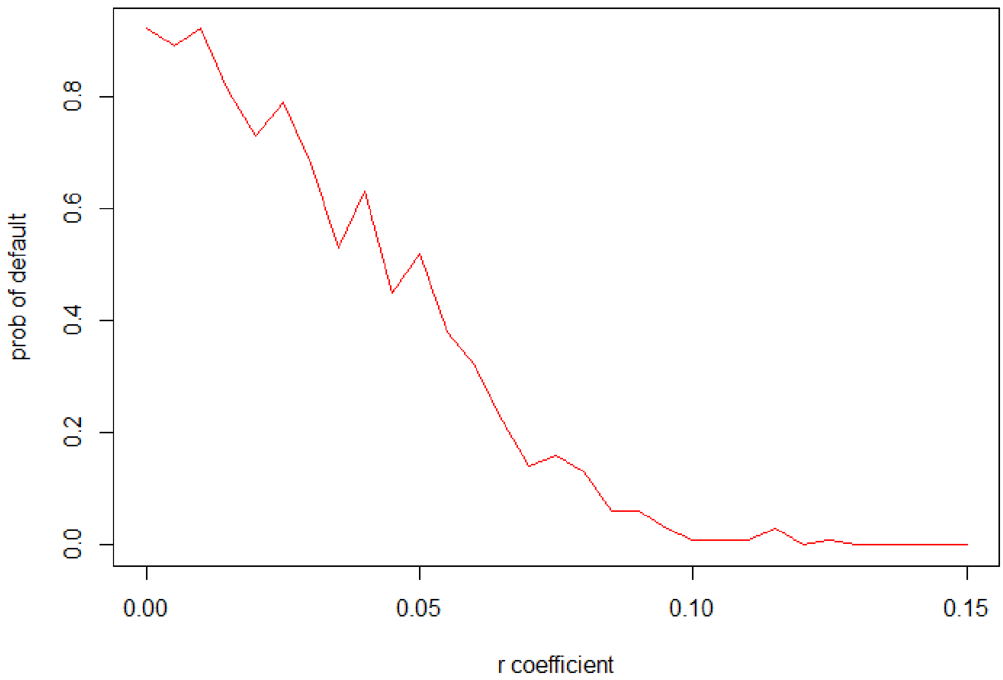
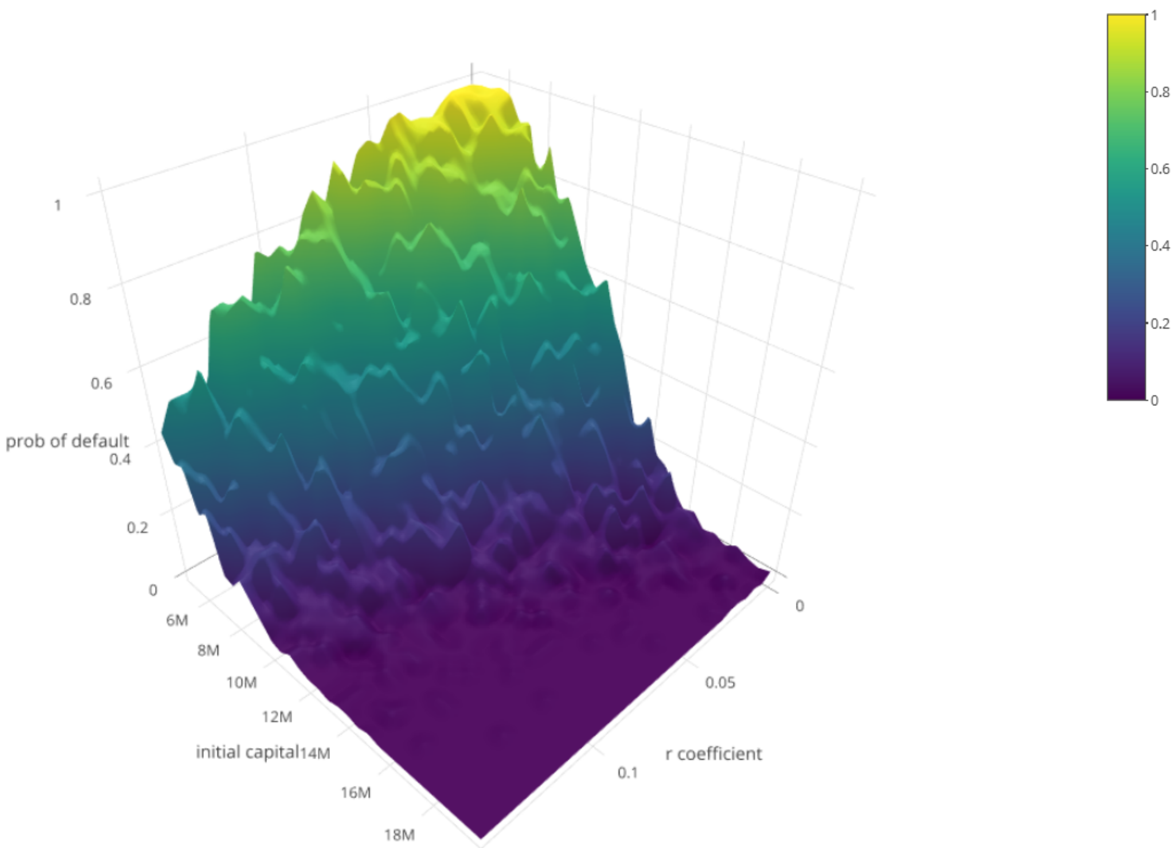


Figure 25. Probability of default, at different starting capitals and r coefficients



9. Appendix - lifetables used

Lifetables used here are featured in tables 10 and 10. They are also available at URL `https://stat.gov.pl/download/gfx/portalinformacyjny/pl/defaultaktualnosci/5470/2/11/1/trwanie_zycia_w_2016.zip` and in the repository `https://github.com/ozonowicz/health-monte-carlo` (file `ttz.csv`)

10. Appendix - R code

The code used for generating the results is available in the repository `https://github.com/ozonowicz/health-monte-carlo`

Table 1. TTZ16-M life tables, used for modeling male lifetime

x	l_x	x	l_x	x	l_x	x	l_x
0	100000	26	98591	51	92126	76	51467
1	99552	27	98494	52	91445	77	48810
2	99529	28	98393	53	90702	78	46076
3	99510	29	98289	54	89892	79	43270
4	99494	30	98179	55	89010	80	40402
5	99481	31	98064	56	88053	81	37488
6	99469	32	97941	57	87017	82	34546
7	99459	33	97810	58	85898	83	31599
8	99449	34	97670	59	84694	84	28671
9	99439	35	97521	60	83401	85	25790
10	99429	36	97361	61	82019	86	22981
11	99418	37	97190	62	80547	87	20274
12	99407	38	97006	63	78986	88	17693
13	99395	39	96806	64	77336	89	15262
14	99380	40	96589	65	75600	90	13001
15	99362	41	96351	66	73781	91	10926
16	99337	42	96089	67	71883	92	9048
17	99301	43	95800	68	69909	93	7376
18	99254	44	95481	69	67862	94	5912
19	99195	45	95128	70	65742	95	4653
20	99124	46	94738	71	63551	96	3592
21	99044	47	94309	72	61288	97	2716
22	98958	48	93836	73	58951	98	2009
23	98869	49	93318	74	56536	99	1451
24	98778	50	92749	75	54042	100	1022
25	98685						

Table 2. TTZ16-K life tables, used for modeling female lifetime

x	l_x	x	l_x	x	l_x	x	l_x
0	100000	26	99281	51	97028	76	75016
1	99645	27	99256	52	96761	77	73012
2	99621	28	99230	53	96464	78	70822
3	99606	29	99203	54	96136	79	68426
4	99597	30	99173	55	95774	80	65807
5	99589	31	99141	56	95375	81	62950
6	99581	32	99107	57	94936	82	59853
7	99573	33	99068	58	94453	83	56521
8	99565	34	99027	59	93925	84	52969
9	99558	35	98982	60	93348	85	49224
10	99552	36	98932	61	92719	86	45323
11	99546	37	98878	62	92036	87	41311
12	99539	38	98818	63	91296	88	37242
13	99531	39	98752	64	90496	89	33174
14	99521	40	98679	65	89632	90	29166
15	99510	41	98597	66	88704	91	25282
16	99496	42	98506	67	87707	92	21577
17	99479	43	98405	68	86640	93	18108
18	99458	44	98291	69	85501	94	14922
19	99436	45	98163	70	84287	95	12057
20	99414	46	98020	71	82993	96	9536
21	99393	47	97862	72	81614	97	7371
22	99373	48	97685	73	80140	98	5558
23	99352	49	97488	74	78561	99	4082
24	99329	50	97270	75	76859	100	2913
25	99306						