#1: CaseMode ≔ Sensitive

#2: InputMode ≔ Word

nondegree wage (section 6 only)

#3: λ ∈ Real (0, ∞)

tuition by college B and college A

#4: ta ∈ Real (0, ∞)

#5: tb ∈ Real (0, ∞)

Time discount factor

#6: δ ∈ Real (0, 1)

interest rate

#7: r ∈ Real (0, ∞)

Number of low-income students

#8: nl ∈ Real (0, ∞)

number of high-income students

#9: nh ∈ Real (0, ∞)

earning parameters college 1 and 2

#10: μ ∈ Real (0, ∞)

consumer ability index a

#11: a ∈ Real [0, 1]

probablity of realizing the earing

#12: ρa ∈ Real (0, 1)

#13: ρb ∈ Real (0, 1)

Eq. (3): Type L rely on loands

#14: δ·ρa·μ·a − δ·ta·(1 + r)

#15: δ·ρb·μ·a − δ·tb·(1 + r)

Eq. (4): Type H: do not need to rely on loans

#16: δ·ρa·μ·a − ta

#17: $\delta \cdot \rho b \cdot \mu \cdot a - tb$

eq (5) left

#18: $\delta \cdot \rho b \cdot \mu \cdot abarl - \delta \cdot tb \cdot (1 + r) = 0$

#19: SOLVE($\delta \cdot \rho b \cdot \mu \cdot abarl - \delta \cdot tb \cdot (1 + r) = 0$, abarl)

#20: $$abarl = \frac{tb \cdot (r + 1)}{\mu \cdot \rho b}$$

Eq. (5) right

#21: $\delta \cdot \rho b \cdot \mu \cdot abarh - tb$

#22: SOLVE($\delta \cdot \rho b \cdot \mu \cdot abarh - tb$, abarh)

#23: $$abarh = \frac{tb}{\delta \cdot \mu \cdot \rho b}$$

Eq. (6) left

#24: $\delta \cdot \rho a \cdot \mu \cdot ahatl - \delta \cdot ta \cdot (1 + r) = \delta \cdot \rho b \cdot \mu \cdot ahatl - \delta \cdot tb \cdot (1 + r)$

#25: SOLVE($\delta \cdot \rho a \cdot \mu \cdot ahatl - \delta \cdot ta \cdot (1 + r) = \delta \cdot \rho b \cdot \mu \cdot ahatl - \delta \cdot tb \cdot (1 + r)$, ahatl)

#26: $$ahatl = \frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)}$$

Eq. (6) right

#27: $\delta \cdot \rho a \cdot \mu \cdot ahath - ta = \delta \cdot \rho b \cdot \mu \cdot ahath - tb$

#28: SOLVE($\delta \cdot \rho a \cdot \mu \cdot ahath - ta = \delta \cdot \rho b \cdot \mu \cdot ahath - tb$, ahath)

#29: $$ahath = \frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)}$$

Eq. (9) Appendix B.1

#30: $Ta = nl \cdot ta \cdot (1 - ahatl) + nh \cdot ta \cdot (1 - ahath)$

#31: $Tb = nl \cdot tb \cdot (ahatl - abarl) + nh \cdot ta \cdot (ahath - abarh)$

#32: $$Ta = nl \cdot ta \cdot \left(1 - \frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)}\right) + nh \cdot ta \cdot \left(1 - \frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)}\right)$$

#33: $$Tb = nl\cdot tb\cdot\left(\frac{(r+1)\cdot(ta-tb)}{\mu\cdot(\rho a-\rho b)}-\frac{tb\cdot(r+1)}{\mu\cdot\rho b}\right)+ nh\cdot tb\cdot\left(\frac{ta-tb}{\delta\cdot\mu\cdot(\rho a-\rho b)}-\frac{tb}{\delta\cdot\mu\cdot\rho b}\right)$$

#34: $$\frac{d}{d\ ta}\left(Ta = nl\cdot ta\cdot\left(1-\frac{(r+1)\cdot(ta-tb)}{\mu\cdot(\rho a-\rho b)}\right)+nh\cdot ta\cdot\left(1-\frac{ta-tb}{\delta\cdot\mu\cdot(\rho a-\rho b)}\right)\right)$$

#35: $$0 = \frac{nh\cdot(2\cdot ta-tb+\delta\cdot\mu\cdot(\rho b-\rho a))+nl\cdot\delta\cdot(r\cdot(2\cdot ta-tb)+2\cdot ta-tb+\sim}{\delta\cdot\mu\cdot(\rho b-\rho a)}\sim \frac{\mu\cdot(\rho b-\rho a))}{}$$

#36: $$0 > \frac{2\cdot(nh+nl\cdot\delta\cdot(r+1))}{\delta\cdot\mu\cdot(\rho b-\rho a)}$$

#37: $$\frac{d}{d\ tb}\left(Tb = nl\cdot tb\cdot\left(\frac{(r+1)\cdot(ta-tb)}{\mu\cdot(\rho a-\rho b)}-\frac{tb\cdot(r+1)}{\mu\cdot\rho b}\right)+nh\cdot tb\cdot\left(\frac{ta-tb}{\delta\cdot\mu\cdot(\rho a-\rho b)}-\frac{tb}{\delta\cdot\mu\cdot\rho b}\right)\right)$$

#38: $$0 = \frac{(nh+nl\cdot\delta\cdot(r+1))\cdot(ta\cdot\rho b-2\cdot tb\cdot\rho a)}{\delta\cdot\mu\cdot\rho b\cdot(\rho a-\rho b)}$$

#39: $$\frac{d}{d\ ta}\frac{d}{d\ ta}\left(Ta = nl\cdot ta\cdot\left(1-\frac{(r+1)\cdot(ta-tb)}{\mu\cdot(\rho a-\rho b)}\right)+nh\cdot ta\cdot\left(1-\frac{ta-tb}{\delta\cdot\mu\cdot(\rho a-\rho b)}\right)\right)$$

#40: $$0 > \frac{2\cdot(nh + nl\cdot\delta\cdot(r + 1))}{\delta\cdot\mu\cdot(\rho b - \rho a)}$$

#41: $$\frac{d}{d\ tb}\frac{d}{d\ tb}\left(Tb = nl\cdot tb\cdot\left(\frac{(r + 1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)} - \frac{tb\cdot(r + 1)}{\mu\cdot\rho b}\right) + nh\cdot tb\cdot\left(\frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)} - \frac{tb}{\delta\cdot\mu\cdot\rho b}\right)\right)$$

#42: $$0 > \frac{2\cdot\rho a\cdot(nh + nl\cdot\delta\cdot(r + 1))}{\delta\cdot\mu\cdot\rho b\cdot(\rho b - \rho a)}$$

#43: $$\text{SOLVE}\left(\left[0 = \frac{nh\cdot(2\cdot ta - tb + \delta\cdot\mu\cdot(\rho b - \rho a)) + nl\cdot\delta\cdot(r\cdot(2\cdot ta - tb) + 2\cdot ta - tb + \sim}{\delta\cdot\mu\cdot(\rho b - \rho a)}\sim \sim \frac{\mu\cdot(\rho b - \rho a))}{}, 0 = \frac{(nh + nl\cdot\delta\cdot(r + 1))\cdot(ta\cdot\rho b - 2\cdot tb\cdot\rho a)}{\delta\cdot\mu\cdot\rho b\cdot(\rho a - \rho b)}\right], [ta, tb]\right)$$

eq. (9)

#44: $$\left[ta = \frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)} \wedge tb = \frac{\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right]$$

eq (10)

#45: $$ta - tb = \frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)} - \frac{\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}$$

$$\#46: \quad ta - tb = \frac{\delta \cdot \mu \cdot (nh + nl) \cdot (\rho a - \rho b) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

Eq. (11)

$$\#47: \quad abarl = \frac{\delta \cdot (nh + nl) \cdot (r + 1) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#48: \quad abarh = \frac{(nh + nl) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

Eq. (12)

$$\#49: \quad ahatl = \frac{\delta \cdot (nh + nl) \cdot (r + 1) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#50: \quad ahath = \frac{(nh + nl) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

Appendix B.3

Eq. (B.7): ahatl – abarl =

$$\#51: \quad \frac{\delta \cdot (nh + nl) \cdot (r + 1) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)} - \frac{\delta \cdot (nh + nl) \cdot (r + 1) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#52: \quad \frac{\delta \cdot \rho a \cdot (nh + nl) \cdot (r + 1)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)} > 0$$

Eq. (B.8): ahath – abarh =

$$\#53: \quad \frac{(nh + nl) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)} - \frac{(nh + nl) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#54: \quad \frac{\rho a \cdot (nh + nl)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)} > 0$$

Eq (13)

$$\#55: \quad Ta = \frac{4 \cdot \delta \cdot \mu \cdot \rho a^2 \cdot (nh + nl)^2 \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)^2}$$

$$\#56: \quad Tb = \frac{\delta \cdot \mu \cdot \rho a \cdot \rho b \cdot (nh + nl)^2 \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)^2}$$

Eq. (14)

$$\#57: \quad Ta - Tb = \frac{4 \cdot \delta \cdot \mu \cdot \rho a^2 \cdot (nh + nl)^2 \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)^2} - \frac{\delta \cdot \mu \cdot \rho a \cdot \rho b \cdot (nh + nl)^2 \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)^2}$$

$$\#58: \quad Ta - Tb = \frac{\delta \cdot \mu \cdot \rho a \cdot (nh + nl)^2 \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

Eq. (15)

$$\#59: \quad n0l = nl \cdot abarl$$

$$\#60: \quad n0l = \frac{nl \cdot \delta \cdot (nh + nl) \cdot (r + 1) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#61: \quad n0h = nh \cdot abarh$$

$$\#62: \quad n0h = \frac{nh \cdot (nh + nl) \cdot (\rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

$$\#63: \quad nal = nl \cdot (1 - ahatl)$$

$$\#64: \quad nal = nl \cdot \left(1 - \frac{\delta \cdot (nh + nl) \cdot (r + 1) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}\right)$$

$$\#65: \quad nah = nh \cdot (1 - ahath)$$

$$\#66: \quad nah = nh \cdot \left(1 - \frac{(nh + nl) \cdot (2 \cdot \rho a - \rho b)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}\right)$$

$$\#67: \quad nbl = nl \cdot (ahatl - abarl)$$

#68: $$nbl = \frac{nl \cdot \delta \cdot \rho a \cdot (nh + nl) \cdot (r + 1)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

#69: $nbh = nh \cdot (ahath - abarh)$

#70: $$nbh = \frac{nh \cdot \rho a \cdot (nh + nl)}{(nh + nl \cdot \delta \cdot (r + 1)) \cdot (4 \cdot \rho a - \rho b)}$$

Eq. (16)

#71: $n0 = n0l + n0h$

#72: $$n0 = \frac{(nh + nl) \cdot (\rho a - \rho b)}{4 \cdot \rho a - \rho b}$$

#73: $na = nal + nah$

#74: $$na = \frac{2 \cdot \rho a \cdot (nh + nl)}{4 \cdot \rho a - \rho b}$$

#75: $nb = nbl + nbh$

#76: $$nb = \frac{\rho a \cdot (nh + nl)}{4 \cdot \rho a - \rho b}$$

verify the above sum up to total population:

#77: $$\frac{(nh + nl) \cdot (\rho a - \rho b)}{4 \cdot \rho a - \rho b} + \frac{2 \cdot \rho a \cdot (nh + nl)}{4 \cdot \rho a - \rho b} + \frac{\rho a \cdot (nh + nl)}{4 \cdot \rho a - \rho b}$$

#78: $$nh + nl$$

*** Section 5: Effects of Student Loans on..

Definition 3 (high or low intererest rate): abarh - abarl =

#79: $$\frac{tb}{\delta \cdot \mu \cdot \rho b} - \frac{tb \cdot (r + 1)}{\mu \cdot \rho b}$$

> 0 if

#80: $$\frac{tb}{\delta \cdot \mu \cdot \rho b} - \frac{tb \cdot (r + 1)}{\mu \cdot \rho b} > 0$$

#81: $$\text{SOLVE}\left(\frac{tb}{\delta\cdot\mu\cdot\rho b} - \frac{tb\cdot(r+1)}{\mu\cdot\rho b} > 0,\ r\right)$$

#82: $$r < \frac{1-\delta}{\delta}$$

#83: $$\text{SOLVE}\left(r < \frac{1-\delta}{\delta},\ \delta\right)$$

#84: $$0 < \delta < \frac{1}{r+1} \vee \frac{1}{r+1} < \delta < 0$$

Result 2a (same as Definition 1 above) [removed]

Result 2b: ahath – ahatl =

#85: $$\frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)} - \frac{(r+1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)}$$

> 0 if

#86: $$\frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)} - \frac{(r+1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)} > 0$$

#87: $$\text{SOLVE}\left(\frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)} - \frac{(r+1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)} > 0,\ r\right)$$

#88: $$\text{IF}\left(\frac{ta - tb}{\rho a - \rho b} < 0,\ r > \frac{1-\delta}{\delta}\right) \vee \text{IF}\left(\frac{ta - tb}{\rho a - \rho b} > 0,\ r < \frac{1-\delta}{\delta}\right)$$

Result 2:

#89: $$\frac{d}{dr}\left(n0l = \frac{nl\cdot\delta\cdot(nh + nl)\cdot(r+1)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r+1))\cdot(4\cdot\rho a - \rho b)}\right)$$

#90: $$0 < \frac{nh\cdot nl\cdot\delta\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r+1))^{2}\cdot(4\cdot\rho a - \rho b)}$$

#91: $$\frac{d}{dr}\left(n0h = \frac{nh\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r+1))\cdot(4\cdot\rho a - \rho b)}\right)$$

$$\#92:\quad 0 > \frac{nh\cdot nl\cdot\delta\cdot(nh + nl)\cdot(\rho b - \rho a)}{(nh + nl\cdot\delta\cdot(r + 1))^{2}\cdot(4\cdot\rho a - \rho b)}$$

$$\#93:\quad \frac{d}{dr}\left(nbl = \frac{nl\cdot\delta\cdot\rho a\cdot(nh + nl)\cdot(r + 1)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right)$$

$$\#94:\quad 0 < \frac{nh\cdot nl\cdot\delta\cdot\rho a\cdot(nh + nl)}{(nh + nl\cdot\delta\cdot(r + 1))^{2}\cdot(4\cdot\rho a - \rho b)}$$

$$\#95:\quad \frac{d}{dr}\left(nbh = \frac{nh\cdot\rho a\cdot(nh + nl)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right)$$

$$\#96:\quad 0 > \frac{nh\cdot nl\cdot\delta\cdot\rho a\cdot(nh + nl)}{(nh + nl\cdot\delta\cdot(r + 1))^{2}\cdot(\rho b - 4\cdot\rho a)}$$

$$\#97:\quad \frac{d}{dr}\left(nal = nl\cdot\left(1 - \frac{\delta\cdot(nh + nl)\cdot(r + 1)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right)\right)$$

$$\#98:\quad 0 > \frac{nh\cdot nl\cdot\delta\cdot(nh + nl)\cdot(\rho b - 2\cdot\rho a)}{(nh + nl\cdot\delta\cdot(r + 1))^{2}\cdot(4\cdot\rho a - \rho b)}$$

$$\#99:\quad \frac{d}{dr}\left(nah = nh\cdot\left(1 - \frac{(nh + nl)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right)\right)$$

$$\#100:\quad 0 < \frac{nh\cdot nl\cdot\delta\cdot(nh + nl)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))^{2}\cdot(4\cdot\rho a - \rho b)}$$

*** Section 6: Measuring the burden of tuition

Equations (17) and (18) leading to Result 3

#101: $ta\cdot(1 + r)$

$$\#102:\quad \frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r)$$

$$\#103:\quad \frac{d}{dr}\left(\frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r)\right)$$

$$\#104: \quad \frac{2\cdot nh\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))^2\cdot(4\cdot\rho a - \rho b)} > 0$$

$$\#105: \quad tb\cdot(1 + r)$$

$$\#106: \quad \frac{\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r)$$

$$\#107: \quad \frac{d}{dr}\left(\frac{\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r)\right)$$

$$\#108: \quad \frac{nh\cdot\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))^2\cdot(4\cdot\rho a - \rho b)} > 0$$

eq (19) ATB (avg tuition burden)

total burden on low income (delayed until graduation)

$$\#109: \quad \delta\cdot(nal\cdot ta\cdot(1 + r) + nbl\cdot tb\cdot(1 + r))$$

total burden on high income

$$\#110: \quad nah\cdot ta + nbh\cdot tb$$

$$\#111: \quad ATB = \frac{\delta\cdot(nal\cdot ta\cdot(1 + r) + nbl\cdot tb\cdot(1 + r)) + nah\cdot ta + nbh\cdot tb}{nal + nbl + nah + nbh}$$

Appendix B.3 Result 4

#112: ATB =

$$\delta\cdot\left(\left(nl\cdot\left(1 - \frac{\delta\cdot(nh + nl)\cdot(r + 1)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right)\right)\cdot\frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + \sim}{(nh + nl\cdot\delta\cdot(r + 1\sim}\right)\sim$$

$$\frac{nl)\cdot(\rho a - \rho b)}{))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r) + \frac{nl\cdot\delta\cdot\rho a\cdot(nh + nl)\cdot(r + 1)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot\frac{\delta\cdot\mu}{(nh + }$$

$$nl\cdot\left(1 - \frac{\delta\cdot(nh + nl)\cdot(r + 1)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\right) + \frac{nl\cdot\delta}{(nh + n}$$

$$\frac{\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}\cdot(1 + r)\right) + \left(nh\cdot\left(1 - \frac{(nh + nl)\cdot(2\cdot\rho}{(nh + nl\cdot\delta\cdot(r + 1))}\right.\right.$$

$$\frac{\cdot\rho a\cdot(nh + nl)\cdot(r + 1)}{l\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)} + nh\cdot\left(1 - \frac{(nh + nl)\cdot(2\cdot\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b}\right.$$

$$\left.\left.\frac{a - \rho b)}{\cdot(4\cdot\rho a - \rho b)}\right)\right)\cdot\frac{2\cdot\delta\cdot\mu\cdot\rho a\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)} + \frac{nh\cdot\rho a\cdot}{(nh + nl\cdot\delta\cdot(r }$$

$$\left.\frac{}{)}\right) + \frac{nh\cdot\rho a\cdot(nh + nl)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}$$

$$\frac{(nh + nl)}{+ 1))\cdot(4\cdot\rho a - \rho b)}\cdot\frac{\delta\cdot\mu\cdot\rho b\cdot(nh + nl)\cdot(\rho a - \rho b)}{(nh + nl\cdot\delta\cdot(r + 1))\cdot(4\cdot\rho a - \rho b)}$$

#113: ATB =

$$\frac{\delta\cdot\mu\cdot(\rho a - \rho b)\cdot(nh^2\cdot(4\cdot\rho a + \rho b) - nh\cdot nl\cdot(r^2\cdot\delta^2\cdot(4\cdot\rho a - 3\cdot\rho b) + 2\cdot r\cdot}{}$$

$$\frac{\delta\cdot(\delta\cdot(4\cdot\rho a - 3\cdot\rho b) - 2\cdot(4\cdot\rho a - \rho b)) + \delta^2\cdot(4\cdot\rho a - 3\cdot\rho b) + 4\cdot\delta\cdot(\rho b - }{3\cdot(nh + nl\cdot\delta\cdot(r + 1))^2\cdot(4\cdot\rho a - \rho b)}$$

$$4\cdot\rho a) + 4\cdot\rho a - 3\cdot\rho b) + nl^2\cdot\delta^2\cdot(r+1)^2\cdot(4\cdot\rho a + \rho b))$$

$$\#114:\ \frac{d}{dr}\left[ATB = \right.$$

$$\frac{\delta\cdot\mu\cdot(\rho a-\rho b)\cdot(nh^2\cdot(4\cdot\rho a+\rho b) - nh\cdot nl\cdot(r^2\cdot\delta^2\cdot(4\cdot\rho a-3\cdot\rho b) + 2\cdot r\cdot\sim}{\sim}$$

$$\frac{\delta\cdot(\delta\cdot(4\cdot\rho a-3\cdot\rho b) - 2\cdot(4\cdot\rho a-\rho b)) + \delta^2\cdot(4\cdot\rho a-3\cdot\rho b) + 4\cdot\delta\cdot(\rho b-\sim}{3\cdot(nh+nl\cdot\delta\cdot(r+1))^2\cdot(4\cdot\rho a-\rho b)}$$

$$\left.\frac{4\cdot\rho a) + 4\cdot\rho a - 3\cdot\rho b) + nl^2\cdot\delta^2\cdot(r+1)^2\cdot(4\cdot\rho a+\rho b))}{}\right]$$

eq B.10

$$\#115:\quad 0 = \frac{2\cdot nh\cdot nl\cdot\delta^2\cdot\mu\cdot(nh+nl)\cdot(\rho b-\rho a)\cdot(4\cdot\rho a-3\cdot\rho b)\cdot(r\cdot\delta+\delta-1)}{3\cdot(nh+nl\cdot\delta\cdot(r+1))^3\cdot(4\cdot\rho a-\rho b)}$$

$$\#116:\ \mathrm{SOLVE}(2\cdot nh\cdot nl\cdot\delta^2\cdot\mu\cdot(nh+nl)\cdot(\rho b-\rho a)\cdot(4\cdot\rho a-3\cdot\rho b)\cdot(r\cdot\delta+\delta-1),\ r)$$

$$\#117:\quad r = \frac{1-\delta}{\delta} \vee 4\cdot\rho a - 3\cdot\rho b = 0$$

$$\#118:\ \frac{d}{dr}\frac{d}{dr}\left[ATB = \right.$$

$$\frac{\delta\cdot\mu\cdot(\rho a-\rho b)\cdot\left(nh^{2}\cdot(4\cdot\rho a+\rho b)-nh\cdot nl\cdot\left(r^{2}\cdot\delta^{2}\cdot(4\cdot\rho a-3\cdot\rho b)+2\cdot r\cdot\left(\frac{\delta\cdot(\delta\cdot(4\cdot\rho a-3\cdot\rho b)-2\cdot(4\cdot\rho a-\rho b))+\delta^{2}\cdot(4\cdot\rho a-3\cdot\rho b)+4\cdot\delta\cdot(\rho b-4\cdot\rho a)+4\cdot\rho a-3\cdot\rho b)+nl^{2}\cdot\delta^{2}\cdot(r+1)^{2}\cdot(4\cdot\rho a+\rho b)}{3\cdot(nh+nl\cdot\delta\cdot(r+1))^{2}\cdot(4\cdot\rho a-\rho b)}\right)\right)}{3\cdot(nh+nl\cdot\delta\cdot(r+1))^{3}\cdot(4\cdot\rho a-\rho b)}$$

#119: $$\frac{2\cdot nh\cdot nl\cdot\delta^{3}\cdot\mu\cdot(nh+nl)\cdot(nh-nl\cdot(2\cdot r\cdot\delta+2\cdot\delta-3))\cdot(\rho b-\rho a)\cdot(4\cdot\rho a-3\cdot\rho b)}{3\cdot(nh+nl\cdot\delta\cdot(r+1))^{4}\cdot(4\cdot\rho a-\rho b)}$$

SOC evaluated at rhat: Eq B.11

$$\frac{2\cdot nh\cdot nl\cdot\delta^{3}\cdot\mu\cdot(nh+nl)\cdot\left(nh-nl\cdot\left(2\cdot\frac{1-\delta}{\delta}\cdot\delta+2\cdot\delta-3\right)\right)\cdot(\rho b-\rho a)\cdot(4\cdot\rho a-3\cdot\rho b)}{3\cdot\left(nh+nl\cdot\delta\cdot\left(\frac{1-\delta}{\delta}+1\right)\right)^{4}\cdot(4\cdot\rho a-\rho b)}$$

#120:

#121: $$\frac{2\cdot nh\cdot nl\cdot\delta^{3}\cdot\mu\cdot(\rho b-\rho a)\cdot(4\cdot\rho a-3\cdot\rho b)}{3\cdot(nh+nl)^{2}\cdot(4\cdot\rho a-\rho b)}<0$$

*** Section 7: Capacity contraints

eq (20)

#122: $ka = nal + nah$

#123: $kb = nbl + nbh$

#124: $ka = nl \cdot (1 - ahatl) + nh \cdot (1 - ahath)$

#125: $kb = nl \cdot (ahatl - abarl) + nh \cdot (ahath - abarh)$

eq (21)

#126: $$ka = nl \cdot \left(1 - \frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)}\right) + nh \cdot \left(1 - \frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)}\right)$$

#127: $$kb = nl \cdot \left(\frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)} - \frac{tb \cdot (r + 1)}{\mu \cdot \rho b}\right) + nh \cdot \left(\frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)} - \frac{tb}{\delta \cdot \mu \cdot \rho b}\right)$$

eq (22) equilibrium tuition under capacity contraints

#128: $$\text{SOLVE}\left(\left[ka = nl \cdot \left(1 - \frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)}\right) + nh \cdot \left(1 - \frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)}\right), kb = nl \cdot \left(\frac{(r + 1) \cdot (ta - tb)}{\mu \cdot (\rho a - \rho b)} - \frac{tb \cdot (r + 1)}{\mu \cdot \rho b}\right) + nh \cdot \left(\frac{ta - tb}{\delta \cdot \mu \cdot (\rho a - \rho b)} - \frac{tb}{\delta \cdot \mu \cdot \rho b}\right)\right], [ta, tb]\right)$$

#129: $$\left[ta = -\frac{\delta \cdot \mu \cdot (ka \cdot \rho a + kb \cdot \rho b - \rho a \cdot (nh + nl))}{nh + nl \cdot \delta \cdot (r + 1)} \wedge tb = -\frac{\delta \cdot \mu \cdot \rho b \cdot (ka + kb - nh - nl)}{nh + nl \cdot \delta \cdot (r + 1)}\right]$$

#130: $$ta - tb = -\frac{\delta \cdot \mu \cdot (ka \cdot \rho a + kb \cdot \rho b - \rho a \cdot (nh + nl))}{nh + nl \cdot \delta \cdot (r + 1)} - -$$

$$\frac{\delta\cdot\mu\cdot\rho b\cdot(ka + kb - nh - nl)}{nh + nl\cdot\delta\cdot(r + 1)}$$

$$\#131: \quad ta - tb = \frac{\delta\cdot\mu\cdot(ka - nh - nl)\cdot(\rho b - \rho a)}{nh + nl\cdot\delta\cdot(r + 1)} > 0$$

need to compute the eql nal, nah, nbl, and nbh under capacity constraints.

$$\#132: \quad nal = nl\cdot\left(1 - \frac{(r + 1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)}\right)$$

$$\#133: \quad nah = nh\cdot\left(1 - \frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)}\right)$$

$$\#134: \quad nbl = nl\cdot\left(\frac{(r + 1)\cdot(ta - tb)}{\mu\cdot(\rho a - \rho b)} - \frac{tb\cdot(r + 1)}{\mu\cdot\rho b}\right)$$

$$\#135: \quad nbh = nh\cdot\left(\frac{ta - tb}{\delta\cdot\mu\cdot(\rho a - \rho b)} - \frac{tb}{\delta\cdot\mu\cdot\rho b}\right)$$

$$\#136: \quad nal = \frac{nl\cdot(ka\cdot\delta\cdot(r + 1) - nh\cdot(r\cdot\delta + \delta - 1))}{nh + nl\cdot\delta\cdot(r + 1)}$$

$$\#137: \quad nah = \frac{nh\cdot(ka + nl\cdot(r\cdot\delta + \delta - 1))}{nh + nl\cdot\delta\cdot(r + 1)}$$

$$\#138: \quad nbl = \frac{kb\cdot nl\cdot\delta\cdot(r + 1)}{nh + nl\cdot\delta\cdot(r + 1)}$$

$$\#139: \quad nbh = \frac{kb\cdot nh}{nh + nl\cdot\delta\cdot(r + 1)}$$

eq (23) ATB under capacity constraints [I don't show the derivations in the paper, so save these]

$$\#140: \quad ATB = \frac{\delta\cdot(nal\cdot ta\cdot(1 + r) + nbl\cdot tb\cdot(1 + r)) + nah\cdot ta + nbh\cdot tb}{nal + nbl + nah + nbh}$$

#141: ATB = –

$$\frac{\delta\cdot\mu\cdot(ka\cdot\rho a\cdot(nh+nl\cdot\delta\cdot(r+1))+ka\cdot(2\cdot kb\cdot\rho b\cdot(nh+nl\cdot\delta\cdot(r+1))^2-\rho a\cdot(nh^2+2\cdot nh\cdot nl\cdot(r\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1)+nl^2\cdot(\delta^2\cdot(r+1))^2)+kb\cdot\rho b\cdot(nh+nl\cdot\delta\cdot(r+1))^2-kb\cdot\rho b\cdot(nh^2+2\cdot nh\cdot nl\cdot(r\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1)+nl^2\cdot\delta^2\cdot(r+1)^2)+nh\cdot nl\cdot\rho a\cdot(nh+nl)\cdot(r\cdot\delta+\delta-1)^2)}{(ka+kb)\cdot(nh+nl\cdot\delta\cdot(r+1))^2}$$

#142: $$\frac{d}{dr}\left[ATB=\frac{\delta\cdot\mu\cdot(ka\cdot\rho a\cdot(nh+nl\cdot\delta\cdot(r+1))+ka\cdot(2\cdot kb\cdot\rho b\cdot(nh+nl\cdot\delta\cdot(r+1))^2-\rho a\cdot(nh^2+2\cdot nh\cdot nl\cdot(r\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1)+nl^2\cdot(\delta^2\cdot(r+1))^2)+kb\cdot\rho b\cdot(nh+nl\cdot\delta\cdot(r+1))^2-kb\cdot\rho b\cdot(nh^2+2\cdot nh\cdot nl\cdot(r\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1)+nl^2\cdot\delta^2\cdot(r+1)^2)+nh\cdot nl\cdot\rho a\cdot(nh+nl)\cdot(r\cdot\delta+\delta-1)^2)}{(ka+kb)\cdot(nh+nl\cdot\delta\cdot(r+1))^2}\right]$$

$$\left(\frac{\begin{array}{l} \cdot(r^2\cdot\delta^2 + r\cdot\delta\cdot(2\cdot\delta - 1) + \delta^2 - \delta + 1) + nl^2\cdot\delta^2\cdot(r + 1)^2) + nh\cdot nl\cdot pa \\ \cdot(nh + nl)\cdot(r\cdot\delta + \delta - 1)^2 \end{array}}{} \right)$$

#143: $0 = -$

$$\frac{\begin{array}{l} 2\cdot nh\cdot nl\cdot\delta^2\cdot\mu\cdot(ka^2\cdot pa + 2\cdot ka\cdot(kb\cdot pb - pa\cdot(nh + nl)) + kb^2\cdot pb - 2\cdot kb \\ (ka + kb)\cdot(nh + nl\cdot\delta\cdot(r + \\ \cdot pb\cdot(nh + nl) + pa\cdot(nh + nl)^2)\cdot(r\cdot\delta + \delta - 1) \end{array}}{1))^3}$$

#144: $\text{SOLVE}(2\cdot nh\cdot nl\cdot\delta^2\cdot\mu\cdot(ka^2\cdot pa + 2\cdot ka\cdot(kb\cdot pb - pa\cdot(nh + nl)) + kb^2\cdot pb -$

$2\cdot kb\cdot pb\cdot(nh + nl) + pa\cdot(nh + nl)^2)\cdot(r\cdot\delta + \delta - 1), r)$

#145: $r = \frac{1 - \delta}{\delta} \vee ka^2\cdot pa + 2\cdot ka\cdot(kb\cdot pb - pa\cdot(nh + nl)) + kb^2\cdot pb -$

$2\cdot kb\cdot pb\cdot(nh + nl) + pa\cdot(nh + nl)^2 = 0$

#146: $\frac{d}{dr}\frac{d}{dr}\left(ATB = -\right.$

$$\left(\frac{\delta\cdot\mu\cdot\left(ka^2\cdot\rho a\cdot\left(nh+nl\cdot\delta^2\cdot(r+1)^2\right)+ka\cdot\left(2\cdot kb\cdot\rho b\cdot\left(nh+nl\cdot\delta^2\cdot(r+1)^2\right)-\rho a\cdot\left(nh^2+2\cdot nh\cdot nl\cdot\left(r^2\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1\right)+nl^2\cdot\left(\delta^2\cdot(r+1)^2\right)\right)+kb^2\cdot\rho b\cdot\left(nh+nl\cdot\delta^2\cdot(r+1)^2\right)-kb\cdot\rho b\cdot\left(nh^2+2\cdot nh\cdot nl\cdot\left(r^2\cdot\delta^2+r\cdot\delta\cdot(2\cdot\delta-1)+\delta^2-\delta+1\right)+nl^2\cdot\delta^2\cdot(r+1)^2\right)+nh\cdot nl\cdot\rho a\cdot\left(nh+nl\right)\cdot\left(r\cdot\delta+\delta-1\right)^2\right)}{\left(ka+kb\right)\cdot\left(nh+nl\cdot\delta\cdot(r+1)\right)^2}\right)$$

#147: $$\frac{2\cdot nh\cdot nl\cdot\delta^3\cdot\mu\cdot\left(ka^2\cdot\rho a+2\cdot ka\cdot\left(kb\cdot\rho b-\rho a\cdot\left(nh+nl\right)\right)+kb^2\cdot\rho b-2\cdot kb\cdot\rho a\cdot\left(nh+nl\right)\cdot\left(ka+kb\right)\cdot\left(nh+nl\cdot\delta\cdot\left(r+1\right)\right)\right)\cdot\left(b\cdot\left(nh+nl\right)+\rho a\cdot\left(nh+nl\right)^2\right)\cdot\left(nl\cdot\left(2\cdot r\cdot\delta+2\cdot\delta-3\right)-nh\right)}{\left(r+1\right)^4}$$

evaluate at rhat

#148: –

$$\frac{2\cdot nh\cdot nl\cdot\delta^{3}\cdot\mu\cdot\left(ka^{2}\cdot\rho a+2\cdot ka\cdot\left(kb\cdot\rho b-\rho a\cdot(nh+nl)\right)+kb^{2}\cdot\rho b-2\cdot kb\cdot\rho b\cdot(nh+nl)+\rho a\cdot(nh+nl)^{2}\right)}{(ka+kb)^{3}\cdot(nh+nl)}$$