Online appendix for a paper entitled:
"Interchange fees in the presence of cashless stores, cashless consumers, and cash-only consumers"
By, Oz Shy, ozshy@ozshy.com, www.ozshy.com

Below, I provide the algebraic derivations for ALL equations in the paper.
If you wish to skip directly to the proof of Result 2, go to line #48 below.

The derivations are made using symbolic algebra software called "Derive for Windows."
I will refer to each equation number in the paper itself.

#1: CaseMode := Sensitive

#2: InputMode := Word

#3: βc :∈ Real (0, ∞)

#4: βm :∈ Real (0, ∞)

#5: bch :∈ Real (0, ∞)

#6: bmh :∈ Real (0, ∞)

#7: nc :∈ Real (0, ∞)

#8: nm :∈ Real (0, ∞)

*** Derivations for section 3 begin

per-trans consumer benefit from paying card [equation (1) in the paper]

#9: $βc·x + φ$

per-trans consumer benefit from paying cash [equation (1) in the paper]

bch

per-trans merchant benefit from paying card [equation (2) in the paper]

#10: $βm·y - φ$

per–trans merchant benefit from paying cash [equation (2) in the paper]

#11: bmh

*** Derivations for section 4 begin

Fraction of consumers who prefer paying cash [equation (3)(left) in the paper]

#12: $$\text{xhat} = \frac{\text{bch} - \phi}{\beta\text{c}}$$

Derivation of equation (3)(right) begins: Fraction of cash–only merchants

#13: $\text{bmh}\cdot(1 - \lambda) = \text{bmh}\cdot\text{xhat} + (\beta\text{m}\cdot\text{yhat} - \phi)\cdot(1 - \text{xhat})$

#14: $\text{SOLVE}(\text{bmh}\cdot(1 - \lambda) = \text{bmh}\cdot\text{xhat} + (\beta\text{m}\cdot\text{yhat} - \phi)\cdot(1 - \text{xhat}), \text{yhat})$

#15: $$\text{yhat} = \frac{\text{bmh}\cdot(\text{xhat} + \lambda - 1) + \phi\cdot(\text{xhat} - 1)}{\beta\text{m}\cdot(\text{xhat} - 1)}$$

#16: $$\text{yhat} = \frac{\text{bmh}\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} + \lambda - 1\right) + \phi\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} - 1\right)}{\beta\text{m}\cdot\left(\frac{\text{bch} - \phi}{\beta\text{c}} - 1\right)}$$

#17: $$\text{yhat} = \frac{\text{bch}\cdot(\text{bmh} + \phi) + \text{bmh}\cdot(\beta\text{c}\cdot(\lambda - 1) - \phi) - \phi\cdot(\beta\text{c} + \phi)}{\beta\text{m}\cdot(\text{bch} - \beta\text{c} - \phi)}$$

Equation (3)(right) in the paper

#18: $$\text{yhat} = \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}$$

Volume of payments [equation (4) in the paper]

#19: $\text{vh} = \text{nc}\cdot\text{xhat}\cdot\text{nm}\cdot(1 - \mu) + \text{nc}\cdot(1 - \lambda - \text{xhat})\cdot\text{nm}\cdot\text{yhat}$

#20: $\text{vd} = \text{nc}\cdot(1 - \text{xhat})\cdot\text{nm}\cdot(1 - \text{yhat}) + \text{nc}\cdot(\text{xhat} - \delta)\cdot\text{nm}\cdot\mu$

Excluded payments [equation (5) in the paper]

#21: $\text{ve} = \text{nc}\cdot\text{nm} - (\text{vh} + \text{vd})$

#22: $$\text{ve} = \text{nc}\cdot\text{nm}\cdot(\text{yhat}\cdot\lambda + \delta\cdot\mu)$$

*** Derivations for section 5.1 and Appendix A

Maximizing volume vd w.r.t φ [equation (A.1) in the paper]

#23: $$\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} - \phi}{\beta c}\right)\cdot\text{nm}\cdot\left(1 - \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}\right) +$$
$$\text{nc}\cdot\left(\frac{\text{bch} - \phi}{\beta c} - \delta\right)\cdot\text{nm}\cdot\mu$$

#24: $$\frac{d}{d\phi}\left(\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} - \phi}{\beta c}\right)\cdot\text{nm}\cdot\left(1 - \frac{\phi^2 - \phi\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \beta c\cdot(\lambda - 1))}{\beta m\cdot(\phi - \text{bch} + \beta c)}\right) +\right.$$

$$nc\cdot\left(\frac{bch-\phi}{\beta c}-\delta\right)\cdot nm\cdot\mu\Bigg)$$

First-order condition [equation (A.2) in the paper]

#25: $$0=\frac{nc\cdot nm\cdot(bch-bmh-\beta c+\beta m\cdot(1-\mu)-2\cdot\phi)}{\beta c\cdot\beta m}$$

Second-order condition [equation (A.3) in the paper]

#26: $$\frac{d}{d\phi}\frac{d}{d\phi}\left(vd=nc\cdot\left(1-\frac{bch-\phi}{\beta c}\right)\cdot nm\cdot\left(1-\frac{\phi^2-\phi\cdot(bch-bmh-\beta c)-bmh\cdot(bch+\beta c\cdot(\lambda-1))}{\beta m\cdot(\phi-bch+\beta c)}\right)+\right.$$

$$\left.nc\cdot\left(\frac{bch-\phi}{\beta c}-\delta\right)\cdot nm\cdot\mu\right)$$

#27: $$0>-\frac{2\cdot nc\cdot nm}{\beta c\cdot\beta m}$$

#28: $$\mathrm{SOLVE}\left(0=\frac{nc\cdot nm\cdot(bch-bmh-\beta c+\beta m\cdot(1-\mu)-2\cdot\phi)}{\beta c\cdot\beta m},\ \phi\right)$$

Interchange fee set by card organization [equation (6) in the paper]

#29: $$\phi\_bar=\frac{bch-bmh-\beta c+\beta m\cdot(1-\mu)}{2}$$

Evaluating xhat and yhat under ϕ_bar [equation (A.4) in the paper]

#30:
$$\text{xhat\_bar} = \frac{\text{bch} - \dfrac{\text{bch} - \text{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}}{\beta c}$$

#31:
$$\text{xhat} = \frac{\text{bch} + \text{bmh} + \beta c + \beta m\cdot(\mu - 1)}{2\cdot\beta c}$$

#32: yhat_bar =

$$\frac{\left(\dfrac{\text{bch} - \text{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}\right)^2 - \dfrac{\text{bch} - \text{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2}\cdot(\text{bch} - \text{bmh} - \beta c) - \text{bmh}\cdot(\text{bch} + \sim}{\beta m\cdot\left(\dfrac{\text{bch} - \text{bmh} - \beta c + \beta m\cdot(1 - \mu)}{2} - \text{bch} + \beta c\right)\sim}$$

$$\frac{\beta c\cdot(\lambda - 1))}{\phantom{xx}}$$

#33: yhat_bar =

$$\frac{bch^2 + 2\cdot bch\cdot(bmh - \beta c) + bmh^2 + 2\cdot bmh\cdot\beta c\cdot(2\cdot\lambda - 1) + (\beta c + \beta m\cdot(1 - \mu))\cdot(\beta c + \beta m\cdot(\mu - 1))}{2\cdot\beta m\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))}$$

Evaluating xhat_bar and yhat_bar when $\lambda=\mu=0$
[in order to be able to use Assumption 2 to prove that xhat_bar and yhat_bar are between 0 and 1, see last sentence in Appendix A]

#34: $$\text{xhat\_bar} = \frac{bch + bmh + \beta c + \beta m\cdot(0 - 1)}{2\cdot\beta c}$$

#35: $$\text{xhat\_bar} = \frac{bch + bmh + \beta c - \beta m}{2\cdot\beta c}$$

#36: yhat_bar =

$$\frac{bch^2 + 2\cdot bch\cdot(bmh - \beta c) + bmh^2 + 2\cdot bmh\cdot\beta c\cdot(2\cdot0 - 1) + (\beta c + \beta m\cdot(1 - 0))\cdot(\beta c + \beta m\cdot(0 - 1))}{2\cdot\beta m\cdot(bch + bmh - \beta c + \beta m\cdot(0 - 1))}$$

#37: $$\text{yhat\_bar} = \frac{bch + bmh - \beta c + \beta m}{2\cdot\beta m}$$

So, Assumption 2 guarantees that xhat_bar and yhat_bar are between 0 and 1 when $\lambda$ and $\mu$ are small close to 0.

*** Derivations for section 5.2

Deriving total consumer benefit [equation (7) in the paper]

First, derive consumer benefits from cash payments only

#38: $wch = vh \cdot bch$

#39: $wch = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bch$

Second, derive consumer benefits from card payments only

#40: $wcd = nc \cdot \int_{xhat}^{1} (\beta c \cdot x + \phi)\, dx \cdot nm \cdot (1 - yhat) + nc \cdot \int_{\delta}^{xhat} (\beta c \cdot x + \phi)\, dx \cdot nm \cdot \mu$

Combining the two yields equation (7) in the paper

#41: $wc = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bch + nc \cdot \int_{xhat}^{1} (\beta c \cdot x + \phi)\, dx \cdot nm \cdot (1 - yhat) +$

$nc \cdot \int_{\delta}^{xhat} (\beta c \cdot x + \phi)\, dx \cdot nm \cdot \mu$

Deriving total merchant benefit [equation (8) in the paper]

First, derive merchant benefits from cash payments only

#42: $wmh = vh \cdot bmh$

#43: $wmh = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bmh$

Second, derive merchant benefit from card payments only

#44: $wmd = nc \cdot (1 - xhat) \cdot nm \cdot \int_{yhat}^{1} (\beta m \cdot y - \phi)\, dy + nc \cdot (xhat - \delta) \cdot nm \cdot \int_{1 - \mu}^{1} (\beta m \cdot y - \phi)\, dy$

combining the two yields equation (8) in paper

#45: $wm = (nc \cdot xhat \cdot nm \cdot (1 - \mu) + nc \cdot (1 - \lambda - xhat) \cdot nm \cdot yhat) \cdot bmh + nc \cdot (1 - xhat) \cdot nm \cdot \int_{yhat}^{1} (\beta m \cdot y - \phi)\, dy +$

$$nc\cdot(xhat - \delta)\cdot nm\cdot\int_{1-\mu}^{1} (\beta m\cdot y - \phi)\, dy$$

total welfare w = wc + wm [equation (9) in the paper]

#46: $$w = (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bch + nc\cdot\int_{xhat}^{1} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot(1 - yhat) + nc\cdot\int_{\delta}^{xhat} (\beta c\cdot x + \phi)\, dx\cdot nm\cdot\mu + (nc\cdot xhat\cdot nm\cdot(1 - \mu) + nc\cdot(1 - \lambda - xhat)\cdot nm\cdot yhat)\cdot bmh + nc\cdot(1 - xhat)\cdot nm\cdot\int_{yhat}^{1} (\beta m\cdot y - \phi)\, dy + nc\cdot(xhat - \delta)\cdot nm\cdot\int_{1-\mu}^{1} (\beta m\cdot y - \phi)\, dy$$

substituting for xhat and yhat, yields total welfare as a function of φ (as well as λ, μ, and δ)

#47: w = -

$$\frac{nc\cdot nm\cdot(bch^{3}\cdot(bmh + \beta m\cdot(\mu - 1) + \phi) + bch^{2}\cdot(bmh^{2} + bmh\cdot(3\cdot\beta c\cdot(\lambda - 1) + 2\cdot\beta m\cdot(\mu - 1) - \phi) + \beta c\cdot(\phi\sim}{\sim}$$

$$\frac{\cdot(2\cdot\lambda - 3) - \beta m\cdot(\mu - 1)) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(1 - \mu) - 2\cdot\phi^{2}) + bch\cdot(2\cdot bmh^{2}\cdot(\beta c\cdot(\lambda - 1) -\sim}{\sim}$$

$$\phi) + bmh\cdot(\beta c^2\cdot(2\cdot\lambda^2 - 4\cdot\lambda + 3) - 2\cdot\beta c\cdot(\beta m\cdot(\mu - 1) + \phi\cdot(\lambda - 1)) - \phi\cdot(4\cdot\beta m\cdot(\mu - 1) + \phi)) + \beta c^2\cdot(\beta\sim$$

$$m\cdot(\delta^2\cdot\mu - 1) - \phi\cdot(2\cdot\lambda - 3)) - \beta c\cdot(\beta m^2\cdot(\delta\cdot\mu\cdot(\mu - 2) + \mu^2 - 2\cdot\mu + 2) + 2\cdot\phi^2\cdot(\lambda - 2)) - \phi\cdot(2\cdot\beta m^2\cdot(\mu\sim$$

$$2\cdot\beta c\cdot\beta m\cdot(bch - \beta c - \phi)$$

$${}^2 - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(\mu - 1) - \phi^2)) + bmh^2\cdot(\beta c^2\cdot(\lambda^2 - 2\cdot\lambda + 1) + 2\cdot\beta c\cdot\phi\cdot(1 - \lambda) + \phi^2) + bmh\cdot(\beta c +\sim$$

$$\phi)\cdot(\beta c^2\cdot(\lambda - 1) - \beta c\cdot\lambda\cdot\phi + \phi\cdot(2\cdot\beta m\cdot(\mu - 1) + \phi)) - (\beta c + \phi)\cdot(\beta c^2\cdot(\beta m\cdot(\delta^2\cdot\mu - 1) + \phi) + \beta c\cdot(\phi^2 -\sim$$

$$\beta m^2\cdot(\delta\cdot\mu\cdot(\mu - 2) + 1)) + \beta m\cdot\phi\cdot(1 - \mu)\cdot(\beta m\cdot(\mu - 1) + \phi)))$$

*** Derivations for section 5.3

Derivation of Result 2(a)

The first-order condition (FOC) for the maximization of total welfare w [differentiating equation (9) in the paper with respect to ϕ]

#48: $$\frac{d}{d\phi}\left(w = -\right.$$

$$\frac{nc\cdot nm\cdot(bch^{3}\cdot(bmh + \beta m\cdot(\mu - 1) + \phi) + bch^{2}\cdot(bmh^{2} + bmh\cdot(3\cdot\beta c\cdot(\lambda - 1) + 2\cdot\beta m\cdot(\mu - 1) - \phi) + \beta c\cdot(\phi\sim}{\sim}$$

$$\frac{\cdot(2\cdot\lambda - 3) - \beta m\cdot(\mu - 1)) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(1 - \mu) - 2\cdot\phi^{2}) + bch\cdot(2\cdot bmh^{2}\cdot(\beta c\cdot(\lambda - 1) -\sim}{\sim}$$

$$\frac{\phi) + bmh\cdot(\beta c^{2}\cdot(2\cdot\lambda^{2} - 4\cdot\lambda + 3) - 2\cdot\beta c\cdot(\beta m\cdot(\mu - 1) + \phi\cdot(\lambda - 1)) - \phi\cdot(4\cdot\beta m\cdot(\mu - 1) + \phi)) + \beta c^{2}\cdot(\beta\sim}{\sim}$$

$$\frac{m\cdot(\delta^{2}\cdot\mu - 1) - \phi\cdot(2\cdot\lambda - 3)) - \beta c\cdot(\beta m^{2}\cdot(\delta\cdot\mu\cdot(\mu - 2) + \mu^{2} - 2\cdot\mu + 2) + 2\cdot\phi^{2}\cdot(\lambda - 2)) - \phi\cdot(2\cdot\beta m^{2}\cdot(\mu\sim}{2\cdot\beta c\cdot\beta m\cdot(bch - \beta c - \phi)\sim}$$

$$\frac{{}^{2} - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 2\cdot\lambda + 1) + 2\cdot\beta c\cdot\phi\cdot(1 - \lambda) + \phi^{2}) + bmh\cdot(\beta c +\sim}{\sim}$$

$$\frac{\phi)\cdot(\beta c^{2}\cdot(\lambda - 1) - \beta c\cdot\lambda\cdot\phi + \phi\cdot(2\cdot\beta m\cdot(\mu - 1) + \phi)) - (\beta c + \phi)\cdot(\beta c^{2}\cdot(\beta m\cdot(\delta^{2}\cdot\mu - 1) + \phi) + \beta c\cdot(\phi^{2} -\sim}{\sim}$$

$$\left.\frac{\beta m^{2}\cdot(\delta\cdot\mu\cdot(\mu - 2) + 1)) + \beta m\cdot\phi\cdot(1 - \mu)\cdot(\beta m\cdot(\mu - 1) + \phi)))}{}\right)$$

#49: $0 = -$

$$nc\cdot nm\cdot(bch^{4} + 2\cdot bch^{3}\cdot(\beta c\cdot(\lambda - 2) - 2\cdot\phi) - bch^{2}\cdot(bmh^{2} + bmh\cdot(2\cdot(\beta m\cdot(\mu - 1) + \phi) - \beta c\cdot\lambda) + 2\cdot\beta c^{2}\cdot\sim$$

$$(2\cdot\lambda - 3) + 4\cdot\beta c\cdot\phi\cdot(\lambda - 3) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) - 5\cdot\phi^{2}) + 2\cdot bch\cdot(bmh^{2}\cdot(\beta c + \phi)\ \sim$$

$$+ bmh\cdot(\beta c^{2}\cdot\lambda\cdot(\lambda - 1) + \beta c\cdot(2\cdot\beta m\cdot(\mu - 1) - \phi\cdot(\lambda - 2)) + 2\cdot\phi\cdot(\beta m\cdot(\mu - 1) + \phi)) + \beta c^{3}\cdot(\lambda - 2) + 2\cdot\beta\sim$$

$$2\cdot\beta c\cdot\beta m\cdot(bch - \beta c\sim$$

$$c^{2}\cdot\phi\cdot(\lambda - 3) + \beta c\cdot(\beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) + \phi^{2}\cdot(\lambda - 5)) + \phi\cdot(\beta m^{2}\cdot(\mu - 1)^{2} + 2\cdot\beta m\cdot\phi\cdot\sim$$

$$- \phi)^{2}$$

$$(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 1) - 2\cdot\beta c\cdot\phi - \phi^{2}) + bmh\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu -\ \sim$$

$$\frac{1) + \phi)) + (\beta c^2 + 2\cdot\beta c\cdot\phi + \phi^2)\cdot(\beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)))}{}$$

The second-order condition (SOC) for $\phi$ to maximize W is

$$\#50: \quad \frac{d}{d\phi}\frac{d}{d\phi}\left(w = -\right.$$

$$\frac{nc\cdot nm\cdot(bch^3\cdot(bmh + \beta m\cdot(\mu - 1) + \phi) + bch^2\cdot(bmh^2 + bmh\cdot(3\cdot\beta c\cdot(\lambda - 1) + 2\cdot\beta m\cdot(\mu - 1) - \phi) + \beta c\cdot(\phi\sim}{}\sim$$

$$\frac{\cdot(2\cdot\lambda - 3) - \beta m\cdot(\mu - 1)) + \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(1 - \mu) - 2\cdot\phi^2) + bch\cdot(2\cdot bmh^2\cdot(\beta c\cdot(\lambda - 1) -\sim}{}\sim$$

$$\frac{\phi) + bmh\cdot(\beta c^2\cdot(2\cdot\lambda^2 - 4\cdot\lambda + 3) - 2\cdot\beta c\cdot(\beta m\cdot(\mu - 1) + \phi\cdot(\lambda - 1)) - \phi\cdot(4\cdot\beta m\cdot(\mu - 1) + \phi)) + \beta c^2\cdot(\beta\sim}{}\sim$$

$$\frac{m\cdot(\delta^2\cdot\mu - 1) - \phi\cdot(2\cdot\lambda - 3)) - \beta c\cdot(\beta m^2\cdot(\delta\cdot\mu\cdot(\mu - 2) + \mu^2 - 2\cdot\mu + 2) + 2\cdot\phi^2\cdot(\lambda - 2)) - \phi\cdot(2\cdot\beta m^2\cdot(\mu\sim}{2\cdot\beta c\cdot\beta m\cdot(bch - \beta c - \phi)}\sim$$

$$\frac{{}^2 - 2\cdot\mu + 1) + \beta m\cdot\phi\cdot(\mu - 1) - \phi^2)) + bmh^2\cdot(\beta c^2\cdot(\lambda^2 - 2\cdot\lambda + 1) + 2\cdot\beta c\cdot\phi\cdot(1 - \lambda) + \phi^2) + bmh\cdot(\beta c +\sim}{}\sim$$

$$\phi)\cdot(\beta c^{2}\cdot(\lambda - 1) - \beta c\cdot\lambda\cdot\phi + \phi\cdot(2\cdot\beta m\cdot(\mu - 1) + \phi)) - (\beta c + \phi)\cdot(\beta c^{2}\cdot(\beta m\cdot(\delta^{2}\cdot\mu - 1) + \phi) + \beta c\cdot(\phi^{2} - \sim$$

$$\left.\beta m^{2}\cdot(\delta\cdot\mu\cdot(\mu - 2) + 1)) + \beta m\cdot\phi\cdot(1 - \mu)\cdot(\beta m\cdot(\mu - 1) + \phi)))\right)$$

#51:

$$nc\cdot nm\cdot(bch^{4} + bch^{3}\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^{2}\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c + \phi\sim$$

$$) - bch\cdot(bmh\cdot(\beta c^{2}\cdot(2\cdot\lambda^{2} - 3) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^{2}) + 4\cdot\beta c^{3} + 3\cdot\beta c^{2}\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi - \beta\sim$$

$$\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^{3}$$

$$m\cdot(\mu - 1)) - \phi^{2}\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)) - bmh^{2}\cdot\beta c^{2}\cdot\lambda^{2} - bmh\cdot(\beta c^{3} + 3\cdot\beta c^{2}\cdot\phi + 3\cdot\beta c\cdot\phi^{2} + \phi^{3}) + (\beta c + \beta\sim$$

$$m\cdot(1 - \mu))\cdot(\beta c^{3} + 3\cdot\beta c^{2}\cdot\phi + 3\cdot\beta c\cdot\phi^{2} + \phi^{3}))$$

Evaluating the SOC at λ=μ=0 yields

#52:

$$\frac{nc\cdot nm\cdot(bch + bmh - \beta c - \beta m)}{\beta c\cdot\beta m}$$

which is < 0 by Assumption 2.

** To prove Result 2(a), evaluate the first-order condition at λ=0 (note that δ is not an argument in the FOC).

#53: $$0 = -\frac{nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) - bmh^2 - 2\cdot bmh\cdot(\beta m\cdot(\mu - 1) + \phi) + \beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu))}{2\cdot\beta c\cdot\beta m}$$

solve for ϕ_star given λ=0

#54: $$\mathrm{SOLVE}(nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) - bmh^2 - 2\cdot bmh\cdot(\beta m\cdot(\mu - 1) + \phi) + \beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)), \phi)$$

#55: $$\phi\_star = \frac{bch - bmh - \beta c + \beta m\cdot(1 - \mu)}{2}$$

and this completes the proof of Result 2(a) because ϕ_star = ϕ_bar [equation (6) in the paper]

** Derivation of Result 2(b)

First, note that λ is not an argument in ϕ_bar given in equation (6) in the paper. Hence, it is sufficient to show that ϕ_star declines with λ evaluated at λ=0 (emergence of cashless consumers).

Second, using the implicit function theorem (IFT), the above first-order condition (lines #48 and #49 above) defines the function ϕ_star(λ). Let 0=F(ϕ_star, λ, μ) be the first order condition, see lines #48 and #49 above. The IFT states that ∂ϕ_star/∂λ is the ratio of (−1)∂F/∂λ divided by ∂F/∂ϕ_star.

The numerator of this ratio is:

#56: $$\frac{\partial(F)}{\partial(\lambda)} = \frac{d}{d\lambda}\left(- \right.$$

$$nc\cdot nm\cdot(bch^{4} + 2\cdot bch^{3}\cdot(\beta c\cdot(\lambda - 2) - 2\cdot\phi) - bch^{2}\cdot(bmh^{2} + bmh\cdot(2\cdot(\beta m\cdot(\mu - 1) + \phi) - \beta c\cdot\lambda) + 2\cdot\beta c^{2}\cdot\sim$$

$$(2\cdot\lambda - 3) + 4\cdot\beta c\cdot\phi\cdot(\lambda - 3) + \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) - 5\cdot\phi^{2}) + 2\cdot bch\cdot(bmh^{2}\cdot(\beta c + \phi)\sim$$

$$+ bmh\cdot(\beta c^{2}\cdot\lambda\cdot(\lambda - 1) + \beta c\cdot(2\cdot\beta m\cdot(\mu - 1) - \phi\cdot(\lambda - 2)) + 2\cdot\phi\cdot(\beta m\cdot(\mu - 1) + \phi)) + \beta c^{3}\cdot(\lambda - 2) + 2\cdot\beta\sim$$

$$2\cdot\beta c\cdot\beta m\cdot(bch - \beta c\sim$$

$$c^{2}\cdot\phi\cdot(\lambda - 3) + \beta c\cdot(\beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) + \phi^{2}\cdot(\lambda - 5)) + \phi\cdot(\beta m^{2}\cdot(\mu - 1)^{2} + 2\cdot\beta m\cdot\phi\cdot\sim$$

$$- \phi)^{2}\sim$$

$$(\mu - 1) - \phi^2)) + bmh^2\cdot(\beta c^2\cdot(\lambda^2 - 1) - 2\cdot\beta c\cdot\phi - \phi^2) + bmh\cdot(\beta c^2 + 2\cdot\beta c\cdot\phi + \phi^2)\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu - \sim$$

$$1) + \phi)) + (\beta c^2 + 2\cdot\beta c\cdot\phi + \phi^2)\cdot(\beta c^2 + 2\cdot\beta c\cdot\phi - \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)))\Bigg)$$

#57: $$\frac{\partial(F)}{\partial(\lambda)} = - \frac{nc\cdot nm\cdot(bch^2 - 2\cdot bch\cdot(\beta c + \phi) + 2\cdot bmh\cdot\beta c\cdot\lambda + (\beta c + \phi)^2)\cdot(2\cdot bch + bmh)}{2\cdot\beta m\cdot(bch - \beta c - \phi)^2}$$

The denominator is:

#58: $$\frac{\partial(F)}{\partial(\phi\_star)} = \frac{d}{d\phi}\Bigg(-$$

$$nc\cdot nm\cdot(bch^4 + 2\cdot bch^3\cdot(\beta c\cdot(\lambda - 2) - 2\cdot\phi) - bch^2\cdot(bmh^2 + bmh\cdot(2\cdot(\beta m\cdot(\mu - 1) + \phi) - \beta c\cdot\lambda) + 2\cdot\beta c^2\cdot\sim$$

$$(2\cdot\lambda - 3) + 4\cdot\beta c\cdot\phi\cdot(\lambda - 3) + \beta m^2\cdot(\mu^2 - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) - 5\cdot\phi^2) + 2\cdot bch\cdot(bmh^2\cdot(\beta c + \phi) \sim$$

$$\frac{+\ bmh\cdot(\beta c^{2}\cdot\lambda\cdot(\lambda - 1) + \beta c\cdot(2\cdot\beta m\cdot(\mu - 1) - \phi\cdot(\lambda - 2)) + 2\cdot\phi\cdot(\beta m\cdot(\mu - 1) + \phi)) + \beta c^{3}\cdot(\lambda - 2) + 2\cdot\beta\sim}{2\cdot\beta c\cdot\beta m\cdot(bch - \beta c\sim}$$

$$\frac{c^{2}\cdot\phi\cdot(\lambda - 3) + \beta c\cdot(\beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(\mu - 1) + \phi^{2}\cdot(\lambda - 5)) + \phi\cdot(\beta m^{2}\cdot(\mu - 1)^{2} + 2\cdot\beta m\cdot\phi\cdot\sim}{-\ \phi)^{2}}$$

$$\frac{(\mu - 1) - \phi^{2})) + bmh^{2}\cdot(\beta c^{2}\cdot(\lambda^{2} - 1) - 2\cdot\beta c\cdot\phi - \phi^{2}) + bmh\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c\cdot\lambda - 2\cdot(\beta m\cdot(\mu - \sim}{}$$

$$\left.\frac{1) + \phi)) + (\beta c^{2} + 2\cdot\beta c\cdot\phi + \phi^{2})\cdot(\beta c^{2} + 2\cdot\beta c\cdot\phi - \beta m^{2}\cdot(\mu^{2} - 2\cdot\mu + 1) + 2\cdot\beta m\cdot\phi\cdot(1 - \mu)))}{}\right)$$

#59: $$\frac{\partial(F)}{\partial(\phi\_star)} =$$

$$\frac{nc\cdot nm\cdot(bch^{4} + bch^{3}\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^{2}\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c +\sim}{}$$

$$\frac{\phi) - bch\cdot(bmh\cdot(\beta c^{2}\cdot(2\cdot\lambda^{2} - 3) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^{2}) + 4\cdot\beta c^{3} + 3\cdot\beta c^{2}\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi - \sim}{\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^{3} \sim}$$

$$\frac{\beta m\cdot(\mu - 1)) - \phi^{2}\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)) - bmh^{2}\cdot\beta c^{2}\cdot\lambda^{2} - bmh\cdot(\beta c^{3} + 3\cdot\beta c^{2}\cdot\phi + 3\cdot\beta c\cdot\phi^{2} + \phi^{3}) + (\beta c + \sim}{\sim}$$

$$\frac{\beta m\cdot(1 - \mu))\cdot(\beta c^{3} + 3\cdot\beta c^{2}\cdot\phi + 3\cdot\beta c\cdot\phi^{2} + \phi^{3}))}{}$$

Therefore, by the implicit function theorem, ∂ϕ_star/∂λ is the ratio of (−1)∂F/∂λ divided by ∂F/∂ϕ_star, hence

#60: $$\frac{\partial(\phi\_star)}{\partial(\lambda)} = -$$

$$nc\cdot nm\cdot(bch^{4} + bch^{3}\cdot(bmh - 4\cdot\beta c + \beta m\cdot(\mu - 1) - 3\cdot\phi) - 3\cdot bch^{2}\cdot(bmh - 2\cdot\beta c + \beta m\cdot(\mu - 1) - \phi)\cdot(\beta c \sim$$

$$- \frac{nc\cdot nm\cdot(bch^{2} - 2\cdot bch\cdot(\beta c + \phi) + 2\cdot bmh\cdot\beta c\cdot\lambda + (\beta c + \phi)^{2})\cdot(2\sim}{2\cdot\beta m\cdot(bch - \beta c - \phi)^{2}}$$

$$\frac{+ \phi) - bch\cdot(bmh\cdot(\beta c^{2}\cdot(2\cdot\lambda^{2} - 3) - 6\cdot\beta c\cdot\phi - 3\cdot\phi^{2}) + 4\cdot\beta c^{3} + 3\cdot\beta c^{2}\cdot(3\cdot\phi - \beta m\cdot(\mu - 1)) + 6\cdot\beta c\cdot\phi\cdot(\phi \sim}{\beta c\cdot\beta m\cdot(bch - \beta c - \phi)^{3}}$$

$$\cdot bch + bmh)$$

$$- \beta m\cdot(\mu - 1)) - \phi^{2}\cdot(3\cdot\beta m\cdot(\mu - 1) - \phi)) - bmh^{2}\cdot\beta c^{2}\cdot\lambda^{2} - bmh\cdot(\beta c^{3} + 3\cdot\beta c^{2}\cdot\phi + 3\cdot\beta c\cdot\phi^{2} + \phi^{3}) + (\beta c \sim$$

$$\overline{\qquad + \beta m\cdot(1 - \mu))\cdot(\beta c^3 + 3\cdot\beta c^2\cdot\phi + 3\cdot\beta c\cdot\phi^2 + \phi^3))}$$

Evaluate at λ=0 yields

#61: $$\frac{\partial(\phi\_star)}{\partial(\lambda)} = \frac{\beta c\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))}$$

< 0 for sufficiently small μ [see Assumption 2 in the paper]. This completes the proof of Result 2(b) because it shows that as λ increases from 0, ϕ_star declines and by Result 2(a) it must be that ϕ_star < ϕ_bar.

** Derivation of Result 2(c)
Fifferentiating #61 with respect to μ

#62: $$\frac{\partial(\phi\_star^2)}{\partial(\mu)\cdot\partial(\lambda)} = \frac{d}{d\mu}\ \frac{\beta c\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))}$$

#63: $$-\ \frac{\beta c\cdot\beta m\cdot(2\cdot bch + bmh)}{2\cdot(bch + bmh - \beta c + \beta m\cdot(\mu - 1))^2} < 0$$

** Derivation of Result 2(d)
Simple, note that δ is not an argument in the first-order condition #48, #49 above and also is not an argument in ϕ_bar in equation (6) in the paper.