COL780: Computer Vision
Max-Flow Min-Cut Theorem

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2015CS10262

September 15,2017

1 Statement

We are given a directed graph $G = (V, E)$, consisting of a source s (node with all outgoing edges) and a sink t (node with all incoming edges). Also, we have a mapping $c : E \to \mathbb{R}^+$, denoted by c_{uv} or $c(u, v)$, which is the maximum capacity of the edge (u, v).
Now, we define flow $f : E \to \mathbb{R}^+$ in the graph satisfying the following constraints:

- Capacity Constaint: $f(u, v) \leq c(u, v)$
- Conservation of Flow: $\forall v \in V \setminus \{s, t\} : \sum_{\{u:(u,v)\in E\}} f(u, v) = \sum_{\{u:(v,u)\in E\}} f(v, u)$.

Also, the value of flow is defined as the net amount of flow leaving the source. Mathematically, it is formulated as:

$$|f| = \sum_{\{v:(s,v)\in E\}} f(s, v) - \sum_{\{v:(v,s)\in E\}} f(v, s)$$

Finally, we define a cut $C = (S, T)$, which is a partition of V in two disjoint sets S, T such that $s \in S$ and $t \in T$. The capacity of the cut if defined as the sum of the capacity of edges going from S to T. Mathematically,

$$cap(C) = \sum_{\{(u,v)\in E, u\in S, v\in T\}} c(u, v)$$

Max-flow min-cut theorem: The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.

2 Proof

In order to show that max-flow is equal to min cut, we will first show that all cuts are always greater or equal to all flows and then proceed to show that there exists a flow which is equal to a cut.
Lemma 1. Given any flow f and any cut C on graph. Then, $|f| \leq cap(C)$
Proof. Let cut $C = (S, T)$. Since, $s \in S$ and $t \notin S$

$$|f| = f_{out}(s) - f_{in}(s) = f_{out}S - f_{in}(S)$$

since nodes other s in S don't contribute to flow. Now, the flows which positively impact $|f|$ are in cut C, therefore

$$|f| \leq \sum_{(u,v)\in\text{edges of cut C}} f(u, v) \leq \sum_{(u,v)\in\text{edges of cut C}} c(u, v) = cap(C)$$

Hence Proved.

Corollary. Let f^* be the maximum flow and C^* be the minimum cut. Then $|f^*| \leq cap(C^*)$.

Now, let us define the notion of augmenting paths. Consider any path P from s to t without considering the direction of edges. Define the f-augment of P to be:

$$aug(P) = \min_{(u,v)\in P} res(u,v)$$

where,

$$res(u,v) = \begin{cases} c(u,v) - f(u,v), & \text{if (u,v) points towards t} \\ f(u,v), & \text{if (u,v) points towards s} \end{cases}$$

A path P is called augmenting path iff it starts from source s and ends at sink t and has a positive f-augment.
Observe that if P is an augmenting path then we can change our flow according to:

$$f'(u,v) = \begin{cases} f(u,v) + aug(P), & \text{if } (u,v) \in P \text{ and (u,v) points towards t} \\ f(u,v) - aug(P), & \text{if } (u,v) \in P \text{ and (u,v) points towards s} \\ f(u,v), & \text{otherwise} \end{cases}$$

and the resulting flow f' will be greater than our previous flow f by value $aug(P)$.

Lemma 2. There exists a flow f and a cut C, such that $|f| = cap(C)$
Proof. Let us start with zero flow f and keep constructing new flow f' from any augmenting path P we can find in the graph. Now we will have a flow f^* such that no augmenting path from s to t is possible.
Now, construct a set S of all nodes u such that there exists a augmenting path from source s to v. Note that sink t cannot be in this set by construction. Let us denote set $\overline{S}$ by T. This also defines a cut $C^* = (S,T)$. We denote set of edges of cut C^* by K i.e., $K = \{(u,v) | u \in S, v \in T, (u,v) \in E\}$

Suppose, for the sake of contradiction, that $\exists (u,v) \in K$ s.t. $f^*(u,v) < c(u,v)$. Now in this case we can extend our set S to include node v because there exists a path from s to v which is augmenting. But this results in a contradiction as set S was maximal set which contained all vertices with augmenting path from s and vertex v was not in the set S. Thus,

$$\forall (u,v) \in K \quad f^*(u,v) = c(u,v)$$

Similarly, $f^*(u,v) = 0$ for all $(v,u) \in \overline{K}$. Now,

$$|f^*| = \sum_{(u,v)\in K} f^*(u,v) - \sum_{(v,u)\in \overline{K}} f^*(v,u) = \sum_{(u,v)\in K} c(u,v) - 0 = cap(C^*)$$

Thus, we have a flow f^* and a cut C^* with equal value.

Max-flow min-cut theorem: The maximum value of an s-t flow is equal to the minimum capacity over all s-t cuts.
Proof: Let the min cut be C^* and the max flow be f^*. By Corollary, we know that:

$$|f^*| \leq cap(C^*)$$

But, from Lemma 2, we know that:

$$\exists f, C \text{ s.t. } |f| = cap(C)$$

Therefore, we must have that:

$$|f^*| = cap(C^*)$$

as f^* is the maximum of all flows and C^* is minimum of all cuts.
Hence Proved.

References

[1] Joseph, Shaun *The Max-Flow Min-Cut Theorem.* The University of Rhode Island, Mathematics Dept:Dec 6, 2007.

[2] *Max-flow min-cut theorem - Wikipedia, the free encyclopedia.* Retrieved from `https://en.wikipedia.org/wiki/Max-flow_min-cut_theorem` ([Online; accessed 15-September-2017])