" Interchange fees, card rewards, and distortionary taxation"
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Below, I provide the algebraic derivations for ALL equations in the paper.

The derivations are made using symbolic algebra software called "Derive for Windows."
I will refer to each equation number in the paper itself.

#1: CaseMode := Sensitive

#2: InputMode := Word

#3: $\beta c :\in \text{Real } (0, \infty)$

#4: $\beta p :\in \text{Real } (0, \infty)$

#5: $bch :\in \text{Real } (0, \infty)$

#6: $bph :\in \text{Real } (0, \infty)$

Number of consumers

#7: $nc :\in \text{Real } (0, \infty)$

Number of merchants selling from physical stores

#8: $mp :\in \text{Real } (0, \infty)$

Number of merchants selling online only

#9: $mo :\in \text{Real } (0, \infty)$

rate of merchant expense tax deduction

#10: $\delta :\in \text{Real } (0, 1)$

tax rate on card rewards

#11: $\tau :\in \text{Real } (0, 1)$

per–trans consumer benefit from paying card [equation (1) in the paper]

#12: $\beta c \cdot x + (1 - \tau) \cdot f$

per–trans consumer benefit from paying cash [equation (1) in the paper]

#13: bch

per–trans p–merchant benefit from receiving a card payment [equation (2) in the paper]

#14: $\beta p \cdot y - (1 - \delta) \cdot f$

per–trans p–merchant benefit from receving a cash payment [equation (2) in the paper]

#15: bph

per–trans e–merchant benefit from paying card [equation (3) in the paper]

#16: $\beta e - (1 - \delta) \cdot f$

per–trans e–merchant benefit from receiving cash payment [equation (3) in the paper]

#17: beh

*** Section 3: Interchange fee set by the card company

#18: $bch = \beta c \cdot xhat + (1 - \tau) \cdot f$

#19: $bph = \beta p \cdot yhat - (1 - \delta) \cdot f$

Equation (4)

#20: SOLVE($bch = \beta c \cdot xhat + (1 - \tau) \cdot f$, xhat)

#21:
$$xhat = \frac{bch + f \cdot (\tau - 1)}{\beta c}$$

#22: SOLVE($bph = \beta p \cdot yhat - (1 - \delta) \cdot f$, yhat)

#23: $$\text{yhat} = \frac{\text{bph} - f\cdot(\delta - 1)}{\beta p}$$

Equation (5): cash volume

#24: $\text{vh} = \text{nc}\cdot\text{xhat}\cdot\text{mp} + \text{nc}\cdot(1 - \text{xhat})\cdot\text{mp}\cdot\text{yhat}$

Equation (6): card volume

#25: $\text{vd} = \text{nc}\cdot(1 - \text{xhat})\cdot\text{mp}\cdot(1 - \text{yhat}) + \text{nc}\cdot\text{me}$

Verify sum vh + vd =

#26: $\text{nc}\cdot\text{xhat}\cdot\text{mp} + \text{nc}\cdot(1 - \text{xhat})\cdot\text{mp}\cdot\text{yhat} + \text{nc}\cdot(1 - \text{xhat})\cdot\text{mp}\cdot(1 - \text{yhat}) + \text{nc}\cdot\text{me}$

#27: $$\text{me}\cdot\text{nc} + \text{mp}\cdot\text{nc}$$

Equation (7) and Appendix A.1
Equations (A.1) to (A.3)

#28: $$\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} + f\cdot(\tau - 1)}{\beta c}\right)\cdot\text{mp}\cdot\left(1 - \frac{\text{bph} - f\cdot(\delta - 1)}{\beta p}\right) + \text{nc}\cdot\text{me}$$

#29: $$\frac{d}{df}\left(\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} + f\cdot(\tau - 1)}{\beta c}\right)\cdot\text{mp}\cdot\left(1 - \frac{\text{bph} - f\cdot(\delta - 1)}{\beta p}\right) + \text{nc}\cdot\text{me}\right)$$

#30: $$0 = -\frac{\text{mp}\cdot\text{nc}\cdot(\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(1 - \tau) + 2\cdot f\cdot(\delta - 1)\cdot(\tau - 1) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1))}{\beta c\cdot\beta p}$$

#31: $$\frac{d}{df}\frac{d}{df}\left(\text{vd} = \text{nc}\cdot\left(1 - \frac{\text{bch} + f\cdot(\tau - 1)}{\beta c}\right)\cdot\text{mp}\cdot\left(1 - \frac{\text{bph} - f\cdot(\delta - 1)}{\beta p}\right) + \text{nc}\cdot\text{me}\right)$$

#32: $$0 > \frac{2\cdot mp\cdot nc\cdot(1-\tau)\cdot(\delta-1)}{\beta c\cdot\beta p}$$

equation (7)

#33: $$\text{SOLVE}\left(0 = -\frac{mp\cdot nc\cdot(bch\cdot(\delta-1) + bph\cdot(1-\tau) + 2\cdot f\cdot(\delta-1)\cdot(\tau-1) + \beta c\cdot(1-\delta) + \beta p\cdot(\tau-1))}{\beta c\cdot\beta p}, f\right)$$

#34: $$fbar = \frac{bch\cdot(\delta-1) + bph\cdot(1-\tau) + \beta c\cdot(1-\delta) + \beta p\cdot(\tau-1)}{2\cdot(1-\tau)\cdot(\delta-1)}$$

equation (8), case where δ=τ=0

#35: $$fbar = \frac{bch\cdot(0-1) + bph\cdot(1-0) + \beta c\cdot(1-0) + \beta p\cdot(0-1)}{2\cdot(1-0)\cdot(0-1)}$$

#36: $$fbar = \frac{bch - bph - \beta c + \beta p}{2}$$

*** Section 4: How taxes affected interchange fees

Result 1 and equation (9), using Assumption 1a

#37: $$\frac{d}{d\delta}\left(fbar = \frac{bch\cdot(\delta-1) + bph\cdot(1-\tau) + \beta c\cdot(1-\delta) + \beta p\cdot(\tau-1)}{2\cdot(1-\tau)\cdot(\delta-1)}\right)$$

#38: $$0 < \frac{\beta p - bph}{2\cdot(\delta-1)^2}$$

#39: $\frac{d}{d\tau}\left(fbar = \frac{bch\cdot(\delta - 1) + bph\cdot(1 - \tau) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot(1 - \tau)\cdot(\delta - 1)}\right)$

#40: $0 > \frac{bch - \beta c}{2\cdot(\tau - 1)^2}$

Equation (10) and Result 2 using Assumption 1a

#41: $(1 - \delta)\cdot fbar$

#42: $(1 - \delta)\cdot\frac{bch\cdot(\delta - 1) + bph\cdot(1 - \tau) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot(1 - \tau)\cdot(\delta - 1)}$

#43: $\frac{d}{d\delta}\left((1 - \delta)\cdot\frac{bch\cdot(\delta - 1) + bph\cdot(1 - \tau) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot(1 - \tau)\cdot(\delta - 1)}\right)$

#44: $\frac{bch - \beta c}{2\cdot(\tau - 1)} > 0$

#45: $(1 - \tau)\cdot fbar$

#46: $(1 - \tau)\cdot\frac{bch\cdot(\delta - 1) + bph\cdot(1 - \tau) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot(1 - \tau)\cdot(\delta - 1)}$

#47: $\frac{d}{d\tau}\left((1 - \tau)\cdot\frac{bch\cdot(\delta - 1) + bph\cdot(1 - \tau) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot(1 - \tau)\cdot(\delta - 1)}\right)$

#48: $\frac{bph - \beta p}{2\cdot(1 - \delta)} < 0$

*** Section 5: Welfare and tax policy

wch (consumer welfare from cash payments)

#49: $wch = vh \cdot bch$

#50: $wch = (nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bch$

wcd (consumer welfare from card payments)

#51: $$wcd = nc \cdot \int_{xhat}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx \cdot mp \cdot (1 - yhat) + nc \cdot me \cdot \int_{0}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx$$

equation (11) consumer welfare

#52: $$wc = (nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bch + \left(nc \cdot \int_{xhat}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx \cdot mp \cdot (1 - yhat) + nc \cdot me \cdot \int_{0}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx\right)$$

#53: $wc = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) -$

$$\frac{nc \cdot (2 \cdot f \cdot (me \cdot (\tau - 1) + mp \cdot (yhat - 1) \cdot (xhat \cdot (\tau - 1) - \tau + 1)) - \beta c \cdot (me + mp \cdot (xhat^2 - 1) \cdot (yhat - 1 \sim}{2} \sim$$

)))

wph (p−merchants welfare from cash payments)

#54: $wph = vh \cdot bph$

#55: $wph = (nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bph$

wpd (p−merchants welfare from card payments)

#56: $wpd = nc \cdot (1 - xhat) \cdot mp \cdot \int_{yhat}^{1} (\beta p \cdot y - (1 - \delta) \cdot f) \, dy$

eqution (12): wp welfare of p−merchants

#57: $wp = (nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bph + nc \cdot (1 - xhat) \cdot mp \cdot \int_{yhat}^{1} (\beta p \cdot y - (1 - \delta) \cdot f) \, dy$

wed = welfare of e−merchants from card payments

#58: $wed = nc \cdot me \cdot (\beta e - (1 - \delta) \cdot f)$

equation (13): Net government revenue from taxing interchaning fees on card payments

#59: $g = (\tau - \delta) \cdot f \cdot vd$

#60: $g = (\tau - \delta) \cdot f \cdot (nc \cdot (1 - xhat) \cdot mp \cdot (1 - yhat) + nc \cdot me)$

total welfare

#61: $w = wc + wp + we + g$

** section 5.1: Optimal volume of card payments

Appendix A.2 and equation (15)

equation (14)

$$\#62: \quad w = \left((nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bch + \left(nc \cdot \int_{xhat}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx \cdot mp \cdot (1 - yhat) + nc \cdot me \cdot \int_{0}^{1} (\beta c \cdot x + (1 - \tau) \cdot f)\, dx\right)\right) + \left((nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bph + nc \cdot (1 - xhat) \cdot mp \cdot \int_{yhat}^{1} (\beta p \cdot y - (1 - \delta) \cdot f)\, dy\right) + nc \cdot me \cdot (\beta e - (1 - \delta) \cdot f) + (\tau - \delta) \cdot f \cdot (nc \cdot (1 - xhat) \cdot mp \cdot (1 - yhat) + nc \cdot me)$$

substituting f = 0 (for first-best optimal) to obtain equation (A.4)

$$\#63: \quad w = \left((nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bch + \left(nc \cdot \int_{xhat}^{1} (\beta c \cdot x + (1 - \tau) \cdot 0)\, dx \cdot mp \cdot (1 - yhat) + nc \cdot me \cdot \int_{0}^{1} (\beta c \cdot x + (1 - \tau) \cdot 0)\, dx\right)\right) + \left((nc \cdot xhat \cdot mp + nc \cdot (1 - xhat) \cdot mp \cdot yhat) \cdot bph + nc \cdot (1 - xhat) \cdot mp \cdot \int_{yhat}^{1} (\beta p \cdot y - (1 - \delta) \cdot 0)\, dy\right) + nc \cdot me \cdot (\beta e - (1 - \delta) \cdot 0) + (\tau - \delta) \cdot 0 \cdot (nc \cdot (1 - xhat) \cdot mp \cdot (1 - yhat) + nc \cdot me)$$

equation (A.4)

#64: $$w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + me \cdot nc \cdot \left(\frac{\beta c}{2} + \beta e\right) + \frac{mp \cdot nc \cdot (xhat^2 \cdot \beta c \cdot (yhat - 1) + xhat \cdot \beta p \cdot (yhat^2 - 1) - yhat^2 \cdot \beta p - yhat \cdot \beta c + \beta c + \beta p)}{2}$$

FOC equations (A.5) and (A.60

#65: $$\frac{d}{d\ xhat}\left(w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + me \cdot nc \cdot \left(\frac{\beta c}{2} + \beta e\right) + \frac{mp \cdot nc \cdot (xhat^2 \cdot \beta c \cdot (yhat - 1) + xhat \cdot \beta p \cdot (yhat^2 - 1) - yhat^2 \cdot \beta p - yhat \cdot \beta c + \beta c + \beta p)}{2}\right)$$

#66: $$0 = - \frac{mp \cdot nc \cdot (2 \cdot bch \cdot (yhat - 1) + 2 \cdot bph \cdot (yhat - 1) + 2 \cdot xhat \cdot \beta c \cdot (1 - yhat) - \beta p \cdot (yhat^2 - 1))}{2}$$

#67: $$\frac{d}{d\ yhat}\left(w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + me \cdot nc \cdot \left(\frac{\beta c}{2} + \beta e\right) + \frac{mp \cdot nc \cdot (xhat^2 \cdot \beta c \cdot (yhat - 1) + xhat \cdot \beta p \cdot (yhat^2 - 1) - yhat^2 \cdot \beta p - yhat \cdot \beta c + \beta c + \beta p)}{2}\right)$$

#68: $$0 = - \frac{mp\cdot nc\cdot (2\cdot bch\cdot (xhat - 1) + 2\cdot bph\cdot (xhat - 1) - xhat^2\cdot \beta c - 2\cdot xhat\cdot yhat\cdot \beta p + 2\cdot yhat\cdot \beta p + \beta c)}{2}$$

Solving the 2 FOCs:

#69: $$\mathrm{SOLVE}\left(0 = - \frac{mp\cdot nc\cdot (2\cdot bch\cdot (yhat - 1) + 2\cdot bph\cdot (yhat - 1) + 2\cdot xhat\cdot \beta c\cdot (1 - yhat) - \beta p\cdot (yhat^2 - 1))}{2}, xhat\right)$$

#70: $$xhat = \frac{2\cdot bch + 2\cdot bph - \beta p\cdot (yhat + 1)}{2\cdot \beta c}$$

#71: $$\mathrm{SOLVE}\left(0 = - \frac{mp\cdot nc\cdot (2\cdot bch\cdot (xhat - 1) + 2\cdot bph\cdot (xhat - 1) - xhat^2\cdot \beta c - 2\cdot xhat\cdot yhat\cdot \beta p + 2\cdot yhat\cdot \beta p + \beta c)}{2}, yhat\right)$$

#72: $$yhat = \frac{2\cdot bch + 2\cdot bph - \beta c\cdot (xhat + 1)}{2\cdot \beta p}$$

#73: $$\mathrm{SOLVE}\left(\left[xhat = \frac{2\cdot bch + 2\cdot bph - \beta p\cdot (yhat + 1)}{2\cdot \beta c}, yhat = \frac{2\cdot bch + 2\cdot bph - \beta c\cdot (xhat + 1)}{2\cdot \beta p}\right], [xhat,\right.$$

yhat])

equation (15)

$$\#74: \quad \left[xstar = \frac{2 \cdot bch + 2 \cdot bph + \beta c - 2 \cdot \beta p}{3 \cdot \beta c} \wedge ystar = \frac{2 \cdot bch + 2 \cdot bph - 2 \cdot \beta c + \beta p}{3 \cdot \beta p} \right]$$

Show that that Assumption 2 ensures that 0 < xstar < 1 and 0 < ystar < 1
Below implied by Assumption 2 first part

$$\#75: \quad 2 \cdot bch + 2 \cdot bph + \beta c - 2 \cdot \beta p > 0$$

$$\#76: \quad \frac{2 \cdot bch + 2 \cdot bph + \beta c - 2 \cdot \beta p}{3 \cdot \beta c} < 1$$

$$\#77: \quad 2 \cdot bch + 2 \cdot bph + \beta c - 2 \cdot \beta p < 3 \cdot \beta c$$

Above implied by Assumption 1a. Below, by Assumption 2, 2nd part

$$\#78: \quad 2 \cdot bch + 2 \cdot bph - 2 \cdot \beta c + \beta p > 0$$

$$\#79: \quad \frac{2 \cdot bch + 2 \cdot bph - 2 \cdot \beta c + \beta p}{3 \cdot \beta p} < 1$$

$$\#80: \quad 2 \cdot bch + 2 \cdot bph - 2 \cdot \beta c + \beta p < 3 \cdot \beta p$$

Above implied by Assumption 1a

Analyzing SOC

$$\#81: \quad \frac{d}{d\ xhat} \frac{d}{d\ xhat} \left(w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + \right.$$

$$me \cdot nc \cdot \left(\frac{\beta c}{2} + \beta e\right) + \frac{mp \cdot nc \cdot (xhat^2 \cdot \beta c \cdot (yhat - 1) + xhat \cdot \beta p \cdot (yhat^2 - 1) - yhat^2 \cdot \beta p - yhat \cdot \beta c + \beta c + \beta p)}{2}\Bigg)$$

#82: $$0 > mp \cdot nc \cdot \beta c \cdot (yhat - 1)$$

#83: $$\frac{d}{d\ yhat}\frac{d}{d\ yhat}\Bigg(w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) +$$

$$me \cdot nc \cdot \left(\frac{\beta c}{2} + \beta e\right) + \frac{mp \cdot nc \cdot (xhat^2 \cdot \beta c \cdot (yhat - 1) + xhat \cdot \beta p \cdot (yhat^2 - 1) - yhat^2 \cdot \beta p - yhat \cdot \beta c + \beta c + \beta p)}{2}\Bigg)$$

#84: $$0 > mp \cdot nc \cdot \beta p \cdot (xhat - 1)$$

developing the Hessian. Below, cross derivative:

#85: $$\frac{d}{d\ yhat}\frac{d}{d\ xhat}\Bigg(w = bch \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) + bph \cdot mp \cdot nc \cdot (yhat - xhat \cdot (yhat - 1)) +$$

$$me\cdot nc\cdot\left(\frac{\beta c}{2}+\beta e\right)+$$

$$\left.\frac{mp\cdot nc\cdot(xhat^2\cdot\beta c\cdot(yhat-1)+xhat\cdot\beta p\cdot(yhat^2-1)-yhat^2\cdot\beta p-yhat\cdot\beta c+\beta c+\beta p)}{2}\right)$$

#86: $$-\ mp\cdot nc\cdot(bch+bph-xhat\cdot\beta c-yhat\cdot\beta p)$$

Hessian equation (A.7)

#87: $$h=(mp\cdot nc\cdot\beta c\cdot(yhat-1))\cdot(mp\cdot nc\cdot\beta p\cdot(xhat-1))-(-\ mp\cdot nc\cdot(bch+bph-xhat\cdot\beta c-yhat\cdot\beta p))^2$$

evaluate at xstar and ystar in equation (15)

#88: $$h=\left(mp\cdot nc\cdot\beta c\cdot\left(\frac{2\cdot bch+2\cdot bph-2\cdot\beta c+\beta p}{3\cdot\beta p}-1\right)\right)\cdot\left(mp\cdot nc\cdot\beta p\cdot\left(\frac{2\cdot bch+2\cdot bph+\beta c-2\cdot\beta p}{3\cdot\beta c}-1\right)\right)-\left(-\ mp\cdot nc\cdot\left(bch+bph-\frac{2\cdot bch+2\cdot bph+\beta c-2\cdot\beta p}{3\cdot\beta c}\cdot\beta c-\frac{2\cdot bch+2\cdot bph-2\cdot\beta c+\beta p}{3\cdot\beta p}\cdot\beta p\right)\right)^2$$

#89: $$h=\frac{mp^2\cdot nc^2\cdot(bch+bph-\beta c-\beta p)^2}{3}>0$$

Equation (16) optimal volume of card payments

#90: $$vd=nc\cdot\left(1-\frac{2\cdot bch+2\cdot bph+\beta c-2\cdot\beta p}{3\cdot\beta c}\right)\cdot mp\cdot\left(1-\frac{2\cdot bch+2\cdot bph-2\cdot\beta c+\beta p}{3\cdot\beta p}\right)+nc\cdot me$$

Result 3 and Appendix A.3

equations (A.8)

#91: $$\frac{2\cdot bch + 2\cdot bph + \beta c - 2\cdot \beta p}{3\cdot \beta c} = \frac{bch + f\cdot(\tau - 1)}{\beta c}$$

#92: $$\frac{2\cdot bch + 2\cdot bph - 2\cdot \beta c + \beta p}{3\cdot \beta p} = \frac{bph - f\cdot(\delta - 1)}{\beta p}$$

setting δ = τ = 0 solving each of the above yields equations (A.9)

#93: $$\text{SOLVE}\left(\frac{2\cdot bch + 2\cdot bph + \beta c - 2\cdot \beta p}{3\cdot \beta c} = \frac{bch + f\cdot(\tau - 1)}{\beta c}, f\right)$$

#94: $$f = \frac{bch - 2\cdot bph - \beta c + 2\cdot \beta p}{3\cdot(1 - \tau)}$$

#95: $$fxstar = \frac{bch - 2\cdot bph - \beta c + 2\cdot \beta p}{3}$$

** Section 5.3: Taxation policy and regulation to induce the optimal number of payments

Result 4a, Appendix A.4, Equation (17)

#96: $$\text{SOLVE}\left(\frac{2\cdot bch + 2\cdot bph - 2\cdot \beta c + \beta p}{3\cdot \beta p} = \frac{bph - f\cdot(\delta - 1)}{\beta p}, \delta\right)$$

#97: $$\delta star = - \frac{2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p}{3\cdot f}$$

#98: $$\text{SOLVE}\left(\frac{2\cdot bch + 2\cdot bph + \beta c - 2\cdot \beta p}{3\cdot \beta c} = \frac{bch + f\cdot(\tau - 1)}{\beta c}, \tau\right)$$

#99: $$\tau star = - \frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p}{3\cdot f}$$

Condition on f ensuring δstar > 0

#100: $2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p < 0$

#101: SOLVE($2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p < 0$, f)

#102: $$f > \frac{2\cdot bch - bph - 2\cdot \beta c + \beta p}{3}$$

Condition on f ensuring δstar < 1 => No reason why doing that, tax rate could exceed 1

#103: $$- \frac{2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p}{3\cdot f} < 1$$

#104: $-(2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p) < 3\cdot f$

#105: SOLVE($-(2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p) < 3\cdot f$, f)

#106: $$2\cdot bch - bph - 2\cdot \beta c + \beta p > 0$$

condition on f ensuring τstar > 0

#107: $bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p < 0$

#108: SOLVE($bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p < 0$, f)

#109: $$f > \frac{bch - 2\cdot bph - \beta c + 2\cdot \beta p}{3}$$

condition on f ensuring τstar < 1 => No reason why doing that, tax rate could exceed 1

$$\#110:\ -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p}{3\cdot f} < 1$$

$$\#111:\ -(bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p) < 3\cdot f$$

#112: SOLVE(− (bch − 2·bph − 3·f − βc + 2·βp) < 3·f, f)

$$\#113:\ bch - 2\cdot bph - \beta c + 2\cdot \beta p > 0$$

Result 4b, inconsistency of equations (17) δstar and τstar with (7) fbar

#114: f =

$$\frac{bch\cdot\left(-\frac{2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p}{3\cdot f} - 1\right) + bph\cdot\left(1 - -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p}{3\cdot f}\right) + \beta c\cdot\sim}{2\cdot\left(1 - -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p}{3\cdot f}\right)\cdot\left(- -\sim\right.}$$

$$\frac{\left(1 - -\frac{2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p}{3\cdot f}\right) + \beta p\cdot\left(-\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot \beta p}{3\cdot f} - 1\right)}{\left.\frac{2\cdot bch - bph - 3\cdot f - 2\cdot \beta c + \beta p}{3\cdot f} - 1\right)}$$

trying to solve for f

#115: SOLVE

$$f = \frac{bch\cdot\left(-\frac{2\cdot bch - bph - 3\cdot f - 2\cdot\beta c + \beta p}{3\cdot f} - 1\right) + bph\cdot\left(1 - -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot\beta p}{3\cdot f}\right) + \beta c\cdot\left(1 - -\frac{2\cdot bch - bph - 3\cdot f - 2\cdot\beta c + \beta p}{3\cdot f}\right) + \beta p\cdot\left(-\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot\beta p}{3\cdot f} - 1\right)}{2\cdot\left(1 - -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot\beta p}{3\cdot f}\right)\cdot\left(- -\frac{2\cdot bch - bph - 3\cdot f - 2\cdot\beta c + \beta p}{3\cdot f} - 1\right)}, f$$

#116:

$$f = 0 \vee \frac{(bch - \beta c)\cdot(bph - \beta p)}{(bch - 2\cdot bph - \beta c + 2\cdot\beta p)\cdot(2\cdot bch - bph - 2\cdot\beta c + \beta p)} = -\frac{1}{9}$$

Equation (18) and Result 5

#117:

$$\delta star - \tau star = -\frac{2\cdot bch - bph - 3\cdot f - 2\cdot\beta c + \beta p}{3\cdot f} - -\frac{bch - 2\cdot bph - 3\cdot f - \beta c + 2\cdot\beta p}{3\cdot f}$$

#118:

$$\delta star - \tau star = -\frac{bch + bph - \beta c - \beta p}{3\cdot f} > 0$$

** Section 5.3: Taxing card rewards without regulating interchange fees

Result 6, Appendix A.5. Recall equation (14)

$$\#119:\ w = \left((nc\cdot xhat\cdot mp + nc\cdot(1 - xhat)\cdot mp\cdot yhat)\cdot bch + \left(nc\cdot\int_{xhat}^{1} (\beta c\cdot x + (1 - \tau)\cdot f)\ dx\cdot mp\cdot(1 - yhat) + nc\cdot me\cdot\int_{0}^{1} (\beta c\cdot x + (1 - \tau)\cdot f)\ dx\right)\right) + \left((nc\cdot xhat\cdot mp + nc\cdot(1 - xhat)\cdot mp\cdot yhat)\cdot bph + nc\cdot(1 - xhat)\cdot mp\cdot\int_{yhat}^{1} (\beta p\cdot y - (1 - \delta)\cdot f)\ dy\right) + nc\cdot me\cdot(\beta e - (1 - \delta)\cdot f) + (\tau - \delta)\cdot f\cdot(nc\cdot(1 - xhat)\cdot mp\cdot(1 - yhat) + nc\cdot me)$$

$$\#120:\ w = bch\cdot mp\cdot nc\cdot(yhat - xhat\cdot(yhat - 1)) + bph\cdot mp\cdot nc\cdot(yhat - xhat\cdot(yhat - 1)) + \frac{nc\cdot(me\cdot(\beta c + 2\cdot\beta e) + mp\cdot(xhat^{2}\cdot\beta c\cdot(yhat - 1) + xhat\cdot\beta p\cdot(yhat^{2} - 1) - yhat^{2}\cdot\beta p - yhat\cdot\beta c + \beta c + \beta p))}{2}$$

subs for xhat and yhat

#121:
$$w = bch\cdot mp\cdot nc\cdot\left(\frac{bph - f\cdot(\delta - 1)}{\beta p} - \frac{bch + f\cdot(\tau - 1)}{\beta c}\cdot\left(\frac{bph - f\cdot(\delta - 1)}{\beta p} - 1\right)\right) +$$

$$bph\cdot mp\cdot nc\cdot\left(\frac{bph - f\cdot(\delta - 1)}{\beta p} - \frac{bch + f\cdot(\tau - 1)}{\beta c}\cdot\left(\frac{bph - f\cdot(\delta - 1)}{\beta p} - 1\right)\right) +$$

$$\frac{nc\cdot\left(me\cdot(\beta c + 2\cdot\beta e) + mp\cdot\left(\left(\left(\frac{bch + f\cdot(\tau - 1)}{\beta c}\right)^2\cdot\beta c\cdot\left(\frac{bph - f\cdot(\delta - 1)}{\beta p} - 1\right) + \frac{bch + f\cdot(\tau - 1)}{\beta c}\cdot\beta p\sim\right.\right.\right.}{2 \quad \sim}$$

$$\frac{\left.\left.\cdot\left(\left(\frac{bph - f\cdot(\delta - 1)}{\beta p}\right)^2 - 1\right) - \left(\frac{bph - f\cdot(\delta - 1)}{\beta p}\right)^2\cdot\beta p - \frac{bph - f\cdot(\delta - 1)}{\beta p}\cdot\beta c + \beta c + \beta p\right)\right)}{}$$

Subst $\delta = \tau = 0$

#122: $w = -$

$$\frac{nc\cdot(bch^2\cdot mp\cdot(bph + f - \beta p) + bch\cdot mp\cdot(bph^2 - 2\cdot bph\cdot(\beta c + \beta p) - f^2 - 2\cdot f\cdot\beta c + \beta p^2) - bph^2\cdot mp\cdot(f +\sim}{\sim}$$

$$\frac{\beta c) - bph\cdot mp\cdot(f^2 - 2\cdot f\cdot \beta p - \beta c^2) + f^2\cdot mp\cdot(\beta c + \beta p) + f\cdot mp\cdot(\beta c^2 - \beta p^2) - me\cdot \beta c\cdot \beta p\cdot(\beta c + 2\cdot \beta e) - \sim}{2\cdot \beta c\cdot \beta p \qquad \sim}$$

$$\frac{mp\cdot \beta c\cdot \beta p\cdot(\beta c + \beta p))}{}$$

FOC w.r.t. f, equation (A.10)

$$\#123: \frac{d}{df}\left(w = -\right.$$

$$\frac{nc\cdot(bch^2\cdot mp\cdot(bph + f - \beta p) + bch\cdot mp\cdot(bph^2 - 2\cdot bph\cdot(\beta c + \beta p) - f^2 - 2\cdot f\cdot \beta c + \beta p^2) - bph^2\cdot mp\cdot(f + \sim}{\sim}$$

$$\frac{\beta c) - bph\cdot mp\cdot(f^2 - 2\cdot f\cdot \beta p - \beta c^2) + f^2\cdot mp\cdot(\beta c + \beta p) + f\cdot mp\cdot(\beta c^2 - \beta p^2) - me\cdot \beta c\cdot \beta p\cdot(\beta c + 2\cdot \beta e) - \sim}{2\cdot \beta c\cdot \beta p \qquad \sim}$$

$$\left.\frac{mp\cdot \beta c\cdot \beta p\cdot(\beta c + \beta p))}{}\right)$$

$$\#124: \qquad 0 = - \frac{mp\cdot nc\cdot(bch^2 - 2\cdot bch\cdot(f + \beta c) - bph^2 + 2\cdot bph\cdot(\beta p - f) + 2\cdot f\cdot(\beta c + \beta p) + \beta c^2 - \beta p^2)}{2\cdot \beta c\cdot \beta p}$$

SOC

#125: $$\frac{d}{df}\frac{d}{df}\left(w = -\frac{nc\cdot(bch^2\cdot mp\cdot(bph + f - βp) + bch\cdot mp\cdot(bph^2 - 2\cdot bph\cdot(βc + βp) - f^2 - 2\cdot f\cdot βc + βp^2) - bph^2\cdot mp\cdot(f + βc) - bph\cdot mp\cdot(f^2 - 2\cdot f\cdot βp - βc^2) + f^2\cdot mp\cdot(βc + βp) + f\cdot mp\cdot(βc^2 - βp^2) - me\cdot βc\cdot βp\cdot(βc + 2\cdot βe) - mp\cdot βc\cdot βp\cdot(βc + βp))}{2\cdot βc\cdot βp}\right)$$

#126: $$0 > \frac{mp\cdot nc\cdot(bch + bph - βc - βp)}{βc\cdot βp}$$

#127: $$\mathrm{SOLVE}\left(0 = -\frac{mp\cdot nc\cdot(bch^2 - 2\cdot bch\cdot(f + βc) - bph^2 + 2\cdot bph\cdot(βp - f) + 2\cdot f\cdot(βc + βp) + βc^2 - βp^2)}{2\cdot βc\cdot βp}, f\right)$$

#128: $$ftilde = \frac{bch - bph - βc + βp}{2}$$

*** Appendix B: Further explorations

Deriving equations (B.1) and (B.2)

#129: $$\text{xhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(\delta - 1) + \beta p\cdot(1 - \tau)}{2\cdot\beta c\cdot(\delta - 1)}$$

#130: $$\text{yhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot\beta p\cdot(\tau - 1)}$$

Using Assumption 1a (for determining the signs), equations (B.3) and Result 8 are

#131: $$\frac{d}{d\delta}\left(\text{xhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(\delta - 1) + \beta p\cdot(1 - \tau)}{2\cdot\beta c\cdot(\delta - 1)}\right)$$

#132: $$0 > \frac{(1 - \tau)\cdot(\text{bph} - \beta p)}{2\cdot\beta c\cdot(\delta - 1)^2}$$

#133: $$\frac{d}{d\delta}\left(\text{yhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot\beta p\cdot(\tau - 1)}\right)$$

#134: $$0 < \frac{\text{bch} - \beta c}{2\cdot\beta p\cdot(\tau - 1)}$$

#135: $$\frac{d}{d\tau}\left(\text{xhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(\delta - 1) + \beta p\cdot(1 - \tau)}{2\cdot\beta c\cdot(\delta - 1)}\right)$$

#136: $$0 < \frac{\text{bph} - \beta p}{2\cdot\beta c\cdot(\delta - 1)}$$

#137: $$\frac{d}{d\tau}\left(\text{yhat} = \frac{\text{bch}\cdot(\delta - 1) + \text{bph}\cdot(\tau - 1) + \beta c\cdot(1 - \delta) + \beta p\cdot(\tau - 1)}{2\cdot\beta p\cdot(\tau - 1)}\right)$$

#138: $$0 > \frac{(1 - \delta)\cdot(bch - \beta c)}{2\cdot\beta p\cdot(\tau - 1)^2}$$

For figure 4 in paper: welfare hat for regulated interchange fee f_R given δ=τ=0. (#52 above)

#139: $$wc = bch\cdot mp\cdot nc\cdot(yhat - xhat\cdot(yhat - 1)) + \frac{nc\cdot(2\cdot f\cdot(me + mp\cdot(xhat - 1)\cdot(yhat - 1)) + \beta c\cdot(me + mp\cdot(xhat^2 - 1)\cdot(yhat - 1)))}{2}$$

#140: $$wp = bph\cdot mp\cdot nc\cdot(yhat - xhat\cdot(yhat - 1)) + \frac{mp\cdot nc\cdot(1 - xhat)\cdot(2\cdot f\cdot(yhat - 1) - yhat^2\cdot\beta p + \beta p)}{2}$$

#141: $$wed = me\cdot nc\cdot(\beta e - f)$$
