Algebraic calculations for a paper titled: "Hotelling's Model with Firms Located Close to Each Other"

#1: CaseMode := Sensitive

#2: InputMode := Word

#3: $\tau :\in \text{Real } (0, \infty)$

#4: $L :\in \text{Real } [1, \infty)$

#5: $pa :\in \text{Real } (0, \infty)$

#6: $pb :\in \text{Real } (0, \infty)$

#7: $xhat :\in \text{Real } [0, L]$

#8: $a :\in \text{Real } [0, 1]$

#9: $b :\in \text{Real } [0, 1]$

Eq (1)

#10: $pa + \tau\cdot(xhat - a)$

#11: $pb + \tau\cdot(1 - b - xhat)$

Eq (2)

#12: $pb + \tau\cdot(1 - b - xhat) = pa + \tau\cdot(xhat - a)$

#13: $\text{SOLVE}(pb + \tau\cdot(1 - b - xhat) = pa + \tau\cdot(xhat - a), xhat)$

#14: $$xhat = \frac{a\cdot\tau - b\cdot\tau - pa + pb + \tau}{2\cdot\tau}$$

#15: $$xhat = \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}$$

Equations (3) and (4), and Appendix A

I will derive equation (4) first disregarding condition (3) which is global deviation

#16: $$\text{profita\_n} = pa\cdot\frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}$$

#17: $$\text{profitb\_n} = pb\cdot\left(1 - \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)$$

#18: $$\frac{d}{d\ pa}\left(\text{profita\_n} = pa\cdot\frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)$$

#19: $$0 = \frac{a\cdot\tau - b\cdot\tau - 2\cdot pa + pb + \tau}{2\cdot\tau}$$

#20: $$\frac{d}{d\ pb}\left(\text{profitb\_n} = pb\cdot\left(1 - \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)\right)$$

#21: $$0 = -\frac{a\cdot\tau - b\cdot\tau - pa + 2\cdot pb - \tau}{2\cdot\tau}$$

#22: $$\text{SOLVE}\left(\left[0 = \frac{a\cdot\tau - b\cdot\tau - 2\cdot pa + pb + \tau}{2\cdot\tau},\ 0 = -\frac{a\cdot\tau - b\cdot\tau - pa + 2\cdot pb - \tau}{2\cdot\tau}\right], [pa, pb]\right)$$

#23: $$\left[pa = \frac{\tau\cdot(a - b + 3)}{3} \wedge pb = -\frac{\tau\cdot(a - b - 3)}{3}\right]$$

#24: $$\text{xhat} = \frac{a - b + 3}{6}$$

#25: $$\text{profita\_n} = \frac{\tau\cdot(a - b + 3)^2}{18}$$

#26: $$\text{profitb\_n} = \frac{\tau\cdot(a - b - 3)\cdot(a - b - 3)}{18}$$

#27: $$\text{profitb\_n} = \frac{\tau\cdot(a - b - 3)^2}{18}$$

back to Appendix A: Deriving (3)
eq (A.1)

#28: $$\frac{\tau\cdot(a - b + 3)^2}{18} \ge - \frac{\tau\cdot(a - b - 3)}{3} - \tau\cdot(1 - b - a)$$

#29: $$\tau\cdot(a - b + 3)^2 \ge 12\cdot\tau\cdot(a + 2\cdot b)$$

eq (A.2)

#30: $$\frac{\tau\cdot(a - b - 3)^2}{18} \ge \frac{\tau\cdot(a - b + 3)}{3} - \tau\cdot(1 - b - a)$$

#31: $$\tau\cdot(a - b - 3)^2 \ge 12\cdot\tau\cdot(2\cdot a + b)$$

Deriving (5a)

#32: $$\tau\cdot(a - b + 3)^2 = 12\cdot\tau\cdot(a + 2\cdot b)$$

#33: $\text{SOLVE}(\tau\cdot(a - b + 3)^2 = 12\cdot\tau\cdot(a + 2\cdot b), b)$

#34: $b = - 6\cdot\sqrt{(a + 6)} + a + 15 \vee b = 6\cdot\sqrt{(a + 6)} + a + 15$

Deriving (5b)

#35: $\tau\cdot(a - b - 3)^2 = 12\cdot\tau\cdot(2\cdot a + b)$

#36: $\text{SOLVE}(\tau\cdot(a - b - 3)^2 = 12\cdot\tau\cdot(2\cdot a + b), b)$

#37: $b = a - 6\cdot\sqrt{a} + 3 \vee b = a + 6\cdot\sqrt{a} + 3$

Explaining Figure 2
(5a), what is b when a=0?

#38: $b = - 6\cdot\sqrt{(0 + 6)} + 0 + 15$

#39: $b = 15 - 6\cdot\sqrt{6}$

#40: $b = 0.3030615433$

5(b): what is a when b=0?

#41: $0 = a - 6\cdot\sqrt{a} + 3$

#42: $\text{SOLVE}(0 = a - 6\cdot\sqrt{a} + 3, a)$

#43: $a = 15 - 6\cdot\sqrt{6} \vee a = 6\cdot\sqrt{6} + 15$

#44: $a = 0.3030615433$

verify instersection at 1/4

#45: $\text{SOLVE}([b = a - 6\cdot\sqrt{a} + 3, b = - 6\cdot\sqrt{(a + 6)} + a + 15], [a, b])$

#46: $$\left[a = \frac{1}{4} \wedge b = \frac{1}{4}\right]$$

eq (6) and Result 2

#47: $$\int_{0}^{0.25} (-6\cdot\sqrt{(a + 6)} + a + 15)\, da$$

#48: $$\int_{0.25}^{15 - 6\cdot\sqrt{6}} (a - 6\cdot\sqrt{a} + 3)\, da$$

#49: $$F = \frac{\int_{0}^{0.25} (-6\cdot\sqrt{(a + 6)} + a + 15)\, da + \int_{0.25}^{15 - 6\cdot\sqrt{6}} (a - 6\cdot\sqrt{a} + 3)\, da}{0.5}$$

#50: $$F = 96\cdot\sqrt{6} - 235$$

#51: $$F = 0.151015307$$

*** section 5 UPE begins

Definition 3 (global property), eq (7)

#52: $pb\_u\cdot(1 - xhat\_u) \geq pa\_u - \tau\cdot(1 - b - a)$

eq (8)

#53: $pa\_u\cdot xhat\_u \geq pb\_u - \tau\cdot(1 - b - a)$

recall xhat

#54: $$xhat = \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}$$

Definition 4 (marginal property): eq (9)-(10), should be ≥0 to satisfy the marginal property

#55: $$\frac{d}{d\ pa}\left(profita\_u = pa\cdot\frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)$$

#56: $$\frac{a\cdot\tau - b\cdot\tau - 2\cdot pa + pb + \tau}{2\cdot\tau} \ge 0$$

#57: $$\frac{d}{d\ pb}\left(profitb\_u = pb\cdot\left(1 - \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)\right)$$

#58: $$-\frac{a\cdot\tau - b\cdot\tau - pa + 2\cdot pb - \tau}{2\cdot\tau} \ge 0$$

*** Section 4: UPE under symmetry a=b=d

eq (11)

#59: $$xhat\_u = -\frac{pa\_u - pb\_u - \tau}{2\cdot\tau}$$

eq (12) From #52 #53,

#60: $$pb\_u\cdot\left(1 - -\frac{pa\_u - pb\_u - \tau}{2\cdot\tau}\right) = pa\_u - \tau\cdot(1 - d - d)$$

eq (13)

#61: $$pa\_u\cdot\left(-\frac{pa\_u - pb\_u - \tau}{2\cdot\tau}\right) = pb\_u - \tau\cdot(1 - d - d)$$

deriving eq (14)

#62: $\mathrm{SOLVE}\left(\left[pb\_u\cdot\left(1 - - \frac{pa\_u - pb\_u - \tau}{2\cdot\tau}\right) = pa\_u - \tau\cdot(1 - d - d),\ pa\_u\cdot\left(- \frac{pa\_u - pb\_u - \tau}{2\cdot\tau}\right) = pb\_u - \tau\cdot(1 - d - d)\right], [pa\_u, pb\_u]\right)$

#63: $\left[pa\_u = 2\cdot\tau\cdot(1 - 2\cdot d) \wedge pb\_u = 2\cdot\tau\cdot(1 - 2\cdot d),\ pa\_u = \frac{\tau\cdot(\sqrt{(1 - 8\cdot d)} + 3)}{2} \wedge pb\_u = \frac{\tau\cdot(3 - \sqrt{(1 - 8\cdot d)})}{2},\ pa\_u = \frac{\tau\cdot(3 - \sqrt{(1 - 8\cdot d)})}{2} \wedge pb\_u = \frac{\tau\cdot(\sqrt{(1 - 8\cdot d)} + 3)}{2}\right]$

#64: $pa\_u = 2\cdot\tau\cdot(1 - 2\cdot d) \wedge pb\_u = 2\cdot\tau\cdot(1 - 2\cdot d)$

#65: $$xhat\_u = \frac{1}{2}$$

#66: $profita\_u = profitb\_u = (2\cdot\tau\cdot(1 - 2\cdot d))\cdot\frac{1}{2}$

#67: $$profita\_u = profitb\_u = \tau\cdot(1 - 2\cdot d)$$

eq (15) marginal conditions from #55 and #56

#68: $\frac{d}{d\ pa}\left(profita\_u = pa\cdot\frac{\tau\cdot(d - d + 1) - pa + 2\cdot\tau\cdot(1 - 2\cdot d)}{2\cdot\tau}\right)$

#69: $$- \frac{4\cdot d\cdot\tau + 2\cdot pa - 3\cdot\tau}{2\cdot\tau}$$

substitute UPE pa yields

#70: $$- \frac{4\cdot d\cdot \tau + 2\cdot(2\cdot \tau\cdot(1 - 2\cdot d)) - 3\cdot \tau}{2\cdot \tau}$$

#71: $$\frac{4\cdot d - 1}{2}$$

#72: $$\frac{d}{d\ pb}\left(profitb\_u = - \frac{pb\cdot(4\cdot d\cdot \tau + pb - 3\cdot \tau)}{2\cdot \tau}\right)$$

#73: $$\frac{4\cdot d - 1}{2}$$

≥ 0 for d>1/4

#74: $$\tau\cdot(1 - 2\cdot d)$$

#75: $$\frac{d}{d\ pb}\left(profitb\_u = pb\cdot\left(1 - \frac{\tau\cdot(d - d + 1) - 2\cdot \tau\cdot(1 - 2\cdot d) + pb}{2\cdot \tau}\right)\right)$$

#76: $$- \frac{4\cdot d\cdot \tau + 2\cdot pb - 3\cdot \tau}{2\cdot \tau}$$

#77: $$\frac{4\cdot d - 1}{2}$$

≥ 0 for d>1/4

*** section 6: UPE under asymmetry a ≠ b

This section is computed using R

*** section proving a=0.9 and b=0 is not an UPEG. eq (17) and (18) in paper

#78: $$pb\cdot\left(1 - \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right) = pa - \tau\cdot(1 - b - a)$$

#79: $$pa\cdot\frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau} = pb - \tau\cdot(1 - b - a)$$

#80: $$pb\cdot\left(1 - \frac{\tau\cdot(0.9 - 0 + 1) - pa + pb}{2\cdot\tau}\right) = pa - \tau\cdot(1 - 0 - 0.9)$$

#81: $$pa\cdot\frac{\tau\cdot(0.9 - 0 + 1) - pa + pb}{2\cdot\tau} = pb - \tau\cdot(1 - 0 - 0.9)$$

#82: $$\text{SOLVE}\left(\left[pb\cdot\left(1 - \frac{\tau\cdot(0.9 - 0 + 1) - pa + pb}{2\cdot\tau}\right) = pa - \tau\cdot(1 - 0 - 0.9),\ pa\cdot\frac{\tau\cdot(0.9 - 0 + 1) - pa + pb}{2\cdot\tau} = pb - \tau\cdot(1 - 0 - 0.9)\right],\ [pa,\ pb]\right)$$

#83: $$\left[pa = 2\cdot\tau \wedge pb = \frac{21\cdot\tau}{20} + \frac{\sqrt{1079}\cdot i\cdot\tau}{20},\ pa = 2\cdot\tau \wedge pb = \frac{21\cdot\tau}{20} - \frac{\sqrt{1079}\cdot i\cdot\tau}{20},\ pa = \frac{\tau}{10} \wedge pb = \frac{\tau}{5}\right]$$

#84: $$[\text{false},\ \text{false},\ pa = 0.1\cdot\tau \wedge pb = 0.2\cdot\tau]$$

#85: $$\frac{d}{d\ pb}\left(profitb\_u = pb\cdot\left(1 - \frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau}\right)\right)$$

#86: $$-\frac{a\cdot\tau - b\cdot\tau - pa + 2\cdot pb - \tau}{2\cdot\tau}$$

$$\#87: \quad - \frac{0.9\cdot\tau - 0\cdot\tau - 0.1\cdot\tau + 2\cdot(0.2\cdot\tau) - \tau}{2\cdot\tau}$$

$$\#88: \quad - \frac{1}{10} < 0$$

which means that firm B can increase its profit by marginally lowering its price.

*** Checking why case I agam (both are bidning for A) yields constant pa
These confirm! the agam case, which alway yields a constant pa.

A global is binding, eq (8)

$$\#89: \quad pa\cdot xhat - (pb - \tau\cdot(1 - b - a)) = 0$$

$$\#90: \quad pa\cdot\frac{\tau\cdot(a - b + 1) - pa + pb}{2\cdot\tau} - (pb - \tau\cdot(1 - b - a)) = 0$$

A marginal is binding, eq (9)

$$\#91: \quad \tau\cdot(1 + a - b) - 2\cdot pa + pb = 0$$

set b=0.25 and a=0 and τ=1

$$\#92: \quad pa\cdot\frac{1\cdot(0 - 0.25 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.25 - 0)) = 0$$

$$\#93: \quad 1\cdot(1 + 0 - 0.25) - 2\cdot pa + pb = 0$$

$$\#94: \quad \mathrm{SOLVE}\left(\left[pa\cdot\frac{1\cdot(0 - 0.25 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.25 - 0)) = 0,\ 1\cdot(1 + 0 - 0.25) - 2\cdot pa + pb\right.\right.$$

$$= 0\Big], [pa, pb]\Big)$$

#95: $\left[pa = 1 \wedge pb = \frac{5}{4}, pa = 3 \wedge pb = \frac{21}{4}\right]$

#96: $[pa = 1 \wedge pb = 1.25, pa = 3 \wedge pb = 5.25]$

set b=0.5 and a=0 and τ=1

#97: $pa\cdot\frac{1\cdot(0 - 0.5 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.5 - 0)) = 0$

#98: $1\cdot(1 + 0 - 0.5) - 2\cdot pa + pb = 0$

#99: $\text{SOLVE}\left(\left[pa\cdot\frac{1\cdot(0 - 0.5 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.5 - 0)) = 0, 1\cdot(1 + 0 - 0.5) - 2\cdot pa + pb = 0\right], [pa, pb]\right)$

#100: $\left[pa = \sqrt{2} + 2 \wedge pb = 2\cdot\sqrt{2} + \frac{7}{2}, pa = 2 - \sqrt{2} \wedge pb = \frac{7}{2} - 2\cdot\sqrt{2}\right]$

#101: $[pa = 3.414213562 \wedge pb = 6.328427124, pa = 0.5857864376 \wedge pb = 0.6715728752]$

set b=0.5 and a=0.2 and τ=1

#102: $pa\cdot\frac{1\cdot(0.2 - 0.5 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.5 - 0.2)) = 0$

#103: $1\cdot(1 + 0.2 - 0.5) - 2\cdot pa + pb = 0$

#104: $\text{SOLVE}\left(\left[pa\cdot\dfrac{1\cdot(0.2 - 0.5 + 1) - pa + pb}{2\cdot 1} - (pb - 1\cdot(1 - 0.5 - 0.2)) = 0,\ 1\cdot(1 + 0.2 - 0.5) - 2\cdot pa + pb = 0\right],\ [pa,\ pb]\right)$

#105: $\left[pa = \sqrt{2} + 2 \wedge pb = 2\cdot\sqrt{2} + \dfrac{33}{10},\ pa = 2 - \sqrt{2} \wedge pb = \dfrac{33}{10} - 2\cdot\sqrt{2}\right]$

#106: $[pa = 3.414213562 \wedge pb = 6.128427124,\ pa = 0.5857864376 \wedge pb = 0.4715728752]$

*** proof of Result 4

Since the global property is binding: (7) and (8) hold with equality

#107: $pb\_ug\cdot(1 - xhat\_ug) = pa\_ug - \tau\cdot(1 - b - a)$

#108: $pa\_ug\cdot xhat\_ug = pb\_ug - \tau\cdot(1 - b - a)$

solving for the UPEG prices:

#109: $\text{SOLVE}([pb\_ug\cdot(1 - xhat\_ug) = pa\_ug - \tau\cdot(1 - b - a),\ pa\_ug\cdot xhat\_ug = pb\_ug - \tau\cdot(1 - b - a)],\ [pa\_ug,\ pb\_ug])$

#110: $\left[pa\_ug = \dfrac{\tau\cdot(a + b - 1)\cdot(xhat\_ug - 2)}{xhat\_ug^2 - xhat\_ug + 1} \wedge pb\_ug = -\dfrac{\tau\cdot(a + b - 1)\cdot(xhat\_ug + 1)}{xhat\_ug^2 - xhat\_ug + 1}\right]$

hence, pa_ug − pb_ug = n

#111: $\dfrac{\tau\cdot(a + b - 1)\cdot(xhat\_ug - 2)}{xhat\_ug^2 - xhat\_ug + 1} - -\dfrac{\tau\cdot(a + b - 1)\cdot(xhat\_ug + 1)}{xhat\_ug^2 - xhat\_ug + 1}$

#112: $$\frac{\tau\cdot(a + b - 1)\cdot(2\cdot xhat\_ug - 1)}{xhat\_ug^2 - xhat\_ug + 1}$$

$\le$ 0 iff

#113: $2\cdot xhat\_ug - 1 \ge 0$

#114: SOLVE($2\cdot xhat\_ug - 1 \ge 0$, xhat_ug)

#115: $$xhat\_ug \ge \frac{1}{2}$$

*** Introduction: Example of UPE for discrete Hotelling model.

#116: $pa\cdot na = (pb - \tau)\cdot(na + nb)$

#117: $pb\cdot nb = (pa - \tau)\cdot(na + nb)$

#118: SOLVE([$pa\cdot na = (pb - \tau)\cdot(na + nb)$, $pb\cdot nb = (pa - \tau)\cdot(na + nb)$], [pa, pb])

#119: $$\left[pa = \frac{\tau\cdot(na + nb)\cdot(na + 2\cdot nb)}{na^2 + na\cdot nb + nb^2} \wedge pb = \frac{\tau\cdot(na + nb)\cdot(2\cdot na + nb)}{na^2 + na\cdot nb + nb^2}\right]$$

pa − pb =

#120: $$\frac{\tau\cdot(na + nb)\cdot(na + 2\cdot nb)}{na^2 + na\cdot nb + nb^2} - \frac{\tau\cdot(na + nb)\cdot(2\cdot na + nb)}{na^2 + na\cdot nb + nb^2}$$

#121: $$\frac{\tau\cdot(na + nb)\cdot(nb - na)}{na^2 + na\cdot nb + nb^2}$$

< 0 if na > nb