Chapter 2
System Models

Abstract This chapter develops representations in state-space notation of the porous-media flow equations derived in Chap. 1. For single-phase flow, the states are grid-block pressures, and for two-phase flow they are grid-block pressures and saturations. The inputs are typically bottom-hole pressures or total well flow rates, the outputs are typically bottom-hole pressures in those wells were the flow rates were prescribed, and phase rates in those wells were the bottom-hole pressures were prescribed. The use of matrix partitioning to describe the different types of inputs leads to a description in terms of nonlinear ordinary-differential and algebraic equations with (state-dependent) system, input, output and direct-throughput matrices. Other topics include generalized state-space representations, linearization, elimination of prescribed pressures, the tracing of stream lines, lift tables, computational aspects, and the derivation of an energy balance for porous-media flow.

2.1 System Equations

2.1.1 Partial-Differential Equations

To describe the physics of fluid flow in a porous medium we generally use *partial-differential equations* (PDEs). Typically the *independent* variables are time, t, and spatial coordinates x, y and z. Furthermore, in reservoir engineering we normally encounter only first-order derivatives in time, but higher-order (typically second-order) derivatives in space. Indicating the, arbitrary, *dependent* variable with a fat dot, $\bullet$, the equations can be represented in general form as[1]

[1] The dependent variables follow from the physics of the problem. In case of multi-phase flow through porous media they are typically pressures, component masses or phase saturations; see Chap. 1. Here we use only a single dependent variable, but in general multiple dependent variables will occur, in which case multiple differential equations are required to describe the problem.

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$$\varepsilon(t,x,y,z,\bullet) \times \frac{\partial(\bullet)}{\partial t} = \varphi(t,x,y,z,\bullet) \times L(\bullet) + \psi(t,x,y,z,\bullet)\,, \tag{2.1}$$

where ε and φ are parameters that may be functions of time and space, L is a spatial *differential operator* and ψ is the *source term*. Note that ε, φ and ψ may be functions of the dependent variable, in which case the equation is nonlinear. The left-hand term in Eq. (2.1) is known as the *accumulation term*, the first term at the right-hand side as the *transport term*. A specific example of the general PDE (2.1) is the mass conservation Eq. 1.4 in Sect. 1.3.1 in which case the spatial difference operator L is the divergence $\nabla(\bullet) \triangleq \partial(\bullet)/\partial x + \partial(\bullet)/\partial y + \partial(\bullet)/\partial z$. In addition to the PDE (2.1), we need to specify the spatial domain Ω and the time domain T on which it is valid. At the boundary Γ of Ω we need to specify boundary conditions, and at a specific point in time an initial condition, to complete the problem formulation. A PDE describes the evolution of the dependent variables in time and space in a continuous fashion, and the solution is therefore specified in an infinitely large number of points. Closed-form solutions of PDEs, i.e. to 'infinite dimensional problems', are generally restricted to simple domains and parameters that are spatially homogeneous. For more realistic geometries we need to solve the equations numerically, which requires some form of *discretization* of the equations, because digital computers can only deal with finite dimensional problems.

2.1.2 Ordinary-Differential Equations

An often followed approach is to first perform a spatial discretization of the PDEs, and only perform the time discretization at a later stage. The initial *semi-discretization* of the equations, i.e. the discretization in space, can be done using the method of finite differences, finite volumes or finite elements. An example of a finite-difference discretization as applied to porous-media flow has been worked out in Chap. 1. All of the discretization methods result in systems of *ordinary-differential equations* (ODEs) which can typically be represented as

$$\begin{cases} \hat{e}_1\left(\bullet_1, \dfrac{d(\bullet_1)}{dt}\right) = \hat{f}_1(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_1), \\ \hat{e}_2\left(\bullet_2, \dfrac{d(\bullet_2)}{dt}\right) = \hat{f}_2(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_2), \\ \qquad\qquad \vdots \\ \hat{e}_n\left(\bullet_n, \dfrac{d(\bullet_n)}{dt}\right) = \hat{f}_n(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_n), \end{cases} \tag{2.2}$$

where the *continuous* dependent variable $\bullet$ and the continuous source term ψ of Eq. (2.1) are now represented with a finite number of *discrete* values $\bullet_i$ and ψ_i, corresponding to discrete points in space, and where $\hat{e}_i$ and $\hat{f}_i$ are, in the general

case, nonlinear functions.[2] Normally the functions $\hat{e}_i$ are linear in the derivatives $d(\bullet_i)/dt$, which makes it possible to transform the system of Eq. (2.2) such that the derivatives in the left-hand side terms are isolated, leading to:

$$\begin{cases} \dfrac{d(\bullet_1)}{dt} = f_1(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_1), \\ \dfrac{d(\bullet_2)}{dt} = f_2(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_2), \\ \qquad \vdots \\ \dfrac{d(\bullet_n)}{dt} = f_n(t, \bullet_1, \bullet_2, \ldots, \bullet_n, \psi_n), \end{cases} \tag{2.3}$$

where the functions f_i are different from the functions $\hat{f}_i$ in Eq. (2.2). Note that Eqs. (2.2) and (2.3) are both coupled systems of ODEs because each of the dependent variables $\bullet_i$ is present in more than one equation. If the functions f_i are linear in the dependent variables, a further simplification is possible that decouples the equations leading to

$$\frac{d(\tilde{\bullet}_i)}{dt} = \tilde{f}_i\left(t, \tilde{\bullet}_i, \tilde{\psi}_i\right), \quad i = 1, 2, \ldots, n \tag{2.4}$$

where the transformed dependent variables $\tilde{\bullet}_i$ are linear combinations of the original variables $\bullet_i$, and $\tilde{f}_i$ are functions again. This decoupling procedure will be addressed in more detail in Sect. 3.1.2 below. In reservoir simulation the functions $\hat{e}_i$ and $\hat{f}_i$ are typically linear in the derivatives and nonlinear in the dependent variables, which at first sight implies that Eq. (2.3) is the most relevant representation. Moreover, system-theoretical results are usually derived using equations of this particular form. However, for large scale computations it is more efficient to use representation (2.2), and in this text we will therefore make use of both formulations.

2.1.3 State-Space Representation

2.1.3.1 State Equations

A more concise form of Eq. (2.3) can be obtained through the use of vector notation. We introduce the vectors $\mathbf{x} = [\,x_1 \quad x_2 \quad \ldots \quad x_n\,]^T$ and $\mathbf{u} = [\,u_1 \quad u_2 \quad \ldots \quad u_m\,]^T$ to represent the discrete values of the dependent variables, instead of $\bullet_i$ and ψ_i

[2] Typically, most of the values $\bullet_i$ are equal to zero in a single equation. For example in the case of one-dimensional single-phase flow modeled with first-order finite differences, the only three non-zero values $\bullet_i$ in the ith equation in a system of n equations with $1 < i < n$ are given by: $\hat{e}_i(\bullet_i, d(\bullet_i)/dt) = \hat{f}_i(\bullet_{i-1}, \bullet_i, \bullet_{i+1}, \psi_i)$.

which we used until now. The reason to use $\mathbf{x}$ and $\mathbf{u}$ is to adhere to the notation convention in the systems-and-control literature. Note that $x_1, x_2, \ldots x_n$ do *not* represent spatial coordinates. Also note that we have indicated that the source term $\mathbf{u}$ has m elements instead of n. This anticipates a situation where many of the source terms are equal to zero, such that $m \ll n$, in which case it may be computationally advantageous to use a shorter vector $\mathbf{u}$. Equations (2.3) can now be written as

$$\dot{\mathbf{x}}(t) = \mathbf{f}(t, \mathbf{u}(t), \dot{\mathbf{x}}(t)) \,, \tag{2.5}$$

where $\mathbf{f}$ is a nonlinear vector function of $\mathbf{x}$, $\mathbf{u}$ and t, and where we have emphasized the dependence of $\mathbf{x}$ and $\mathbf{u}$ on t by writing $\mathbf{x}(t)$ and $\mathbf{u}(t)$. In reservoir simulation, the equations are usually nonlinear but with coefficients that do not depend on time directly, such that we can write

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{u}(t), \mathbf{x}(t)) \,. \tag{2.6}$$

In the special case that $\mathbf{f}$ is a linear function of $\mathbf{x}$ and $\mathbf{u}$, we can use a vector-matrix notation and write Eq. (2.6) as a *linear time-varying* (LTV) vector differential equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \,, \tag{2.7}$$

where the coefficients of the $n \times n$ matrix $\mathbf{A}$ and the $n \times m$ matrix $\mathbf{B}$ may still be functions of t. The matrices $\mathbf{A}$ and $\mathbf{B}$ are usually referred to as the *system matrix* and the *input matrix* respectively.[3] In the case that $\mathbf{A}$ and $\mathbf{B}$ are independent of t, we obtain a *linear time-invariant* (LTI) equation given by

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \,. \tag{2.8}$$

From now on we will mostly not explicitly indicate the dependence on time of the variables, and we will write, e.g., $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x})$ instead of $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{u}(t), \mathbf{x}(t))$. First-order systems of equations such as (2.5), (2.6), (2.7) and (2.8) are referred to as *state equations* in the systems-and-control literature. In this representation, the elements of vector $\mathbf{x}$ are the state variables which completely define the dynamic state of the system. A continuous sequence of values of $\mathbf{x}$ over a certain time interval is often referred to as a *trajectory* in state space. In reservoir engineering applications it is sometimes preferred to start from Eqs. (2.2) rather than from Eq. (2.3), even if the functions e_i are linear. We will refer to equations of the type of Eq. (2.2) as *generalized state equations*.[4] In LTI form they can be written as

$$\hat{\mathbf{E}}\dot{\mathbf{x}} = \hat{\mathbf{A}}\mathbf{x} + \hat{\mathbf{B}}\mathbf{u} \,. \tag{2.9}$$

[3] An alternative name for the input matrix is *distribution matrix* because it distributes the inputs $\mathbf{u}$ over the states $\mathbf{x}$.

[4] Sometimes this form of generalized state equations is referred to as a *descriptor system*.

2.1.3.2 Output Equations

In addition to Eqs. (2.5) to (2.9), which define the relationship between the input vector $\mathbf{u}$ and the state vector $\mathbf{x}$ of a dynamic system, we can also define a relationship between an output vector $\mathbf{y}$ and the state $\mathbf{x}$. Moreover, the output may to some extent also depend directly on the input $\mathbf{u}$, such that we can write

$$\mathbf{y} = \mathbf{h}(\mathbf{u}, \mathbf{x}) \,, \tag{2.10}$$

for the nonlinear case or

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \,, \tag{2.11}$$

for the linear case, where $\mathbf{C}$ is known as the *output matrix* and $\mathbf{D}$ as the *direct-throughput matrix*. If the output vector $\mathbf{y}$ has p elements, the matrices $\mathbf{C}$ and $\mathbf{D}$ have dimensions $p \times n$ and $p \times m$, respectively.

2.1.3.3 Implicit Nonlinear Equations

In addition to the general nonlinear system functions (2.6) and (2.10) we will sometimes use even more general nonlinear functions

$$\mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0} \,, \tag{2.12}$$

$$\mathbf{j}(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \mathbf{0} \,, \tag{2.13}$$

where $\mathbf{g}$ and $\mathbf{j}$ are nonlinear vector-valued functions.[5] Note that the explicit Eqs. (2.6) and (2.10) can always simply be expressed in the implicit form of Eqs. (2.12) and (2.13), i.e.

$$\mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}) = \dot{\mathbf{x}} - \mathbf{f}(\mathbf{u}, \mathbf{x}) \,, \tag{2.14}$$

$$\mathbf{j}(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \mathbf{y} - \mathbf{h}(\mathbf{u}, \mathbf{x}) \,. \tag{2.15}$$

The reverse is not always true, i.e. it may not be possible to derive an explicit expression $\mathbf{f}$ for a given implicit representation $\mathbf{g}$. However, usually the implicit representation may be solved numerically for $\dot{\mathbf{x}}$, typically using some form of time discretization and an iterative algorithm. In that case we can still conceptually write the nonlinear equations in their explicit forms (2.6) and (2.10) which is often preferred for analysis purposes. In most cases the functions $\mathbf{f}$, $\mathbf{g}$, $\mathbf{h}$ and $\mathbf{j}$ are to be interpreted as numerical operations, e.g. $\mathbf{g}$ could represent a complete reservoir simulator. Detailed examples of the state variable description of reservoir systems will be discussed below.

[5] System equations expressed as $\mathbf{g}(\ldots) = \mathbf{0}$ are sometimes referred to as equations in *residual* form.

2.1.3.4 Error Terms

In systems-and-control applications it is customary to introduce *error terms* to account for the fact that a system description is only an approximation of reality. For example we can write

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} + \boldsymbol{\varepsilon}\,, \tag{2.16}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} + \boldsymbol{\eta}\,, \tag{2.17}$$

for the linear case, where $\boldsymbol{\varepsilon}$ is called the *model error* and $\boldsymbol{\eta}$ the *measurement error*. Both are random variables[6] which are often, although not necessarily, taken as zero-mean Gaussian. As a result of the random error terms, $\mathbf{x}$ and $\mathbf{y}$ also become random variables such that to completely quantify them it will be necessary to specify their probability distributions as functions of time. In the special case of linear equations, Gaussian error terms will result in Gaussian states and outputs which can be completely specified by their first and second moments (mean values and covariance matrices). In the more general, nonlinear, case it will be necessary to specify higher moments or ensembles of representative *realizations* of $\mathbf{x}$ and $\mathbf{y}$. In reservoir simulation it is not customary to introduce error terms that are additive to the states (as in Eq. (2.16)). Instead it is much more common to consider the parameters of the system equations, in particular the grid-block permeabilities, as uncertain. Typically these parameter uncertainties are considered to be so large that they dominate the model errors. Measurement errors are normally introduced in computer-assisted history matching; see e.g. Oliver et al. (2008). In this text we will not make use of error terms in the system description.

2.1.4 Linearized Equations

2.1.4.1 Jacobians

To analyze the nature of nonlinear system equations $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x})$, or to approximate their solution through numerical computation, it is usually necessary to linearize them around a point in state-input space. Using a Taylor expansion and starting from Eq. (2.6) we can write:

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x}) \approx \mathbf{f}\left(\mathbf{u}^0, \mathbf{x}^0\right) + \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{u}}\left(\mathbf{u} - \mathbf{u}^0\right) + \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{x}}\left(\mathbf{x} - \mathbf{x}^0\right), \tag{2.18}$$

where we have neglected terms of second order and higher, and applied the usual short-cut notation

[6] The random model errors are also referred to as *random input*, or as a *stochastic forcing term*.

$$\frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{x}} \triangleq \left.\frac{\partial \mathbf{f}(\mathbf{u}, \mathbf{x})}{\partial \mathbf{x}}\right|_{\mathbf{u}=\mathbf{u}^0,\, \mathbf{x}=\mathbf{x}^0} . \tag{2.19}$$

Defining

$$\bar{\mathbf{u}} \triangleq \mathbf{u} - \mathbf{u}^0 , \tag{2.20}$$

$$\bar{\mathbf{x}} \triangleq \mathbf{x} - \mathbf{x}^0 , \tag{2.21}$$

we can rewrite Eq. (2.18) as

$$\dot{\bar{\mathbf{x}}} + \dot{\mathbf{x}}^0 \approx \mathbf{f}\left(\mathbf{u}^0, \mathbf{x}^0\right) + \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{u}} \bar{\mathbf{u}} + \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{x}} \bar{\mathbf{x}} , \tag{2.22}$$

which, because

$$\dot{\mathbf{x}}^0 = \mathbf{f}\left(\mathbf{u}^0, \mathbf{x}^0\right) , \tag{2.23}$$

can be reduced to the *linearized system equations*

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\left(\mathbf{u}^0, \mathbf{x}^0\right)\bar{\mathbf{x}} + \bar{\mathbf{B}}\left(\mathbf{u}^0, \mathbf{x}^0\right)\bar{\mathbf{u}} , \tag{2.24}$$

where the *Jacobian matrices*[7] $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are defined as

$$\bar{\mathbf{A}}\left(\mathbf{u}^0, \mathbf{x}^0\right) \triangleq \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{x}} , \tag{2.25}$$

$$\bar{\mathbf{B}}\left(\mathbf{u}^0, \mathbf{x}^0\right) \triangleq \frac{\partial \mathbf{f}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{u}} . \tag{2.26}$$

In a similar fashion we can linearize a nonlinear output function $\mathbf{y} = \mathbf{h}(\mathbf{u}, \mathbf{x})$ to obtain

$$\bar{\mathbf{y}} = \bar{\mathbf{C}}\left(\mathbf{u}^0, \mathbf{x}^0\right)\bar{\mathbf{x}} + \bar{\mathbf{D}}\left(\mathbf{u}^0, \mathbf{x}^0\right)\bar{\mathbf{u}} , \tag{2.27}$$

where the Jacobians $\bar{\mathbf{C}}$ and $\bar{\mathbf{D}}$ are defined as

$$\bar{\mathbf{C}}\left(\mathbf{u}^0, \mathbf{x}^0\right) \triangleq \frac{\partial \mathbf{h}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{x}} , \tag{2.28}$$

$$\bar{\mathbf{D}}\left(\mathbf{u}^0, \mathbf{x}^0\right) \triangleq \frac{\partial \mathbf{h}(\mathbf{u}^0, \mathbf{x}^0)}{\partial \mathbf{u}} . \tag{2.29}$$

If the system and output equations are given in implicit form $\mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}) = \mathbf{0}$ and $\mathbf{j}(\mathbf{u}, \mathbf{x}, \mathbf{y}) = \mathbf{0}$ we obtain linearized equations in terms of Jacobians

$$\bar{\hat{\mathbf{A}}} \triangleq \frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}{\partial \mathbf{x}} , \tag{2.30}$$

[7] Usually simply referred to as *Jacobians*.

$$\bar{\hat{\mathbf{B}}} \triangleq \frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}{\partial \mathbf{u}}, \tag{2.31}$$

$$\bar{\hat{\mathbf{E}}} \triangleq \frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}})}{\partial \dot{\mathbf{x}}}, \tag{2.32}$$

$$\bar{\hat{\mathbf{C}}} \triangleq \frac{\partial \mathbf{j}(\mathbf{u}, \mathbf{x}, \mathbf{y})}{\partial \mathbf{x}} \tag{2.33}$$

$$\bar{\hat{\mathbf{D}}} \triangleq \frac{\partial \mathbf{j}(\mathbf{u}, \mathbf{x}, \mathbf{y})}{\partial \mathbf{u}}, \tag{2.34}$$

$$\bar{\hat{\mathbf{F}}} \triangleq \frac{\partial \mathbf{g}(\mathbf{u}, \mathbf{x}, \mathbf{y})}{\partial \mathbf{y}}, \tag{2.35}$$

where we have dropped the superscripts 0 for clarity.

2.1.4.2 Secant and Tangent Matrices

In reservoir simulation one often encounters systems $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x})$ that can be expressed in the form[8]

$$\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}. \tag{2.36}$$

In that case we obtain the linearized Eq. (2.24) with Jacobians defined as

$$\bar{\mathbf{A}}\left(\mathbf{u}^0, \mathbf{x}^0\right) \triangleq \mathbf{A}\left(\mathbf{x}^0\right) + \frac{\partial \mathbf{A}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{x}^0 + \frac{\partial \mathbf{B}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{u}^0, \tag{2.37}$$

$$\bar{\mathbf{B}}\left(\mathbf{x}^0\right) \triangleq \mathbf{B}\left(\mathbf{x}^0\right). \tag{2.38}$$

If we linearize the state equations along all points of a given trajectory $(\mathbf{x}^0(t), \mathbf{u}^0(t))$ in state-input space, the resulting model is referred to as the *tangent-linear* approximation of the nonlinear model, or simply the tangent-linear model. The Jacobians $\bar{\mathbf{A}}$ and $\bar{\mathbf{B}}$ are therefore also referred to as the *tangent matrices* of the system. Note that the matrices $\mathbf{A}$ and $\mathbf{B}$ are not tangent matrices because they do not describe the system dynamics tangent to the state trajectory. Instead they can be interpreted as *secant matrices*; see Fig. 2.1.

[8] In the systems-and-control literature this is known as a *control-affine* nonlinear equation. An affine function is a linear function plus a translation. Control affine functions are an important topic of study in nonlinear control theory.

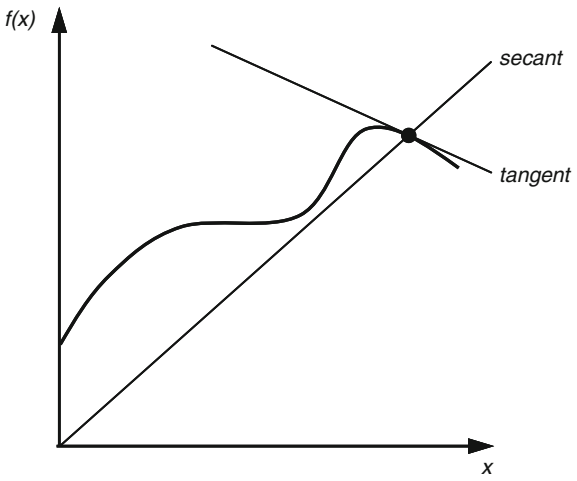

Fig. 2.1 The secant and the tangent to a function $f(x)$ in a point $(x^0, f(x^0))$

2.1.4.3 Generalized State-Space Form*

In reservoir simulation we often encounter systems that can be expressed in the generalized state-space form

$$\hat{\mathbf{E}}(\mathbf{x})\dot{\mathbf{x}} = \hat{\mathbf{A}}(\mathbf{x})\mathbf{x} + \hat{\mathbf{B}}(\mathbf{x})\mathbf{u}\,. \tag{2.39}$$

In that case we can linearize around a point $(\mathbf{u}^0, \mathbf{x}^0, \dot{\mathbf{x}}^0)$ to obtain the linearized equations

$$\bar{\hat{\mathbf{E}}}(\mathbf{x}^0)\dot{\bar{\mathbf{x}}} = \bar{\hat{\mathbf{A}}}(\mathbf{u}^0, \mathbf{x}^0, \dot{\mathbf{x}}^0)\bar{\mathbf{x}} + \bar{\hat{\mathbf{B}}}(\mathbf{x}^0)\bar{\mathbf{u}}\,, \tag{2.40}$$

with Jacobians defined as

$$\bar{\hat{\mathbf{A}}}(\mathbf{u}^0, \mathbf{x}^0, \dot{\mathbf{x}}^0) \triangleq \hat{\mathbf{A}}(\mathbf{x}^0) + \frac{\partial \hat{\mathbf{A}}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{x}^0 + \frac{\partial \hat{\mathbf{B}}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{u}^0 - \frac{\partial \hat{\mathbf{E}}(\mathbf{x}^0)}{\partial \mathbf{x}}\dot{\mathbf{x}}^0\,, \tag{2.41}$$

$$\bar{\hat{\mathbf{B}}}(\mathbf{x}^0) \triangleq \hat{\mathbf{B}}(\mathbf{x}^0)\,, \tag{2.42}$$

$$\bar{\hat{\mathbf{E}}}(\mathbf{x}^0) \triangleq \hat{\mathbf{E}}(\mathbf{x}^0)\,. \tag{2.43}$$

However, for analysis purposes it is normally more useful to bring this equation in the linearized ordinary state-space form (2.24):

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x}) = \underbrace{\left(\hat{\mathbf{E}}(\mathbf{x})\right)^{-1}\hat{\mathbf{A}}(\mathbf{x})}_{\mathbf{A}(\mathbf{x})}\mathbf{x} + \underbrace{\left(\hat{\mathbf{E}}(\mathbf{x})\right)^{-1}\hat{\mathbf{B}}(\mathbf{x})}_{\mathbf{B}(\mathbf{x})}\mathbf{u}\,, \tag{2.44}$$

in which case the Jacobians $\bar{\mathbf{A}} = \partial\mathbf{f}(\mathbf{u}, \mathbf{x})/\partial\mathbf{x}$ and $\bar{\mathbf{B}} = \partial\mathbf{f}(\mathbf{u}, \mathbf{x})/\partial\mathbf{u}$ can be obtained as

$$\bar{\mathbf{A}}(\mathbf{u}^0,\mathbf{x}^0) \triangleq \left(\hat{\mathbf{E}}(\mathbf{x}^0)\right)^{-1} \left\{ \begin{array}{l} \hat{\mathbf{A}}(\mathbf{x}^0) + \left[\dfrac{\partial \hat{\mathbf{A}}(\mathbf{x}^0)}{\partial \mathbf{x}} - \dfrac{\partial \hat{\mathbf{E}}(\mathbf{x}^0)}{\partial \mathbf{x}} \left(\hat{\mathbf{E}}(\mathbf{x}^0)\right)^{-1} \hat{\mathbf{A}}(\mathbf{x}^0)\right] \mathbf{x}^0 \\ \quad + \left[\dfrac{\partial \hat{\mathbf{B}}(\mathbf{x}^0)}{\partial \mathbf{x}} - \dfrac{\partial \hat{\mathbf{E}}(\mathbf{x}^0)}{\partial \mathbf{x}} \left(\hat{\mathbf{E}}(\mathbf{x}^0)\right)^{-1} \hat{\mathbf{B}}(\mathbf{x}^0)\right] \mathbf{u}^0 \end{array} \right\}, \tag{2.45}$$

$$\bar{\mathbf{B}}(\mathbf{x}^0) \triangleq \left(\hat{\mathbf{E}}(\mathbf{x}^0)\right)^{-1} \mathbf{B}(\mathbf{x}^0)\,. \tag{2.46}$$

Alternatively we can write the generalized state Eq. (2.39) in implicit form

$$\hat{\mathbf{g}}(\mathbf{u},\mathbf{x},\dot{\mathbf{x}}) \triangleq \hat{\mathbf{E}}(\mathbf{x})\dot{\mathbf{x}} - \hat{\mathbf{A}}(\mathbf{x})\mathbf{x} - \hat{\mathbf{B}}(\mathbf{x})\mathbf{u} = \mathbf{0}\,, \tag{2.47}$$

and use implicit differentiation to obtain the Jacobian $\bar{\mathbf{A}}$ related to the ordinary state-space representation. I.e., because

$$\frac{d\hat{\mathbf{g}}}{d\mathbf{x}} = \frac{\partial \hat{\mathbf{g}}}{\partial \mathbf{x}} + \frac{\partial \hat{\mathbf{g}}}{\partial \dot{\mathbf{x}}}\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = \mathbf{0}\,, \tag{2.48}$$

we have

$$\frac{\partial \dot{\mathbf{x}}}{\partial \mathbf{x}} = -\left(\frac{\partial \hat{\mathbf{g}}}{\partial \dot{\mathbf{x}}}\right)^{-1}\frac{\partial \hat{\mathbf{g}}}{\partial \mathbf{x}}\,, \tag{2.49}$$

and because $\bar{\mathbf{A}} = \partial \mathbf{f}(\mathbf{u},\mathbf{x})/\partial \mathbf{x} = \partial \dot{\mathbf{x}}/\partial \mathbf{x}$ we find that

$$\bar{\mathbf{A}}(\mathbf{u}^0,\mathbf{x}^0,\dot{\mathbf{x}}^0) = \left(\hat{\mathbf{E}}(\mathbf{x}^0)\right)^{-1}\left[\hat{\mathbf{A}}(\mathbf{x}^0) + \frac{\partial \hat{\mathbf{A}}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{x}^0 + \frac{\partial \hat{\mathbf{B}}(\mathbf{x}^0)}{\partial \mathbf{x}}\mathbf{u}^0 - \frac{\partial \hat{\mathbf{E}}(\mathbf{x}^0)}{\partial \mathbf{x}}\dot{\mathbf{x}}^0\right], \tag{2.50}$$

which, with the aid of Eqs. (2.41) and (2.43), can also be expressed as

$$\bar{\mathbf{A}}(\mathbf{u}^0,\mathbf{x}^0,\dot{\mathbf{x}}^0) = \left(\bar{\hat{\mathbf{E}}}(\mathbf{x}^0)\right)^{-1}\bar{\hat{\mathbf{A}}}(\mathbf{u}^0,\mathbf{x}^0,\dot{\mathbf{x}}^0)\,. \tag{2.51}$$

2.2 Single-Phase Flow

2.2.1 System Equations

As a first application, we consider flow of a weakly-compressible single-phase liquid through a porous medium. The derivation of the governing PDEs and the semi-discretization has been presented in Chap. 1. We used a finite difference discretization, but the following theory is equally applicable to results derived with

other semi-discretization methods. Use of any of the methods produces a system of ODEs that can be written in matrix form as:

$$\mathbf{V}\dot{\mathbf{p}} + \mathbf{T}\mathbf{p} = \mathbf{q}\,. \tag{2.52}$$

Here $\mathbf{V}$ and $\mathbf{T}$ are matrices with entries that depend on dynamic and static reservoir properties, $\mathbf{p}$ is a vector of pressures and $\mathbf{q}$ is a vector of volumetric flow rates. $\mathbf{V}$ is a diagonal matrix known as the *accumulation matrix* and $\mathbf{T}$ is a symmetric banded matrix, known as the *transmissibility matrix*. The flow rates $\mathbf{q}$ correspond to flow into or out of the reservoir, i.e. to wells, and are expressed in m^3/s. Positive values imply injection and negative values imply production. Because usually only a few grid blocks are penetrated by wells, only a few elements of $\mathbf{q}$ have a non-zero value. In the case of a reservoir modeled with n grid blocks and produced with m wells, $\mathbf{V}$ and $\mathbf{T}$ would be $n \times n$ matrices, and $\mathbf{p}$ and $\mathbf{q}$ would be $n \times 1$ vectors, of which $\mathbf{q}$ would have m non-zero entries. Equation (2.52) can be re-casted in state variable form (2.8) through definition of

$$\mathbf{u} \triangleq \mathbf{L}_{uq}\mathbf{q}\,, \tag{2.53}$$

$$\mathbf{x} \triangleq \mathbf{p}\,. \tag{2.54}$$

In single-phase flow the state variables $\mathbf{x}$ are just identical to the pressures $\mathbf{p}$. The vector $\mathbf{u}$ represents the inputs to the system, which are in our case the non-zero elements of the flow rate vector $\mathbf{q}$. The matrix $\mathbf{L}_{uq}$ is therefore a *location matrix*, also known as a *selection matrix* which contains only ones and zeros at the appropriate places. The inverse relationship is given by

$$\mathbf{q} = \mathbf{L}_{qu}\mathbf{u}\,, \tag{2.55}$$

where

$$\mathbf{L}_{qu} = \mathbf{L}_{uq}^{T}\,, \tag{2.56}$$

Substitution of relationships (2.53) and (2.55) in Eq. (2.52) results in the generalized state-space representation (2.9) with matrices $\hat{\mathbf{A}}$, $\hat{\mathbf{B}}$ and $\hat{\mathbf{E}}$ given by:

$$\hat{\mathbf{A}} \triangleq -\mathbf{T}\,, \tag{2.57}$$

$$\hat{\mathbf{B}} \triangleq \mathbf{L}_{qu}\,, \tag{2.58}$$

$$\hat{\mathbf{E}} \triangleq \mathbf{V}\,. \tag{2.59}$$

The ordinary state-space form (2.8) is obtained by defining the matrices $\mathbf{A}$ and $\mathbf{B}$ as[9]

[9] In a numerical implementation, the inverse $\mathbf{V}^{-1}$ of the diagonal matrix $\mathbf{V}$ can be computed very efficiently by just taking the reciprocals of the diagonal elements.

$$\mathbf{A} \triangleq -\mathbf{V}^{-1}\mathbf{T}\,, \tag{2.60}$$

$$\mathbf{B} \triangleq \mathbf{V}^{-1}\mathbf{L}_{qu}\,. \tag{2.61}$$

If we choose the output vector $\mathbf{y}$ to consist of only those pressures that are accessible to measurements, the matrix $\mathbf{C}$ is therefore also a selection matrix. Matrix $\mathbf{D}$ is zero because there is no direct dependency of the output on the input. In reality the outputs are usually surface measurements of the tubing-head pressure, and therefore we should include a description of the dynamic behavior of the well between the reservoir and the surface. However, as a first assumption, we neglect well dynamics and assume that the wells are equipped with permanent down hole gauges (PDGs) to measure the pressures. In the case of a reservoir modeled with n grid blocks and produced with m wells, of which m_p contain PDGs, matrices $\mathbf{A}$ and $\mathbf{B}$ have dimension $n \times n$, matrix $\mathbf{C}$ dimension $m_p \times n$, and vectors $\mathbf{u}$, $\mathbf{x}$ and $\mathbf{y}$ dimensions $m \times 1$, $n \times 1$ and $m_p \times 1$ respectively. Matrix Eq. (2.8) represents a system of linear first order ODEs with constant coefficients, i.e. an LTI system. Starting from an initial value $\breve{\mathbf{x}}$, the ODEs for $\mathbf{x}$ can be integrated in time, and because the equations are linear the solution can be expressed analytically. Alternatively, the integration can be performed numerically as will be discussed in Chap. 3.

2.2.2 Example 1 Continued—Location Matrix

Reconsidering the six-grid-block example introduced in Sect. 1.3.3, the location matrix $\mathbf{L}_{uq}$ as defined in Eq. (2.53) is given by

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{L}_{uq}} \underbrace{\begin{bmatrix} q_1 \\ 0 \\ 0 \\ 0 \\ 0 \\ q_6 \end{bmatrix}}_{\mathbf{q}}\,. \tag{2.62}$$

If the output $\mathbf{y}$ consists of the pressures in the two wells, the output matrix $\mathbf{C} = \mathbf{L}_{qu} = \mathbf{L}_{uq}^T$ is given by:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}}_{\mathbf{x}}\,. \tag{2.63}$$

2.2.3 Prescribed Pressures and Flow Rates

Until now we have assumed that the source terms, i.e. the flow rates in the wells, were the input variables, and that their values can be prescribed as a function of time. However, it is also possible to control the system by prescribing the state variables, i.e. the pressures in the wells. Note that it is not possible to prescribe both pressure and flow rate in a well; either one of them should be fixed and the other left free, or a mathematical relationship between them should be specified which may be in algebraic or differential form. The most commonly used method in reservoir engineering is through the definition of a *well model*, which is an algebraic relationship between the grid-block pressure and the well flow rate. Alternatively, one of the pressures may be prescribed directly, resulting in a reduction of the length of the state vector with one element. We will discuss both methods in the following sections. In order to take into account prescribed pressures and flow rates in a structured way it is convenient to re-order the variables in Eq. (2.52) such that the prescribed and the non-prescribed, free, values are grouped. In addition, we take the opportunity to make a distinction between prescribed flow rates in grid blocks with and without wells.[10] We can formally describe the re-ordering with the aid of a *permutation matrix* $\mathbf{L}$ as:

$$\mathbf{p}^* \triangleq \begin{bmatrix} \mathbf{p}_1^* \\ \mathbf{p}_2^* \\ \mathbf{p}_3^* \end{bmatrix} \equiv \mathbf{L}_{p*p}\mathbf{p}\,, \tag{2.64}$$

where $\mathbf{p}_1^*$ are the pressures in the grid blocks that are not penetrated by a well, i.e. where the source terms $\mathbf{q}_1^*$ are equal to zero, $\mathbf{p}_2^*$ are the pressures in the blocks where the source terms $\mathbf{q}_2^*$ are prescribed as well flow rates, and $\mathbf{p}_3^*$ are the pressures in the blocks where the source terms $\mathbf{q}_3^*$ are obtained through prescription of the bottom-hole pressures in the wells. Similarly we can write

$$\mathbf{q}^* \triangleq \begin{bmatrix} \mathbf{0} \\ \mathbf{q}_2^* \\ \mathbf{q}_3^* \end{bmatrix} \equiv \mathbf{L}_{q*q}\mathbf{q}\,, \tag{2.65}$$

where we choose

$$\mathbf{L}_{q*q} = \mathbf{L}_{p*p}\,, \tag{2.66}$$

which means that we re-order the elements of $\mathbf{p}$ and $\mathbf{q}$ in Eq. (2.52) identically, i.e. we interchange the rows of the equations. The permutation matrix $\mathbf{L}_{p*p} = \mathbf{L}_{q*q}$ is an identity matrix with interchanged rows. Permutation matrices are orthogonal, which implies that

[10] In grid blocks that are not penetrated by a well the prescribed flow rates are of course equal to zero.

$$\mathbf{L}_{p*p}\mathbf{L}^{T}{}_{p*p} = \mathbf{I}\,. \tag{2.67}$$

The inverse relationships corresponding to expressions (2.64) and (2.65) are therefore given by

$$\mathbf{p} = \mathbf{L}_{pp*}\mathbf{p}^{*}\,, \tag{2.68}$$

$$\mathbf{q} = \mathbf{L}_{qq*}\mathbf{q}^{*}\,, \tag{2.69}$$

where

$$\mathbf{L}_{pp*} = \mathbf{L}_{qq*} = \mathbf{L}^{T}_{p*p} = \mathbf{L}^{T}_{q*q}\,. \tag{2.70}$$

Substitution of Eqs. (2.68) and (2.69) in Eq. (2.52) and reorganizing the terms results in

$$\mathbf{V}^{*}\dot{\mathbf{p}}^{*} + \mathbf{T}^{*}\mathbf{p}^{*} = \mathbf{q}^{*}\,, \tag{2.71}$$

where $\mathbf{T}^{*}$ and $\mathbf{V}^{*}$ are given by

$$\mathbf{T}^{*} = \mathbf{L}_{q*q}\mathbf{T}\mathbf{L}_{pp*}\,, \tag{2.72}$$

$$\mathbf{V}^{*} = \mathbf{L}_{q*q}\mathbf{V}\mathbf{L}_{pp*}\,. \tag{2.73}$$

Equation (2.71) can be written in partitioned form as:

$$\begin{bmatrix} \mathbf{V}^{*}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}^{*}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}^{*}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}^{*}_{1} \\ \dot{\mathbf{p}}^{*}_{2} \\ \dot{\mathbf{p}}^{*}_{3} \end{bmatrix} + \begin{bmatrix} \mathbf{T}^{*}_{11} & \mathbf{T}^{*}_{12} & \mathbf{T}^{*}_{13} \\ \mathbf{T}^{*}_{21} & \mathbf{T}^{*}_{22} & \mathbf{T}^{*}_{23} \\ \mathbf{T}^{*}_{31} & \mathbf{T}^{*}_{32} & \mathbf{T}^{*}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}^{*}_{1} \\ \mathbf{p}^{*}_{2} \\ \mathbf{p}^{*}_{3} \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{q}^{*}_{2} \\ \mathbf{q}^{*}_{3} \end{bmatrix}. \tag{2.74}$$

Note that the diagonal structure of matrix $\mathbf{V}$ has been maintained in $\mathbf{V}^{*}$. We can also apply the partitioning to the state-space representation, in which case we may choose not to partition $\mathbf{u}$ and $\mathbf{y}$, or to partition them also. We choose to partition them, according to

$$\mathbf{u}^{*} \triangleq \mathbf{L}_{u^{*}u}\mathbf{u}\,, \tag{2.75}$$

$$\mathbf{y}^{*} \triangleq \mathbf{L}_{y^{*}y}\mathbf{y}\,, \tag{2.76}$$

where details of the partitioning are left open for the moment. Substitution of Eqs. (2.68), (2.69) and the inverse of Eq. (2.75) in Eqs. (2.53) and (2.54) results in

$$\mathbf{u}^{*} = \mathbf{L}_{u^{*}q^{*}}\mathbf{q}^{*}\,, \tag{2.77}$$

$$\mathbf{x}^{*} = \mathbf{p}^{*}\,, \tag{2.78}$$

where

$$\mathbf{L}_{u^{*}q^{*}} = \mathbf{L}_{u^{*}u}\mathbf{L}_{uq}\mathbf{L}_{qq^{*}}\,, \tag{2.79}$$

$$\mathbf{x}^{*} = \mathbf{L}_{p^{*}p}\mathbf{x}\,. \tag{2.80}$$

The partitioned state-space representation can then be written as

$$\dot{\mathbf{x}}^* = \mathbf{A}^*\mathbf{x}^* + \mathbf{B}^*\mathbf{u}^*\,, \tag{2.81}$$

$$\mathbf{y}^* = \mathbf{C}^*\mathbf{x}^*\,, \tag{2.82}$$

where

$$\mathbf{A}^* \triangleq -\left(\mathbf{V}^*\right)^{-1}\mathbf{T}^* = \mathbf{L}_{q*q}\mathbf{A}\mathbf{L}_{pp*}\,, \tag{2.83}$$

$$\mathbf{B}^* \triangleq \left(\mathbf{V}^*\right)^{-1}\mathbf{L}_{q^*u^*} = \mathbf{L}_{q*q}\mathbf{B}\mathbf{L}_{uu*}\,, \tag{2.84}$$

$$\mathbf{C}^* \triangleq \mathbf{L}_{y*y}\mathbf{C}\mathbf{L}_{pp*}\,. \tag{2.85}$$

The re-ordering of vector and matrix elements using permutation matrices as described above is a formal technique. It results in partitioned vectors and matrices that allow for a structured handling of prescribed pressures. However, for a numerical implementation it is not essential to actually perform the re-ordering. In the following we will therefore omit the star superscripts and simply work with partitioned matrices without the use of permutation matrices.

2.2.4 Well Models

2.2.4.1 Prescribed Bottom-Hole Pressures and Well Flow Rates

The standard approach in reservoir simulation to prescribe bottom-hole pressures is through the definition of a *well model*. In that case the flow rate q in the grid block where we want to prescribe the pressure is defined as

$$q = J_{well}\left(\breve{p}_{well} - p\right)\,, \tag{2.86}$$

where $\breve{p}_{well}$ is the prescribed bottom-hole pressure, p is the grid-block pressure and J_{well} is called the well index or productivity index. The well index is a function of the grid-block geometry and reflects the effect of near-well flow which is normally not properly represented by the finite-difference discretization because the grid block dimensions are usually much larger than the well diameter; see also Sect. 1.3.6. Note that, in line with our convention, a negative flow rate indicates production. Use of Eq. (2.86) can be interpreted as specifying an algebraic relationship between the state variable (i.e. the pressure) and the source term (i.e. the flow rate) in the grid block that contains the well. Equation (2.86) can be generalized for multiple wells to

$$\mathbf{q}_3 = \mathbf{J}_3\left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3\right)\,, \tag{2.87}$$

where $\mathbf{J}_3$ is a diagonal matrix of well indices J_{well}, and $\breve{\mathbf{p}}_{well}$ is a vector of prescribed bottom-hole pressures. In a similar fashion, we can write

$$\mathbf{q}_2 = \breve{\mathbf{q}}_{well}\,, \tag{2.88}$$

where $\breve{\mathbf{q}}_{well}$ are the prescribed well rates. If we combine Eqs. (2.87) and (2.88) with Eq. (2.74), and reorganize terms, we obtain

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} + \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \mathbf{J}_3 \breve{\mathbf{p}}_{well} \end{bmatrix}. \tag{2.89}$$

An important aspect of the introduction of the well model is that, compared to matrix $\mathbf{T}$ in Eq. (2.52), the transmissibility matrix in Eq. (2.89) has elements added to its main diagonal. We will discuss the consequences of this addition in Chap. 3.

2.2.4.2 Free Bottom-Hole Pressures and Well Flow Rates

The flow rates $\bar{\mathbf{q}}_{well} = \mathbf{q}_3$ in the wells where the bottom-hole pressures have been prescribed can be obtained directly from Eq. (2.87) as

$$\bar{\mathbf{q}}_{well} = \mathbf{J}_3 \left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3 \right). \tag{2.90}$$

To compute the bottom-hole pressures $\bar{\mathbf{p}}_{well}$ in the wells where the flow rates have been prescribed we need an additional diagonal matrix $\mathbf{J}_{q,2}$ of well indices J_q. We can then write

$$\breve{\mathbf{q}}_{well} = \mathbf{J}_2 (\bar{\mathbf{p}}_{well} - \mathbf{p}_2)\,, \tag{2.91}$$

from which we obtain

$$\bar{\mathbf{p}}_{well} = \mathbf{p}_2 + \mathbf{J}_2^{-1} \breve{\mathbf{q}}_{well}. \tag{2.92}$$

2.2.4.3 State-Space Representation

If we define the (partitioned) state, input and output vectors

$$\mathbf{x} \triangleq \mathbf{p} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}, \tag{2.93}$$

$$\mathbf{u} \triangleq \begin{bmatrix} \breve{\mathbf{q}}_{well} \\ \breve{\mathbf{p}}_{well} \end{bmatrix}, \tag{2.94}$$

$$\mathbf{y} \triangleq \begin{bmatrix} \bar{\mathbf{p}}_{well} \\ \bar{\mathbf{q}}_{well} \end{bmatrix}, \tag{2.95}$$

equations (2.89), (2.90) and (2.92) can be rewritten in state-space form as

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}\,, \tag{2.96}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}\,, \tag{2.97}$$

where the matrices are defined as

$$\mathbf{A} \triangleq -\begin{bmatrix} \mathbf{V}_{11}^{-1}\mathbf{T}_{11} & \mathbf{V}_{11}^{-1}\mathbf{T}_{12} & \mathbf{V}_{11}^{-1}\mathbf{T}_{13} \\ \mathbf{V}_{22}^{-1}\mathbf{T}_{21} & \mathbf{V}_{22}^{-1}\mathbf{T}_{22} & \mathbf{V}_{22}^{-1}\mathbf{T}_{23} \\ \mathbf{V}_{33}^{-1}\mathbf{T}_{31} & \mathbf{V}_{33}^{-1}\mathbf{T}_{32} & \mathbf{V}_{33}^{-1}(\mathbf{T}_{33}+\mathbf{J}_3) \end{bmatrix}, \tag{2.98}$$

$$\mathbf{B} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{V}_{22}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{33}^{-1}\mathbf{J}_3 \end{bmatrix}, \tag{2.99}$$

$$\mathbf{C} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{J}_3 \end{bmatrix}, \tag{2.100}$$

$$\mathbf{D} \triangleq \begin{bmatrix} \mathbf{J}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_3 \end{bmatrix}. \tag{2.101}$$

2.2.5 Example 1 Continued—Well Model

In Example 1, discussed in Sec. 2.2.2, we fixed the flow rates in both wells. Here, we fix the bottom-hole pressure of the producer in grid block 6 as: $p_{well} = 28.00 \times 10^6$ Pa (4061 psi), while we choose an injection rate in block 1 as $q_1 = 0.01$ m^3/s (864 m^3/d, 5434 bpd), where we use the convention that positive flow rates indicate injection. Because we only have one well with a prescribed pressure and one with a prescribed rate, we have

$$\breve{\mathbf{p}}_{well} = \left[28.00 \times 10^6\right], \tag{2.102}$$

$$\breve{\mathbf{q}}_{well} = [0.01]\,. \tag{2.103}$$

Correspondingly, the matrices $\mathbf{J}_3$ and $\mathbf{J}_2$ contain only one element each. Using the data for the near-well permeabilities as derived in Sect. 1.3.6 they become

$$\mathbf{J}_3 = \left[3.72 \times 10^{-9}\right], \tag{2.104}$$

$$\mathbf{J}_2 = \left[3.72 \times 10^{-8}\right]. \tag{2.105}$$

2.2.6 Elimination of Prescribed Pressures*

An alternative way to implement a prescribed pressure is through directly prescribing the grid-block pressure. This means that one of the state variables is fixed, and may be eliminated from the system equations. To illustrate this method, we start again from the partitioned system Eq. (2.74). We indicate prescribed values with a '˘' above the variable, and free values with a '–':

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \\ \dot{\breve{\mathbf{p}}}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{p}}_1 \\ \bar{\mathbf{p}}_2 \\ \breve{\mathbf{p}}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_2 \\ \bar{\mathbf{q}}_3 \end{bmatrix}. \tag{2.106}$$

From the first two rows of matrix Eq. (2.106) we find the system of ODEs

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} \end{bmatrix} \begin{bmatrix} \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \end{bmatrix} = -\begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{p}}_1 \\ \bar{\mathbf{p}}_2 \end{bmatrix} \underbrace{- \begin{bmatrix} \mathbf{T}_{13} \\ \mathbf{T}_{23} \end{bmatrix} \breve{\mathbf{p}}_3 + \begin{bmatrix} 0 \\ \breve{\mathbf{q}}_2 \end{bmatrix}}_{\text{prescribed}}. \tag{2.107}$$

Because we eliminated the prescribed pressures, the length of the pressure vector has been reduced. From the third row of Eq. (2.106) it follows that

$$\bar{\mathbf{q}}_3 = \begin{bmatrix} \mathbf{T}_{31} & \mathbf{T}_{32} \end{bmatrix} \begin{bmatrix} \bar{\mathbf{p}}_1 \\ \bar{\mathbf{p}}_2 \end{bmatrix} + \underbrace{\mathbf{T}_{33}\breve{\mathbf{p}}_3 + \mathbf{V}_{33}\dot{\breve{\mathbf{p}}}}_{\text{prescribed}}, \tag{2.108}$$

where $\bar{\mathbf{q}}_3$ represents the free flow rates in the wells where the pressures have been prescribed. Apparently the price to pay for the reduced length of the pressure vector is an increase in the number of input parameters to compute the free flow rates in case of time-varying prescribed pressures. Equations (2.107) and (2.108) can be rewritten in partitioned state-space form, as in Eqs. (2.96) and (2.97), through definition of

$$\mathbf{A} \triangleq -\begin{bmatrix} \mathbf{V}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} \\ \mathbf{T}_{21} & \mathbf{T}_{22} \end{bmatrix}, \tag{2.109}$$

$$\mathbf{x} \triangleq \begin{bmatrix} \dot{\bar{\mathbf{p}}}_1 \\ \dot{\bar{\mathbf{p}}}_2 \end{bmatrix}, \tag{2.110}$$

$$\mathbf{B} \triangleq \begin{bmatrix} \mathbf{0} & -\mathbf{V}_{11}^{-1}\mathbf{T}_{13} & \mathbf{0} \\ \mathbf{V}_{22}^{-1} & -\mathbf{V}_{22}^{-1}\mathbf{T}_{23} & \mathbf{0} \end{bmatrix}, \tag{2.111}$$

$$\mathbf{u} \triangleq \begin{bmatrix} \breve{\mathbf{q}}_2 \\ \breve{\mathbf{p}}_3 \\ \dot{\breve{\mathbf{p}}}_3 \end{bmatrix}, \tag{2.112}$$

$$\mathbf{y} \triangleq \begin{bmatrix} \bar{\mathbf{p}}_2 \\ \bar{\mathbf{q}}_3 \end{bmatrix}, \tag{2.113}$$

$$\mathbf{C} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{T}_{31} & \mathbf{T}_{32} \end{bmatrix}, \tag{2.114}$$

$$\mathbf{D} \triangleq \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{T}_{33} & \mathbf{V}_{33} \end{bmatrix}. \tag{2.115}$$

As before, we have chosen the output vector $\mathbf{y}$ such that it contains the free pressures and flow rates in the wells. However the input vector $\mathbf{u}$ now not only contains the prescribed flow rates and pressures in the wells, but also the time derivatives of the pressures. This technique to eliminate the prescribed state variables is mainly of theoretical value, and is not commonly used, if at all, in reservoir engineering applications.

2.2.7 System Energy*

The *energy balance* in flow through porous media is governed by three components:

1) *Potential energy* in the form of compressed fluids inside compressed rock and in the form of elevated fluid mass.
2) *Energy dissipation* caused by resistance to fluids flowing through the pore network.
3) *Energy transport* through the system boundaries in the form of *work* done by injecting or producing fluids under a pressure differential in the wells.

Note that we do not consider *kinetic energy*, because of the assumption that inertia forces may be neglected due to the very low flow velocities inside the pores.[11] Moreover, we will not take into account the effect of elevation on the potential energy because we restrict the theory and examples to two-dimensional horizontal reservoirs where gravity forces can be neglected.[12] Finally, we maintain our earlier

[11] I.e. we do not consider kinetic energy at a macroscopic level. We do take into account energy dissipation, which is the change of mechanical energy into thermal energy, or heat, and which at an atomic level can be interpreted as kinetic energy again.

[12] Elevation-related potential energy plays an important role in well-bore flow . Most reservoirs have enough potential energy, at least initially, to *lift* the oil to surface naturally in the production wells. This lift effect is in most cases caused by the difference in density between oil and water,

assumption of isothermal conditions, which implies that the heat generated by energy dissipation is instantaneously transferred to the surroundings (i.e. to outside the reservoir boundaries) such that there is no increase in reservoir temperature.

2.2.7.1 Potential Energy*

Starting from the assumption of a small, pressure-independent, total compressibility c_t, the fluid volume $\bar{V}(p)$ in a single grid block with volume V is expressed as[13]

$$\bar{V}(p) = V\phi_0[1 + c_t(p - p_0)]\,, \tag{2.116}$$

where p_0 is a reference pressure, and ϕ_0 the corresponding porosity. The difference in potential energy when the fluids in the grid block experience a pressure increase from p_0 to p can therefore be expressed as

$$E_{pot}(p) = \int_{p_0}^{p} \frac{\partial \bar{V}(\pi)}{\partial \pi} \pi d\pi = \int_{p_0}^{p} V\phi_0 c_t \pi d\pi = \frac{1}{2} V\phi_0 c_t \left(p^2 - p_0^2\right). \tag{2.117}$$

If we choose the reference value for E_{pot} as zero at the reference pressure (which we may conveniently take to be the initial reservoir pressure p_R), we can compute the total potential energy in a reservoir model through summation over the grid blocks according to

$$E_{pot,tot}(t) = \sum_{i=1}^{n_{gb}} \frac{1}{2} V_i \phi_i c_t p_i^2(t)\,, \tag{2.118}$$

where n_{gb} is the total number of grid blocks, and where V_i, ϕ_i and p_i are the grid-block volumes, porosities and pressures respectively, with only the pressures being a function of time.

2.2.7.2 Dissipation Energy*

The energy dissipated per unit time by resistance to flow through a grid-block boundary can be expressed as

$$\frac{dE_{dis}}{dt} = -\tilde{q}\Delta p = T\Delta p^2\,, \tag{2.119}$$

(Footnote 12 continued)
such that in an oil-filled well that drains a hydrostatically-pressured reservoir the oil will be lifted to surface because of elevation-related potential energy.

[13] For a detailed derivation of pressure-related potential energy for the case of pressure-dependent rock and fluid compressibilities, see Hubbert (1940).

where $\tilde{q}$ is the volumetric flow rate, Δp is the pressure difference between the two grid-block centers, and T is the grid-block transmissibility as defined in Eq. (1.30). In addition to dissipation at the grid-block boundaries, a large amount of energy is dissipated in the near-well-bore region where large pressure gradients are present. The energy dissipated per unit time by resistance to flow in a well grid block can be expressed as

$$\frac{dE_{dis}}{dt} = q_{well}\left(p_{well} - p_{gb}\right) = J_{well}\left(p_{well} - p_{gb}\right)^2, \tag{2.120}$$

where q_{well} is the well flow rate (positive for injection), p_{well} is the flowing bottom-hole pressure, p_{gb} is the well-grid-block pressure and J_{well} is the well index, as defined in Eq. (1.44). The total amount of energy dissipated per unit time in a reservoir model is therefore obtained through summation over all grid-block connectivities and all wells as

$$\frac{dE_{dis,tot}(t)}{dt} = -\sum_{j=1}^{n_{con}} T_j(t)\Delta p_j^2(t) + \sum_{k=1}^{n_{well}} J_{well,k}(t)\left[p_{well,k}(t) - p_{gb,k}(t)\right]^2, \tag{2.121}$$

where n_{con} is the number of connectivities, n_{well} is the number of wells, and where the transmissibilities, well indices and pressure drops may be functions of time.

2.2.7.3 Work Done by Wells*

The *power* (i.e. the work per unit time[14]) delivered by fluids injected into the reservoir can be expressed as

$$P_{well} = q_{well}\, p_{well}. \tag{2.122}$$

The same equation holds for production wells, where we use the convention that flow rates in the producers have negative values. The total power delivered by all injectors and producers to the reservoir is therefore given by

$$P_{well,tot}(t) = \sum_{k=1}^{n_{well}} q_{well,k}(t) \times p_{well,k}(t)\,. \tag{2.123}$$

2.2.7.4 Energy Balance*

The total energy balance for the reservoir over a time interval $\Delta t = t_2 - t_1$ can now be expressed as

[14] Energy delivered to a system through mechanical or hydraulic action is often referred to as *work*. Work (or energy) per unit time is then called *power*. In strict SI units time is expressed in s (seconds), energy and work in J (Joule) and power in W (Watt), such that 1 W is equal to 1 J/s.

$$\underbrace{E_{pot,tot}(t_2) - E_{pot,tot}(t_1)}_{\text{accumulation}} + \underbrace{\int_{t_1}^{t_2} \frac{dE_{dis,tot}(t)}{dt}\, dt}_{\textit{dissipation}} = \underbrace{\int_{t_1}^{t_2} P_{well,tot}(t)\, dt}_{\textit{source}} \,. \tag{2.124}$$

Note that the potential energy accumulation has simply been expressed as the difference between the values at the begin and end times which illustrates that potential energy is not dependent on the pressure history, i.e. it is path-independent. This is unlike the energy lost by dissipation and gained by work, which are both (pressure) path-dependent as indicated by the integrals which need to be evaluated over the entire time interval. Alternatively, the energy balance per unit time, i.e. the *power balance*, can be expressed as

$$\frac{dE_{pot,tot}(t)}{dt} + \frac{dE_{dis,tot}(t)}{dt} = P_{well,tot}(t)\,. \tag{2.125}$$

or, according to Eqs. (2.118), (2.121) and (2.123), as

$$\begin{aligned} &\sum_{i=1}^{n_{gb}} V_i \phi_i c_t p_i(t) \frac{dp_i(t)}{dt} + \sum_{j=1}^{n_{con}} T_j(t) \Delta p_j^2(t) + \sum_{k=1}^{n_{well}} J_{well,k}(t) \left[p_{well,k}(t) - p_{gb,k}(t)\right]^2 \\ &\quad = \sum_{k=1}^{n_{well}} q_{well,k}(t) \times p_{well,k}(t). \end{aligned} \tag{2.126}$$

Expression (2.126) can also be written in matrix–vector notation as

$$\begin{aligned} &\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} \\ &+ \begin{bmatrix} (\mathbf{p}_1 - \mathbf{p}_{av}) & (\mathbf{p}_2 - \mathbf{p}_{av}) & (\mathbf{p}_3 - \mathbf{p}_{av}) \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} & (\bar{\mathbf{p}}_{well} - \mathbf{p}_2) & \left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3\right) \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \bar{\mathbf{p}}_{well} - \mathbf{p}_2 \\ \breve{\mathbf{p}}_{well} - \mathbf{p}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \bar{\mathbf{p}}_{well} & \bar{\mathbf{q}}_{well} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \breve{\mathbf{p}}_{well} \end{bmatrix}, \end{aligned} \tag{2.127}$$

where we have used the partitioned vectors and matrices as introduced in Sects. 2.2.3 and 2.2.4. The vector $\mathbf{p}_{av}$ represents the time-dependent average reservoir pressure defined as

$$\mathbf{p}_{av}(t) \triangleq \frac{1}{n_{gb}} \sum_{i=1}^{n_{gb}} p_i(t) \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \tag{2.128}$$

with a number of elements as appropriate to match those of $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. The equivalence of Eqs. (2.126) and (2.127) can be confirmed by inspection of the matrices $\mathbf{V}$, $\mathbf{T}$ and $\mathbf{J}_3$, and the underlying matrices as defined in Eq. (1.31).

2.2.7.5 Minimum Energy Interpretation*

With the aid of relationships (2.90) and (2.91) we can rewrite the power balance (2.127) as

$$\begin{aligned} &\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} \\ &+ \begin{bmatrix} (\mathbf{p}_1 - \mathbf{p}_{av}) & (\mathbf{p}_2 - \mathbf{p}_{av}) & (\mathbf{p}_3 - \mathbf{p}_{av}) \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \\ &+ \begin{bmatrix} \mathbf{0} & (\bar{\mathbf{p}}_{well} - \mathbf{p}_2) & \left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3\right) \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \bar{\mathbf{q}}_{well} \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \bar{\mathbf{p}}_{well} & \bar{\mathbf{q}}_{well} \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \breve{\mathbf{p}}_{well} \end{bmatrix}, \end{aligned} \tag{2.129}$$

from which follows

$$\begin{aligned} &\begin{bmatrix} \mathbf{p}_1 & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} \\ &+ \begin{bmatrix} (\mathbf{p}_1 - \mathbf{p}_{av}) & (\mathbf{p}_2 - \mathbf{p}_{av}) & (\mathbf{p}_3 - \mathbf{p}_{av}) \end{bmatrix} \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{0} & \mathbf{p}_2 & \mathbf{p}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \bar{\mathbf{q}}_{well} \end{bmatrix}. \end{aligned} \tag{2.130}$$

Equation (2.130) is again an expression for the power balance in the system, but now expressed solely in terms of the state variables[15] $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$. It can be written compactly as $dE_{sys}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)/dt = P_{well}(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3)$, where E_{sys} is the system energy as governed by the state variables proper. Equation (2.130) is a single scalar equation in the n_{gb} state variables, and therefore does not have a unique solution. However, using thermodynamic arguments it can be argued that all natural systems tend to organize themselves in such a way that they minimize the amount of energy required to maintain equilibrium between internal and external forces. In our particular case of a system comprising flow through a porous medium this implies that the values of the state variables $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$ will be such that the power flow through the system becomes minimal i.e. that the first derivatives of P_{well} with respect to the state variables, and therefore also the first derivatives of dE_{sys}/dt with respect to the state variables, become zero. Taking derivatives of Eq. (2.130), setting the results equal to zero and combining them in matrix form results in

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \bar{\mathbf{q}}_{well} \end{bmatrix} . \tag{2.131}$$

Here we made use of the fact that

$$\frac{d}{d\mathbf{p}}[(\mathbf{p} - \mathbf{p}_{av})\mathbf{T}\mathbf{p}] = 2\mathbf{T}\mathbf{p} - \underbrace{\frac{d\mathbf{p}_{av}}{d\mathbf{p}}}_{\mathbf{I}} \mathbf{T}\mathbf{p} - \underbrace{\mathbf{T}\mathbf{p}_{av}}_{\mathbf{0}} = \mathbf{T}\mathbf{p} , \tag{2.132}$$

where we used the compact notation $\mathbf{p} = [(\mathbf{p}_1)^T (\mathbf{p}_2)^T (\mathbf{p}_3)^T]^T$, etc., as introduced in Sects. 2.2.3, leaving out the superscripted stars for clarity. Using Eq. (2.90), we can finally rewrite Eq. (2.131) as

$$\begin{bmatrix} \mathbf{V}_{11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{33} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \end{bmatrix} + \begin{bmatrix} \mathbf{T}_{11} & \mathbf{T}_{12} & \mathbf{T}_{13} \\ \mathbf{T}_{21} & \mathbf{T}_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & \mathbf{T}_{32} & \mathbf{T}_{33} + \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3 \end{bmatrix} \begin{bmatrix} \mathbf{0} \\ \breve{\mathbf{q}}_{well} \\ \breve{\mathbf{p}}_{well} \end{bmatrix} . \tag{2.133}$$

Equation (2.133) is completely identical to system Eq. (2.89) which was derived from balance equations for mass and momentum only. The alternative way to derive the system equations using the concept of energy, as described for porous-medium flow in this section, is frequently used in the fields of theoretical and applied mechanics; see e.g. Langhaar (1962) and Lanczos (1970). Closely related are other *energy methods* also known as *variational methods*, which are used to

[15] In comparison, Eq. (2.127) was also a function of the well-bore pressures $\breve{\mathbf{p}}_{well}$ and $\bar{\mathbf{p}}_{well}$.

compute approximate solutions for applied mechanics problems in complex-shaped domains. In particular, they often form the basis to derive numerical approximations using the finite-element method; see e.g. Zienckiewicz and Taylor (1989). Direct use of energy methods in porous-media flow does not seem to have an advantage over the conventional direct methods, and has therefore scarcely been described in the literature. An exception is the paper by Karney and Seneviratne (1991) who propose to use energy concepts for adaptive time step control in numerical simulation.

2.3 Two-Phase Flow

2.3.1 System Equations

2.3.1.1 Nonlinear Equations

As a next step we consider a simplified description of two-phase flow of oil and water, as derived in some detail in Sect. 1.4. We start from Eq. (1.129),

$$\underbrace{\begin{bmatrix} \mathbf{V}_{wp}(\mathbf{s}) & \mathbf{V}_{ws} \\ \mathbf{V}_{op}(\mathbf{s}) & \mathbf{V}_{os} \end{bmatrix}}_{\mathbf{V}} \begin{bmatrix} \dot{\mathbf{p}} \\ \dot{\mathbf{s}} \end{bmatrix} + \underbrace{\begin{bmatrix} \mathbf{T}_w(\mathbf{s}) & \mathbf{0} \\ \mathbf{T}_o(\mathbf{s}) & \mathbf{0} \end{bmatrix}}_{\mathbf{T}} \begin{bmatrix} \mathbf{p} \\ \mathbf{s} \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{F}_w(\mathbf{s}) \\ \mathbf{F}_o(\mathbf{s}) \end{bmatrix}}_{\mathbf{F}} \mathbf{q}_{well,t} \,, \tag{2.134}$$

where $\mathbf{p}$ and $\mathbf{s}$ are vectors of pressures p_o and water saturations S_w in the grid-block centers, $\mathbf{V}$ is the accumulation matrix (with entries that are functions of the porosity ϕ, and the oil, water and rock compressibilities c_o, c_w and c_r), $\mathbf{T}$ is the transmissibility matrix (with entries that are functions of the rock permeabilities k, the oil and water relative permeabilities k_{ro} and k_{rw} and the oil and water viscosities μ_o and μ_w), $\mathbf{F}$ is the fractional-flow matrix (with entries that have functional dependencies similar to those of $\mathbf{T}$), and $\mathbf{q}_{well,t}$ is a vector of total well flow rates with non-zero values in those elements that correspond to grid blocks penetrated by a well. The nonlinearities in Eq. (2.134) result from various sources; see also Sect. 1.4.10. If the oil and water phases have different compressibilities, the accumulation terms $\mathbf{V}_{wp}(\mathbf{s})$ and $\mathbf{V}_{op}(\mathbf{s})$ are (weak) functions of the saturations because the liquid compressibility is a saturation-weighted average of the oil and water compressibilities. Moreover, the porosity and compressibility values in these terms may be weak functions of pressure, but we do not take this effect into account in the examples in this text. The transmissibility terms $\mathbf{T}_w(\mathbf{s})$ and $\mathbf{T}_o(\mathbf{s})$ are much stronger functions of the saturations, because the relative permeabilities for oil and water are strongly saturation-dependent. The viscosities may also be weakly pressure-dependent, but, yet again, the pressure dependency is disregarded in the examples. Finally, matrices $\mathbf{F}_o(\mathbf{s})$ and $\mathrm{F}_w(\mathbf{s})$ contain saturation-dependent terms that relate the oil and water flow rates in the wells to the total flow rates.

2.3.1.2 Well Model

In practice the source terms are often not the flow rates in the wells but rather the pressures. This can be accounted for by rewriting Eq. (2.134) in partitioned form as

$$\left[\begin{array}{ccc|ccc} \mathbf{V}_{wp,11} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{ws,11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{wp,22} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{ws,22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{wp,33} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{ws,33} \\ \hline \mathbf{V}_{op,11} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{os,11} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{V}_{op,22} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{os,22} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{op,33} & \mathbf{0} & \mathbf{0} & \mathbf{V}_{os,33} \end{array}\right] \left[\begin{array}{c} \dot{\mathbf{p}}_1 \\ \dot{\mathbf{p}}_2 \\ \dot{\mathbf{p}}_3 \\ \hline \dot{\mathbf{s}}_1 \\ \dot{\mathbf{s}}_2 \\ \dot{\mathbf{s}}_3 \end{array}\right] +$$

$$\left[\begin{array}{ccc|ccc} \mathbf{T}_{w,11} & \mathbf{T}_{w,12} & \mathbf{T}_{w,13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{w,21} & \mathbf{T}_{w,22} & \mathbf{T}_{w,23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{w,31} & \mathbf{T}_{w,32} & \mathbf{T}_{w,33} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{T}_{o,11} & \mathbf{T}_{o,12} & \mathbf{T}_{o,13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{o,21} & \mathbf{T}_{o,22} & \mathbf{T}_{o,23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{o,31} & \mathbf{T}_{o,32} & \mathbf{T}_{o,33} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right] \left[\begin{array}{c} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \hline \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{array}\right] = \left[\begin{array}{ccc|ccc} \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{F}_{w,22} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{w,33} & & & \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{F}_{o,22} & \mathbf{0} & & & \\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{o,33} & & & \end{array}\right] \left[\begin{array}{c} \mathbf{0} \\ \breve{\mathbf{q}}_{well,t} \\ \mathbf{J}_3\left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3\right) \end{array}\right]. \tag{2.135}$$

Here, the elements of vector $\mathbf{p}_1$ are the pressures in those grid blocks that are not penetrated by a well. The elements of $\mathbf{p}_2$ are the pressures in the blocks where the source terms are prescribed total well flow rates $\breve{\mathbf{q}}_{well,t}$, and those of $\mathbf{p}_3$ are the pressures in the blocks where the source terms are obtained through prescription of the bottom-hole pressures $\breve{\mathbf{p}}_{well}$ with the aid of a diagonal matrix of well indices $\mathbf{J}_3$. To compute the oil and water flow rates in the wells with prescribed pressures we use the well model

$$\left[\begin{array}{c} \bar{\mathbf{q}}_{well,w} \\ \bar{\mathbf{q}}_{well,o} \end{array}\right] = \left[\begin{array}{c} \mathbf{F}_{w,33} \\ \mathbf{F}_{o,33} \end{array}\right] \mathbf{J}_3\left(\breve{\mathbf{p}}_{well} - \mathbf{p}_3\right). \tag{2.136}$$

To compute the bottom-hole pressures $\bar{\mathbf{p}}_{well}$ in the wells with prescribed total flow rates we need an additional diagonal matrix $\mathbf{J}_{q,2}$ of well indices such that

$$\breve{\mathbf{q}}_{well,t} = \mathbf{J}_2(\bar{\mathbf{p}}_{well} - \mathbf{p}_2), \tag{2.137}$$

from which we obtain

$$\bar{\mathbf{p}}_{well} = \mathbf{J}_2^{-1}\breve{\mathbf{q}}_{well,t} - \mathbf{p}_2. \tag{2.138}$$

2.3.1.3 State-Space Form

To bring these equations in state-space form we define the state vector $\mathbf{x}$, input vector $\mathbf{u}$ and output vector $\mathbf{y}$ as

$$\mathbf{u} \triangleq \begin{bmatrix} \breve{\mathbf{q}}_{well,t} \\ \breve{\mathbf{p}}_{well} \end{bmatrix}, \tag{2.139}$$

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{p} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \\ \hline \mathbf{s}_1 \\ \mathbf{s}_2 \\ \mathbf{s}_3 \end{bmatrix}, \tag{2.140}$$

$$\mathbf{y} \triangleq \begin{bmatrix} \bar{\mathbf{p}}_{well} \\ \bar{\mathbf{q}}_{well,w} \\ \bar{\mathbf{q}}_{well,o} \end{bmatrix}. \tag{2.141}$$

Equations (2.135), (2.136) and (2.138) can then be rewritten in nonlinear state-space form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{u}, \mathbf{x}), \tag{2.142}$$

$$\mathbf{y} = \mathbf{h}(\mathbf{u}, \mathbf{x}), \tag{2.143}$$

where the functions $\mathbf{f}$ and $\mathbf{h}$ are defined as

$$\mathbf{f} \triangleq \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \tag{2.144}$$

$$\mathbf{h} \triangleq \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}, \tag{2.145}$$

with state-dependent secant matrices $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$, $\mathbf{C}(\mathbf{x})$ and $\mathbf{D}(\mathbf{x})$ given by

$$\mathbf{A} \triangleq -\mathbf{V}^{-1} \underbrace{\left[\begin{array}{ccc|ccc} \mathbf{T}_{w,11} & \mathbf{T}_{w,12} & \mathbf{T}_{w,13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{w,21} & \mathbf{T}_{w,22} & \mathbf{T}_{w,23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{w,31} & \mathbf{T}_{w,32} & \mathbf{T}_{w,33} + \mathbf{F}_{w,33}\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{T}_{o,11} & \mathbf{T}_{o,12} & \mathbf{T}_{o,13} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{o,21} & \mathbf{T}_{o,22} & \mathbf{T}_{o,23} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{o,31} & \mathbf{T}_{o,32} & \mathbf{T}_{o,33} + \mathbf{F}_{o,33}\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right]}_{-\hat{\mathbf{A}}} \tag{2.146}$$

$$\mathbf{B} \triangleq \mathbf{V}^{-1} \underbrace{\left[\begin{array}{cc} \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{w,22} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{w,33}\mathbf{J}_3 \\ \hline \mathbf{0} & \mathbf{0} \\ \mathbf{F}_{o,22} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{o,33}\mathbf{J}_3 \end{array}\right]}_{\hat{\mathbf{B}}}, \tag{2.147}$$

$$\mathbf{C} \triangleq \left[\begin{array}{ccc|ccc} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{F}_{w,33}\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{F}_{o,33}\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \tag{2.148}$$

$$\mathbf{D} \triangleq \left[\begin{array}{cc} \mathbf{J}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{w,33}\mathbf{J}_3 \\ \mathbf{0} & \mathbf{F}_{o,33}\mathbf{J}_3 \end{array}\right]. \tag{2.149}$$

We note that the explicit representations (2.142) and (2.143) are primarily of theoretical interest because they allow direct application of concepts from systems-and-control theory. For computational purposes it is usually required to express the system equations in fully-implicit (residual) state-space form

$$\mathbf{g}(\mathbf{u}, \mathbf{x}, \dot{\mathbf{x}}) = \hat{\mathbf{E}}\dot{\mathbf{x}} - \hat{\mathbf{A}}\mathbf{x} - \hat{\mathbf{B}}\mathbf{u} = \mathbf{0}\,, \tag{2.150}$$

where $\hat{\mathbf{E}} = \mathbf{V}$ and where $\hat{\mathbf{A}}$ and $\hat{\mathbf{B}}$ are have been defined in Eqs. (2.146) and (2.147). The computation of the inverse of $\mathbf{V}$ as required in the explicit form can be performed efficiently by using the analytical expression for the inverse of a 2×2 block matrix with diagonal blocks of equal size[16]:

$$\left[\begin{array}{cc} \mathbf{V}_{wp} & \mathbf{V}_{ws} \\ \mathbf{V}_{op} & \mathbf{V}_{os} \end{array}\right]^{-1} = \left[\begin{array}{cc} \tilde{\mathbf{V}}^{-1}\mathbf{V}_{os} & -\tilde{\mathbf{V}}^{-1}\mathbf{V}_{ws} \\ -\tilde{\mathbf{V}}^{-1}\mathbf{V}_{op} & \tilde{\mathbf{V}}^{-1}\mathbf{V}_{wp} \end{array}\right], \tag{2.151}$$

where

$$\tilde{\mathbf{V}} = \mathbf{V}_{wp}\mathbf{V}_{os} - \mathbf{V}_{ws}\mathbf{V}_{op}\,. \tag{2.152}$$

Because the four sub-matrices of $\mathbf{V}$ are diagonal, $\tilde{\mathbf{V}}$ and the four sub-matrices of $\mathbf{V}^{-1}$ are also diagonal. Moreover, the inverse $\tilde{\mathbf{V}}^{-1}$ can be obtained simply by

[16] The general expression for the inverse of a 2×2 block matrix is given by $\left[\begin{smallmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix} \tilde{\mathbf{V}}_1^{-1} & -\mathbf{V}_{11}^{-1}\mathbf{V}_{12}\tilde{\mathbf{V}}_2^{-1} \\ -\mathbf{V}_{22}^{-1}\mathbf{V}_{21}\tilde{\mathbf{V}}_1^{-1} & \tilde{\mathbf{V}}_2^{-1} \end{smallmatrix}\right]$, where $\tilde{\mathbf{V}}_1 = \mathbf{V}_{11} - \mathbf{V}_{12}\mathbf{V}_{22}^{-1}\mathbf{V}_{21}$ and $\tilde{\mathbf{V}}_2 = \mathbf{V}_{22} - \mathbf{V}_{11}^{-1}\mathbf{V}_{12}$ are the *Schur complements* of $\mathbf{V}_{11}$ and $\mathbf{V}_{22}$ respectively; see e.g. Friedland (1986), pp. 479–481. Using the property that equally sized diagonal matrices commute, we can derive Eq. (2.151) from this more general expression.

taking the reciprocals of the diagonal elements. However, we re-emphasize that there is no need to perform the inverse operation if the equations serve as a basis for computation, and that the explicit state-space form (2.142) is only required for analysis of the system-theoretical properties of the equations.

2.3.1.4 Extended Output vector

In the formulation discussed so far we considered system outputs in the sense of response signals, and we tacitly assumed that all system inputs were known. However, in reality, both inputs and outputs have to be measured and we can therefore also define an extended output vector that contains all measured signals. For example it may be required to know the oil and water flow rates in those wells were the total flow rates have been prescribed, which leads to

$$\begin{bmatrix} \tilde{\breve{\mathbf{q}}}_{well,w} \\ \tilde{\breve{\mathbf{q}}}_{well,o} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{w,22} \\ \mathbf{F}_{o,22} \end{bmatrix} \breve{\mathbf{q}}_{well,t} \,, \tag{2.153}$$

where we have added tildes to indicate that the variables are measurements rather than real prescribed variables. Moreover, we may want to include measurements of the prescribed pressure $\breve{\mathbf{p}}_{well}$ in the output. The extended output vector then becomes

$$\mathbf{y} \triangleq \begin{bmatrix} \bar{\mathbf{p}}_{well} \\ \bar{\mathbf{q}}_{well,w} \\ \bar{\mathbf{q}}_{well,o} \\ \hdashline \breve{\mathbf{p}}_{well} \\ \tilde{\breve{\mathbf{q}}}_{well,w} \\ \tilde{\breve{\mathbf{q}}}_{well,o} \end{bmatrix}, \tag{2.154}$$

where the elements above and below the dotted line represent measurements related to output and input variables respectively.[17] In this case the matrices $\mathbf{C}$ and $\mathbf{D}$ can be expressed as

$$\mathbf{C} \triangleq \left[\begin{array}{ccc|ccc} \mathbf{0} & \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{F}_{w,33}\,\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{F}_{o,33}\,\mathbf{J}_3 & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hdashline \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{array}\right], \tag{2.155}$$

[17] This distinction is not very clear cut. For example to compute the oil and water 'input' rates, we make use of the fractional flows around the wells which are a direct function of the saturations, i.e. of state variables. In this sense the rates also contain indirect output information on the saturations around the wells.

$$\mathbf{D} \triangleq \left[\begin{array}{cc} \mathbf{J}_2^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{F}_{w,33}\,\mathbf{J}_3 \\ \mathbf{0} & \mathbf{F}_{o,33}\,\mathbf{J}_3 \\ \hdashline \mathbf{0} & \mathbf{I} \\ \mathbf{F}_{w,22} & \mathbf{0} \\ \mathbf{F}_{o,22} & \mathbf{0} \end{array}\right]. \tag{2.156}$$

2.3.2 Well Operating Constraints

During the operation of an oil field it often occurs that wells or groups of wells are operated on liquid constraints because the surface facilities are not capable of processing more than a certain throughput of gas, oil and water. Water injection wells are often operated on pressure constraints to avoid or limit fracturing of the formation around the wells. Production wells are often constrained to operate at a tubing-head pressure above a certain minimum, as determined by the working pressure of the first separator plus some additional pressure to displace the fluids through the flow line to that separator. Moreover, during the producing life of a field the well operating constraints may change because of changes in reservoir pressure and well-bore stream composition. In practice the control of tubing head pressures or phase rates is often done indirectly, through adjusting valve settings or changing out chokes and monitoring the resulting pressure of flow rate response. Methods for well control vary drastically. At the high end we find sophisticated remotely controlled valves with remotely observed pressure gauges and multi-phase flow meters. At the low end we have manual change out of chokes, and infrequent, say monthly, observations of well head pressures and measurements of the well phase rates by temporarily re-routing the well through a test separator. In reservoir simulation we can prescribe pressures or flow rates, but, in addition we may specify constraints in the form of maximum or minimum values for pressures and flow rates in wells or groups of wells. During the simulation the conditions may change such that a well changes from being operated at a prescribed rate to being operated at a prescribed pressure or vice versa. For example if a production well is operated at a prescribed total liquid rate, reservoir depletion may cause the bottom-hole pressure required to maintain this flow rate to gradually drop until it reaches the minimum pressure required to lift the well-bore fluid to surface, i.e. until it reaches its minimum pressure constraint. From that moment on the well needs to be operated at a prescribed bottom-hole pressure. Most reservoir simulators therefore allow the user to define minimum and maximum constraints for pressures and phase rates and automatically determine the *most constraining constraint* at any moment in time during the simulation. Examples of operating constraints as implemented in reservoir simulation will be discussed in Chap. 3.

2.3.3 Computational Aspects

In this section we discuss some general computational aspects of the numerical implementation of the two-phase system equations.

- Most (sub-)matrices considered so far are *sparse*: the accumulation sub-matrices are diagonal, the transmissibility sub-matrices are penta-diagonally banded with two sub diagonals, and the fractional-flow and well index sub-matrices are sparse diagonal. Most of the matrix elements are therefore equal to zero, and this property may be used to significantly reduce computer memory usage.
- As mentioned before, the re-ordering of vector and matrix elements with permutation matrices as used in Sects. 2.2.3 to Sects. 2.2.6 is not essential in a numerical implementation. There is no computational need to e.g. re-group state or input vector elements; it is merely a convenient notation. In a numerical implementation we may simply use (sparse) matrices with elements that correspond to the relevant state or input variables at the appropriate locations.
- Computation of an element of a transmissibility sub-matrix corresponding to a specific grid block involves computing the transmissibilities for flow to or from the four neighboring grid blocks. Therefore, assembling the transmissibility (sub–)matrices requires knowledge of the *connectivities* of the grid blocks. This knowledge is often administered with the aid of a *connectivity* table, an $n_{con} \times 2$ matrix of which each row corresponds to a connectivity between a pair of grid blocks with grid-block numbers stored in the first and second column positions. For a rectangular model with $n_x \times n_y$ grid blocks, we have

$$n_{con} = (n_x - 1)n_y + \big(n_y - 1\big)n_x\,, \tag{2.157}$$

  and for the 2×3 reservoir used in Examples 1 and 2 the 7×2 connectivity table $\mathbf{L}_{con}$ is given by (see also Table 1.2)

$$\mathbf{L}_{con} = \begin{bmatrix} 1 & 2 \\ 1 & 4 \\ 2 & 3 \\ 2 & 5 \\ 3 & 6 \\ 4 & 5 \\ 5 & 6 \end{bmatrix}, \tag{2.158}$$

- The elements in the two-phase state vector $\mathbf{x} = [\mathbf{p}^T \mathbf{s}^T]^T$ have different physical dimensions and strongly varying magnitudes. If we express the pressures in Pa, they are in the order of 10^6–10^7, whereas the saturation values remain, by definition, between 0 and 1. As a result the elements of the transmissibility matrix $\mathbf{T}$, and therefore of the system matrix $\mathbf{A}$, will also have strongly varying

magnitude. This may influence the accuracy of the result when solving a system of equations $\mathbf{A}\ \mathbf{x} = \mathbf{b}$, as will be required in Chap. 3 to simulate the response of the system[18]. The reason for the inaccuracy is in the finite precision representation of the matrix elements in any numerical implementation. We may avoid this problem by scaling the elements of $\mathbf{x}$ such that the difference in magnitude between the pressure and saturation values becomes much smaller. This can be done by dividing the first n_{gb} columns of $\mathbf{A}$, which multiply the first n_{gb} 'pressure' elements of $\mathbf{x}$, by a factor

$$f_{scal} = \max(\mathbf{p})\,. \tag{2.159}$$

and multiply the corresponding elements of $\mathbf{x}$ with f_{scal} after the equations have been solved. In addition, we may also scale the elements of the right-hand side $\mathbf{b}$ by dividing the first n_{gb} rows of $\mathbf{b}$ and the corresponding rows of $\mathbf{A}$ by f_{scal}.

- In an injection well we have $\mathbf{q}_t = \mathbf{q}_w$, and we expect that soon after injection has started the fractional flows for water and oil close to the well will approach one and zero respectively. However, before injection starts, the initial condition for the saturation is usually equal to the connate-water saturation, which means that the fractional flows for water and oil are zero and one respectively. In theory, it would then be impossible to ever inject water. This paradox is usually circumvented by simply specifying a fractional flow equal to one for every injection well.

2.3.4 Lift Tables*

Until now, we have considered prescribed pressures in the form of flowing bottom-hole pressures p_{well}. In most cases, however, it is not the bottom-hole pressure that is controlled but the pressure at the top of the well, which is usually referred to as the flowing tubing-head pressure, p_{tf}. The difference in pressure between top and bottom of a well is governed by the multi-phase flow behavior of the well-bore fluids. Various techniques have been developed to compute well-bore pressure drops, ranging from empirical correlations to complex mechanistic models; see e.g. Brill and Mukherjee 1999. Typically, the tubing-head pressure can be computed for given fluid properties, well-bore geometry, oil, gas and water flow rates, and bottom-hole pressure. Conversely, the bottom-hole pressure may be computed for a given tubing-head pressure. The computation is performed with the aid of a *well-bore simulator* that numerically integrates a one-dimensional averaged version of the multi-phase flow equations along the well bore. Especially in the case of complex mechanistic multi-phase flow models these computations may be too

[18] Here, $\mathbf{b}$ is an arbitrary right-hand side.

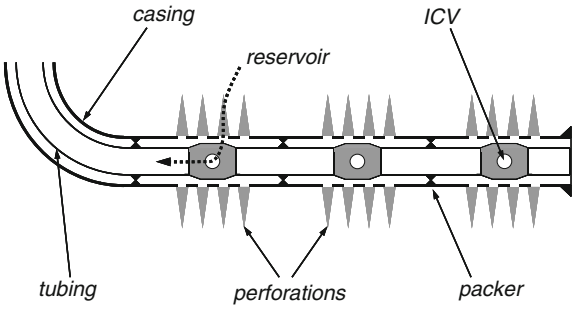

Fig. 2.2 Schematic representation of a smart horizontal well equipped with three ICVs. These allow for individual control of the inflow from the reservoir, through the perforations and the ICVs, into the tubing. The packers form flow barriers in the annular space between the casing and the tubing

time consuming to perform every time step of the reservoir simulator. An alternative approach is then to perform a large number of well-bore flow simulations up-front to generate a multi-dimensional table, known as a lift table or *flow performance* table, which can be used as a look-up table by the reservoir simulator. Usually the four entries for a lift table are the tubing-head pressure, and the oil, gas and water rates, all expressed at standard conditions.[19] Typically each of the entries is described with a small number of points, say 5 in which case the table has $5^4 = 625$ points that correspond to the same number of bottom-hole pressures. For intermediate values of the entries a linear or higher-order interpolation is used to compute the corresponding bottom-hole pressure, which is much faster than performing a full well-bore flow simulation. Sometimes a higher number of points is needed, at the cost of a longer pre-processing time, e.g. to prevent convergence problems during the numerical solution of the reservoir equations.

2.3.5 Control Valves*

In addition to specifying inputs in the form of prescribed well flow rates or pressures, either down hole or at surface, it is also possible to prescribe the opening of control valves. In particular the use of *interval control valves* (ICVs) is becoming increasingly popular. ICVs are mounted in an inner tube (the *tubing*) inside an outer tube (the *casing*) which penetrates the reservoir; see Fig. 2.2. The role of these valves is to control the inflow from individual reservoir compartments, allowing one, e.g., to shut-in a zone that experiences a too-high water production.

[19] Even if the reservoir is above bubble point such that it contains only oil and water and no free gas, the flow in the well bore will be three-phase because associated gas will be released from the oil as the well-bore pressure decreases at increasing elevations above the reservoir. Alternatively, the lift table entries can be chosen as tubing-head pressure, oil rate, gas-oil ratio, and water-cut. Whatever the choice of the table entries, it is assumed that the fluid properties at standard conditions and the well-bore geometry do not change during the reservoir simulation.

The usual way to represent the control action of such a valve is to specify a dimensionless valve opening $0 \leq \alpha \leq 1$ and modify the well inflow Eq. (2.86) according to

$$q = \alpha J_{well}\left(\breve{p}_{well} - p\right). \tag{2.160}$$

To compute the oil and water flow rates in wells with prescribed valve settings we can modify the two-phase well model (2.136). Introducing a subscript 4 to indicate wells controlled with ICVs this leads to

$$\begin{bmatrix} \bar{\mathbf{q}}_{well,w} \\ \bar{\mathbf{q}}_{well,o} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{w,44} \\ \mathbf{F}_{o,44} \end{bmatrix} \mathbf{J}_\alpha \boldsymbol{\alpha}, \tag{2.161}$$

where α is a vector of valve settings, and J_α is a modified well index matrix defined as

$$\mathbf{J}_\alpha \triangleq \begin{bmatrix} J_{well,4,1}\left(p_{well} - p_{4,1}\right) & 0 & \cdots & 0 \\ 0 & J_{well,4,2}\left(p_{well} - p_{4,2}\right) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & J_{well,4,m}\left(p_{well} - p_{4,m}\right) \end{bmatrix}. \tag{2.162}$$

Note that $\mathbf{J}_\alpha$ is a function of both $\mathbf{p}$ and $\mathbf{s}$. If we define the input and state vectors,

$$\mathbf{u} \triangleq \boldsymbol{\alpha} \tag{2.163}$$

$$\mathbf{x} \triangleq \begin{bmatrix} \mathbf{p} \\ \mathbf{s} \end{bmatrix} = \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_4 \\ \hline \mathbf{s}_1 \\ \mathbf{s}_4 \end{bmatrix}, \tag{2.164}$$

where subscripts 1 and 4 refer to grid blocks without wells and grid blocks containing an ICV respectively, and where it has been assumed that all ICVs operate at the same, fixed, well pressure p_{well}, the matrices $\mathbf{A}$ and $\mathbf{B}$ become (c.f. Eqs. (2.146) and (2.147)):

$$\mathbf{A} \triangleq -\mathbf{V}^{-1} \left[\begin{array}{cc|cc} \mathbf{T}_{w,11} & \mathbf{T}_{w,14} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{w,41} & \mathbf{T}_{w,44} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{T}_{o,11} & \mathbf{T}_{o,14} & \mathbf{0} & \mathbf{0} \\ \mathbf{T}_{o,41} & \mathbf{T}_{o,44} & \mathbf{0} & \mathbf{0} \end{array}\right], \tag{2.165}$$

$$\mathbf{B} \triangleq \mathbf{V}^{-1} \begin{bmatrix} \mathbf{0} \\ \mathbf{F}_{w,44}\mathbf{J}_\alpha \\ \mathbf{0} \\ \mathbf{F}_{o,44}\,\mathbf{J}_\alpha \end{bmatrix}. \tag{2.166}$$

In this form the system equations $\dot{\mathbf{x}} = \mathbf{A}(\mathbf{x})\mathbf{x} + \mathbf{B}(\mathbf{x})\mathbf{u}$ are still control-affine.[20] If we introduce the option to control both the bottom-hole pressures and the valve settings, the inputs become nonlinear in $\mathbf{u}$ and the control-affine property is lost.

2.3.6 Streamlines*

As discussed in Chap. 1, the governing equations for flow through porous media consist of a mass-balance equation in combination with Darcy's law which describes the relationship between spatial pressure gradients and fluid velocities. After spatial discretization, Darcy's law can be interpreted as an equation relating pressure differences between adjacent grid blocks to the Darcy velocities (volumetric fluxes) at the corresponding grid-block boundaries. This discrete form of Darcy's law can be expressed as

$$\boldsymbol{v}_t = \mathbf{S}\mathbf{p}\,, \tag{2.167}$$

where $\mathbf{p}$ is an $n_{gb} \times 1$ vector of pressures at the grid-block centers with n_{gb} the number of grid blocks, $\boldsymbol{v}_t$ is an $n_{con} \times 1$ vector of total Darcy velocities at the grid-block boundaries with n_{con} the number of connectivities, i.e. the number of grid-block boundaries, and $\mathbf{S}$ is an $n_{con} \times n_{gb}$ matrix of transmissibility coefficients. Expressions for the elements of $\mathbf{S}$ are given in detail in Eq. (1.144) in Sect. 1.4.12. Given the velocity vector v_t, we can now simply visualize the trajectory of a fluid particle starting from its entrance into the reservoir at an injection well, all the way until it leaves again via a producer. These trajectories are known as *streamlines* and they can be computed using a procedure due to Pollock (1988). Consider a two-dimensional reservoir model with total Darcy velocities at the grid-block boundaries given by the vector $\boldsymbol{v}_t$. The corresponding total interstitial velocities are then given by

$$\tilde{\boldsymbol{v}}_t = \frac{\boldsymbol{v}_t}{\phi}\,. \tag{2.168}$$

Assuming a linear change in velocities in the x and y directions we can define the velocity gradients g_x and g_y for a single grid block as

$$g_x = \frac{\tilde{v}_{x_0+\Delta x} - \tilde{v}_{x_0}}{\Delta x}\,, \tag{2.169}$$

[20] See the footnote on page 46.

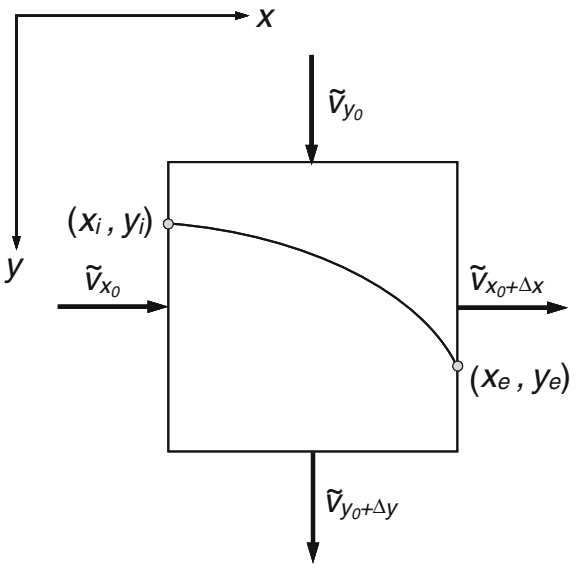

Fig. 2.3 Grid block with velocity vectors at the boundaries and a streamline from entrance point (x_i, y_i) to exit point (x_e, y_e)

$$g_y = \frac{\tilde{v}_{y_0+\Delta y} - \tilde{v}_{y_0}}{\Delta y}\,, \tag{2.170}$$

where Δx and Δy are the grid-block dimensions, where we dropped the subscript t from the velocities for clarity, and where we used subscripts x_0, y_0, $x_0 + \Delta x$, and $y_0 + \Delta y$ to indicate the four relevant elements out of the m elements of $\tilde{\boldsymbol{v}}_t$; see Fig. 2.3. In case of positive velocity components, a fluid particle will enter the grid block either at the left or at the top. We indicate the location of the entrance point as (x_i, y_i), where it should be understood that either $x_i = x_0$ or $y_i = y_0$ (or both, in the special case that the particle enters at the corner). The particle will now travel along a curved path until it reaches the exit point (x_e, y_e), and its velocity at an arbitrary point (x, y) inside the grid block has components

$$\tilde{v}_x = \tilde{v}_{x_0} + g_x(x - x_0)\,, \tag{2.171}$$

$$\tilde{v}_y = \tilde{v}_{y_0} + g_y(y - y_0)\,. \tag{2.172}$$

Note that at the exit point either $x_e = x_0 + \Delta x$ or $y_e = y_0 + \Delta x$, except when the particle leaves at the corner. Because we have, by definition,

$$\tilde{v}_x = \frac{dx}{dt}\,, \tag{2.173}$$

it follows that

$$dt = \frac{1}{\tilde{v}_x} dx\,, \tag{2.174}$$

which can be integrated to obtain the time $\Delta\tau$ to travel the distance $x_e - x_i$:

$$\int_0^{\Delta\tau} dt = \int_{x_i}^{x_e} \frac{1}{\tilde{v}_x} dx = \int_{x_i}^{x_e} \frac{1}{\tilde{v}_{x_0} + g_x(x - x_0)} dx = \frac{1}{g_x} \ln[\tilde{v}_{x_0} + g_x(x - x_0)]\Big|_{x_i}^{x_e}, \quad (2.175)$$

from which we obtain

$$\Delta\tau = \frac{1}{g_x} \ln\left[\frac{\tilde{v}_{x_0} + g_x(x_e - x_0)}{\tilde{v}_{x_0} + g_x(x_i - x_0)}\right]. \quad (2.176)$$

The travel time in the y direction will of course be identical and we can therefore also write

$$\Delta\tau = \frac{1}{g_y} \ln\left[\frac{\tilde{v}_{y_0} + g_y(y_e - y_0)}{\tilde{v}_{y_0} + g_y(y_i - y_0)}\right]. \quad (2.177)$$

We do not know in advance whether the particle will exit at the right or at the bottom of the grid block, but we do know that it must be one of the two (or both, in case of an exit at the corner). To determine the correct exit boundary we should first compute the travel times from Eqs. (2.176) and (2.177) using $x_e - x_0 = \Delta x$ and $y_e - y_0 = \Delta y$ respectively, i.e. for the maximum possible travel distance:

$$\Delta\tau_x = \frac{1}{g_x} \ln\left[\frac{\tilde{v}_{x_0} + g_x\Delta x}{\tilde{v}_{x_0} + g_x(x_i - x_0)}\right], \quad (2.178)$$

$$\Delta\tau_y = \frac{1}{g_y} \ln\left[\frac{\tilde{v}_{y_0} + g_y\Delta y}{\tilde{v}_{y_0} + g_y(y_i - y_0)}\right], \quad (2.179)$$

and the determine the correct grid-block travel time as

$$\Delta\tau = \min\left(\Delta\tau_x, \Delta\tau_y\right). \quad (2.180)$$

In the case that $\Delta\tau = \tau_y$, we have $y_e = y_0 + \Delta x$, and can we solve for x_e from Eq. (2.176) as

$$x_e = x_0 + \frac{1}{g_x}\{[\tilde{v}_{x_0} + g_x(x_i - x_0)]\exp(g_x\Delta\tau) - \tilde{v}_{x_0}\}. \quad (2.181)$$

Similarly, if $\Delta\tau = \tau_x$, we can solve from Eq. (2.177) for y_e as

$$y_e = y_0 + \frac{1}{g_y}\left\{\left[\tilde{v}_{y_0} + g_y(y_i - y_0)\right]\exp\left(g_y\Delta\tau\right) - \tilde{v}_{y_0}\right\}. \quad (2.182)$$

The exit point then forms the entry point of the next grid block and we can repeat the procedure to trace the stream line until it reaches one of the producers. If we sum the travel times over all grid blocks we obtain the *arrival time* for a streamline

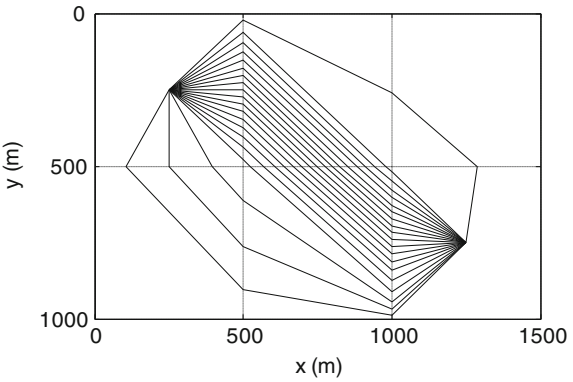

Fig. 2.4 Streamlines for steady state flow in Example 1. Note: The tracing algorithm described above computes parabolic trajectories for the streamlines in each grid block. However, the streamlines have been plotted more coarsely as straight lines between the entry and exit points in the grid blocks

which indicates the moment in time at which a 'virtual' particle travelling with speed $\tilde{v}_t$ along the streamline would reach the producer (assuming it starts at the injector at time zero). A related quantity is the *time-of-flight* of a virtual particle required to reach a specific point along a streamline, which is equal to the summation of grid-block travel times from the injector until that point. Figure 2.4 displays a streamline plot computed for Example 1 after steady state conditions have been reached. If, for a given total number of streamlines, we choose the fraction starting from each injector in proportion to the fraction of total water injected, the distance between streamlines becomes an inverse measure for the flow per unit surface area (which is also known as the *flux*). In other words, the closer the streamlines, the higher the flux. Apart from providing a powerful means to visualize reservoir flow, streamlines can also be used during numerical simulation, as will be briefly discussed in Chap. 3. For a much more in-depth treatment of streamline methods we refer to the classic papers of Bratvedt, Gimse and Tegnander (1996), Batycky, Blunt and Thiele (1997), King and Datta Gupta (King and Datta-Gupta 1998) and to the text book of Datta-Gupta and King (2007).

*2.3.7 System Energy**

In analogy to the single-phase case discussed in Sects. 2.2.7 we can formulate the energy balance per unit time, i.e. the power balance, for the two-phase case. The power balance can be expressed in terms of potential energy rate, dissipation rate and work, each related to both oil and water flow. Using matrix–vector notation this results in Eq. (2.183) below, where we applied the vector and matrix partitioning as introduced in Sects. 2.2.3 to distinguish between gridblocks without wells, gridblocks with wells where the flow rates are prescribed, and those with wells were the bottom-hole pressures are prescribed. We note that the presence of gravity forces and capillary pressures would make the expression for the power balance more complex. Using a similar reasoning as in Sects. 2.2.7 we can recover

system Eq. (2.135) by first simplifying Eq. (2.183) such that the well index matrices $\mathbf{J}_2$ and $\mathbf{J}_3$ are eliminated, then take derivatives with respect to the state variables $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$, set the result equal to zero, and finally re-introduce the well indices for the prescribed pressures. In Sect. 3.4.5 we will present a numerical example that illustrates the relative importance of the various terms in the power balance.

$$
\begin{aligned}
&\underbrace{[\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3\,]\left\{\begin{bmatrix}\mathbf{V}_{wp,11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_{wp,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{wp,33}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{p}}_1\\ \dot{\mathbf{p}}_2\\ \dot{\mathbf{p}}_3\end{bmatrix}+\begin{bmatrix}\mathbf{V}_{ws,11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_{ws,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{ws,33}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{s}}_1\\ \dot{\mathbf{s}}_2\\ \dot{\mathbf{s}}_3\end{bmatrix}\right\}}_{\frac{dE_{pot}}{dt}\big|_w}\\
&+\underbrace{[\,\mathbf{p}_1 \quad \mathbf{p}_2 \quad \mathbf{p}_3\,]\left\{\begin{bmatrix}\mathbf{V}_{op,11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_{op,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{op,33}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{p}}_1\\ \dot{\mathbf{p}}_2\\ \dot{\mathbf{p}}_3\end{bmatrix}+\begin{bmatrix}\mathbf{V}_{os,11} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{V}_{os,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{V}_{os,33}\end{bmatrix}\begin{bmatrix}\dot{\mathbf{s}}_1\\ \dot{\mathbf{s}}_2\\ \dot{\mathbf{s}}_3\end{bmatrix}\right\}}_{\frac{dE_{pot}}{dt}\big|_o}\\
&+\underbrace{[\,(\mathbf{p}_1-\mathbf{p}_{av}) \quad (\mathbf{p}_2-\mathbf{p}_{av}) \quad (\mathbf{p}_3-\mathbf{p}_{av})\,]\begin{bmatrix}\mathbf{T}_{w,11} & \mathbf{T}_{w,12} & \mathbf{T}_{w,13}\\ \mathbf{T}_{w,21} & \mathbf{T}_{w,22} & \mathbf{T}_{w,23}\\ \mathbf{T}_{w,31} & \mathbf{T}_{w,32} & \mathbf{T}_{w,33}\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}}_{\frac{dE_{dis,gb}}{dt}\big|_w}\\
&+\underbrace{[\,(\mathbf{p}_1-\mathbf{p}_{av}) \quad (\mathbf{p}_2-\mathbf{p}_{av}) \quad (\mathbf{p}_3-\mathbf{p}_{av})\,]\begin{bmatrix}\mathbf{T}_{o,11} & \mathbf{T}_{o,12} & \mathbf{T}_{o,13}\\ \mathbf{T}_{o,21} & \mathbf{T}_{o,22} & \mathbf{T}_{o,23}\\ \mathbf{T}_{o,31} & \mathbf{T}_{o,32} & \mathbf{T}_{o,33}\end{bmatrix}\begin{bmatrix}\mathbf{p}_1\\ \mathbf{p}_2\\ \mathbf{p}_3\end{bmatrix}}_{\frac{dE_{dis,gb}}{dt}\big|_o}\\
&+\underbrace{\left[\,\mathbf{0} \quad (\bar{\mathbf{p}}_{well}-\mathbf{p}_2) \quad \left(\breve{\mathbf{p}}_{well}-\mathbf{p}_3\right)\right]\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_{w,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{w,33}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{p}}_{well}-\mathbf{p}_2\\ \breve{\mathbf{p}}_{well}-\mathbf{p}_3\end{bmatrix}}_{\frac{dE_{dis,well}}{dt}\big|_w}\\
&+\underbrace{\left[\,0 \quad (\bar{\mathbf{p}}_{well}-\mathbf{p}_2) \quad \left(\breve{\mathbf{p}}_{well}-\mathbf{p}_3\right)\right]\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_{o,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{o,33}\end{bmatrix}\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{J}_3\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \bar{\mathbf{p}}_{well}-\mathbf{p}_2\\ \breve{\mathbf{p}}_{well}-\mathbf{p}_3\end{bmatrix}}_{\frac{dE_{dis,well}}{dt}\big|_o}\\
&=\underbrace{[\,\mathbf{0} \quad \bar{\mathbf{p}}_{well} \quad \bar{\mathbf{q}}_{well}\,]\begin{bmatrix}0 & 0 & 0\\ 0 & \mathbf{F}_{w,22} & 0\\ 0 & 0 & \mathbf{F}_{w,33}\end{bmatrix}\begin{bmatrix}0\\ \breve{\mathbf{q}}_{well}\\ \breve{\mathbf{p}}_{well}\end{bmatrix}}_{P_{well}|_w}+\underbrace{[\,\mathbf{0} \quad \bar{\mathbf{p}}_{well} \quad \bar{\mathbf{q}}_{well}\,]\begin{bmatrix}\mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{F}_{o,22} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{F}_{o,33}\end{bmatrix}\begin{bmatrix}\mathbf{0}\\ \breve{\mathbf{q}}_{well}\\ \breve{\mathbf{p}}_{well}\end{bmatrix}}_{P_{well}|_o}.
\end{aligned}
\tag{2.183}
$$

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