1) Stationary means that the statistical characteristics such as mean and variance of the data you analyze do not change over time and remain constant.

It is useful because, since these characteristics/parameters do not change over time, it simplifies the modeling process. It is also easier to forecast and predict the future with a stationary time series based on past data having constant variance, so you do not have to account for them.

2) Sample autocorrelation $\hat{\rho}(h)$ measures the linear dependence between a time series and its lagged version.

For a time series dataset $(x_t)_{t=1}^{n}$, we get the sample mean $\bar{x}$, its sample auto-covariance function $\hat{\gamma}(h)$, and sample autocorrelation function $\hat{\rho}(h)$ by the following:

$$\bar{x} = \frac{1}{n}\sum_{t=1}^{n} x_t,$$

$$\hat{\gamma}(h) = \frac{1}{n}\sum_{t=1}^{n-|h|}(x_t - \bar{x})(x_{t+h} - \bar{x}), \quad |h| < n,$$

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \quad |h| < n.$$

$\hat{\rho}(h)$ is not meaningful for large values of h because as h increases, the number of data points used to calculate $\hat{\rho}(h)$ decreases. For large values of h, truncated time series become very short, and the resulting estimate of the autocorrelation becomes unreliable.

It is not meaningful for nonstationary processes because the designation of ACF assumes stationary. Nonstationary processes' changing statistical properties over time makes the fixed correlation between time steps irrelevant. Also it can lead to correlations that are not actually true (which is not a genuine correlation).

4) Solution for the AR(2) Process

$$X_t - 0.8X_{t-1} + 0.2X_{t-2} = W_t,$$

we have the characteristic polynomial:

$$\phi(z) = 1 - 0.8z + 0.2z^2 = \left(1 - \frac{z}{r_1}\right)\left(1 - \frac{z}{r_2}\right).$$

Using R, we find the roots with:

```
polyroot(c(0.2, -0.8, 1))
```

This yields:

$$r_1^{-1} = 0.4 + 0.2i, \quad r_2^{-1} = 0.4 - 0.2i,$$

which are complex conjugates with magnitudes $|r_1|$ and $|r_2|$ greater than 1, indicating that the process is causal.

The autocorrelation function is:

$$\rho_X(h) = c_1 r^h \cos(\theta h) + c_2 r^h \sin(\theta h), \quad h \geq 0,$$

where $r \approx 0.447$ and $\theta \approx 0.464$ radians. (from

```
c <- 0.4 + 0.2i
Mod(c)
Arg(c)
```

in R)

Initial Conditions

For $h = 0$:

$$\rho_X(0) = c_1 + c_2 = 1.$$

For $h = 1$:

$$\rho_X(1) = c_1 \cdot 0.447 \cos(0.464) + c_2 \cdot 0.447 \sin(0.464).$$

Using the AR(2) equation:

$$\rho_X(1) + 0.8 \cdot 1 + 0.2\rho_X(1) = 0,$$

we find:

$$1.2\rho_X(1) = -0.8,$$

$$\rho_X(1) = -\frac{0.8}{1.2} \approx -0.667.$$

This gives us the system of equations:

$$\begin{cases} c_1 + c_2 = 1, \\ 0.4c_1 + 0.6c_2 \approx -0.667. \end{cases}$$

in matrix form:

$$\begin{bmatrix} 1 & 1 \\ 0.4 & 0.6 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.667 \end{bmatrix}.$$

```
r.inv <- c(0.4 + 0.2i, 0.4 - 0.2i)
phi <- c(0.8, -0.2)
A <- matrix(c(1, r.inv[1], 1, r.inv[2]), 2, 2)
b <- c(1, phi[1] / (1 - phi[2]))
d <- solve(A, b)
rho.X <- NULL
for (h in 1:30) {
  rho.X[h] <- d[1] * r.inv[1]^h + d[2] * r.inv[2]^h
}
```

```
+   rho.X[h] <- d[1] * r.inv[1]^h + d[2] * r.inv[2]^h
+ }
> print(rho.X)
 [1]  6.666667e-01+0i  3.333333e-01+0i  1.333333e-01+0i  4.000000e-02+0i  5.333333e-03+0i
 [6] -3.733333e-03+0i -4.053333e-03+0i -2.496000e-03+0i -1.186133e-03+0i -4.497067e-04+0i
[11] -1.225387e-04+0i -8.089600e-06+0i  1.803605e-05+0i  1.604676e-05+0i  9.230199e-06+0i
[16]  4.174807e-06+0i  1.493806e-06+0i  3.600832e-07+0i -1.069460e-08+0i -8.057232e-08+0i
[21] -6.231893e-08+0i -3.374068e-08+0i -1.452876e-08+0i -4.874871e-09+0i -9.941451e-10+0i
[26]  1.796582e-10+0i  3.425556e-10+0i  2.381128e-10+0i  1.219791e-10+0i  4.996075e-11+0i
> rho.X <- ARMAacf(ar = phi, lag.max = 30)
```

```
> rho.X <- ARMAacf(ar = phi, lag.max = 30)
> print(rho.X)
            0             1             2             3             4             5             6
 1.000000e+00  6.666667e-01  3.333333e-01  1.333333e-01  4.000000e-02  5.333333e-03 -3.733333e-03
            7             8             9            10            11            12            13
-4.053333e-03 -2.496000e-03 -1.186133e-03 -4.497067e-04 -1.225387e-04 -8.089600e-06  1.803605e-05
           14            15            16            17            18            19            20
 1.604676e-05  9.230199e-06  4.174807e-06  1.493806e-06  3.600832e-07 -1.069460e-08 -8.057232e-08
           21            22            23            24            25            26            27
-6.231893e-08 -3.374068e-08 -1.452876e-08 -4.874871e-09 -9.941451e-10  1.796582e-10  3.425556e-10
```

```
rho.X <- Re(rho.X)
plot(rho.X, type = 'h', xlim = c(0,30), ylim = c(-1,1),
xlab = '')
abline(h = 0)
points(0,1, type = 'h')
```

```
alpha <- Mod(c2[1])       0.8333333
Theta <- Arg(c2[1])      -0.9272952
beta <- Mod(r.inv2[1])      0.4472136
Phi <- Arg(r.inv2[1])      0.4636476
```

Given the calculated values:

$$\alpha = 0.8333333, \quad \Theta = -0.9272952, \quad \beta = 0.4472136, \quad \Phi = 0.4636476,$$

the autocorrelation function $\rho_X(h)$ for $h \geq 0$ is:

$$\rho_X(h) = 2\alpha\beta^h \cos(h\Phi + \Theta),$$

or, up to rounding errors:

$$\rho_X(h) = 2(0.833)(0.447)^h \cos(0.464h - 0.927), \quad h \geq 0.$$

5) For the AR(3) process:

$$X_t = 0.6X_{t-1} - 0.37X_{t-2} + 0.21X_{t-3} = W_t,$$

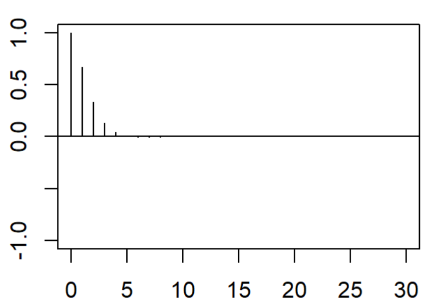


Figure 1: 4th question ACF

The characteristic polynomial is:

$$\phi(z) = 1 - 0.6z - 0.37z^2 + 0.21z^3.$$

Solving for the roots using:

```
polyroot(c(0.21, -0.37, -0.6, 1))
```

yields the following roots:

$$0.5 + 0i, \quad -0.6 + 0i, \quad 0.7 + 0i.$$

The polynomial $\phi(z)$ can be factored using its roots:

$$\phi(z) = (1 - 0.5z)(1 + 0.6z)(1 - 0.7z).$$

$$\rho_X(h) = c_1(0.5)^h + c_2(-0.6)^h + c_3(0.7)^h, \quad h \geq 0.$$

$$\rho_X(1) = 0.6 + 0.37\rho_X(1) - 0.21\rho_X(2),$$

$$\rho_X(2) = 0.6\rho_X(1) + 0.37 - 0.21\rho_X(1).$$

$$\begin{bmatrix} 0.37 - 1 & -0.21 \\ 0.6 - 0.21 & -1 \end{bmatrix} \begin{bmatrix} \rho_X(1) \\ \rho_X(2) \end{bmatrix} = \begin{bmatrix} -0.6 \\ -0.37 \end{bmatrix}.$$

```
A <- matrix(c(0.37 - 1, -0.21,
              0.6 - 0.21, -1),
            nrow = 2, byrow = TRUE)
b <- c(-0.6, -0.37)
rho <- solve(A, b)
rho
```

$$\rho_X(1) = 0.7336705$$

$$\rho_X(2) = 0.6561315$$

$$1 = \rho_X(0) = c_1 + c_2 + c_3,$$

$$0.7337 \approx \rho_X(1) = 0.5c_1 - 0.6c_2 + 0.7c_3,$$

$$0.6561 \approx \rho_X(2) = 0.25c_1 + 0.36c_2 + 0.49c_3.$$

```
A <- matrix(c(1, 1, 1,
              0.5, -0.6, 0.7,
              0.25, 0.36, 0.49),
            nrow = 3, byrow = TRUE)

b <- c(1, 0.7337, 0.6561)
c_values <- solve(A, b)
c_values
```

$$c_1 = -0.73968182$$

$$c_2 = 0.08787413$$

$$c_3 = 1.65180769$$

$$\rho_X(h) = -0.740(0.5)^h + 0.088(-0.6)^h + 1.652(0.7)^h, \quad h \geq 0.$$

6)

$$X_t = 0.2X_{t-1} + 0.08X_{t-2} + W_t + 0.7W_{t-1}.$$

With R code:

```
# The Durbin-Levinson Algorithm

n <- 1000
phi <- c(0.2, 0.08)
th <- 0.7

ACF <- ARMAacf(ar = phi, ma = th, lag.max = n)[-1]

phi.matrix <- matrix(NA, n, n)
phi.matrix[1, 1] <- ACF[1]

for (j in 2:n) {
  num <- ACF[j] - sum(phi.matrix[j - 1, (j - 1):1] * ACF[1:(j - 1)])
  denom <- 1 - sum(phi.matrix[j - 1, 1:(j - 1)] * ACF[1:(j - 1)])
  phi.matrix[j, j] <- num / denom
  for (k in 1:(j - 1)) {
    term.1 <- phi.matrix[j - 1, k]
    term.2 <- phi.matrix[j, j] * phi.matrix[j - 1, j - k]
    phi.matrix[j, k] <- term.1 - term.2
  }
}

m <- 5
PACF <- NULL

for (h in 1:n) {
  PACF[h] <- phi.matrix[h, h]
}
PACF[1:m]
#verifying
ARMAacf(ar = phi, ma = th, lag.max = m, pacf = TRUE)

j=3
phi.matrix[j,1:j]
```

```
+ }
> PACF[1:m]
[1]  0.6188030 -0.2903282  0.1897431 -0.1286372  0.0886776
> ARMAacf(ar = phi, ma = th, lag.max = m, pacf = TRUE)
[1]  0.6188030 -0.2903282  0.1897431 -0.1286372  0.0886776
>
```

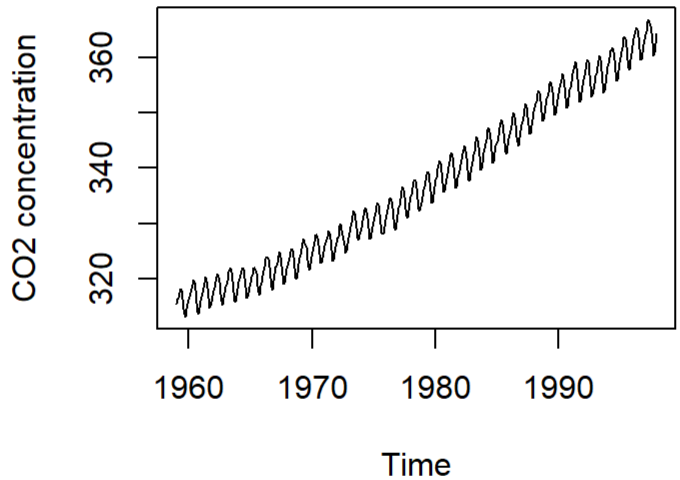
Figure 2: AR(2,1)

results are: 0.8535467 -0.4418303 0.1897431

$$P(X_4 \mid X_1, X_2, X_3) \approx 0.854X_3 - 0.442X_2 + 0.190X_1.$$

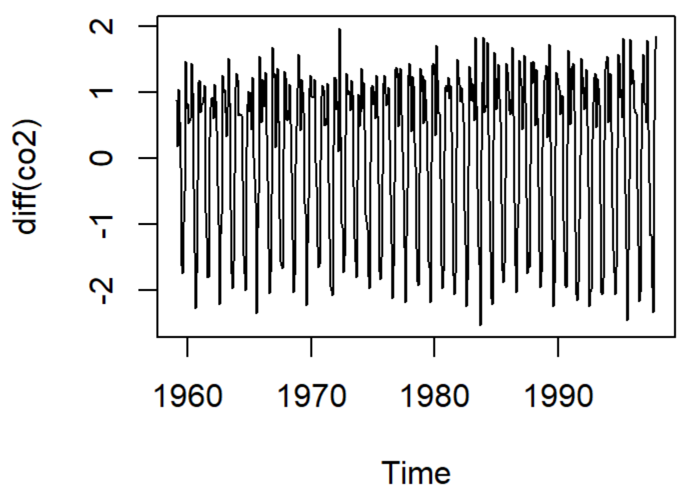
3) #original plot

plot(co2, ylab='CO2 concentration')



#there is a trend, so I applied diff

plot(diff(co2))



#there is still seasonal effect, so I applied lag 12 for 12 months

```
plot(diff(co2, lag = 12))
```