Online appendix for a paper entitled:
"Interchange fees in the presence of cashless stores, cashless consumers, and cash-only consumers"
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Below, I provide the algebraic derivations for ALL equations in the paper.
If you wish to skip directly to the proof of Result 2, go to line #48 below.

The derivations are made using symbolic algebra software called "Derive for Windows."
I will refer to each equation number in the paper itself.

#1: CaseMode := Sensitive

#2: InputMode := Word

#3: $\beta c :\in \text{Real } (0, \infty)$

#4: $\beta m :\in \text{Real } (0, \infty)$

#5: $bch :\in \text{Real } (0, \infty)$

#6: $bmh :\in \text{Real } (0, \infty)$

#7: $nc :\in \text{Real } (0, \infty)$

#8: $nm :\in \text{Real } (0, \infty)$

*** Derivations for section 3 begin

per-trans consumer benefit from paying card [equation (1) in the paper]

#9: $\beta c \cdot x + \phi$

per-trans consumer benefit from paying cash [equation (1) in the paper]

bch

per-trans merchant benefit from paying card [equation (2) in the paper]

#10: $\beta m \cdot y - \phi$

per–trans merchant benefit from paying cash [equation (2) in the paper]

#11: bmh

*** Derivations for section 4 begin

Fraction of consumers who prefer paying cash [equation (3)(left) in the paper]

#12: $$\text{xhat} = \frac{\text{bch} - \phi}{\beta\text{c}}$$

Derivation of equation (3)(right) begins: Fraction of cash–only merchants

#13: $\text{bmh}\cdot(1 - \lambda) = \text{bmh}\cdot\text{xhat} + (\beta\text{m}\cdot\text{yhat} - \phi)\cdot(1 - \text{xhat})$

#14: $\text{SOLVE}(\text{bmh}\cdot(1 - \lambda) = \text{bmh}\cdot\text{xhat} + (\beta\text{m}\cdot\text{yhat} - \phi)\cdot(1 - \text{xhat}), \text{yhat})$

#15: $$\text{yhat} = \frac{\text{bmh}\cdot(\text{xhat} + \lambda - 1) + \phi\cdot(\text{xhat} - 1)}{\beta\text{m}\cdot(\text{xhat} - 1)}$$

#16: $$\text{yhat} = \frac{\text{bmh}\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} + \lambda - 1\right) + \phi\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} - 1\right)}{\beta\text{m}\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} - 1\right)}$$

#17: $$\text{yhat} = \frac{\text{bch}\cdot(\text{bmh} + \phi) + \text{bmh}\cdot(\beta\text{c}\cdot(\lambda - 1) - \phi) - \phi\cdot(\beta\text{c} + \phi)}{\beta\text{m}\cdot(\text{bch} - \beta\text{c} - \phi)}$$

Equation (3)(right) in the paper

#18: $$\text{yhat} = \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}$$

Volume of payments [equation (4) in the paper]

#19: $\text{vh} = \text{nc}\cdot\text{xhat}\cdot\text{nm}\cdot(1 - \mu) + \text{nc}\cdot(1 - \lambda - \text{xhat})\cdot\text{nm}\cdot\text{yhat}$

#20: $\text{vd} = \text{nc}\cdot(1 - \text{xhat})\cdot\text{nm}\cdot(1 - \text{yhat}) + \text{nc}\cdot(\text{xhat} - \delta)\cdot\text{nm}\cdot\mu$

Excluded payments [equation (5) in the paper]

#21: $\text{ve} = \text{nc}\cdot\text{nm} - (\text{vh} + \text{vd})$

#22: $$\text{ve} = \text{nc}\cdot\text{nm}\cdot(\text{yhat}\cdot\lambda + \delta\cdot\mu)$$

*** Derivations for section 5.1 and Appendix A

Maximizing volume vd w.r.t ϕ [equation (A.1) in the paper]

#23: $$\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} - \phi}{\beta c}\right)\cdot\text{nm}\cdot\left(1 - \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}\right) + \text{nc}\cdot\left(\frac{\text{bch} - \phi}{\beta c} - \delta\right)\cdot\text{nm}\cdot\mu$$

#24: $$\frac{d}{d\phi}\left(\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} - \phi}{\beta c}\right)\cdot\text{nm}\cdot\left(1 - \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}\right) +\right.$$

$$nc\cdot\left(\frac{bch - \phi}{\beta c} - \delta\right)\cdot nm\cdot\mu\Bigg)$$

First-order condition [equation (A.2) in the paper]

#25: $$0 = \frac{nc\cdot nm\cdot(bch - bmh - \beta c + \beta m\cdot(1 - \mu) - 2\cdot\phi)}{\beta c\cdot\beta m}$$

Second-order condition [equation (A.3) in the paper]

#26: $$\frac{d}{d\phi}\frac{d}{d\phi}\left(vd = nc\cdot\left(1 - \frac{bch - \phi}{\beta c}\right)\cdot nm\cdot\left(1 - \frac{\phi^2 - \phi\cdot(bch - bmh - \beta c) - bmh\cdot(bch + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - bch + \beta c)}\right) + nc\cdot\left(\frac{bch - \phi}{\beta c} - \delta\right)\cdot nm\cdot\mu\right)$$

#27: $$0 > -\frac{2\cdot nc\cdot nm}{\beta c\cdot\beta m}$$

#28: $$\text{SOLVE}\left(0 = \frac{nc\cdot nm\cdot(bch - bmh - \beta c + \beta m\cdot(1 - \mu) - 2\cdot\phi)}{\beta c\cdot\beta m}, \phi\right)$$

Interchange fee set by card organization [equation (6) in the paper]

#29: $$\phi\_bar = \frac{bch - bmh - \beta c + \beta m\cdot(1 - \mu)}{2}$$

xhat and yhat under φ_bar [equation (A.4) in the paper]

#30: $$\mathrm{xhat} = \frac{\mathrm{bch} - \dfrac{\mathrm{bch} - \mathrm{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}}{\beta c}$$

#31: $$\mathrm{xhat} = \frac{\mathrm{bch} + \mathrm{bmh} + \beta c + \beta m\cdot(\mu - 1)}{2\cdot\beta c}$$

#32: $$\mathrm{yhat} = \frac{\left(\dfrac{\mathrm{bch} - \mathrm{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}\right)^2 - \dfrac{\mathrm{bch} - \mathrm{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}\cdot(\mathrm{bch} - \mathrm{bmh} - \beta c) - \mathrm{bmh}\cdot(\mathrm{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot\left(\dfrac{\mathrm{bch} - \mathrm{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2} - \mathrm{bch} + \beta c\right)}$$

#33: $$\mathrm{yhat} = \frac{\mathrm{bch}^2 + 2\cdot\mathrm{bch}\cdot(\mathrm{bmh} - \beta c) + \mathrm{bmh}^2 + 2\cdot\mathrm{bmh}\cdot\beta c\cdot(2\cdot\lambda - 1) + (\beta c + \beta m\cdot(1 - \mu))\cdot(\beta c + \beta m\cdot(\mu - 1))}{2\cdot\beta m\cdot(\mathrm{bch} + \mathrm{bmh} - \beta c + \beta m\cdot(\mu - 1))}$$

evaluate xhat and yhat when λ=μ=0 [which should collapse Assumption 2]

$$\#34: \quad xhat = \frac{bch + bmh + βc + βm·(0 - 1)}{2·βc}$$

$$\#35: \quad xhat = \frac{bch + bmh + βc - βm}{2·βc}$$

$$\#36: \quad yhat = \frac{bch^2 + 2·bch·(bmh - βc) + bmh^2 + 2·bmh·βc·(2·0 - 1) + (βc + βm·(1 - 0))·(βc + βm·(0 - 1))}{2·βm·(bch + bmh - βc + βm·(0 - 1))}$$

$$\#37: \quad yhat = \frac{bch + bmh - βc + βm}{2·βm}$$

*** Derivations for section 5.2

Deriving total consumer benefit [equation (7) in the paper]

First, derive consumer benefits from cash payments only

$$\#38: \quad wch = vh·bch$$

$$\#39: \quad wch = (nc·xhat·nm·(1 - μ) + nc·(1 - λ - xhat)·nm·yhat)·bch$$

Second, derive consumer benefits from card payments only

$$\#40: \quad wcd = nc·\int_{xhat}^{1} (βc·x + φ)\,dx·nm·(1 - yhat) + nc·\int_{δ}^{xhat} (βc·x + φ)\,dx·nm·μ$$

Combining the two yields equation (7) in the paper

#41: $wc = (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bch + nc\cdot\int_{xhat}^{1} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot(1 - yhat) + nc\cdot\int_{\delta}^{xhat} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot\mu$

Deriving total merchant benefit [equation (8) in the paper]

First, derive merchant benefits from cash payments only

#42: $wmh = vh\cdot bmh$

#43: $wmh = (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bmh$

Second, derive merchant benefit from card payments only

#44: $wmd = nc\cdot(1 - xhat)\cdot nm\cdot\int_{yhat}^{1} (\beta m\cdot y - \phi)\, dy + nc\cdot(xhat - \delta)\cdot nm\cdot\int_{1 - \mu}^{1} (\beta m\cdot y - \phi)\, dy$

combine the two yields equation (8) in paper

#45: $wm = (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bmh + nc\cdot(1 - xhat)\cdot nm\cdot\int_{yhat}^{1} (\beta m\cdot y - \phi)\, dy + nc\cdot(xhat - \delta)\cdot nm\cdot\int_{1 - \mu}^{1} (\beta m\cdot y - \phi)\, dy$

total welfare w = wc + wm

#46: $w = (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bch + nc\cdot\int_{xhat}^{1} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot(1 - yhat) +$

$$nc\cdot\int_{\delta}^{xhat} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot\mu + (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bmh + nc\cdot(1 -$$

$$xhat)\cdot nm\cdot\int_{yhat}^{1} (\beta m\cdot y - \phi)\, dy + nc\cdot(xhat - \delta)\cdot nm\cdot\int_{1 - \mu}^{1} (\beta m\cdot y - \phi)\, dy$$

substituting for xhat and yhat, yields total welfare as a function of φ (as well as λ, μ, and δ)

#47: $w = -$

$$nc\cdot nm\cdot(bch^{3}\cdot(bmh + \beta m\cdot(\mu - 1) + \phi) + bch^{2}\cdot(bmh^{2} + bmh\cdot(3\cdot\beta c\cdot(\lambda - 1) + 2\cdot\beta m\cdot(\mu - 1) - \phi) + \beta c\cdot(\phi\sim$$

$$\cdot(2\cdot\lambda - 3) - \beta m\cdot(\mu - 1)) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(1 - \mu) - 2\cdot\phi^{2}) + bch\cdot(2\cdot bmh^{2}\cdot(\beta c\cdot(\lambda - 1) -\sim$$

$$\phi) + bmh\cdot(\beta c^{2}\cdot(2\cdot\lambda^{2} - 4\cdot\lambda + 3) - 2\cdot\beta c\cdot(\beta m\cdot(\mu - 1) + \phi\cdot(\lambda - 1)) - \phi\cdot(4\cdot\beta m\cdot(\mu - 1) + \phi)) + \beta c^{2}\cdot(\beta\sim$$

$$m\cdot(\delta^{2}\cdot\mu - 1) - \phi\cdot(2\cdot\lambda - 3)) - \beta c\cdot(\beta m^{2}\cdot(\delta\cdot\mu\cdot(\mu - 2) + \mu^{2} - 2\cdot\mu + 2) + 2\cdot\phi^{2}\cdot(\lambda - 2)) - \phi\cdot(2\cdot\beta m^{2}\cdot(\mu\sim$$

$$\overline{\qquad 2\cdot\beta c\cdot\beta m\cdot(bch - \beta c - \phi) \qquad}$$

$$^{2} - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 2\cdot\lambda + 1) + 2\cdot\beta c\cdot\phi\cdot(1 - \lambda) + \phi^{2}) + bmh\cdot(\beta c + \sim$$

$$\phi)\cdot(\beta c^{2}\cdot(\lambda - 1) - \beta c\cdot\lambda\cdot\phi + \phi\cdot(2\cdot\beta m\cdot(\mu - 1) + \phi)) - (\beta c + \phi)\cdot(\beta c^{2}\cdot(\beta m\cdot(\delta^{2}\cdot\mu - 1) + \phi) + \beta c\cdot(\phi^{2} - \sim$$

$$\beta m^{2}\cdot(\delta\cdot\mu\cdot(\mu - 2) + 1)) + \beta m\cdot\phi\cdot(1 - \mu)\cdot(\beta m\cdot(\mu - 1) + \phi)))$$

*** Derivations for section 5.3

Derivation of Result 2(a)

The first-order condition for the maximization of total welfare w (define in equation (9) in the paper) is:

#48: $0 = -$

$$nc\cdot nm\cdot(bch^{4} + 2\cdot bch^{3}\cdot(\beta c\cdot(\lambda - 2) - 2\cdot\phi) - bch^{2}\cdot(bmh^{2} + bmh\cdot(2\cdot(\beta m\cdot(\mu - 1) + \phi) - \beta c\cdot\lambda) + 2\cdot\beta c^{2}\cdot \sim$$

$$(2\cdot\lambda - 3) + 4\cdot\beta c\cdot\phi\cdot(\lambda - 3) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) - 5\cdot\phi^{2}) + 2\cdot bch\cdot(bmh^{2}\cdot(\beta c + \phi) \sim$$

$$+ bmh\cdot(\beta c^{2}\cdot\lambda\cdot(\lambda - 1) + \beta c\cdot(2\cdot\beta m\cdot(\mu - 1) - \phi\cdot(\lambda - 2)) + 2\cdot\phi\cdot(\beta m\cdot(\mu - 1) + \phi)) + \beta c^{3}\cdot(\lambda - 2) + 2\cdot\beta\sim$$

$$2\cdot\beta c\cdot\beta m\cdot(bch - \beta c\sim$$

$$c^{2}\cdot\phi\cdot(\lambda - 3) + \beta c\cdot(\beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) + \phi^{2}\cdot(\lambda - 5)) + \phi\cdot(\beta m^{2}\cdot(\mu - 1)^{2} + 2\cdot\beta m\cdot\phi\cdot\sim$$

$$- \phi)^{2}$$

$$(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 1) - 2\cdot\beta c\cdot\phi - \phi^{2}) + bmh\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu - \sim$$

$$1) + \phi)) + (\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi - \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)))$$

to prove Result 2(a), evaluate the first-order condition at $\lambda=0$ (note that δ is not an argument in the FOC).

#49: $0 = -$

$$\frac{nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) - bmh^2 - 2\cdot bmh\cdot(\beta m\cdot(\mu - 1) + \phi) + \beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1\sim}{2\cdot\beta c\cdot\beta m}\sim$$

$$\overline{) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu))}$$

solve for ϕ_star given λ=0

#50: $\text{SOLVE}(nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) - bmh^2 - 2\cdot bmh\cdot(\beta m\cdot(\mu - 1) + \phi) + \beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu$

$+ 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)), \phi)$

#51: $$\phi\_star = \frac{bch - bmh - \beta c + \beta m\cdot(1 - \mu)}{2}$$

and this completes the proof of Result 2(a) because ϕ_star = ϕ_bar [equation (6) in the paper]

Derivation of Result 2(b)

First, note that λ is not an argument in ϕ_bar given in equation (6) in the paper. Hence, it is sufficient to show that ϕ_star declines with λ evaluated at λ=0 (emergence of cashless consumers).

Second, the above first-order condition line #48 above defines an implicit function theorem ϕ_star(λ). Let 0=F(ϕ_star, λ, μ) be the first order condition, see line#48 above. The IFT states that ∂ϕ_star/∂λ is the ratio of (-1)∂F/∂λ divided by ∂F/∂ϕ_star.

#52: $$\frac{d}{d\lambda}\left(-\right.$$

$$nc\cdot nm\cdot(bch^{4} + 2\cdot bch^{3}\cdot(\beta c\cdot(\lambda - 2) - 2\cdot\phi) - bch^{2}\cdot(bmh^{2} + bmh\cdot(2\cdot(\beta m\cdot(\mu - 1) + \phi) - \beta c\cdot\lambda) + 2\cdot\beta c^{2}\cdot \sim$$

$$(2\cdot\lambda - 3) + 4\cdot\beta c\cdot\phi\cdot(\lambda - 3) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) - 5\cdot\phi^{2}) + 2\cdot bch\cdot(bmh^{2}\cdot(\beta c + \phi) \sim$$

$$+ bmh\cdot(\beta c^{2}\cdot\lambda\cdot(\lambda - 1) + \beta c\cdot(2\cdot\beta m\cdot(\mu - 1) - \phi\cdot(\lambda - 2)) + 2\cdot\phi\cdot(\beta m\cdot(\mu - 1) + \phi)) + \beta c^{3}\cdot(\lambda - 2) + 2\cdot\beta \sim$$

$$2\cdot\beta c\cdot\beta m\cdot(bch - \beta c \sim$$

$$c^{2}\cdot\phi\cdot(\lambda - 3) + \beta c\cdot(\beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) + \phi^{2}\cdot(\lambda - 5)) + \phi\cdot(\beta m^{2}\cdot(\mu - 1)^{2} + 2\cdot\beta m\cdot\phi\cdot \sim$$

$$- \phi)^{2}$$

$$(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 1) - 2\cdot\beta c\cdot\phi - \phi^{2}) + bmh\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu - \sim$$

$$1) + \phi)) + (\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi - \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu))) \Bigg)$$

#53: $$\frac{\partial(F)}{\partial(\lambda)} = - \frac{nc \cdot nm \cdot (bch^2 - 2 \cdot bch \cdot (\beta c + \phi) + 2 \cdot bmh \cdot \beta c \cdot \lambda + (\beta c + \phi)^2) \cdot (2 \cdot bch + bmh)}{2 \cdot \beta m \cdot (bch - \beta c - \phi)^2}$$

#54: $$\frac{d}{d\phi}\left(- \right.$$

$$\frac{nc \cdot nm \cdot (bch^4 + 2 \cdot bch^3 \cdot (\beta c \cdot (\lambda - 2) - 2 \cdot \phi) - bch^2 \cdot (bmh^2 + bmh \cdot (2 \cdot (\beta m \cdot (\mu - 1) + \phi) - \beta c \cdot \lambda) + 2 \cdot \beta c^2 \cdot \sim}{\sim}$$

$$\frac{(2 \cdot \lambda - 3) + 4 \cdot \beta c \cdot \phi \cdot (\lambda - 3) + \beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) - 5 \cdot \phi^2) + 2 \cdot bch \cdot (bmh^2 \cdot (\beta c + \phi) \sim}{\sim}$$

$$\frac{+ bmh \cdot (\beta c^2 \cdot \lambda \cdot (\lambda - 1) + \beta c \cdot (2 \cdot \beta m \cdot (\mu - 1) - \phi \cdot (\lambda - 2)) + 2 \cdot \phi \cdot (\beta m \cdot (\mu - 1) + \phi)) + \beta c^3 \cdot (\lambda - 2) + 2 \cdot \beta \sim}{2 \cdot \beta c \cdot \beta m \cdot (bch - \beta c \sim}$$

$$\frac{c^2 \cdot \phi \cdot (\lambda - 3) + \beta c \cdot (\beta m^2 \cdot (\mu^2 - 2 \cdot \mu + 1) + 2 \cdot \beta m \cdot \phi \cdot (\mu - 1) + \phi^2 \cdot (\lambda - 5)) + \phi \cdot (\beta m^2 \cdot (\mu - 1)^2 + 2 \cdot \beta m \cdot \phi \cdot \sim}{- \phi)^2 \sim}$$

$$\frac{(\mu - 1) - \phi^2)) + bmh^2\cdot(\beta c^2\cdot(\lambda^2 - 1) - 2\cdot\beta c\cdot\phi - \phi^2) + bmh\cdot(\beta c^2 + 2\cdot\beta c\cdot\phi + \phi^2)\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu - \sim}{}$$

$$\left.\frac{1) + \phi)) + (\beta c^2 + 2\cdot\beta c\cdot\phi + \phi^2)\cdot(\beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)))}{}\right)$$

#55:

$$\frac{nc\cdot nm\cdot(bch^4 + bch^3\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^2\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c + \phi\sim}{}$$

$$\frac{) - bch\cdot(bmh\cdot(\beta c^2\cdot(2\cdot\lambda^2 - 3) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^2) + 4\cdot\beta c^3 + 3\cdot\beta c^2\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi - \beta\sim}{\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^3}$$

$$\frac{m\cdot(\mu - 1)) - \phi^2\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)) - bmh^2\cdot\beta c^2\cdot\lambda^2 - bmh\cdot(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3) + (\beta c + \beta\sim}{}$$

$$\frac{m\cdot(1 - \mu))\cdot(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3))}{}$$

#56: $$\frac{\partial(F)}{\partial(\phi\_star)} =$$

$$\frac{nc\cdot nm\cdot\left(bch^4 + bch^3\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^2\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c + \phi) - bch\cdot\left(bmh\cdot\left(\beta c^2\cdot\left(2\cdot\lambda^2 - 3\right) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^2\right) + 4\cdot\beta c^3 + 3\cdot\beta c^2\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi - \beta m\cdot(\mu - 1)) - \phi^2\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)\right) - bmh^2\cdot\beta c^2\cdot\lambda^2 - bmh\cdot\left(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3\right) + (\beta c + \beta m\cdot(1 - \mu))\cdot\left(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3\right)\right)}{\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^3}$$

Therefore, by the implicit function theorem, ∂φ_star/∂λ is the ratio of (−1)∂F/∂λ divided by ∂F/∂φ_star, hence

#57: $-$

$$nc\cdot nm\cdot(bch^4 + bch^3\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^2\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c \sim$$

$$- \frac{nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) + 2\cdot bmh\cdot\beta c\cdot\lambda + (\beta c + \phi)^2)\cdot(2\sim}{2\cdot\beta m\cdot(bch - \beta c - \phi)^2}$$

$$\frac{+ \phi) - bch\cdot(bmh\cdot(\beta c^2\cdot(2\cdot\lambda^2 - 3) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^2) + 4\cdot\beta c^3 + 3\cdot\beta c^2\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi \sim}{\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^3}$$

$$\frac{\cdot bch + bmh)}{\ }$$

$$\frac{\ }{- \beta m\cdot(\mu - 1)) - \phi^2\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)) - bmh^2\cdot\beta c^2\cdot\lambda^2 - bmh\cdot(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3) + (\beta c}$$

$$\frac{\ }{+ \beta m\cdot(1 - \mu))\cdot(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3))}$$

evaluate at λ=0

#58:

$$\frac{\partial(\phi\_star)}{\partial(\lambda)} = \frac{\beta c\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))}$$

< 0 for sufficiently small μ [see Assumption 2 in the paper]. This completes the proof of Result 2(b) because it shows that as λ increases from 0, ϕ_star declines and by Result 2(a) it must be that ϕ_star < ϕ_bar.

Derivation of Result 2(c)
Just differentiate #58 with respect to μ

#59: $$\frac{d}{d\mu} \frac{\beta c\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))}$$

#60: $$- \frac{\beta c\cdot\beta m\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))^2} < 0$$

Derivation of Result 2(d)
Simple, note that δ is not an argument in the first-order condition #48 above and not an argument in ϕ_bar in equation (6) in the paper.