cc_compete_2025_mm_dd Reformulation of credit card competition

#1: CaseMode := Sensitive

#2: InputMode := Word

Number of banks

#3: k :∈ Real (0, ∞)

direct and indirect card demand parameters

#4: H :∈ Real (0, ∞)

#5: F :∈ Real (0, ∞)

#6: G :∈ Real (0, ∞)

#7: α :∈ Real (0, ∞)

#8: β :∈ Real (0, ∞)

#9: γ :∈ Real (0, ∞)

#10: Na :∈ Real (0, ∞)

#11: Nb :∈ Real (0, ∞)

#12: nak :∈ Real (0, ∞)

#13: nbk :∈ Real (0, ∞)

#14: m :∈ Real (0, ∞)

#15: M :∈ Real (0, ∞)

#16: μ :∈ Real (0, ∞)

eq (1): merchant participation

#17: $m = M - \mu \cdot (ia + ib)$

#18: $ia :\in \text{Real } (0, \infty)$

#19: $ib :\in \text{Real } (0, \infty)$

#20: $ra :\in \text{Real } (0, \infty)$

#21: $rb :\in \text{Real } (0, \infty)$

eq (2) direct demand for cards A and B

#22: $na = H \cdot m + F \cdot ra - G \cdot rb$

#23: $nb = H \cdot m + F \cdot rb - G \cdot ra$

na > 0 if

#24: $H \cdot m + F \cdot ra - G \cdot rb > 0$

#25: SOLVE($H \cdot m + F \cdot ra - G \cdot rb > 0$, ra)

#26: $$ra > \frac{G \cdot rb - H \cdot m}{F}$$

nb > 0 if

#27: $H \cdot m + F \cdot rb - G \cdot ra > 0$

#28: SOLVE($H \cdot m + F \cdot rb - G \cdot ra > 0$, rb)

#29: $$rb > \frac{G \cdot ra - H \cdot m}{F}$$

inverting demand. Define,

#30: $\alpha = H$

#31: $$\beta = \frac{F}{F^2 - G^2}$$

#32: $$\gamma = \frac{G}{F^2 - G^2}$$

#33: $$\mathrm{SOLVE}\left(\left[\beta = \frac{F}{F^2 - G^2}, \gamma = \frac{G}{F^2 - G^2}\right], [F, G]\right)$$

#34: $$\left[F = \frac{\beta}{\beta^2 - \gamma^2} \wedge G = \frac{\gamma}{\beta^2 - \gamma^2} \wedge F^2 - G^2 \neq 0\right]$$

#35: $$na = \alpha \cdot m + \frac{\beta}{\beta^2 - \gamma^2} \cdot ra - \frac{\gamma}{\beta^2 - \gamma^2} \cdot rb$$

#36: $$nb = \alpha \cdot m + \frac{\beta}{\beta^2 - \gamma^2} \cdot rb - \frac{\gamma}{\beta^2 - \gamma^2} \cdot ra$$

#37: $$\mathrm{SOLVE}\left(\left[na = \alpha \cdot m + \frac{\beta}{\beta^2 - \gamma^2} \cdot ra - \frac{\gamma}{\beta^2 - \gamma^2} \cdot rb, nb = \alpha \cdot m + \frac{\beta}{\beta^2 - \gamma^2} \cdot rb - \frac{\gamma}{\beta^2 - \gamma^2} \cdot ra\right], [ra, rb]\right)$$

eq (3)

#38: $$[ra = - m \cdot \alpha \cdot (\beta + \gamma) + na \cdot \beta + nb \cdot \gamma \wedge rb = - m \cdot \alpha \cdot (\beta + \gamma) + na \cdot \gamma + nb \cdot \beta]$$

Assumption 1

#39: $$\alpha < \alpha max = \frac{1}{6\cdot\mu\cdot(\beta + \gamma)}$$

eq (4) cards' number of users max problem

#40: $ta = na\cdot m$

#41: $tb = nb\cdot m$

eq (5): profits of bank k

#42: $profitk = (ia - ra)\cdot m\cdot nak + (ib - rb)\cdot m\cdot nbk$

#43: $profitk = (ia - (- m\cdot\alpha\cdot(\beta + \gamma) + na\cdot\beta + nb\cdot\gamma))\cdot m\cdot nak + (ib - (- m\cdot\alpha\cdot(\beta + \gamma) + na\cdot\gamma + nb\cdot\beta))\cdot m\cdot nbk$

#44: $profitk = (ia - (- m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot m\cdot nak + (ib - (- m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot m\cdot nbk$

*** Section 4: Equilibrium

eq (6) and Appendix A (banks' profit-max interchange fees)

eq (A.1) FOC

#45: $$\frac{d}{d\ nak}\left(profitk = (ia - (- m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot m\cdot nak + (ib - (- m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot m\cdot nbk\right)$$

#46: $$0 = m\cdot(ia + m\cdot\alpha\cdot(\beta + \gamma) - na\_notk\cdot\beta - 2\cdot nak\cdot\beta - \gamma\cdot(nb\_notk + 2\cdot nbk))$$

#47: $$\frac{d}{d\ nbk}\left(profitk = (ia - (-m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot m\cdot nak + (ib - (-m\cdot\alpha\cdot(\beta + \gamma) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot m\cdot nbk\right)$$

#48: $$0 = m\cdot(ib + m\cdot\alpha\cdot(\beta + \gamma) - na\_notk\cdot\gamma - 2\cdot nak\cdot\gamma - nb\_notk\cdot\beta - 2\cdot nbk\cdot\beta)$$

rewrite FOC for symmetric banks

#49: $$0 = m\cdot(ia - k\cdot(nak\cdot\beta + nbk\cdot\gamma) + m\cdot\alpha\cdot(\beta + \gamma) - nak\cdot\beta - nbk\cdot\gamma)$$

#50: $$0 = m\cdot(ib - k\cdot(nak\cdot\gamma + nbk\cdot\beta) + m\cdot\alpha\cdot(\beta + \gamma) - nak\cdot\gamma - nbk\cdot\beta)$$

eq (6)

#51: $$\mathrm{SOLVE}([0 = m\cdot(ia - k\cdot(nak\cdot\beta + nbk\cdot\gamma) + m\cdot\alpha\cdot(\beta + \gamma) - nak\cdot\beta - nbk\cdot\gamma),\ 0 = m\cdot(ib - k\cdot(nak\cdot\gamma + nbk\cdot\beta) + m\cdot\alpha\cdot(\beta + \gamma) - nak\cdot\gamma - nbk\cdot\beta)],\ [nak,\ nbk])$$

#52: $$\left[nak = \frac{ia\cdot\beta - ib\cdot\gamma + m\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{(k + 1)\cdot(\beta^2 - \gamma^2)} \wedge nbk = \frac{ia\cdot\gamma - ib\cdot\beta + m\cdot\alpha\cdot(\beta + \gamma)\cdot(\gamma - \beta)}{(k + 1)\cdot(\gamma^2 - \beta^2)}\right]$$

multiplying by k to obtain aggregate numbers of A and B cards

#53: $$na = k\cdot\frac{ia\cdot\beta - ib\cdot\gamma + m\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{(k + 1)\cdot(\beta^2 - \gamma^2)}$$

#54: $$nb = k\cdot\frac{ia\cdot\gamma - ib\cdot\beta + m\cdot\alpha\cdot(\beta + \gamma)\cdot(\gamma - \beta)}{(k + 1)\cdot(\gamma^2 - \beta^2)}$$

eq (7): number of cards expressed as functions of ia and ib only (after m is substituted).

#55: $$na = \frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) + ia\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)) - ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma))}{(k + 1)\cdot(\beta^2 - \gamma^2)}$$

#56: $$nb = \frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) - ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma) + ib\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)))}{(k + 1)\cdot(\beta^2 - \gamma^2)}$$

eq (8) Appendix B: Card org interchange fees (max ta and max tb)

eq (B.1)

#57: $$ta = \frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) + ia\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)) - ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma))}{(k + 1)\cdot(\beta^2 - \gamma^2)}\cdot(M -$$

$$\mu\cdot(ia + ib))$$

eq (B.2) FOC

#58: $$\frac{d}{d\ ia}\left(ta = \right.$$

$$\frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) + ia\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)) - ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma))}{(k + 1)\cdot(\beta^2 - \gamma^2)}\cdot(M -$$

$$\mu\cdot(ia + ib))\Bigg)$$

#59:

$$0 = \frac{k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) - \mu\cdot(2\cdot ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + ib\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)))}{(k + 1)\cdot(\gamma^2 - \beta^2)}$$

eq (B.3)

#60:

$$tb = \frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) - ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma) + ib\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)))}{(k + 1)\cdot(\beta^2 - \gamma^2)}\cdot(M - \mu\cdot(ia + ib))$$

eq (B.4) FOC

#61:

$$\frac{d}{d\ ib}\Bigg(tb =$$

$$\frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) - ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma) + ib\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)))}{(k + 1)\cdot(\beta^2 - \gamma^2)}\cdot(M -$$

$$\left.\mu\cdot(ia + ib))\right)$$

#62: $0 =$

$$\frac{k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) - \mu\cdot(ia\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + 2\cdot ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma\sim}{(k + 1)\cdot(\gamma^2 - \beta^2)}$$

$$\underline{) - \beta)))}$$

eq (8) ia and ib set by card org

#63: $\text{SOLVE}\left(\left[0 = \right.\right.$

$$\frac{k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) - \mu\cdot(2\cdot ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + ib\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma\sim}{(k + 1)\cdot(\gamma^2 - \beta^2)}$$

$$\frac{) - 1)))}{}, 0 =$$

$$\frac{k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) - \mu\cdot(ia\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + 2\cdot ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma\sim}{(k + 1)\cdot(\gamma^2 - \beta^2)}$$

$$\left.\frac{) - \beta)))}{}\right], [ia, ib]\Big)$$

#64: $$\left[ia = \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} \wedge ib = \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}\right]$$

eq (B.5): SOC

SOC

#65: $$\frac{d}{d\ ia}\frac{d}{d\ ia}\Big(ta =$$

$$\frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) + ia\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)) - ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma))}{(k + 1)\cdot(\beta^2 - \gamma^2)}\cdot(M -$$

$$\mu\cdot(ia + ib))\Big)$$

#66: $$\frac{2\cdot k\cdot \mu\cdot (\alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma) - \beta)}{(k + 1)\cdot (\beta^2 - \gamma^2)}$$

#67: $$\frac{d}{d\ ib}\frac{d}{d\ ib}\left(tb = \frac{k\cdot (M\cdot \alpha\cdot (\beta + \gamma)\cdot (\beta - \gamma) - ia\cdot (\alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma) + \gamma) + ib\cdot (\beta - \alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma)))}{(k + 1)\cdot (\beta^2 - \gamma^2)}\cdot (M - \mu\cdot (ia + ib))\right)$$

#68: $$\frac{2\cdot k\cdot \mu\cdot (\alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma) - \beta)}{(k + 1)\cdot (\beta^2 - \gamma^2)}$$

SOC < 0 if

#69: $\alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma) - \beta < 0$

#70: $\mathrm{SOLVE}(\alpha\cdot \mu\cdot (\beta + \gamma)\cdot (\beta - \gamma) - \beta < 0, \alpha)$

#71: $$\mathrm{IF}\left(\mu\cdot (\beta - \gamma) < 0, \alpha > \frac{\beta}{\mu\cdot (\beta + \gamma)\cdot (\beta - \gamma)}\right) \vee \mathrm{IF}\left(\mu\cdot (\beta - \gamma) > 0, \alpha < \frac{\beta}{\mu\cdot (\beta + \gamma)\cdot (\beta - \gamma)}\right)$$

#72: $$\alpha < \frac{\beta}{\mu\cdot (\beta + \gamma)\cdot (\beta - \gamma)}$$

is the above larger than αmax in Assumption 1? [yes]

#73: $$\frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} - \frac{\beta}{3\cdot\mu\cdot(\beta^2 - \gamma^2)}$$

#74: $$\frac{2\cdot\beta}{3\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} > 0$$

eq (9): equilibrium number of merchants m

#75: $$m = \frac{M\cdot(\gamma - \beta)}{4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma}$$

eq (10): equilibrium number of cards

#76: $$na = \frac{M\cdot k\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

#77: $$nb = \frac{M\cdot k\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

> 0 if [2 conditions] [yes]

#78: $\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta < 0$

#79: $\mathrm{SOLVE}(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta < 0, \alpha)$

#80: $$\mathrm{IF}\left(\mu\cdot(\beta - \gamma) < 0, \alpha > \frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \mathrm{IF}\left(\mu\cdot(\beta - \gamma) > 0, \alpha < \frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$$

#81: $\alpha < \dfrac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$

larger than αmax if [yes]

#82: $\dfrac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} - \dfrac{\beta}{3\cdot\mu\cdot(\beta^2 - \gamma^2)}$

#83: $\dfrac{2\cdot\beta}{3\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} > 0$

#84: $4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0$

#85: $\mathrm{SOLVE}(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0, \alpha)$

#86: $\mathrm{IF}\left(\mu\cdot(\beta - \gamma) < 0, \alpha > \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \mathrm{IF}\left(\mu\cdot(\beta - \gamma) > 0, \alpha < \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$

#87: $\alpha < \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$

The above is larger than αmax if [yes]

#88: $\dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} - \dfrac{\beta}{3\cdot\mu\cdot(\beta^2 - \gamma^2)}$

#89: $\dfrac{5\cdot\beta - 3\cdot\gamma}{12\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} > 0$

eq (11) equilibrium rewards

#90: $$ra = - M\cdot\left(\frac{\alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta + \gamma)^2 - \beta^2 - \beta\cdot\gamma}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} + \frac{\beta - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}\right)$$

#91: $$ra = - M\cdot\left(- \frac{k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}\right)$$

#92: $$rb = - M\cdot\left(\frac{\alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta + \gamma)^2 - \beta^2 - \beta\cdot\gamma}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} + \frac{\beta - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}\right)$$

#93: $$rb = - M\cdot\left(- \frac{k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}\right)$$

#94: $$ra = \frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma))}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

#95: $$rb = \frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma))}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

$>$ 0 if

#96: $$k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) < 0$$

#97: $\text{SOLVE}(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) < 0, \alpha)$

#98: $$\text{IF}\left(\mu\cdot(\beta - \gamma)\cdot(2\cdot k + 1) < 0, \alpha > \frac{k\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot k + 1)}\right) \vee \text{IF}\left(\mu\cdot(\beta - \gamma)\cdot(2\cdot k + 1) > 0, \alpha < \frac{k\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot k + 1)}\right)$$

#99: $$\alpha < \frac{k\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot k + 1)}$$

#100: $$\frac{d}{dk}\ \frac{k\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot k + 1)}$$

#101: $$\frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot k + 1)^2} > 0$$

so set k=1

#102: $$\alpha < \frac{1\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)\cdot(2\cdot 1 + 1)}$$

#103: $$\alpha < \frac{\beta}{3\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$$

which is exactly Assumption 1

eq (12) profit margin

#104: $ga = ia - ra$

#105: $gb = ib - rb$

#106: $$ga = \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} - \frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma))}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

#107: $$ga = \frac{M\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

#108: $$gb = \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} - \frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta) + \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma))}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

#109: $$gb = \frac{M\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

> 0 if [2 conditions]

#110: $\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta < 0$

#111: $\text{SOLVE}(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta < 0, \alpha)$

#112: $$\text{IF}\left(\mu\cdot(\beta - \gamma) < 0, \alpha > \frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \text{IF}\left(\mu\cdot(\beta - \gamma) > 0, \alpha < \frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$$

#113: $$\alpha < \frac{\beta}{\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$$

#114: $4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0$

#115: SOLVE($4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0$, α)

#116: $$\mathrm{IF}\left(\mu\cdot(\beta - \gamma) < 0,\ \alpha > \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \mathrm{IF}\left(\mu\cdot(\beta - \gamma) > 0,\ \alpha < \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$$

#117: $$\alpha < \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$$

*** Section 5: Equilibrium versus optimum

eq (13): combined total number of payments and Appendix C

#118: $ta + tb = m\cdot na + m\cdot nb$

#119: $$ta + tb = (M - \mu\cdot(ia + ib))\cdot\frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) + ia\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)) - ib\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma))}{(k + 1)\cdot(\beta^2 - \gamma^2)} + (M - \mu\cdot(ia + ib))\cdot\frac{k\cdot(M\cdot\alpha\cdot(\beta + \gamma)\cdot(\beta - \gamma) - ia\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) + \gamma) + ib\cdot(\beta - \alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)))}{(k + 1)\cdot(\beta^2 - \gamma^2)}$$

#120: $$ta + tb = \frac{k\cdot(M - \mu\cdot(ia + ib))\cdot(2\cdot M\cdot\alpha\cdot(\beta + \gamma) + (1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma))\cdot(ia + ib))}{(k + 1)\cdot(\beta + \gamma)}$$

set i_star = ia = ib

#121: $$ta + tb = \frac{2\cdot k\cdot(M - 2\cdot i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}$$

#122: $$\frac{d}{d\ i\_star}\left(ta + tb = \frac{2\cdot k\cdot(M - 2\cdot i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}\right)$$

eq (C.1)

#123: $$0 = - \frac{2\cdot k\cdot(M\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + 4\cdot i\_star\cdot\mu\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}$$

SOC

#124: $$\frac{d}{d\ i\_star}\ \frac{d}{d\ i\_star}\left(ta + tb = \frac{2\cdot k\cdot(M - 2\cdot i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}\right)$$

eq (C.2)

#125: $$\frac{8\cdot k\cdot\mu\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{(k + 1)\cdot(\beta + \gamma)}$$

< 0 if [follows from Assumption 1)

#126: $2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$

#127: SOLVE($2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$, α)

#128: $$\alpha < \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}$$

#129: SOLVE(M·(4·α·μ·(β + γ) − 1) + 4·i_star·μ·(1 − 2·α·μ·(β + γ)), i_star)

eq (14)

#130: $$i\_star = \frac{M\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{4\cdot\mu\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

eq (15) optimal m

#131: $$m\_star = \frac{M}{2\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma))}$$

> 0 if [yes, Assumption 1]

#132: 2·(1 − 2·α·μ·(β + γ)) > 0

#133: SOLVE(2·(1 − 2·α·μ·(β + γ)) > 0, α)

#134: $$\mathrm{IF}\left(\mu < 0,\ \alpha > \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}\right) \vee \mathrm{IF}\left(\mu > 0,\ \alpha < \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}\right)$$

#135: $$\alpha < \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}$$

eq (16): optimal number of cards

#136: $$na\_star = \frac{M\cdot k}{4\cdot\mu\cdot(k + 1)\cdot(\beta + \gamma)}$$

#137: $$nb\_star = \frac{M \cdot k}{4 \cdot \mu \cdot (k + 1) \cdot (\beta + \gamma)}$$

eq (17) optimal rewards

#138: $$ra\_star = - M \cdot \left(\frac{1}{4 \cdot \mu \cdot (k + 1)} - \frac{1}{4 \cdot \mu \cdot (2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1)} - \frac{1}{2 \cdot \mu} \right)$$

#139: $$ra\_star = \frac{M \cdot (k \cdot (4 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1) + 2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma))}{4 \cdot \mu \cdot (k + 1) \cdot (2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1)}$$

#140: $$rb\_star = - M \cdot \left(\frac{1}{4 \cdot \mu \cdot (k + 1)} - \frac{1}{4 \cdot \mu \cdot (2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1)} - \frac{1}{2 \cdot \mu} \right)$$

#141: $$rb\_star = \frac{M \cdot (k \cdot (4 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1) + 2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma))}{4 \cdot \mu \cdot (k + 1) \cdot (2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1)}$$

> 0 if

#142: $k \cdot (4 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1) + 2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) < 0$

#143: $\mathrm{SOLVE}(k \cdot (4 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1) + 2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) < 0, \alpha)$

#144: $$\mathrm{IF}\left(\mu \cdot (2 \cdot k + 1) < 0, \alpha > \frac{k}{2 \cdot \mu \cdot (\beta + \gamma) \cdot (2 \cdot k + 1)} \right) \vee \mathrm{IF}\left(\mu \cdot (2 \cdot k + 1) > 0, \alpha < \frac{k}{2 \cdot \mu \cdot (\beta + \gamma) \cdot (2 \cdot k + 1)} \right)$$

set k=1 => New Assumption 1!

#145: $$\alpha < \frac{1}{6 \cdot \mu \cdot (\beta + \gamma)}$$

Result 4 and Appendix D

eq (D.1): Result 4a i − i*=

#146: $\dfrac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)} - \dfrac{M\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{4\cdot\mu\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$

#147: $\dfrac{M\cdot(\beta + \gamma)}{4\cdot\mu\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$

> 0 if

#148: $4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0$

#149: $\mathrm{SOLVE}(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0,\ \alpha)$

#150: $\mathrm{IF}\left(\mu\cdot(\beta - \gamma) < 0,\ \alpha > \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \mathrm{IF}\left(\mu\cdot(\beta - \gamma) > 0,\ \alpha < \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$

#151: $\alpha < \dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$

is this larger than Assumption 1? [yes]

#152: $\dfrac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} - \dfrac{1}{6\cdot\mu\cdot(\beta + \gamma)}$

#153: $\dfrac{7\cdot\beta - \gamma}{12\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} > 0$

also implied:

#154: $2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$

#155: $\mathrm{SOLVE}(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0,\ \alpha)$

#156: $$\text{IF}\left(\mu < 0,\ \alpha > \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}\right) \vee \text{IF}\left(\mu > 0,\ \alpha < \frac{1}{2\cdot\mu\cdot(\beta + \gamma)}\right)$$

eq (D.2): Result 4b m − m* =

#157: $$\frac{M\cdot(\gamma - \beta)}{4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma} - \frac{M}{2\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma))}$$

#158: $$\frac{M\cdot(\beta + \gamma)}{2\cdot(1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma))\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$$

< 0 if

#159: $4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0$

#160: $\text{SOLVE}(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma < 0,\ \alpha)$

#161: $$\text{IF}\left(\mu\cdot(\beta - \gamma) < 0,\ \alpha > \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right) \vee \text{IF}\left(\mu\cdot(\beta - \gamma) > 0,\ \alpha < \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}\right)$$

#162: $$\alpha < \frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)}$$

is this larger than αmax? [yes]

#163: $$\frac{3\cdot\beta - \gamma}{4\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} - \frac{1}{6\cdot\mu\cdot(\beta + \gamma)}$$

#164: $$\frac{7\cdot\beta - \gamma}{12\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma)} > 0$$

eq (D.3): Result 4b r − r* =

#165: $$\frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-\beta)+\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma))}{\mu\cdot(k+1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-3\cdot\beta+\gamma)}-\frac{M\cdot(k\cdot(4\cdot\alpha\cdot\mu\cdot(\beta+\gamma)-1)+2\cdot\alpha\cdot\mu\cdot(\beta+\gamma))}{4\cdot\mu\cdot(k+1)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta+\gamma)-1)}$$

#166: $$\frac{M\cdot(k+2\cdot\alpha\cdot\mu\cdot(\beta+\gamma))\cdot(\beta+\gamma)}{4\cdot\mu\cdot(k+1)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta+\gamma)-1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-3\cdot\beta+\gamma)}$$

> 0 by Assumption 1 (same as the above explanation)

eq (D.4): Result 4b n – n* =

#167: $$\frac{M\cdot k\cdot(\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-\beta)}{\mu\cdot(k+1)\cdot(\beta+\gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-3\cdot\beta+\gamma)}-\frac{M\cdot k}{4\cdot\mu\cdot(k+1)\cdot(\beta+\gamma)}$$

#168: $$-\frac{M\cdot k}{4\cdot\mu\cdot(k+1)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta+\gamma)\cdot(\beta-\gamma)-3\cdot\beta+\gamma)}$$

> 0 by Assumption 1 (same explanation)

**Subsection 5.3: No IFs and no rewards verus optimum

set ia=ib=0 into merchant's participation (1) in paper

#169: $m = M$

set ra=rb=0 into the direct demand (1) in paper

#170: na_none = $H\cdot M = \alpha\cdot M$

#171: ta_none = $M\cdot H\cdot M$

#172: tb_none = $M\cdot H\cdot M$

compare with equilibrium, eq (9) and (10) in paper

#173: $M\cdot\alpha\cdot M - \dfrac{M\cdot(\gamma - \beta)}{4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma}\cdot\dfrac{M\cdot k\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)}$

#174: $$M\cdot\alpha\cdot M - \frac{M^2\cdot k\cdot(\gamma - \beta)\cdot(\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - \beta)}{\mu\cdot(k + 1)\cdot(\beta + \gamma)\cdot(4\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\cdot(\beta - \gamma) - 3\cdot\beta + \gamma)^2}$$

Not solvable. Resort to simulations (added to Figure 3 in paper).

*** Section 6: Extension: Card-specific merchant acceptance

eq (19): Modified merchant acceptance functions

#175: $ma = M - \mu\cdot ia$

#176: $mb = M - \mu\cdot ib$

eq (20): Modified inverse demand

#177: $na = H\cdot ma + F\cdot ra - G\cdot rb$

#178: $nb = H\cdot mb + F\cdot rb - G\cdot ra$

#179: SOLVE([na = H·ma + F·ra − G·rb, nb = H·mb + F·rb − G·ra], [ra, rb])

#180: $$\left[ra = \frac{F\cdot(H\cdot ma - na) + G\cdot(H\cdot mb - nb)}{G^2 - F^2} \wedge rb = \frac{F\cdot(H\cdot mb - nb) + G\cdot(H\cdot ma - na)}{G^2 - F^2}\right]$$

Define

#181: $\alpha = H$

#182: $\beta = \dfrac{F}{F^2 - G^2}$

#183: $\gamma = \dfrac{G}{F^2 - G^2}$

#184: $\mathrm{SOLVE}\left(\left[\beta = \dfrac{F}{F^2 - G^2}, \gamma = \dfrac{G}{F^2 - G^2}\right], [F, G]\right)$

#185: $\left[F = \dfrac{\beta}{\beta^2 - \gamma^2} \wedge G = \dfrac{\gamma}{\beta^2 - \gamma^2} \wedge F^2 - G^2 \neq 0\right]$

#186: $[ra = - ma\cdot\alpha\cdot\beta - mb\cdot\alpha\cdot\gamma + na\cdot\beta + nb\cdot\gamma \wedge rb = - ma\cdot\alpha\cdot\gamma - mb\cdot\alpha\cdot\beta + na\cdot\gamma + nb\cdot\beta]$

#187: $[ra = - \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + na\cdot\beta + nb\cdot\gamma \wedge rb = - \alpha\cdot(ma\cdot\gamma + mb\cdot\beta) + na\cdot\gamma + nb\cdot\beta]$

eq (22): Profit of bank k

#188: $profitk = (ia - ra)\cdot ma\cdot nak + (ib - rb)\cdot mb\cdot nbk$

#189: $profitk = (ia - (- \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot ma\cdot nak + (ib - (- \alpha\cdot(ma\cdot\gamma + mb\cdot\beta) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot mb\cdot nbk$

Derivations of (22) and (23) and Appendix E

#190: $\dfrac{d}{d\ nak}\ (profitk = (ia - (- \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot ma\cdot nak + (ib -$

$(-\ \alpha\cdot(ma\cdot\gamma + mb\cdot\beta) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot mb\cdot nbk)$

eq (E.1): FOCa

#191: $0 = ia\cdot ma + ma^2\cdot\alpha\cdot\beta + ma\cdot(mb\cdot\alpha\cdot\gamma - na\_notk\cdot\beta - 2\cdot nak\cdot\beta - \gamma\cdot(nb\_notk + nbk)) - mb\cdot nbk\cdot\gamma$

#192: $\frac{d}{d\ nbk}$ $(profitk = (ia - (-\ \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot ma\cdot nak + (ib -$

$(-\ \alpha\cdot(ma\cdot\gamma + mb\cdot\beta) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot mb\cdot nbk)$

eq (E.2): FOCb

#193: $0 = ib\cdot mb + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\gamma) + mb\cdot(mb\cdot\alpha\cdot\beta - na\_notk\cdot\gamma - nak\cdot\gamma - nb\_notk\cdot\beta - 2\cdot nbk\cdot\beta)$

SOC

#194: $\frac{d}{d\ nak}\ \frac{d}{d\ nak}$ $(profitk = (ia - (-\ \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot ma\cdot nak +$

$(ib - (-\ \alpha\cdot(ma\cdot\gamma + mb\cdot\beta) + (nak + na\_notk)\cdot\gamma + (nbk + nb\_notk)\cdot\beta))\cdot mb\cdot nbk)$

#195: $0 > -\ 2\cdot ma\cdot\beta$

#196: $\frac{d}{d\ nbk}\ \frac{d}{d\ nbk}$ $(profitk = (ia - (-\ \alpha\cdot(ma\cdot\beta + mb\cdot\gamma) + (nak + na\_notk)\cdot\beta + (nbk + nb\_notk)\cdot\gamma))\cdot ma\cdot nak +$

$(ib - (- \alpha \cdot (ma \cdot \gamma + mb \cdot \beta) + (nak + na\_notk) \cdot \gamma + (nbk + nb\_notk) \cdot \beta)) \cdot mb \cdot nbk)$

#197: $$0 > - 2 \cdot mb \cdot \beta$$

#198: $\frac{d}{d\ nbk} \frac{d}{d\ nak} (profitk = (ia - (- \alpha \cdot (ma \cdot \beta + mb \cdot \gamma) + (nak + na\_notk) \cdot \beta + (nbk + nb\_notk) \cdot \gamma)) \cdot ma \cdot nak +$

$(ib - (- \alpha \cdot (ma \cdot \gamma + mb \cdot \beta) + (nak + na\_notk) \cdot \gamma + (nbk + nb\_notk) \cdot \beta)) \cdot mb \cdot nbk)$

#199: $$- ma \cdot \gamma - mb \cdot \gamma$$

#200: $\frac{d}{d\ nak} \frac{d}{d\ nbk} (profitk = (ia - (- \alpha \cdot (ma \cdot \beta + mb \cdot \gamma) + (nak + na\_notk) \cdot \beta + (nbk + nb\_notk) \cdot \gamma)) \cdot ma \cdot nak +$

$(ib - (- \alpha \cdot (ma \cdot \gamma + mb \cdot \beta) + (nak + na\_notk) \cdot \gamma + (nbk + nb\_notk) \cdot \beta)) \cdot mb \cdot nbk)$

#201: $$- ma \cdot \gamma - mb \cdot \gamma$$

det Hessian

#202: $H = (- 2 \cdot ma \cdot \beta) \cdot (- 2 \cdot mb \cdot \beta) - (- ma \cdot \gamma - mb \cdot \gamma) \cdot (- ma \cdot \gamma - mb \cdot \gamma)$

#203: $$H = - ma^2 \cdot \gamma^2 + ma \cdot mb \cdot (4 \cdot \beta^2 - 2 \cdot \gamma^2) - mb^2 \cdot \gamma^2$$

set ma=mb=m

#204: $$H = m^2 \cdot (4 \cdot \beta^2 - 4 \cdot \gamma^2) > 0$$

modifying FOC: na_notk = (k-1) nak, etc

#205: $0 = ia\cdot ma + ma^2\cdot\alpha\cdot\beta + ma\cdot(mb\cdot\alpha\cdot\gamma - ((k - 1)\cdot nak)\cdot\beta - 2\cdot nak\cdot\beta - \gamma\cdot((k - 1)\cdot nbk + nbk)) - mb\cdot nbk\cdot\gamma$

#206: $0 = ia\cdot ma - k\cdot ma\cdot(nak\cdot\beta + nbk\cdot\gamma) + ma^2\cdot\alpha\cdot\beta + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\beta) - mb\cdot nbk\cdot\gamma$

#207: $0 = ib\cdot mb + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\gamma) + mb\cdot(mb\cdot\alpha\cdot\beta - ((k - 1)\cdot nak)\cdot\gamma - nak\cdot\gamma - ((k - 1)\cdot nbk)\cdot\beta - 2\cdot nbk\cdot\beta)$

#208: $0 = ib\cdot mb - k\cdot mb\cdot(nak\cdot\gamma + nbk\cdot\beta) + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\gamma) + mb\cdot\beta\cdot(mb\cdot\alpha - nbk)$

eq (23) and (24)

#209: $\text{SOLVE}\left(\left[0 = ia\cdot ma - k\cdot ma\cdot(nak\cdot\beta + nbk\cdot\gamma) + ma^2\cdot\alpha\cdot\beta + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\beta) - mb\cdot nbk\cdot\gamma,\ 0 = ib\cdot mb - k\cdot mb\cdot(nak\cdot\gamma + nbk\cdot\beta) + ma\cdot(mb\cdot\alpha\cdot\gamma - nak\cdot\gamma) + mb\cdot\beta\cdot(mb\cdot\alpha - nbk)\right], [nak, nbk]\right)$

#210: $\Big[nak =$

$$\frac{mb\cdot(ia\cdot ma\cdot\beta\cdot(k + 1) - ib\cdot\gamma\cdot(k\cdot ma + mb) + \alpha\cdot(k\cdot ma^2\cdot(\beta^2 - \gamma^2) + ma^2\cdot\beta^2 + ma\cdot mb\cdot\gamma\cdot(\beta - \gamma) - mb^2\cdot\beta\cdot\sim}{k^2\cdot ma\cdot mb\cdot(\beta^2 - \gamma^2) - k\cdot(ma^2\cdot\gamma^2 - 2\cdot ma\cdot mb\cdot\beta^2 + mb^2\cdot\gamma^2) + ma\cdot mb\cdot(\beta^2 - \gamma^2)\quad\sim}$$

$\frac{\gamma))}{} \wedge nbk = -$

$$\left.\frac{ma\cdot(ia\cdot\gamma\cdot(k\cdot mb + ma) - ib\cdot mb\cdot\beta\cdot(k + 1) - \alpha\cdot(k\cdot mb^2\cdot(\beta^2 - \gamma^2) - ma^2\cdot\beta\cdot\gamma + ma\cdot mb\cdot\gamma\cdot(\beta - \gamma) + mb^2\cdot\beta^2))}{k^2\cdot ma\cdot mb\cdot(\beta^2 - \gamma^2) - k\cdot(ma^2\cdot\gamma^2 - 2\cdot ma\cdot mb\cdot\beta^2 + mb^2\cdot\gamma^2) + ma\cdot mb\cdot(\beta^2 - \gamma^2)}\right]$$

eq (25) max number of transactions (card org)

#211: $ta = Na\cdot ma$

#212: $tb = Nb\cdot mb$

Substituting for Na and Nb from (23) and (24)

#213: ta =

$$\left(k\cdot\frac{mb\cdot(ia\cdot ma\cdot\beta\cdot(k + 1) - ib\cdot\gamma\cdot(k\cdot ma + mb) + \alpha\cdot(k\cdot ma^2\cdot(\beta^2 - \gamma^2) + ma^2\cdot\beta^2 + ma\cdot mb\cdot\gamma\cdot(\beta - \gamma) - mb^2\cdot\beta\cdot\gamma))}{k^2\cdot ma\cdot mb\cdot(\beta^2 - \gamma^2) - k\cdot(ma^2\cdot\gamma^2 - 2\cdot ma\cdot mb\cdot\beta^2 + mb^2\cdot\gamma^2) + ma\cdot mb\cdot(\beta^2 - \gamma^2)}\right)\cdot ma$$

#214:
$$tb = \left(k\cdot\left(-\frac{ma\cdot(ia\cdot\gamma\cdot(k\cdot mb + ma) - ib\cdot mb\cdot\beta\cdot(k + 1) - \alpha\cdot(k\cdot mb^{2}\cdot(\beta^{2} - \gamma^{2}) - ma^{2}\cdot\beta\cdot\gamma + ma\cdot mb\cdot\gamma\cdot(\beta - \gamma) + mb^{2}\cdot\beta^{2}))}{k^{2}\cdot ma\cdot mb\cdot(\beta^{2} - \gamma^{2}) - k\cdot(ma^{2}\cdot\gamma^{2} - 2\cdot ma\cdot mb\cdot\beta^{2} + mb^{2}\cdot\gamma^{2}) + ma\cdot mb\cdot(\beta^{2} - \gamma^{2})}\right)\right)\cdot mb$$

Substituting for ma and mb from (19)

#215:
$$ta = \left(k\cdot\frac{(M - \mu\cdot ib)\cdot(ia\cdot(M - \mu\cdot ia)\cdot\beta\cdot(k + 1) - ib\cdot\gamma\cdot(k\cdot(M - \mu\cdot ia) + (M - \mu\cdot ib)) + \alpha\cdot(k\cdot(M - \mu\cdot ia)^{2}\cdot(\beta^{2} - \gamma^{2}) + (M - \mu\cdot ia)^{2}\cdot\beta^{2} + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) - (M - \mu\cdot ib)^{2}\cdot\beta\cdot\gamma))}{k^{2}\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2}) - k\cdot((M - \mu\cdot ia)^{2}\cdot\gamma^{2} - 2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\beta^{2} + (M - \mu\cdot ib)^{2}\cdot\gamma^{2}) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2})}\right)\cdot(M - \mu\cdot ia)$$

$$\#216:\quad tb = \left(k\cdot\left(-\right.\right.$$

$$\frac{(M - \mu\cdot ia)\cdot(ia\cdot\gamma\cdot(k\cdot(M - \mu\cdot ib) + (M - \mu\cdot ia)) - ib\cdot(M - \mu\cdot ib)\cdot\beta\cdot(k + 1) - \alpha\cdot(k\cdot(M - \mu\cdot ib)^{2}\cdot(\beta^{2} -\sim}{k^{2}\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2}) - k\cdot((M - \mu\cdot ia)^{2}\cdot\gamma^{2} - 2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot\sim}$$

$$\left.\left.\frac{\gamma^{2}) - (M - \mu\cdot ia)^{2}\cdot\beta\cdot\gamma + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) + (M - \mu\cdot ib)^{2}\cdot\beta^{2}))}{ib)\cdot\beta^{2} + (M - \mu\cdot ib)^{2}\cdot\gamma^{2}) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2})}\right)\right)\cdot(M - \mu\cdot ib)$$

FOCs

$$\#217:\quad \frac{d}{d\ ia}\left(ta = \right.$$

$$\left(k\cdot\frac{(M - \mu\cdot ib)\cdot(ia\cdot(M - \mu\cdot ia)\cdot\beta\cdot(k + 1) - ib\cdot\gamma\cdot(k\cdot(M - \mu\cdot ia) + (M - \mu\cdot ib)) + \alpha\cdot(k\cdot(M - \mu\cdot ia)^{2}\cdot(\beta\sim}{k^{2}\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2}) - k\cdot((M - \mu\cdot ia)^{2}\cdot\gamma^{2} - 2\cdot(M - \mu\cdot ia)\cdot(M -\sim}\right.$$

$$\left.\left.\frac{{}^{2} - \gamma^{2}) + (M - \mu\cdot ia)^{2}\cdot\beta^{2} + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) - (M - \mu\cdot ib)^{2}\cdot\beta\cdot\gamma))}{\mu\cdot ib)\cdot\beta^{2} + (M - \mu\cdot ib)^{2}\cdot\gamma^{2}) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2})}\right)\cdot(M - \mu\cdot ia)\right)$$

#218: 0 =

$$k\cdot(M^{4}\cdot(k^{2} + 2\cdot k + 1)\cdot(\beta^{2} - \gamma^{2})\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^{2} + \beta\cdot\gamma - \gamma^{2}) - \beta) + M^{3}\cdot\mu\cdot(k \sim$$

$$+ 1)\cdot(\gamma^{2} - \beta^{2})\cdot(2\cdot ia\cdot(k^{2}\cdot(3\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - 2\cdot\beta) + k\cdot(\alpha\cdot\mu\cdot(6\cdot\beta^{2} + \beta\cdot\gamma - 3\cdot\gamma^{2}) - 4\cdot\beta) + \alpha\cdot\mu\cdot(3\cdot\beta^{2} \sim$$

$$+ \beta\cdot\gamma - \gamma^{2}) - 2\cdot\beta) + ib\cdot(k^{2}\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + k\cdot(2\cdot\alpha\cdot\mu\cdot(2\cdot\beta^{2} + \beta\cdot\gamma - 3\cdot\gamma^{2}) - 2\cdot\beta + \sim$$

$$\gamma) + 2\cdot\alpha\cdot\mu\cdot(\beta^{2} + \beta\cdot\gamma - \gamma^{2}) - \beta)) + M^{2}\cdot\mu^{2}\cdot(ia^{2}\cdot(k^{3}\cdot(\beta^{2} - \gamma^{2})\cdot(6\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - 5\cdot\beta) + k^{2}\cdot(\alpha\cdot\mu\cdot(\beta \sim$$

$$^{2} - \gamma^{2})\cdot(18\cdot\beta^{2} + \beta\cdot\gamma - 10\cdot\gamma^{2}) - \beta\cdot(15\cdot\beta^{2} - 14\cdot\gamma^{2})) + k\cdot(\alpha\cdot\mu\cdot(18\cdot\beta^{4} + 2\cdot\beta^{3}\cdot\gamma - 23\cdot\beta^{2}\cdot\gamma^{2} - \beta\cdot\gamma^{3} + \sim$$

$$6\cdot\gamma^{4}) - \beta\cdot(15\cdot\beta^{2} - 14\cdot\gamma^{2})) + (\beta^{2} - \gamma^{2})\cdot(\alpha\cdot\mu\cdot(6\cdot\beta^{2} + \beta\cdot\gamma - \gamma^{2}) - 5\cdot\beta)) + 2\cdot ia\cdot ib\cdot(k^{3}\cdot(\beta^{2} - \gamma^{2})\cdot(3\sim$$

$$\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - 2\cdot\beta + \gamma) + k^{2}\cdot(\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2})\cdot(9\cdot\beta^{2} + 2\cdot\beta\cdot\gamma - 8\cdot\gamma^{2}) - 6\cdot\beta^{3} + \gamma\cdot(2\cdot\beta^{2} + 6\cdot\beta\cdot\gamma - \gamma\sim$$

$${}^{2})) + k\cdot(\alpha\cdot\mu\cdot(9\cdot\beta^{4} + 4\cdot\beta^{3}\cdot\gamma - 16\cdot\beta^{2}\cdot\gamma^{2} - 5\cdot\beta\cdot\gamma^{3} + 6\cdot\gamma^{4}) - \beta\cdot(6\cdot\beta^{2} - \beta\cdot\gamma - 6\cdot\gamma^{2})) + (\beta^{2} - \gamma^{2})\cdot(\alpha\cdot\sim$$

$$\mu\cdot(3\cdot\beta^{2} + 2\cdot\beta\cdot\gamma - 2\cdot\gamma^{2}) - 2\cdot\beta)) + ib^{2}\cdot\gamma\cdot(k^{3}\cdot(\beta^{2} - \gamma^{2}) + k^{2}\cdot(\alpha\cdot\mu\cdot(\beta - 4\cdot\gamma)\cdot(\beta^{2} - \gamma^{2}) + 2\cdot\beta^{2} + \beta\cdot\gamma\sim$$

$$(M^{2}\cdot(k^{2} + 2\cdot k + 1)\cdot\sim$$

$$- 4\cdot\gamma^{2}) + k\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^{3} - 5\cdot\beta^{2}\cdot\gamma - \beta\cdot\gamma^{2} + 6\cdot\gamma^{3}) + \beta^{2} + \beta\cdot\gamma - 3\cdot\gamma^{2}) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^{2} - \gamma^{2}))) - M\sim$$

$$(\beta^{2} - \gamma^{2}) + M\cdot\mu\cdot(ia + ib)\cdot(k^{2} + 2\cdot k + 1)\cdot(\gamma^{2} - \beta^{2}) - \mu^{2}\cdot(ia^{2}\cdot k\cdot\gamma^{2} - ia\cdot ib\cdot(k^{2}\cdot(\beta^{2} - \gamma^{2}) + 2\cdot k\cdot\beta^{2}\sim$$

$${}^{3}\cdot\mu^{3}\cdot(2\cdot ia^{3}\cdot(k^{2} + 2\cdot k + 1)\cdot(\beta^{2} - \gamma^{2})\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1)) + ia^{2}\cdot ib\cdot(k^{3}\cdot(\beta^{2} - \gamma\sim$$

$$+ \beta^{2} - \gamma^{2}) + ib^{2}\cdot k\cdot\gamma^{2}))^{2}\sim$$

$$^{2})\cdot(6\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - 5\cdot\beta + \gamma) + k^{2}\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2})\cdot(9\cdot\beta^{2} + \beta\cdot\gamma - 4\cdot\gamma^{2}) - \beta\cdot(15\cdot\beta^{2} - 2\cdot\beta\cdot\gamma - 11\sim$$

$$\cdot\gamma^{2})) + k\cdot(2\cdot\alpha\cdot\mu\cdot(9\cdot\beta^{4} + 2\cdot\beta^{3}\cdot\gamma - 11\cdot\beta^{2}\cdot\gamma^{2} - \beta\cdot\gamma^{3} + 3\cdot\gamma^{4}) - \beta\cdot(15\cdot\beta^{2} - \beta\cdot\gamma - 11\cdot\gamma^{2})) + (\beta^{2} - \gamma^{2})\sim$$

$$\cdot(2\cdot\alpha\cdot\mu\cdot(3\cdot\beta^{2} + \beta\cdot\gamma - \gamma^{2}) - 5\cdot\beta)) + 2\cdot ia\cdot ib^{2}\cdot\gamma\cdot(k^{3}\cdot(\beta^{2} - \gamma^{2}) + k^{2}\cdot(\alpha\cdot\mu\cdot(\beta - 4\cdot\gamma)\cdot(\beta^{2} - \gamma^{2}) + 2\cdot(\sim$$

$$\beta^{2} + \beta\cdot\gamma - \gamma^{2})) + k\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^{3} - 5\cdot\beta^{2}\cdot\gamma - 4\cdot\beta\cdot\gamma^{2} + 3\cdot\gamma^{3}) + \beta\cdot(\beta + 2\cdot\gamma)) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^{2} - \gamma^{2}))\sim$$

$$- ib^{3}\cdot k\cdot\gamma^{3}\cdot(2\cdot k - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) + 3)) - \mu^{4}\cdot(ia^{4}\cdot k\cdot\gamma^{2}\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1)) - \sim$$

$$2\cdot ia^{3}\cdot ib\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^{2} - \gamma^{2}) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1))\cdot(k^{2}\cdot(\beta^{2} - \gamma^{2}) + 2\cdot k\cdot\beta^{2} + \beta^{2} - \gamma^{2}) - ia^{2}\cdot ib^{2}\cdot\gamma\cdot(\sim$$

$$k^{3}\cdot(\beta^{2} - \gamma^{2}) + k^{2}\cdot(\alpha\cdot\mu\cdot(\beta - 4\cdot\gamma)\cdot(\beta^{2} - \gamma^{2}) + \beta\cdot(2\cdot\beta + 3\cdot\gamma)) + k\cdot\beta\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^{2} - 5\cdot\beta\cdot\gamma - \gamma^{2}) + \beta + \sim$$

$$3\cdot\gamma) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^{2} - \gamma^{2})) + 2\cdot ia\cdot ib^{3}\cdot k\cdot\gamma^{3}\cdot(k + \alpha\cdot\mu\cdot(\beta - \gamma)) + ib^{4}\cdot k\cdot\gamma^{3}\cdot(1 - \alpha\cdot\beta\cdot\mu)))\cdot(ib\cdot\mu \sim$$

$$- M)$$

$$\#219:\ \frac{d}{d\ ib}\left(tb = \left(k\cdot\left(-\right.\right.\right.$$

$$\frac{(M - \mu\cdot ia)\cdot(ia\cdot\gamma\cdot(k\cdot(M - \mu\cdot ib) + (M - \mu\cdot ia)) - ib\cdot(M - \mu\cdot ib)\cdot\beta\cdot(k + 1) - \alpha\cdot(k\cdot(M - \mu\cdot ib)^{2}\cdot(\beta^{2} - \sim}{k^{2}\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2}) - k\cdot((M - \mu\cdot ia)^{2}\cdot\gamma^{2} - 2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot \sim}$$

$$\left.\left.\left.\frac{\gamma^{2}) - (M - \mu\cdot ia)^{2}\cdot\beta\cdot\gamma + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) + (M - \mu\cdot ib)^{2}\cdot\beta^{2}))}{ib)\cdot\beta^{2} + (M - \mu\cdot ib)^{2}\cdot\gamma^{2}) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^{2} - \gamma^{2})}\right)\right)\cdot(M - \mu\cdot ib)\right)$$

#220: $0 =$

$$k\cdot(M^4\cdot(k^2 + 2\cdot k + 1)\cdot(\beta^2 - \gamma^2)\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta) + M^3\cdot\mu\cdot(k \sim$$

$$+ 1)\cdot(\gamma^2 - \beta^2)\cdot(ia\cdot(k^2\cdot(\beta - \gamma)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + k\cdot(2\cdot\alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) - 2\cdot\beta + \gamma) + 2 \sim$$

$$\cdot\alpha\cdot\mu\cdot(\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta) + 2\cdot ib\cdot(k^2\cdot(3\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta) + k\cdot(\alpha\cdot\mu\cdot(6\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) - 4\cdot\beta) \sim$$

$$+ \alpha\cdot\mu\cdot(3\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)) + M^2\cdot\mu^2\cdot(ia^2\cdot\gamma\cdot(k^3\cdot(\beta^2 - \gamma^2) + k^2\cdot(\alpha\cdot\mu\cdot(\beta - 4\cdot\gamma)\cdot(\beta^2 - \gamma^2) + 2\cdot \sim$$

$$\beta^2 + \beta\cdot\gamma - 4\cdot\gamma^2) + k\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^3 - 5\cdot\beta^2\cdot\gamma - \beta\cdot\gamma^2 + 6\cdot\gamma^3) + \beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^2 - \gamma \sim$$

```
2               3    2    2           2    2                2         2    2      2             2~
 )) + 2·ia·ib·(k  ·(β  − γ )·(3·α·μ·(β  − γ ) − 2·β + γ) + k  ·(α·μ·(β  − γ )·(9·β  + 2·β·γ − 8·γ ~
──────────────────────────────────────────────────────────────────────────────────────────────────~
                                                                                                  ~
                                                                                                  ~

      3        2            2                4      3        2 2        3      4         2 ~
) − 6·β  + γ·(2·β  + 6·β·γ − γ )) + k·(α·μ·(9·β  + 4·β ·γ − 16·β ·γ  − 5·β·γ  + 6·γ ) − β·(6·β  ~
──────────────────────────────────────────────────────────────────────────────────────────────────~
                                                                                                  ~
                                                                                                  ~

          2       2    2           2              2              2   3   2    2           2     ~
− β·γ − 6·γ )) + (β  − γ )·(α·μ·(3·β  + 2·β·γ − 2·γ ) − 2·β)) + ib ·(k ·(β  − γ )·(6·α·μ·(β  − γ~
──────────────────────────────────────────────────────────────────────────────────────────────────~
                                                                                                  ~
                                                                                                  ~

2          2        2    2        2             2            2       2                 4     3 ~
 ) − 5·β) + k ·(α·μ·(β  − γ )·(18·β  + β·γ − 10·γ ) − β·(15·β  − 14·γ )) + k·(α·μ·(18·β  + 2·β ·~
──────────────────────────────────────────────────────────────────────────────────────────────────~
                                                                                      2   2       ~
                                                                                   (M ·(k  + 2·k + 1)·~

        2  2      3      4          2       2       2    2           2          2               ~
γ − 23·β ·γ  − β·γ  + 6·γ ) − β·(15·β  − 14·γ )) + (β  − γ )·(α·μ·(6·β  + β·γ − γ ) − 5·β))) + M~
──────────────────────────────────────────────────────────────────────────────────────────────────~
  2    2                        2               2    2      2    2    2           2   2    2      2~
(β  − γ ) + M·μ·(ia + ib)·(k  + 2·k + 1)·(γ  − β ) − μ ·(ia ·k·γ  − ia·ib·(k ·(β  − γ ) + 2·k·β ~

  3    3    3                                2         3   2    2     2                    2    2 ~
·μ ·(ia ·k·γ ·(2·k − 2·α·μ·(β + γ) + 3) − 2·ia ·ib·γ·(k ·(β  − γ ) + k ·(α·μ·(β − 4·γ)·(β  − γ )~
──────────────────────────────────────────────────────────────────────────────────────────────────~
    2    2      2    2  2                                                                         ~
 + β  − γ ) + ib ·k·γ ))                                                                          ~
```

$$+ 2\cdot(\beta^2 + \beta\cdot\gamma - \gamma^2)) + k\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^3 - 5\cdot\beta^2\cdot\gamma - 4\cdot\beta\cdot\gamma^2 + 3\cdot\gamma^3) + \beta\cdot(\beta + 2\cdot\gamma)) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^2 \sim$$

$$- \gamma^2)) - ia\cdot ib^2\cdot(k^3\cdot(\beta^2 - \gamma^2)\cdot(6\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 5\cdot\beta + \gamma) + k^2\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2)\cdot(9\cdot\beta^2 + \beta\cdot\gamma - 4\cdot \sim$$

$$\gamma^2) - \beta\cdot(15\cdot\beta^2 - 2\cdot\beta\cdot\gamma - 11\cdot\gamma^2)) + k\cdot(2\cdot\alpha\cdot\mu\cdot(9\cdot\beta^4 + 2\cdot\beta^3\cdot\gamma - 11\cdot\beta^2\cdot\gamma^2 - \beta\cdot\gamma^3 + 3\cdot\gamma^4) - \beta\cdot(15\cdot\beta^2 \sim$$

$$- \beta\cdot\gamma - 11\cdot\gamma^2)) + (\beta^2 - \gamma^2)\cdot(2\cdot\alpha\cdot\mu\cdot(3\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 5\cdot\beta)) + 2\cdot ib^3\cdot(k^2 + 2\cdot k + 1)\cdot(\gamma^2 - \beta^2)\cdot(k \sim$$

$$\cdot(\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1))) + \mu^4\cdot(ia^4\cdot k\cdot\gamma^3\cdot(\alpha\cdot\beta\cdot\mu - 1) - 2\cdot ia^3\cdot ib\cdot k\cdot\gamma^3\cdot(k + \alpha\cdot\mu\cdot(\beta - \sim$$

$$\gamma)) + ia^2\cdot ib^2\cdot\gamma\cdot(k^3\cdot(\beta^2 - \gamma^2) + k^2\cdot(\alpha\cdot\mu\cdot(\beta - 4\cdot\gamma)\cdot(\beta^2 - \gamma^2) + \beta\cdot(2\cdot\beta + 3\cdot\gamma)) + k\cdot\beta\cdot(\alpha\cdot\mu\cdot(2\cdot\beta^2 - \sim$$

$$\frac{5\cdot\beta\cdot\gamma - \gamma^2) + \beta + 3\cdot\gamma) + \alpha\cdot\mu\cdot(\beta - \gamma)\cdot(\beta^2 - \gamma^2)) + 2\cdot ia\cdot ib^3\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1\sim}{\sim}$$

$$\frac{))\cdot(k^2\cdot(\beta^2 - \gamma^2) + 2\cdot k\cdot\beta^2 + \beta^2 - \gamma^2) - ib^4\cdot k\cdot\gamma^2\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \beta\cdot(\alpha\cdot\beta\cdot\mu - 1))))\cdot(ia\cdot\mu\sim}{\sim}$$

$$\frac{- M)}{}$$

set ia=ib in FOCa

#221: 0 =

$$\frac{k\cdot(M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta) - i\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma)\sim}{(k + 1)^2\cdot(\beta + \gamma)\cdot(\gamma - \beta)\sim}$$

$$\frac{+ \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta))}{}$$

#222: $\text{SOLVE}\left(0 = \dfrac{k\cdot(M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta) - i\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta))}{(k + 1)^2\cdot(\beta + \gamma)\cdot(\gamma - \beta)}, i\right)$

eq (26)

#223: $$i = \frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta)}{\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)} \vee k = 0$$

set ia=ib in FOCb

#224: 0 =

$$\frac{k\cdot(M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-\beta)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-\beta)-i\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-2\cdot\beta+\gamma)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-2\cdot\beta))}{(k+1)^2\cdot(\beta+\gamma)\cdot(\gamma-\beta)}$$

#225: $$\text{SOLVE}\left(0=\frac{k\cdot(M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-\beta)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-\beta)-i\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-2\cdot\beta+\gamma)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-2\cdot\beta))}{(k+1)^2\cdot(\beta+\gamma)\cdot(\gamma-\beta)}, i\right)$$

again, eq (26)

#226: $$i=\frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-\beta)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-\beta)}{\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2-\gamma^2)-2\cdot\beta+\gamma)+\alpha\cdot\mu\cdot(2\cdot\beta^2+\beta\cdot\gamma-\gamma^2)-2\cdot\beta)} \vee k=0$$

$i > 0$ if (2 conditions). Numerator < 0 if

#227: $k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta < 0$

#228: $\mathrm{SOLVE}(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta < 0, \alpha)$

#229: $\mathrm{IF}\left(2\cdot k\cdot(\beta^2 - \gamma^2) + 2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2 < 0, \alpha > \dfrac{\beta\cdot(k + 1)}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}\right) \vee \mathrm{IF}\left(2\cdot k\cdot(\beta^2 - \gamma^2) + 2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2 > 0, \alpha < \dfrac{\beta\cdot(k + 1)}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}\right)$

#230: $\alpha < \dfrac{\beta\cdot(k + 1)}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$

Need to check how k affects the above inequality => show that the RHS increases with k

#231: $\dfrac{d}{dk} \dfrac{\beta\cdot(k + 1)}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$

#232: $\dfrac{\beta\cdot\gamma}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)^2} > 0$

so set k=1 because that's where RHS is minimized

#233: $\alpha < \dfrac{\beta\cdot(1 + 1)}{\mu\cdot(\beta + \gamma)\cdot(2\cdot 1\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$

Below, is part of Assumption 2:

#234: $$\alpha < \frac{2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)}$$

2nd condition to verify that i > 0 (show that denominator is < 0):

#235: $$k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta < 0$$

#236: $$\mathrm{SOLVE}(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta < 0,\ \alpha)$$

#237: $$\mathrm{IF}\left(2\cdot k\cdot(\beta^2 - \gamma^2) + 2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2 < 0,\ \alpha > \frac{k\cdot(2\cdot\beta - \gamma) + 2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}\right) \vee \mathrm{IF}\left(2\cdot k\cdot(\beta^2 - \gamma^2) + 2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2 > 0,\ \alpha < \frac{k\cdot(2\cdot\beta - \gamma) + 2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}\right)$$

#238: $$\alpha < \frac{k\cdot(2\cdot\beta - \gamma) + 2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$$

Show that the RHS increases with K. Then, can set K=1 to minimize the RHS.

#239: $$\frac{d}{dk}\ \frac{k\cdot(2\cdot\beta - \gamma) + 2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$$

#240: $$\frac{\gamma^2}{\mu\cdot(\beta + \gamma)\cdot(2\cdot k\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)^2} > 0$$

so set k=1

#241: $\alpha < \dfrac{1\cdot(2\cdot\beta - \gamma) + 2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(2\cdot1\cdot(\beta - \gamma) + 2\cdot\beta - \gamma)}$

#242: $\alpha < \dfrac{4\cdot\beta - \gamma}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)}$

the above is implied by the condition above it because:

#243: $\dfrac{4\cdot\beta - \gamma}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)} - \dfrac{2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)}$

#244: $\dfrac{2\cdot\beta - \gamma}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)} > 0$

Hence sufficient condition for i > 0 is:

#245: $\alpha < \dfrac{2\cdot\beta}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)}$

eq (27): total payments

#246: $ta + tb =$

$$\left(k\cdot\frac{(M - \mu\cdot ib)\cdot(ia\cdot(M - \mu\cdot ia)\cdot\beta\cdot(k + 1) - ib\cdot\gamma\cdot(k\cdot(M - \mu\cdot ia) + (M - \mu\cdot ib)) + \alpha\cdot(k\cdot(M - \mu\cdot ia)^2\cdot(\beta\sim}{k^2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^2 - \gamma^2) - k\cdot((M - \mu\cdot ia)^2\cdot\gamma^2 - 2\cdot(M - \mu\cdot ia)\cdot(M -\sim}\right.$$

$$\left.\frac{^2 - \gamma^2) + (M - \mu\cdot ia)^2\cdot\beta^2 + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) - (M - \mu\cdot ib)^2\cdot\beta\cdot\gamma))}{\mu\cdot ib)\cdot\beta^2 + (M - \mu\cdot ib)^2\cdot\gamma^2) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^2 - \gamma^2)}\right)\cdot(M - \mu\cdot ia) +$$

$$\left(k\cdot\left(-\right.\right.$$

$$\frac{(M - \mu\cdot ia)\cdot(ia\cdot\gamma\cdot(k\cdot(M - \mu\cdot ib) + (M - \mu\cdot ia)) - ib\cdot(M - \mu\cdot ib)\cdot\beta\cdot(k + 1) - \alpha\cdot(k\cdot(M - \mu\cdot ib)^2\cdot(\beta^2 - \sim}{k^2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^2 - \gamma^2) - k\cdot((M - \mu\cdot ia)^2\cdot\gamma^2 - 2\cdot(M - \mu\cdot ia)\cdot(M - \mu\cdot\sim}$$

$$\left.\left.\frac{\gamma^2) - (M - \mu\cdot ia)^2\cdot\beta\cdot\gamma + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot\gamma\cdot(\beta - \gamma) + (M - \mu\cdot ib)^2\cdot\beta^2))}{ib)\cdot\beta^2 + (M - \mu\cdot ib)^2\cdot\gamma^2) + (M - \mu\cdot ia)\cdot(M - \mu\cdot ib)\cdot(\beta^2 - \gamma^2)}\right)\right)\cdot(M - \mu\cdot ib)$$

#247: ta + tb =

$$\frac{k\cdot(M - ia\cdot\mu)\cdot(M - ib\cdot\mu)\cdot(2\cdot M^2\cdot\alpha\cdot(k + 1)\cdot(\beta^2 - \gamma^2) + M\cdot(ia + ib)\cdot(k + 1)\cdot(\gamma - \beta)\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma)\sim}{M^2\cdot(k^2 + 2\cdot k + 1)\cdot(\beta^2 - \gamma^2) + M\cdot\mu\cdot(ia + ib)\cdot\sim}$$

$$\frac{- 1) + \mu\cdot(ia^2\cdot(k\cdot(\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + (\beta - \gamma)\cdot(\alpha\cdot\beta\cdot\mu - 1)) + 2\cdot ia\cdot ib\cdot\gamma\cdot(k + \alpha\cdot\mu\cdot(\beta - \gamma)) + ib^2\cdot \sim}{(k^2 + 2\cdot k + 1)\cdot(\gamma^2 - \beta^2) - \mu^2\cdot(ia^2\cdot k\cdot\gamma^2 - ia\cdot ib\cdot(k^2\cdot(\beta^2 - \gamma^2) + 2\cdot k\cdot\beta^2 + \beta^2 - \gamma^2) + ib^2\cdot k\cdot\gamma^2) \sim}$$

$$\underline{(k\cdot(\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + (\beta - \gamma)\cdot(\alpha\cdot\beta\cdot\mu - 1))))}$$

Derivation of eq (28): Optimum: max ta+tb and Appendix F

set i_star = ia = ib

#248: $$ta + tb = \frac{2\cdot k\cdot(M - i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - \alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}$$

#249: $$\frac{d}{d\ i\_star}\left(ta + tb = \frac{2\cdot k\cdot(M - i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - \alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}\right)$$

eq (F.1)

#250: $$0 = - \frac{2\cdot k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + 2\cdot i\_star\cdot\mu\cdot(1 - \alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}$$

#251: $$\frac{d}{d\ i\_star}\ \frac{d}{d\ i\_star}\left(ta + tb = \frac{2\cdot k\cdot(M - i\_star\cdot\mu)\cdot(M\cdot\alpha\cdot(\beta + \gamma) + i\_star\cdot(1 - \alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}\right)$$

#252: $$\frac{4\cdot k\cdot\mu\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{(k + 1)\cdot(\beta + \gamma)}$$

< 0 if [yes, by Assumption 1]

#253: $\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$

#254: SOLVE($\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0, \alpha$)

#255: $$\alpha < \frac{1}{\mu\cdot(\beta + \gamma)}$$

#256: $$\text{SOLVE}\left(0 = - \frac{2\cdot k\cdot(M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1) + 2\cdot i\_star\cdot\mu\cdot(1 - \alpha\cdot\mu\cdot(\beta + \gamma)))}{(k + 1)\cdot(\beta + \gamma)}, i\_star\right)$$

eq (28): Optimal interchange fees

#257: $$i\_star = \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{2\cdot\mu\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

Result 4 and Appendix G

subtracting i_star from eql i

#258: $$\frac{M\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - \beta) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - \beta)}{\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)} - \frac{M\cdot(2\cdot\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}{2\cdot\mu\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

eq (G.1):

#259: $$\Delta i = \frac{M\cdot\gamma\cdot(k + \alpha\cdot\mu\cdot(\beta + \gamma))}{2\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

#260: $$\frac{d}{dk} \frac{M\cdot\gamma\cdot(k + \alpha\cdot\mu\cdot(\beta + \gamma))}{2\cdot\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

eq (G.2)

#261: $$\frac{M\cdot\gamma\cdot(\beta - \alpha\cdot\mu\cdot(\beta^2 - \gamma^2))}{\mu\cdot(k\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)^2} > 0$$

so set k=1 to obtain the minimal value for checking whether Δi > 0:

#262: $$\frac{M\cdot\gamma\cdot(1 + \alpha\cdot\mu\cdot(\beta + \gamma))}{2\cdot\mu\cdot(1\cdot(2\cdot\alpha\cdot\mu\cdot(\beta^2 - \gamma^2) - 2\cdot\beta + \gamma) + \alpha\cdot\mu\cdot(2\cdot\beta^2 + \beta\cdot\gamma - \gamma^2) - 2\cdot\beta)\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)}$$

eq (G.3)

#263: $$\frac{M\cdot\gamma\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) + 1)}{2\cdot\mu\cdot(\alpha\cdot\mu\cdot(\beta + \gamma) - 1)\cdot(\alpha\cdot\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) - 4\cdot\beta + \gamma)}$$

> 0 if [2 conditions]

#264: $\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$

#265: SOLVE($\alpha\cdot\mu\cdot(\beta + \gamma) - 1 < 0$, α)

#266: $$\alpha < \frac{1}{\mu\cdot(\beta + \gamma)}$$

#267: $\alpha\cdot\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) - 4\cdot\beta + \gamma < 0$

#268: SOLVE($\alpha\cdot\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2) - 4\cdot\beta + \gamma < 0$, α)

#269: $\text{IF}\left(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2 < 0,\ \alpha > \dfrac{4\cdot\beta - \gamma}{\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2)}\right) \vee \text{IF}\left(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2 > 0,\ \alpha < \dfrac{4\cdot\beta - \gamma}{\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2)}\right)$

#270: $\alpha < \dfrac{4\cdot\beta - \gamma}{\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2)}$

which one is binding?

#271: $\dfrac{4\cdot\beta - \gamma}{\mu\cdot(4\cdot\beta^2 + \beta\cdot\gamma - 3\cdot\gamma^2)} - \dfrac{1}{\mu\cdot(\beta + \gamma)}$

#272: $\dfrac{2\cdot\gamma}{\mu\cdot(\beta + \gamma)\cdot(4\cdot\beta - 3\cdot\gamma)} > 0$

Hence the binding condition is

#273: $\alpha < \dfrac{1}{\mu\cdot(\beta + \gamma)}$

#274: $\dfrac{d}{d\ ib}\dfrac{d}{d\ ia}\left(ta + tb = \dfrac{k\cdot(M - \mu\cdot(ia + ib))\cdot(2\cdot M\cdot\alpha\cdot(\beta + \gamma) + (1 - 2\cdot\alpha\cdot\mu\cdot(\beta + \gamma))\cdot(ia + ib))}{(k + 1)\cdot(\beta + \gamma)}\right)$

#275: $$\frac{2\cdot k\cdot \mu\cdot (2\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)}{(k + 1)\cdot (\beta + \gamma)}$$

#276: $$\frac{d}{d\ ia}\ \frac{d}{d\ ib}\left(ta + tb = \frac{k\cdot (M - \mu\cdot (ia + ib))\cdot (2\cdot M\cdot \alpha\cdot (\beta + \gamma) + (1 - 2\cdot \alpha\cdot \mu\cdot (\beta + \gamma))\cdot (ia + ib))}{(k + 1)\cdot (\beta + \gamma)}\right)$$

#277: $$\frac{2\cdot k\cdot \mu\cdot (2\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)}{(k + 1)\cdot (\beta + \gamma)}$$

H =

#278: $$\left(- \frac{k\cdot (M\cdot (4\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1) + 2\cdot \mu\cdot (1 - 2\cdot \alpha\cdot \mu\cdot (\beta + \gamma))\cdot (ia + ib))}{(k + 1)\cdot (\beta + \gamma)}\right)\cdot \left(- \frac{k\cdot (M\cdot (4\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1) + 2\cdot \mu\cdot (1 - 2\cdot \alpha\cdot \mu\cdot (\beta + \gamma))\cdot (ia + ib))}{(k + 1)\cdot (\beta + \gamma)}\right) - \frac{2\cdot k\cdot \mu\cdot (2\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)}{(k + 1)\cdot (\beta + \gamma)}\cdot \frac{2\cdot k\cdot \mu\cdot (2\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)}{(k + 1)\cdot (\beta + \gamma)}$$

#279: $$\frac{k^2\cdot (M^2\cdot (4\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)^2 + 4\cdot M\cdot \mu\cdot (1 - 2\cdot \alpha\cdot \mu\cdot (\beta + \gamma))\cdot (ia + ib)\cdot (4\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1) + 4\cdot \mu^2\cdot (ia^2 + 2\cdot ia\cdot ib + ib^2 - 1)\cdot (2\cdot \alpha\cdot \mu\cdot (\beta + \gamma) - 1)^2)}{(k + 1)^2\cdot (\beta + \gamma)^2}$$

#280: $$- \frac{4 \cdot k^2 \cdot \mu^2 \cdot (2 \cdot \alpha \cdot \mu \cdot (\beta + \gamma) - 1)^2}{(k + 1)^2 \cdot (\beta + \gamma)^2} < 0$$

bad! H < 0 => minimum!