COL780 - Computer Vision
Minor 1

Suyash Agrawal
2015CS10262

September 12, 2017

Question 5

Expanding given appromixation equation we get:

$$u = u_0 + xu_x + yu_y$$
$$v = v_0 + xv_x + yv_y$$

The Optical flow equation is:

$$uI_x + vI_y + I_t = 0$$

Putting the given equation in optical flow equation, we get:

$$I_x(u_0 + xu_x + yu_y) + I_y(v_0 + xv_x + yv_y) = -I_t$$

Now, we take a region around the point (x, y) and we assume that optical flow assumptions hold in this region.
Thus, using the above equation, we can transform our case into a least square problem as follows:
Define matrix **A** as:

$$\mathbf{A} = \begin{bmatrix} I_x(p1) & x_{p1}I_x(p1) & y_{p1}I_x(p1) & I_y(p1) & x_{p1}I_y(p1) & y_{p1}I_y(p1) \\ I_x(p2) & x_{p2}I_x(p2) & y_{p2}I_x(p2) & I_y(p2) & x_{p2}I_y(p2) & y_{p2}I_y(p2) \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ I_x(pn) & x_{pn}I_x(pn) & y_{pn}I_x(pn) & I_y(pn) & x_{pn}I_y(pn) & y_{pn}I_y(pn) \end{bmatrix}$$

and $\vec{u}$ and $\vec{b}$ as:

$$\vec{u} = \begin{bmatrix} u_0 \\ u_x \\ u_y \\ v_0 \\ v_x \\ v_y \end{bmatrix} \qquad \vec{b} = -\begin{bmatrix} I_t(p1) \\ I_t(p2) \\ \vdots \\ I_t(pn) \end{bmatrix}$$

and our least square problem becomes

$$\min \quad \|\mathbf{A}\vec{u} - \vec{b}\|^2$$

whose solution comes out to be:

$$\vec{u} = (A^T A)^{-1} A^T b$$

The assumption that we made in this case are:

1. The values of the parameter don't change for the n points near (x, y).
2. Matrix $A^T A$ is invertible.

Question 6

The velocity of the object is:

$$V = -U - \Omega \times P$$

which, when expanded, translates to:

$$\begin{aligned}
V_1 = \frac{dX}{dt} &= -U_1 - \Omega_2 Z + \Omega_3 Y \\
V_2 = \frac{dY}{dt} &= -U_2 - \Omega_3 Z + \Omega_1 Z \\
V_3 = \frac{dZ}{dt} &= -U_3 - \Omega_1 Y + \Omega_2 X
\end{aligned}$$

Also, it is given that:

$$x = \frac{fX}{Z} \quad y = \frac{fY}{Z}$$

and by vector calculus:

$$u = \frac{dx}{dt} \qquad v = \frac{dy}{dt}$$

and by putting value of x from above:

$$\begin{aligned}
u &= \frac{d(\frac{fX}{Z})}{dt} \\
&= f\frac{Z\frac{dX}{dt} - X\frac{dZ}{dt}}{Z^2} \\
&= f\frac{ZV_1 - XV_3}{Z^2} \\
&= \frac{f}{Z}(-U_1 - \Omega_2 Z + \Omega_3 Y) - \frac{x}{Z}(-U_3 - \Omega_1 Y + \Omega_2 X) \\
&= -U_1\frac{f}{Z} - \Omega_2 f + \Omega_3 y - x(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})
\end{aligned}$$

Therefore,

$$u = -U_1\frac{f}{Z} - \Omega_2 f + \Omega_3 y - x(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})$$

Similary, putting value of y, we get:

$$\begin{aligned}
v &= \frac{d(\frac{fY}{Z})}{dt} \\
&= f\frac{Z\frac{dY}{dt} - Y\frac{dZ}{dt}}{Z^2} \\
&= f\frac{ZV_2 - YV_3}{Z^2} \\
&= \frac{f}{Z}(-U_2 - \Omega_3 Z + \Omega_1 Z) - \frac{y}{Z}(-U_3 - \Omega_1 Y + \Omega_2 X) \\
&= -U_2\frac{f}{Z} - \Omega_3 x + \Omega_1 f - y(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})
\end{aligned}$$

Therefore,

$$v = -U_2\frac{f}{Z} - \Omega_3 x + \Omega_1 f - y(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})$$

Hence Proved.

Question 7

From previous question, it is given that:

$$u = -U_1\frac{f}{Z} - \Omega_2 f + \Omega_3 y - x(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})$$
$$v = -U_2\frac{f}{Z} - \Omega_3 x + \Omega_1 f - y(-\frac{U_3}{Z} - \frac{\Omega_1 y}{f} + \frac{\Omega_2 x}{f})$$

Now, since we are taking first order linear approximation of u and y:

$$u = -U_1\frac{f}{Z} - \Omega_2 f + \Omega_3 y + x\frac{U_3}{Z} + O(x^2, xy, y^2)$$
$$v = -U_2\frac{f}{Z} + \Omega_1 f - \Omega_3 x + y\frac{U_3}{Z} + O(x^2, xy, y^2)$$

After taking partial derivative of u w.r.t. x we get:

$$\begin{aligned}\frac{\partial u}{\partial x} &= U_1 f\frac{\frac{\partial Z}{\partial x}}{Z^2} + 0 + 0 + \frac{U_3}{Z} - xU_3\frac{\frac{\partial Z}{\partial x}}{Z^2} \\ &= U_1 f\frac{Z_x}{Z^2} + \frac{U_3}{Z} + \textit{(tending to 0 terms)}\end{aligned}$$
$$\boxed{\therefore u_x = U_1 f\frac{Z_x}{Z^2} + \frac{U_3}{Z}}$$

Similarly, for others:

$$\begin{aligned}\frac{\partial u}{\partial y} &= U_1 f\frac{\frac{\partial Z}{\partial y}}{Z^2} + 0 + \Omega_3 + 0 - xU_3\frac{\frac{\partial Z}{\partial y}}{Z^2} \\ &= U_1 f\frac{Z_y}{Z^2} + \Omega_3 + \textit{(tending to 0 terms)}\end{aligned}$$
$$\boxed{\therefore u_y = U_1 f\frac{Z_y}{Z^2} + \Omega_3}$$
$$\begin{aligned}\frac{\partial v}{\partial x} &= U_2 f\frac{\frac{\partial Z}{\partial x}}{Z^2} + 0 - \Omega_3 + 0 - yU_3\frac{\frac{\partial Z}{\partial x}}{Z^2} \\ &= U_2 f\frac{Z_x}{Z^2} - \Omega_3 + \textit{(tending to 0 terms)}\end{aligned}$$
$$\boxed{\therefore v_x = U_2 f\frac{Z_x}{Z^2} - \Omega_3}$$
$$\begin{aligned}\frac{\partial v}{\partial y} &= U_2 f\frac{\frac{\partial Z}{\partial y}}{Z^2} + 0 + 0 + \frac{U_3}{Z} - yU_3\frac{\frac{\partial Z}{\partial y}}{Z^2} \\ &= U_2 f\frac{Z_y}{Z^2} + \frac{U_3}{Z} + \textit{(tending to 0 terms)}\end{aligned}$$
$$\boxed{\therefore v_y = U_2 f\frac{Z_y}{Z^2} + \frac{U_3}{Z}}$$

Given:

$$V_z = \frac{U_3}{Z}, \quad V_x = \frac{U_1}{Z}, \quad V_y = \frac{U_2}{Z}, \quad Z_X = \frac{fZ_x}{Z} \quad \textit{and} \quad Z_Y = \frac{fZ_y}{Z}$$

We get:

$$\begin{aligned}
u_x &= U_1 f\frac{Z_x}{Z^2} + \frac{U_3}{Z} &&= V_z + V_x Z_X \\
u_y &= U_1 f\frac{Z_y}{Z^2} + \Omega_3 &&= \Omega_3 + V_x Z_Y \\
v_x &= U_2 f\frac{Z_x}{Z^2} - \Omega_3 &&= -\Omega_3 + V_y Z_X \\
v_y &= U_2 f\frac{Z_y}{Z^2} + \frac{U_3}{Z} &&= V_z + V_y Z_Y
\end{aligned}$$

Hence Proved.

Question 8

Given statements:

$$\begin{aligned}
div\mathbf{v} &= (u_x + v_y) \\
curl\mathbf{v} &= -(u_y - v_x) \\
(def\mathbf{v})cos2\mu &= (u_x - v_y) \\
(def\mathbf{v})sin2\mu &= (u_y + v_x)
\end{aligned}$$

Putting values in the equation:

$$\frac{div\mathbf{v}}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{curl\mathbf{v}}{2}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \frac{def\mathbf{v}}{2}\begin{bmatrix}cos2\mu & sin2\mu\\sin2\mu & -cos2\mu\end{bmatrix}$$

we get :

$$\begin{aligned}
&= \frac{(u_x+v_y)}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{(v_x-u_y)}{2}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \frac{def\mathbf{v}}{2}\begin{bmatrix}cos2\mu & sin2\mu\\sin2\mu & -cos2\mu\end{bmatrix} \\
&= \begin{bmatrix}\frac{(u_x+v_y)}{2} & 0\\0 & \frac{(u_x+v_y)}{2}\end{bmatrix} + \begin{bmatrix}0 & \frac{(u_y-v_x)}{2}\\\frac{(v_x-u_y)}{2} & 0\end{bmatrix} + \begin{bmatrix}(\frac{def\mathbf{v}}{2})cos2\mu & (\frac{def\mathbf{v}}{2})sin2\mu\\(\frac{def\mathbf{v}}{2})sin2\mu & -(\frac{def\mathbf{v}}{2})cos2\mu\end{bmatrix} \\
&= \begin{bmatrix}\frac{(u_x+v_y)}{2} & \frac{(u_y-v_x)}{2}\\\frac{(u_y-v_x)}{2} & \frac{(v_y+u_x)}{2}\end{bmatrix} + \begin{bmatrix}\frac{(u_x-v_y)}{2} & \frac{(u_y+v_x)}{2}\\\frac{(u_y+v_x)}{2} & \frac{(v_y-u_x)}{2}\end{bmatrix} \\
&= \begin{bmatrix}u_x & u_y\\v_x & v_y\end{bmatrix}
\end{aligned}$$

Therefore,

$$\begin{bmatrix}u_x & u_y\\v_x & v_y\end{bmatrix} = \frac{div\mathbf{v}}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \frac{curl\mathbf{v}}{2}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} + \frac{def\mathbf{v}}{2}\begin{bmatrix}cos2\mu & sin2\mu\\sin2\mu & -cos2\mu\end{bmatrix}$$

Hence Verified.

Question 9

We know that $\hat{\mathbf{Q}}$ is the unit vector in the camera look-at direction. Therefore,

$$\hat{\mathbf{Q}} = (0, 0, 1)$$

Also, $\vec{\mathbf{U}}$ is the translational velocity of the object. Therefore,

$$\begin{aligned}
\vec{\mathbf{U}} &= (U_1, U_2, U_3) \\
\vec{\mathbf{A}} &= \frac{\vec{\mathbf{U}} - (\vec{\mathbf{U}} \cdot \hat{\mathbf{Q}})\hat{\mathbf{Q}}}{Z} \\
&= \frac{(U_1, U_2, U_3) - (0, 0, U_3)}{Z} \\
\therefore \vec{\mathbf{A}} &= \frac{(U_1, U_2, 0)}{Z}
\end{aligned}$$

Next, we write ∇Z as:

$$\begin{aligned}
\nabla Z &= (\frac{\partial Z}{\partial x}, \frac{\partial Z}{\partial y}, 0) = (Z_x, Z_y, 0) \\
\therefore \vec{\mathbf{F}} &= \frac{f}{Z}\nabla Z = \frac{f}{Z}(Z_x, Z_y, 0)
\end{aligned}$$

Also, from *Question 8* we have:

$$\begin{aligned}
div\mathbf{v} &= (u_x + v_y) \\
curl\mathbf{v} &= -(u_y - v_x) \\
(def\mathbf{v})cos2\mu &= (u_x - v_y) \\
(def\mathbf{v})sin2\mu &= (u_y + v_x)
\end{aligned}$$

And from *Question 7* we have:

$$\begin{aligned}
u_x &= U_1 f\frac{Z_x}{Z^2} + \frac{U_3}{Z} \\
u_y &= U_1 f\frac{Z_y}{Z^2} + \Omega_3 \\
v_x &= U_2 f\frac{Z_x}{Z^2} - \Omega_3 \\
v_y &= U_2 f\frac{Z_y}{Z^2} + \frac{U_3}{Z}
\end{aligned}$$

Putting the values, we get:

$$\begin{aligned}
div\mathbf{v} &= 2\frac{U_3}{Z} + \frac{f}{Z^2}(U_1 Z_x + U_2 Z_y) \\
curl\mathbf{v} &= -2\Omega_3 + \frac{f}{Z^2}(U_2 Z_x - U_1 Z_y) \\
(def\mathbf{v})cos2\mu &= \frac{f}{Z^2}(U_1 Z_x - U_2 Z_y) \\
(def\mathbf{v})sin2\mu &= \frac{f}{Z^2}(U_1 Z_y + U_2 Z_x)
\end{aligned}$$

Now let us evaluate:

$$
\begin{aligned}
&2\frac{\vec{\mathbf{U}}\cdot\hat{\mathbf{Q}}}{Z}+\vec{\mathbf{F}}\cdot\vec{\mathbf{A}}\\
=&2\frac{(U_1,U_2,U_3)\cdot(0,0,1)}{Z}+\frac{f}{Z}(Z_x,Z_y,0)\cdot\frac{(U_1,U_2,0)}{Z}\\
=&2\frac{U_3}{Z}+\frac{f}{Z^2}(Z_xU_1+Z_yU_2)\\
=&div\mathbf{v}
\end{aligned}
$$

$$
\boxed{\therefore\, div\mathbf{v}=2\frac{\vec{\mathbf{U}}\cdot\hat{\mathbf{Q}}}{Z}+\vec{\mathbf{F}}\cdot\vec{\mathbf{A}}}
$$

Similarly, for others:

$$
\begin{aligned}
&-2\vec{\mathbf{\Omega}}\cdot\hat{\mathbf{Q}}+\left|(\vec{\mathbf{F}}\times\vec{\mathbf{A}})\right|\\
=&-2(\Omega_1,\Omega_2,\Omega_3)\cdot(0,0,1)+\left|\frac{f}{Z}(Z_x,Z_y,0)\times\frac{(U_1,U_2,0)}{Z}\right|\\
=&-2\Omega_3+\left|\frac{f}{Z^2}(0,0,U_2Z_x-U_1Z_y)\right|\\
=&-2\Omega_3+\frac{f}{Z^2}(U_2Z_x-U_1Z_y)\\
=&curl\mathbf{v}
\end{aligned}
$$

$$
\boxed{\therefore\, curl\mathbf{v}=-2\vec{\mathbf{\Omega}}\cdot\hat{\mathbf{Q}}+\left|(\vec{\mathbf{F}}\times\vec{\mathbf{A}})\right|}
$$

Now :

$$
\begin{aligned}
def\mathbf{v}&=\sqrt{((def\mathbf{v})cos2\mu)^2+((def\mathbf{v})sin2\mu)^2}\\
&=\frac{f}{Z^2}\sqrt{U_1^2Z_x^2+U_2^2Z_y^2+U_2^2Z_x^2+U_1^2Z_y^2}\\
&=\frac{f}{Z^2}\sqrt{U_1^2+U_2^2}\sqrt{Z_x^2+Z_y^2}\\
&=\left|\frac{(U_1,U_2,0)}{Z}\right|+\left|\frac{f}{Z}(Z_x,Z_y,0)\right|\\
&=\left|\vec{\mathbf{F}}\right|\left|\vec{\mathbf{A}}\right|
\end{aligned}
$$

$$
\boxed{\therefore\, def\mathbf{v}=\left|\vec{\mathbf{F}}\right|\left|\vec{\mathbf{A}}\right|}
$$

Finally,

$$\begin{aligned}
tan2\mu &= \frac{(def\mathbf{v})sin2\mu}{(def\mathbf{v})cos2\mu} \\
&= \frac{\frac{f}{Z^2}(U_1Z_y + U_2Z_x)}{\frac{f}{Z^2}(U_1Z_x - U_2Z_y)} \\
&= \frac{(U_1Z_y + U_2Z_x)}{(U_1Z_x - U_2Z_y)} \\
&= \frac{(\frac{Z_y}{Z_x} + \frac{U_2}{U_1})}{(1 - \frac{U_2}{U_1}\frac{Z_y}{Z_x})}
\end{aligned}$$

And we already have:

$$tan(\angle A) = \frac{U_2}{U_1} \qquad tan(\angle F) = \frac{Z_y}{Z_x}$$

Therefore,

$$\begin{aligned}
tan2\mu &= \frac{tan(\angle A) + tan(\angle F)}{1 - tan(\angle A)tan(\angle F)} \\
&= \tan(\angle A + \angle F) \\
2\mu &= \angle A + \angle F \\
&\boxed{\therefore \mu = \frac{(\angle A + \angle F)}{2}}
\end{aligned}$$

Hence Proved.

Question 10

(a) Deformation can be used to encode the surface orientation as it gives us an idea of how object deforms with small change in object position, thus revealing information about the orientation. Similarly, divergence is a measure of how objects scale as they move towards camera, and thus high divergence implies that velocity is high and thus time of collision will be shorter. Also, these components are unaffected by viewer rotations such as panning or tilting of the camera unlike point image velocities, which change considerably.

(b) If $\vec{\mathbf{A}} = 0$ then $div\mathbf{v} = 2\frac{\vec{\mathbf{U}}\cdot\hat{\mathbf{Q}}}{Z}$. Also, $\vec{\mathbf{U}} \cdot \hat{\mathbf{Q}}$ encodes the velocity of object in direction of camera and Z is distance of object. Thus, time of collision is $\frac{Z}{\vec{\mathbf{U}}\cdot\hat{\mathbf{Q}}}$ which is $\frac{2}{div\mathbf{v}}$. Thus, if $\vec{\mathbf{A}} = 0$ then we can determine time of collision.
Also, in general motion, the time of collision is bounded by:

$$\frac{2}{div\mathbf{v} + def\mathbf{v}} \leq \mathbf{t_c} \leq \frac{2}{div\mathbf{v} - def\mathbf{v}}$$

(c) It can be seen that shallow objects that are near to camera will produce same affect as deep structures which are far away from camera, because in pinhole model the motion registered are actually scaled components of actual motion ($x = f\frac{X}{Z}$) and thus a large motion which is far away will approximate to small motion that is near (as their ratio will be of same order).

(d) I think that egomotion can be used to gain some knowledge about the surface orientation, assuming that the object in question is actually constant. This is because small changes in the direction of camera look-at direction can be used to map the surface gradient of the object, and thus can give us an idea of the orientation of surface of the object.

Declaration of Originality

I hereby claim that the work presented here is my own and not copied from anywhere. Though, while writing these answers I had discussions with Aman Agrawal (2015CS10210) , Ankesh Gupta (2015CS10435) and Saket Dingliwal (2015CS10254).
These discussions were mainly focused on clearing doubts on the meaning of question and clearing diagramatic representation of the given situation. Also, I referred to a paper cited in the References for help in understanding the context and background of question 6.

References

[1] Roberto Cipolla and Andrew Blake. *Image divergence and deformation from closed curves*. Int. Journal of Robotics Research, 16(1):77-96, 1997.