

Online appendix for a paper entitled:

"Interchange fees in the presence of cashless stores, cashless consumers, and cash-only consumers"

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Below, I provide the algebraic derivations for ALL equations in the paper.

If you wish to skip directly to the proof of Result 2, go to line #48 below.

The derivations are made using symbolic algebra software called "Derive for Windows."

I will refer to each equation number in the paper itself.

#1: CaseMode := Sensitive

#2: InputMode := Word

#3: $\beta_c \in \text{Real } (0, \infty)$

#4: $\beta_m \in \text{Real } (0, \infty)$

#5: $b_{ch} \in \text{Real } (0, \infty)$

#6: $b_{mh} \in \text{Real } (0, \infty)$

#7: $n_c \in \text{Real } (0, \infty)$

#8: $n_m \in \text{Real } (0, \infty)$

*** Derivations for section 3 begin

per-trans consumer benefit from paying card [equation (1) in the paper]

#9: $\beta_c \cdot x + \phi$

per-trans consumer benefit from paying cash [equation (1) in the paper]

b_{ch}

per-trans merchant benefit from paying card [equation (2) in the paper]

#10: $\beta_m \cdot y - \phi$

per-trans merchant benefit from paying cash [equation (2) in the paper]

#11: b_{mh}

*** Derivations for section 4 begin

Fraction of consumers who prefer paying cash [equation (3)(left) in the paper]

#12:
$$\hat{x} = \frac{b_{ch} - \phi}{\beta_c}$$

Derivation of equation (3)(right) begins: Fraction of cash-only merchants

#13: $b_{mh} \cdot (1 - \lambda) = b_{mh} \cdot \hat{x} + (\beta_m \cdot \hat{y} - \phi) \cdot (1 - \hat{x})$

#14: SOLVE($b_{mh} \cdot (1 - \lambda) = b_{mh} \cdot \hat{x} + (\beta_m \cdot \hat{y} - \phi) \cdot (1 - \hat{x})$, \hat{y})

#15:
$$\hat{y} = \frac{b_{mh} \cdot (\hat{x} + \lambda - 1) + \phi \cdot (\hat{x} - 1)}{\beta_m \cdot (\hat{x} - 1)}$$

#16:
$$\hat{y} = \frac{b_{mh} \cdot \left(\frac{b_{ch} - \phi}{\beta_c} + \lambda - 1 \right) + \phi \cdot \left(\frac{b_{ch} - \phi}{\beta_c} - 1 \right)}{\beta_m \cdot \left(\frac{b_{ch} - \phi}{\beta_c} - 1 \right)}$$

#17:
$$\hat{y} = \frac{b_{ch} \cdot (b_{mh} + \phi) + b_{mh} \cdot (\beta_c \cdot (\lambda - 1) - \phi) - \phi \cdot (\beta_c + \phi)}{\beta_m \cdot (b_{ch} - \beta_c - \phi)}$$

Equation (3)(right) in the paper

$$\#18: \quad y_{\text{hat}} = \frac{\phi^2 - \phi \cdot (b_{\text{ch}} - b_{\text{mh}} - \beta c) - b_{\text{mh}} \cdot (b_{\text{ch}} + \beta c \cdot (\lambda - 1))}{\beta m \cdot (\phi - b_{\text{ch}} + \beta c)}$$

Volume of payments [equation (4) in the paper]

$$\#19: \quad v_{\text{h}} = n_{\text{c}} \cdot x_{\text{hat}} \cdot n_{\text{m}} \cdot (1 - \mu) + n_{\text{c}} \cdot (1 - \lambda - x_{\text{hat}}) \cdot n_{\text{m}} \cdot y_{\text{hat}}$$

$$\#20: \quad v_{\text{d}} = n_{\text{c}} \cdot (1 - x_{\text{hat}}) \cdot n_{\text{m}} \cdot (1 - y_{\text{hat}}) + n_{\text{c}} \cdot (x_{\text{hat}} - \delta) \cdot n_{\text{m}} \cdot \mu$$

Excluded payments [equation (5) in the paper]

$$\#21: \quad v_{\text{e}} = n_{\text{c}} \cdot n_{\text{m}} - (v_{\text{h}} + v_{\text{d}})$$

$$\#22: \quad v_{\text{e}} = n_{\text{c}} \cdot n_{\text{m}} \cdot (y_{\text{hat}} \cdot \lambda + \delta \cdot \mu)$$

*** Derivations for section 5.1 and Appendix A

Maximizing volume v_{d} w.r.t ϕ [equation (A.1) in the paper]

$$\#23: \quad v_{\text{d}} = n_{\text{c}} \cdot \left(1 - \frac{b_{\text{ch}} - \phi}{\beta c}\right) \cdot n_{\text{m}} \cdot \left(1 - \frac{\phi^2 - \phi \cdot (b_{\text{ch}} - b_{\text{mh}} - \beta c) - b_{\text{mh}} \cdot (b_{\text{ch}} + \beta c \cdot (\lambda - 1))}{\beta m \cdot (\phi - b_{\text{ch}} + \beta c)}\right) +$$

$$n_{\text{c}} \cdot \left(\frac{b_{\text{ch}} - \phi}{\beta c} - \delta\right) \cdot n_{\text{m}} \cdot \mu$$

$$\#24: \quad \frac{d}{d\phi} \left(v_{\text{d}} = n_{\text{c}} \cdot \left(1 - \frac{b_{\text{ch}} - \phi}{\beta c}\right) \cdot n_{\text{m}} \cdot \left(1 - \frac{\phi^2 - \phi \cdot (b_{\text{ch}} - b_{\text{mh}} - \beta c) - b_{\text{mh}} \cdot (b_{\text{ch}} + \beta c \cdot (\lambda - 1))}{\beta m \cdot (\phi - b_{\text{ch}} + \beta c)}\right) + \right.$$

$$nc \cdot \left(\frac{bch - \phi}{\beta_c} - \delta \right) \cdot nm \cdot \mu$$

First-order condition [equation (A.2) in the paper]

$$\#25: \quad 0 = \frac{nc \cdot nm \cdot (bch - bmh - \beta_c + \beta_m \cdot (1 - \mu) - 2 \cdot \phi)}{\beta_c \cdot \beta_m}$$

Second-order condition [equation (A.3) in the paper]

$$\#26: \quad \frac{d}{d\phi} \frac{d}{d\phi} \left(vd = nc \cdot \left(1 - \frac{bch - \phi}{\beta_c} \right) \cdot nm \cdot \left(1 - \frac{\phi^2 - \phi \cdot (bch - bmh - \beta_c) - bmh \cdot (bch + \beta_c \cdot (\lambda - 1))}{\beta_m \cdot (\phi - bch + \beta_c)} \right) + \right. \\ \left. nc \cdot \left(\frac{bch - \phi}{\beta_c} - \delta \right) \cdot nm \cdot \mu \right)$$

$$\#27: \quad 0 > - \frac{2 \cdot nc \cdot nm}{\beta_c \cdot \beta_m}$$

$$\#28: \quad \text{SOLVE} \left(0 = \frac{nc \cdot nm \cdot (bch - bmh - \beta_c + \beta_m \cdot (1 - \mu) - 2 \cdot \phi)}{\beta_c \cdot \beta_m}, \phi \right)$$

Interchange fee set by card organization [equation (6) in the paper]

$$\#29: \quad \phi_bar = \frac{bch - bmh - \beta_c + \beta_m \cdot (1 - \mu)}{2}$$

xhat and yhat under ϕ_bar [equation (A.4) in the paper]

$$\#30: \quad xhat = \frac{bch - \frac{bch - bmh - \beta c + \beta m \cdot (1 - \mu)}{2}}{\beta c}$$

$$\#31: \quad xhat = \frac{bch + bmh + \beta c + \beta m \cdot (\mu - 1)}{2 \cdot \beta c}$$

$$\#32: \quad yhat =$$

$$\frac{\left(\frac{bch - bmh - \beta c + \beta m \cdot (1 - \mu)}{2} \right)^2 - \frac{bch - bmh - \beta c + \beta m \cdot (1 - \mu)}{2} \cdot (bch - bmh - \beta c) - bmh \cdot (bch + \beta c)}{\beta m \cdot \left(\frac{bch - bmh - \beta c + \beta m \cdot (1 - \mu)}{2} - bch + \beta c \right)}$$

$$\frac{\beta c \cdot (\lambda - 1))}{\beta c \cdot (\lambda - 1))}$$

$$\#33: \quad yhat = \frac{bch^2 + 2 \cdot bch \cdot (bmh - \beta c) + bmh^2 + 2 \cdot bmh \cdot \beta c \cdot (2 \cdot \lambda - 1) + (\beta c + \beta m \cdot (1 - \mu)) \cdot (\beta c + \beta m \cdot (\mu - 1))}{2 \cdot \beta m \cdot (bch + bmh - \beta c + \beta m \cdot (\mu - 1))}$$

evaluate xhat and yhat when $\lambda=\mu=0$ [which should collapse Assumption 2]

$$\#34: \quad xhat = \frac{bch + bmh + \beta_c + \beta_m \cdot (0 - 1)}{2 \cdot \beta_c}$$

$$\#35: \quad xhat = \frac{bch + bmh + \beta_c - \beta_m}{2 \cdot \beta_c}$$

$$\#36: \quad yhat = \frac{bch^2 + 2 \cdot bch \cdot (bmh - \beta_c) + bmh^2 + 2 \cdot bmh \cdot \beta_c \cdot (2 \cdot 0 - 1) + (\beta_c + \beta_m \cdot (1 - 0)) \cdot (\beta_c + \beta_m \cdot (0 - 1))}{2 \cdot \beta_m \cdot (bch + bmh - \beta_c + \beta_m \cdot (0 - 1))}$$

$$\#37: \quad yhat = \frac{bch + bmh - \beta_c + \beta_m}{2 \cdot \beta_m}$$

*** Derivations for section 5.2

Deriving total consumer benefit [equation (7) in the paper]

First, derive consumer benefits from cash payments only

$$\#38: \quad wch = v_h \cdot bch$$

$$\#39: \quad wch = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bch$$

Second, derive consumer benefits from card payments only

$$\#40: \quad wcd = nc \cdot \int_{xhat}^1 (\beta_c \cdot x + \phi) dx \cdot nm \cdot (1 - yhat) + nc \cdot \int_{\delta}^{xhat} (\beta_c \cdot x + \phi) dx \cdot nm \cdot \mu$$

Combining the two yields equation (7) in the paper

$$\begin{aligned} \#41: \quad wc = & (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bch + nc \cdot \int_{xhat}^1 (\beta c \cdot x + \phi) dx \cdot nm \cdot (1 - yhat) + \\ & nc \cdot \int_{\delta}^{xhat} (\beta c \cdot x + \phi) dx \cdot nm \cdot \mu \end{aligned}$$

Deriving total merchant benefit [equation (8) in the paper]

First, derive merchant benefits from cash payments only

$$\#42: \quad wmh = vh \cdot bmh$$

$$\#43: \quad wmh = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bmh$$

Second, derive merchant benefit from card payments only

$$\#44: \quad wmd = nc \cdot (1 - xhat) \cdot nm \cdot \int_{yhat}^1 (\beta m \cdot y - \phi) dy + nc \cdot (xhat - \delta) \cdot nm \cdot \int_{1 - \mu}^1 (\beta m \cdot y - \phi) dy$$

combine the two yields equation (8) in paper

$$\begin{aligned} \#45: \quad wm = & (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bmh + nc \cdot (1 - xhat) \cdot nm \cdot \int_{yhat}^1 (\beta m \cdot y - \phi) dy + \\ & nc \cdot (xhat - \delta) \cdot nm \cdot \int_{1 - \mu}^1 (\beta m \cdot y - \phi) dy \end{aligned}$$

total welfare $w = wc + wm$

$$\#46: \quad w = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bch + nc \cdot \int_{xhat}^1 (\beta c \cdot x + \phi) dx \cdot nm \cdot (1 - yhat) +$$

$$\begin{aligned}
& \frac{2}{-2 \cdot \mu + 1) + \beta m \cdot \phi \cdot (\mu - 1) - \phi^2)} + b m h^2 \cdot (\beta c^2 \cdot (\lambda^2 - 2 \cdot \lambda + 1) + 2 \cdot \beta c \cdot \phi \cdot (1 - \lambda) + \phi^2) + b m h \cdot (\beta c + \phi) \cdot (\beta c^2 \cdot (\lambda - 1) - \beta c \cdot \lambda \cdot \phi + \phi \cdot (2 \cdot \beta m \cdot (\mu - 1) + \phi)) - (\beta c + \phi) \cdot (\beta c^2 \cdot (\beta m \cdot (\delta^2 \cdot \mu - 1) + \phi) + \beta c \cdot (\phi^2 - \beta m \cdot (\delta \cdot \mu \cdot (\mu - 2) + 1)) + \beta m \cdot \phi \cdot (1 - \mu) \cdot (\beta m \cdot (\mu - 1) + \phi)))}
\end{aligned}$$

*** Derivations for section 5.3

Derivation of Result 2(a)

The first-order condition (FOC) for the maximization of total welfare w [see equation (9) in the paper] is:

$$\begin{aligned}
\#48: \quad \frac{d}{d\phi} \left(w = - \right. \\
& \frac{n c \cdot n m \cdot (b c h^3 \cdot (b m h + \beta m \cdot (\mu - 1) + \phi) + b c h^2 \cdot (b m h^2 + b m h \cdot (3 \cdot \beta c \cdot (\lambda - 1) + 2 \cdot \beta m \cdot (\mu - 1) - \phi) + \beta c \cdot (\phi^2 - (2 \cdot \lambda - 3) - \beta m \cdot (\mu - 1)) + \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + \beta m \cdot \phi \cdot (1 - \mu) - 2 \cdot \phi^2) + b c h \cdot (2 \cdot b m h^2 \cdot (\beta c \cdot (\lambda - 1) - \beta m \cdot (\delta \cdot \mu \cdot (\mu - 2) + 1)) + \beta m \cdot \phi \cdot (1 - \mu) \cdot (\beta m \cdot (\mu - 1) + \phi)))}{\left. \right)
\end{aligned}$$

$$\begin{aligned}
& \frac{\phi) + b m h \cdot (\beta c^2 \cdot (2 \cdot \lambda^2 - 4 \cdot \lambda + 3) - 2 \cdot \beta c \cdot (\beta m \cdot (\mu - 1) + \phi \cdot (\lambda - 1)) - \phi \cdot (4 \cdot \beta m \cdot (\mu - 1) + \phi)) + \beta c^2 \cdot (\beta m \cdot (\delta^2 \cdot \mu - 1) - \phi \cdot (2 \cdot \lambda - 3)) - \beta c \cdot (\beta m^2 \cdot (\delta \cdot \mu \cdot (\mu - 2) + \mu^2 - 2 \cdot \mu + 2) + 2 \cdot \phi^2 \cdot (\lambda - 2)) - \phi \cdot (2 \cdot \beta m^2 \cdot (\mu - 1) - 2 \cdot \beta c \cdot \beta m \cdot (b c h - \beta c - \phi))}{2 \cdot \beta c \cdot \beta m \cdot (b c h - \beta c - \phi)} \\
& \frac{- 2 \cdot \mu + 1) + \beta m \cdot \phi \cdot (\mu - 1) - \phi^2)}{+ b m h \cdot (\beta c^2 \cdot (\lambda^2 - 2 \cdot \lambda + 1) + 2 \cdot \beta c \cdot \phi \cdot (1 - \lambda) + \phi^2) + b m h \cdot (\beta c + \phi) \cdot (\beta c^2 \cdot (\lambda - 1) - \beta c \cdot \lambda \cdot \phi + \phi \cdot (2 \cdot \beta m \cdot (\mu - 1) + \phi)) - (\beta c + \phi) \cdot (\beta c^2 \cdot (\beta m \cdot (\delta^2 \cdot \mu - 1) + \phi) + \beta c \cdot (\phi^2 - \beta m \cdot (\delta \cdot \mu \cdot (\mu - 2) + 1)) + \beta m \cdot \phi \cdot (1 - \mu) \cdot (\beta m \cdot (\mu - 1) + \phi)))} \\
& \left. \frac{\beta m^2 \cdot (\delta \cdot \mu \cdot (\mu - 2) + 1)) + \beta m \cdot \phi \cdot (1 - \mu) \cdot (\beta m \cdot (\mu - 1) + \phi))}{\right)
\end{aligned}$$

#49: 0 = -

$$\frac{n c \cdot n m \cdot (b c h^4 + 2 \cdot b c h^3 \cdot (\beta c \cdot (\lambda - 2) - 2 \cdot \phi) - b c h^2 \cdot (b m h^2 + b m h \cdot (2 \cdot (\beta m \cdot (\mu - 1) + \phi) - \beta c \cdot \lambda) + 2 \cdot \beta c^2 \cdot (\mu - 1) - \phi))}{\sim}$$

$$\begin{aligned}
& \frac{nc \cdot nm \cdot (bch \cdot (bmh + \beta m \cdot (\mu - 1) + \phi) + bch \cdot (bmh^2 + bmh \cdot (3 \cdot \beta c \cdot (\lambda - 1) + 2 \cdot \beta m \cdot (\mu - 1) - \phi) + \beta c \cdot (\phi \cdot (2 \cdot \lambda - 3) - \beta m \cdot (\mu - 1)) + \beta m \cdot (\mu^2 - 2 \cdot \mu + 1) + \beta m \cdot \phi \cdot (1 - \mu) - 2 \cdot \phi^2) + bch \cdot (2 \cdot bmh^2 \cdot (\beta c \cdot (\lambda - 1) - \phi) + bmh \cdot (\beta c \cdot (2 \cdot \lambda^2 - 4 \cdot \lambda + 3) - 2 \cdot \beta c \cdot (\beta m \cdot (\mu - 1) + \phi \cdot (\lambda - 1)) - \phi \cdot (4 \cdot \beta m \cdot (\mu - 1) + \phi)) + \beta c \cdot (\beta m \cdot (\delta^2 \cdot \mu - 1) - \phi \cdot (2 \cdot \lambda - 3)) - \beta c \cdot (\beta m^2 \cdot (\delta \cdot \mu \cdot (\mu - 2) + \mu^2 - 2 \cdot \mu + 2) + 2 \cdot \phi^2 \cdot (\lambda - 2)) - \phi \cdot (2 \cdot \beta m^2 \cdot (\mu - 2 \cdot \mu + 1) + \beta m \cdot \phi \cdot (\mu - 1) - \phi^2)) + bmh^2 \cdot (\beta c \cdot (\lambda^2 - 2 \cdot \lambda + 1) + 2 \cdot \beta c \cdot \phi \cdot (1 - \lambda) + \phi^2) + bmh \cdot (\beta c \cdot (\phi \cdot (\beta c \cdot (\lambda - 1) - \beta c \cdot \lambda \cdot \phi + \phi \cdot (2 \cdot \beta m \cdot (\mu - 1) + \phi)) - (\beta c + \phi) \cdot (\beta c \cdot (\beta m \cdot (\delta^2 \cdot \mu - 1) + \phi) + \beta c \cdot (\phi \cdot (\beta m \cdot (\delta \cdot \mu \cdot (\mu - 2) + 1)) + \beta m \cdot \phi \cdot (1 - \mu) \cdot (\beta m \cdot (\mu - 1) + \phi)))$$

$$\begin{aligned}
 \#51: & \frac{nc \cdot nm \cdot (bch^4 + bch^3 \cdot (bmh - 4 \cdot \beta c + \beta m \cdot (\mu - 1) - 3 \cdot \phi) - 3 \cdot bch^2 \cdot (bmh - 2 \cdot \beta c + \beta m \cdot (\mu - 1) - \phi) \cdot (\beta c + \phi \\
 &) - bch \cdot (bmh \cdot (\beta c^2 \cdot (2 \cdot \lambda^2 - 3) - 6 \cdot \beta c \cdot \phi - 3 \cdot \phi^2) + 4 \cdot \beta c^3 + 3 \cdot \beta c^2 \cdot (3 \cdot \phi - \beta m \cdot (\mu - 1)) + 6 \cdot \beta c \cdot \phi \cdot (\phi - \beta \\
 & \beta c \cdot \beta m \cdot (bch - \beta c - \phi)^3 \\
 & m \cdot (\mu - 1)) - \phi^2 \cdot (3 \cdot \beta m \cdot (\mu - 1) - \phi)) - bmh \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3) + (\beta c + \beta \\
 & m \cdot (1 - \mu)) \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3))}{\beta c \cdot \beta m}
 \end{aligned}$$

Evaluating the SOC at $\lambda=\mu=0$ yields

$$\#52: \frac{nc \cdot nm \cdot (bch + bmh - \beta c - \beta m)}{\beta c \cdot \beta m}$$

which is < 0 by Assumption 2.

** To prove Result 2(a), evaluate the first-order condition at $\lambda=0$ (note that δ is not an argument in the FOC).

$$\#53: 0 = -$$

$$\frac{nc \cdot nm \cdot (bch^2 - 2 \cdot bch \cdot (\beta c + \phi) - bmh^2 - 2 \cdot bmh \cdot (\beta m \cdot (\mu - 1) + \phi) + \beta c^2 + 2 \cdot \beta c \cdot \phi - \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1))}{2 \cdot \beta c \cdot \beta m} \\) + 2 \cdot \beta m \cdot \phi \cdot (1 - \mu))$$

solve for ϕ_{star} given $\lambda=0$

#54: $\text{SOLVE}(nc \cdot nm \cdot (bch^2 - 2 \cdot bch \cdot (\beta c + \phi) - bmh^2 - 2 \cdot bmh \cdot (\beta m \cdot (\mu - 1) + \phi) + \beta c^2 + 2 \cdot \beta c \cdot \phi - \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (1 - \mu)), \phi)$

#55:
$$\phi_{\text{star}} = \frac{bch - bmh - \beta c + \beta m \cdot (1 - \mu)}{2}$$

and this completes the proof of Result 2(a) because $\phi_{\text{star}} = \phi_{\text{bar}}$ [equation (6) in the paper]

** Derivation of Result 2(b)

First, note that λ is not an argument in ϕ_{bar} given in equation (6) in the paper. Hence, it is sufficient to show that ϕ_{star} declines with λ evaluated at $\lambda=0$ (emergence of cashless consumers).

Second, using the implicit function theorem (IFT), the above first-order condition (lines #48 and #49 above) defines the function $\phi_{\text{star}}(\lambda)$. Let $0=F(\phi_{\text{star}}, \lambda, \mu)$ be the first order condition, see lines #48 and #49 above. The IFT states that $\partial \phi_{\text{star}} / \partial \lambda$ is the ratio of $(-1) \partial F / \partial \lambda$ divided by $\partial F / \partial \phi_{\text{star}}$. The numerator of this ratio is:

#56:
$$\frac{d}{d\lambda} \left[- \right]$$

$$\begin{aligned}
& \frac{nc \cdot nm \cdot (bch^4 + 2 \cdot bch^3 \cdot (\beta c \cdot (\lambda - 2) - 2 \cdot \phi) - bch^2 \cdot (bmh^2 + bmh \cdot (2 \cdot (\beta m \cdot (\mu - 1) + \phi) - \beta c \cdot \lambda) + 2 \cdot \beta c \cdot \phi^2))}{(2 \cdot \lambda - 3) + 4 \cdot \beta c \cdot \phi \cdot (\lambda - 3) + \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) - 5 \cdot \phi^2} + 2 \cdot bch \cdot (bmh^2 \cdot (\beta c + \phi) \\
& + bmh \cdot (\beta c^2 \cdot \lambda \cdot (\lambda - 1) + \beta c \cdot (2 \cdot \beta m \cdot (\mu - 1) - \phi \cdot (\lambda - 2)) + 2 \cdot \phi \cdot (\beta m \cdot (\mu - 1) + \phi)) + \beta c^3 \cdot (\lambda - 2) + 2 \cdot \beta c \cdot \phi \cdot (\lambda - 1) + \beta c \cdot \phi^2 \cdot (\lambda - 3) + \beta c \cdot (\beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) + \phi^2 \cdot (\lambda - 5)) + \phi \cdot (\beta m^2 \cdot (\mu - 1)^2 + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) - \phi^2) \\
& - \phi^2) + bmh^2 \cdot (\beta c^2 \cdot (\lambda^2 - 1) - 2 \cdot \beta c \cdot \phi - \phi^2) + bmh \cdot (\beta c^2 + 2 \cdot \beta c \cdot \phi + \phi^2) \cdot (\beta c \cdot \lambda - 2 \cdot (\beta m \cdot (\mu - 1) + \phi)) + (\beta c^2 + 2 \cdot \beta c \cdot \phi + \phi^2) \cdot (\beta c^2 + 2 \cdot \beta c \cdot \phi - \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (1 - \mu))) \Bigg)
\end{aligned}$$

$$\#57: \quad \frac{\partial(F)}{\partial(\lambda)} = - \frac{nc \cdot nm \cdot (bch^2 - 2 \cdot bch \cdot (\beta c + \phi) + 2 \cdot bmh \cdot \beta c \cdot \lambda + (\beta c + \phi)^2) \cdot (2 \cdot bch + bmh)}{2 \cdot \beta m \cdot (bch - \beta c - \phi)^2}$$

The denominator is:

$$\#58: \quad \frac{d}{d\phi} \left[- \frac{nc \cdot nm \cdot (bch^4 + 2 \cdot bch^3 \cdot (\beta c \cdot (\lambda - 2) - 2 \cdot \phi) - bch^2 \cdot (bmh^2 + bmh \cdot (2 \cdot (\beta m \cdot (\mu - 1) + \phi) - \beta c \cdot \lambda) + 2 \cdot \beta c^2 \cdot (2 \cdot \lambda - 3) + 4 \cdot \beta c \cdot \phi \cdot (\lambda - 3) + \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) - 5 \cdot \phi^2) + 2 \cdot bch \cdot (bmh^2 \cdot (\beta c + \phi) + bmh \cdot (\beta c^2 \cdot \lambda \cdot (\lambda - 1) + \beta c \cdot (2 \cdot \beta m \cdot (\mu - 1) - \phi \cdot (\lambda - 2)) + 2 \cdot \phi \cdot (\beta m \cdot (\mu - 1) + \phi)) + \beta c^3 \cdot (\lambda - 2) + 2 \cdot \beta c^2 \cdot \beta m \cdot (bch - \beta c - \phi)) + \phi \cdot (\beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) + \phi^2 \cdot (\lambda - 5)) + \phi \cdot (\beta m^2 \cdot (\mu - 1)^2 + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) - \phi^2)}{2 \cdot \beta c \cdot \beta m \cdot (bch - \beta c - \phi)^2} \right]$$

$$\begin{aligned}
& \frac{(\mu - 1) - \phi^2)}{+ b m h^2 \cdot (\beta c^2 \cdot (\lambda^2 - 1) - 2 \cdot \beta c \cdot \phi - \phi^2)} + b m h^2 \cdot (\beta c^2 + 2 \cdot \beta c \cdot \phi + \phi^2) \cdot (\beta c \cdot \lambda - 2 \cdot (\beta m \cdot (\mu - 1) + \phi)) + (\beta c^2 + 2 \cdot \beta c \cdot \phi + \phi^2) \cdot (\beta c^2 + 2 \cdot \beta c \cdot \phi - \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (1 - \mu)) \Bigg) \\
\#59: & \frac{n c \cdot n m \cdot (b c h^4 + b c h^3 \cdot (b m h - 4 \cdot \beta c + \beta m \cdot (\mu - 1) - 3 \cdot \phi) - 3 \cdot b c h^2 \cdot (b m h - 2 \cdot \beta c + \beta m \cdot (\mu - 1) - \phi) \cdot (\beta c + \phi^2) - b c h \cdot (b m h \cdot (\beta c^2 \cdot (2 \cdot \lambda^2 - 3) - 6 \cdot \beta c \cdot \phi - 3 \cdot \phi^2) + 4 \cdot \beta c^3 + 3 \cdot \beta c^2 \cdot (3 \cdot \phi - \beta m \cdot (\mu - 1)) + 6 \cdot \beta c \cdot \phi \cdot (\phi - \beta c \cdot \beta m \cdot (b c h - \beta c - \phi)^3) - m \cdot (\mu - 1)) - \phi^2 \cdot (3 \cdot \beta m \cdot (\mu - 1) - \phi)) - b m h^2 \cdot \beta c^2 \cdot \lambda^2 - b m h^3 \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3) + (\beta c + \beta m \cdot (1 - \mu)) \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3))}{\beta c \cdot \beta m \cdot (b c h - \beta c - \phi)^3}
\end{aligned}$$

$$\begin{aligned}
 \#60: \quad \frac{\partial(F)}{\partial(\phi_{\text{star}})} = & \frac{nc \cdot nm \cdot (bch^4 + bch^3 \cdot (bmh - 4 \cdot \beta c + \beta m \cdot (\mu - 1) - 3 \cdot \phi) - 3 \cdot bch^2 \cdot (bmh - 2 \cdot \beta c + \beta m \cdot (\mu - 1) - \phi) \cdot (\beta c + \phi) - bch \cdot (bmh \cdot (\beta c^2 \cdot (2 \cdot \lambda^2 - 3) - 6 \cdot \beta c \cdot \phi - 3 \cdot \phi^2) + 4 \cdot \beta c^3 + 3 \cdot \beta c^2 \cdot (3 \cdot \phi - \beta m \cdot (\mu - 1)) + 6 \cdot \beta c \cdot \phi \cdot (\phi - \beta c \cdot \beta m \cdot (bch - \beta c - \phi)^3 \\
 & \beta m \cdot (\mu - 1)) - \phi^2 \cdot (3 \cdot \beta m \cdot (\mu - 1) - \phi)) - bmh \cdot (\beta c^2 \cdot \lambda^2 - bmh \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3) + (\beta c + \phi)^3 \\
 & \beta m \cdot (1 - \mu)) \cdot (\beta c^3 + 3 \cdot \beta c^2 \cdot \phi + 3 \cdot \beta c \cdot \phi^2 + \phi^3))}{
 \end{aligned}$$

Therefore, by the implicit function theorem, $\partial\phi_{\text{star}}/\partial\lambda$ is the ratio of $(-1)\partial F/\partial\lambda$ divided by $\partial F/\partial\phi_{\text{star}}$, hence

$$\#63: \frac{d}{d\mu} \frac{\beta_c \cdot (2 \cdot bch + bmh)}{2 \cdot (bch + bmh - \beta_c + \beta_m \cdot (\mu - 1))}$$

$$\#64: - \frac{\beta_c \cdot \beta_m \cdot (2 \cdot bch + bmh)}{2 \cdot (bch + bmh - \beta_c + \beta_m \cdot (\mu - 1))^2} < 0$$

** Derivation of Result 2(d)

Simple, note that δ is not an argument in the first-order condition #48, #49 above and not an argument in ϕ_{bar} in equation (6) in the paper.