Problem Session 1: Probability Review

AA 228 / CS 238

Autumn 2025

Slides adapted from Emily Molins, with example problems from Algorithms for Decision Making by Mykel Kochenderfer, Tim Wheeler, Kyle Wray and the CS109 Reader by Chris Piech.

Stanford University

Hello! Today, we will:

1. Solidify our foundations of **probability**, **conditional and joint probabilities**, **Bayes' theorem** and **Bayesian inference**

2. Refresh our memory on some useful **probability distributions**

3. Review a few **example problems** to get ourselves up to speed and ready for exploring Baysian Networks shortly in the course.

Slides adapted from Emily Molins, with example problems contributed

Lot’s of handwritten math, feel free to follow along!

Detailed notes + worked solutions

https://cs.stanford.edu/~houjun/probability.pdf

Probability?

"P(E) is a measure of the chance of E occurring."

“P(E) is a measure of the chance of E occurring.”

“Dice rolls a 1”

“Dice rolls a 2”

“Dice rolls a 3”

…

So many different events
for the same “thing”

Random variables to the rescue!

Random Variable

A variable that probabilistically takes on different values

X

$$P(X = 1) = \frac{1}{6} \qquad P(X = 3) = \frac{1}{6} \qquad P(X = 5) = \frac{1}{6}$$

$$P(X = 2) = \frac{1}{6} \qquad P(X = 4) = \frac{1}{6} \qquad P(X = 6) = \frac{1}{6}$$

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$$P(\underbrace{X = 2}) = \frac{1}{6} \qquad P(\underbrace{X = 4}_{event}) = \frac{1}{6} \qquad P(X = 6) = \frac{1}{6}$$

"Distribution"

Random Variable

A variable that probabilistically takes on different values

$X \sim D$ “X is a D random variable”

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$X \sim D$ "X is a D random variable"

$$P(x^1) = \frac{1}{6} \qquad P(x^3) = \frac{1}{6} \qquad P(x^5) = \frac{1}{6}$$

$$P(x^2) = \frac{1}{6} \qquad P(\underbrace{x^4}_{event}) = \frac{1}{6} \qquad P(x^6) = \frac{1}{6}$$

"Distribution"

Key Axioms

$$0 \leq P(x^j) \leq 1, \forall X, \forall j$$

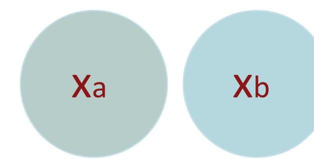
All probabilities are numbers between 0 and 1.

$$P(x^1 \vee \ldots \vee x^n) = 1$$

The sample space is the set of all possible outcomes.

$$P(x^a) + P(x^b) = P(x^a \vee x^b)$$
$$if\ x^a \wedge x^b = F$$

Mutual Exclusivity.



Key Results

$$0 \leq P(x^j) \leq 1, \forall X, \forall j$$

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$$P(x^1 \vee \ldots \vee x^n) = 1$$

The sample space is the set of all possible outcomes.

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Mutual Exclusivity.

$$P(\neg x^j) = 1 - P(x^j)$$

Probability of complements

$$x^a \Rightarrow x^b, P(x^a) \leq P(x^b)$$

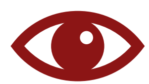
Probability of subsets

$$P(x^a) + P(x^b) - P(x^a \wedge x^b) = P(x^a \vee x^b)$$

Inclusion Exclusion

Two events:

x, loosing contact

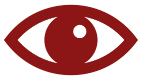


y, sensor failure

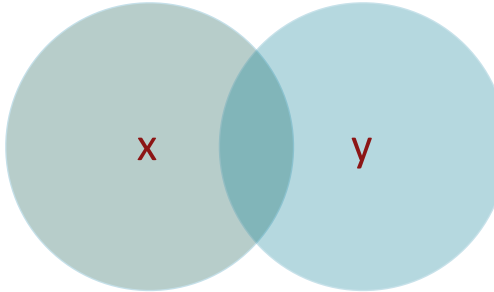
What’s the probability of us loosing contact given we had a sensor failure?

Two events:

x, loosing contact



y, sensor failure

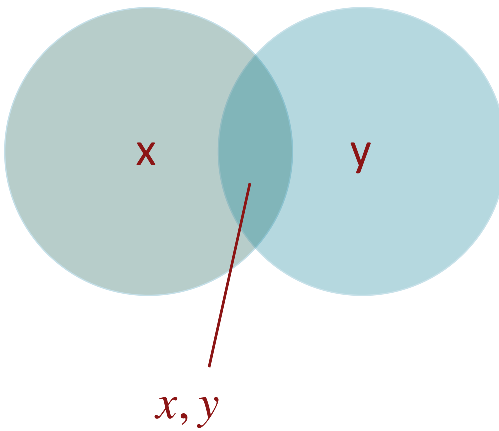


What’s the probability of us loosing contact given we had a sensor failure?

Conditional Probabilities

x, loosing contact y, sensor failure

What's the probability of us loosing contact given we had a sensor failure?



$$P(x \mid y) := \frac{P(x, y)}{P(y)}$$

Probability Chain Rule

$$P(x^1, x^2 \ldots, x^n) = P(x^n \mid x^1, x^2 \ldots x^{n-1}) P(x^1, x^2 \ldots x^{n-1})$$

Bayes Inference

x: email has word “gold” in subject

y: email is spam

$$P(x \mid y) \qquad\qquad P(y \mid x)$$

Easy to measure | More useful

Bayes Theorem

$$P(y \mid x) = \frac{P(x \mid y)P(y)}{P(x)}$$

Independence

$$P(x \mid y) = P(x), \textit{if } x \perp y$$

$$P(x) \cdot P(y) = P(x, y), \textit{if } x \perp y$$

Law of Total Probability

$$P(x) = \sum_{y \in Y} P(x, y)$$

$$P(x) = \sum_{y \in Y} P(x \mid y) P(y)$$

What’s the probability that a patient has breast cancer, given they have a positive mammogram result?

- Patient has 8% chance of developing breast cancer
- Mammogram has a true positive rate of 95%
- Mammogram has a false positive rate of 7%

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about 54.13%

There are **three** doors, prize behind **one**, dreaded midterm behind the other two.

Probability of prize being behind each door is equivalent. You are playing against a host, who is playing rationally.

You picked door 1.
Host reveals that door 3 had a midterm behind it.

Should you switch your choice to door 2?

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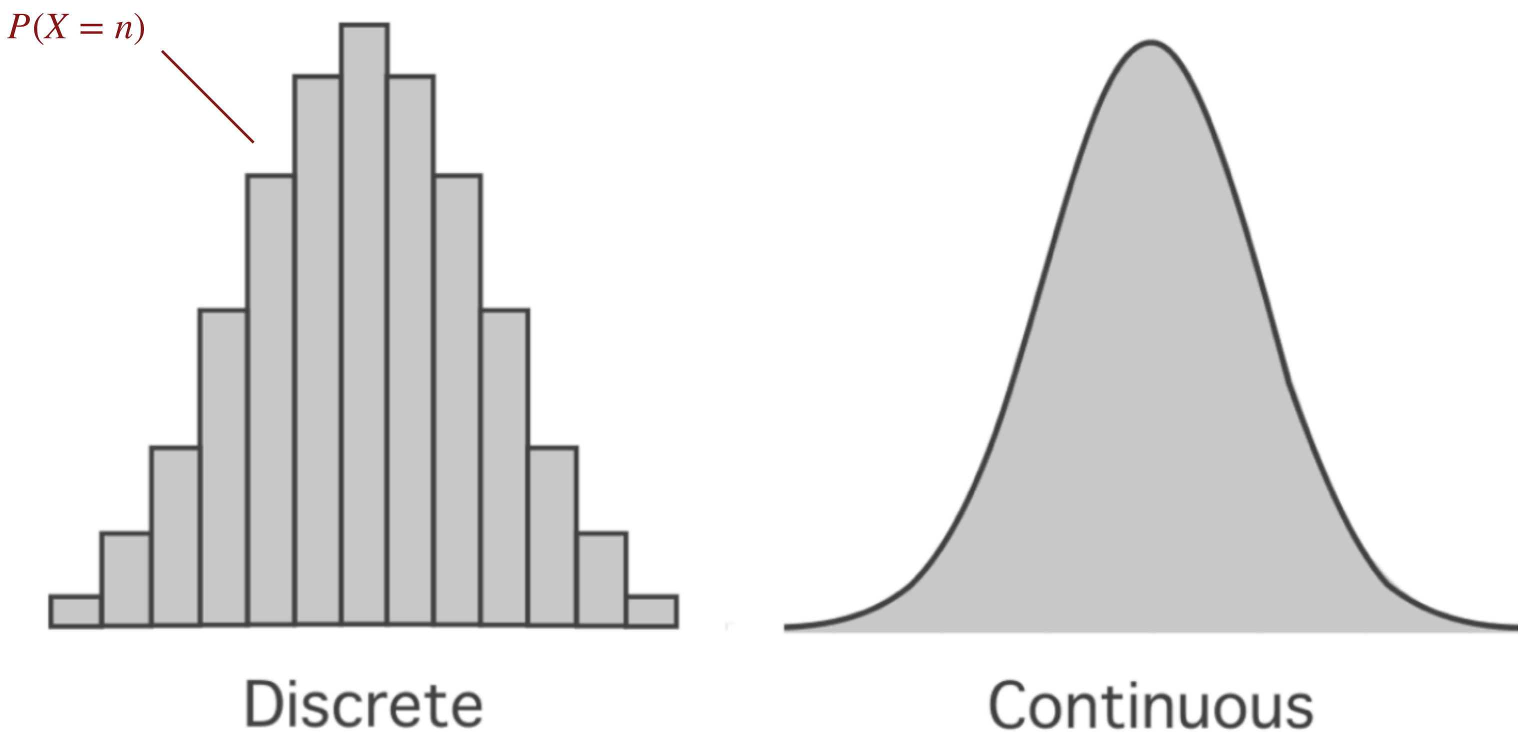
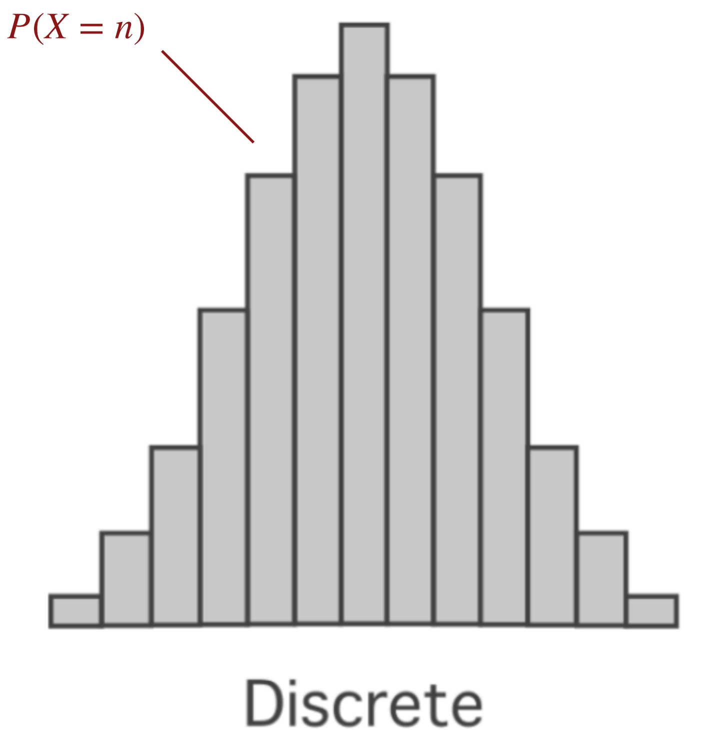
$$P(p^1 \mid h^3) = \frac{1}{3} < P(p^2 \mid h^3) = \frac{2}{3}$$

you should.

Probability Distributions



Discrete Distributions & Probability Mass Function (PDF)



$$\sum_{x \in X} P(x) = 1$$

Where P is our **probability mass function**, which is an assignment of probabilities to outcomes.

"What's the likelihood of the high tomorrow at stanford being exactly 82.9239820 deg F?"

Basically… none.

Continuous Distributions & Probability Density Functions (PDF)

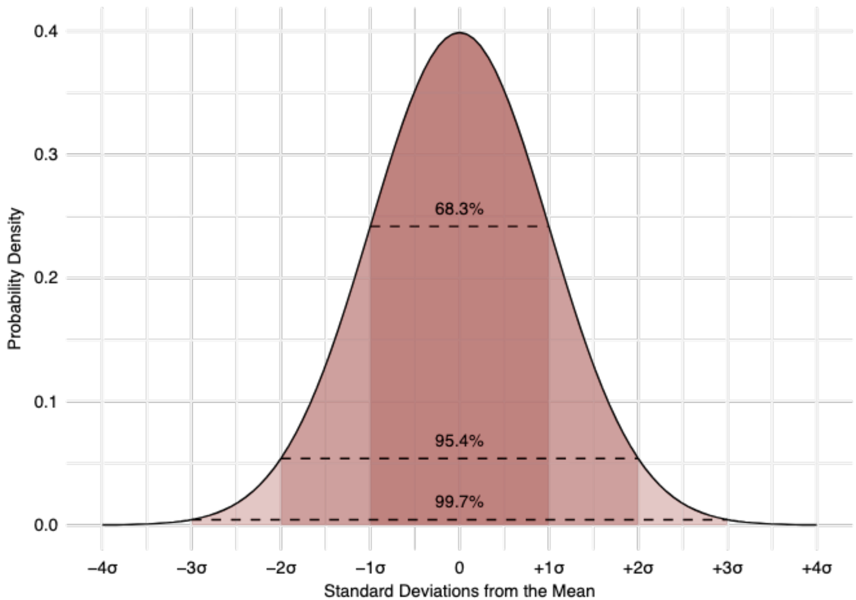


Photo: wikimedia commons

$$P(a \leq X \leq b) = \int_a^b f(x)\, dX$$

We often ask for P(X<x), so we also define a **probability mass function**

$$F(x) = P[X \leq x] = \int_{-\infty}^{x} f(z)\, dZ$$

PDF gives probability change, CDF gives probability.

First and second moments

Expected value: the “mean” of a random variable

$$E[X] = \int_{-\infty}^{\infty} x f(x)\, dX$$

Variance

$$Var[X] = E[X^2] - [E[X]]^2$$

Some Useful Distributions

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Gaussian

Notation

$$X \sim \mathcal{N}(\mu, \sigma^2)$$

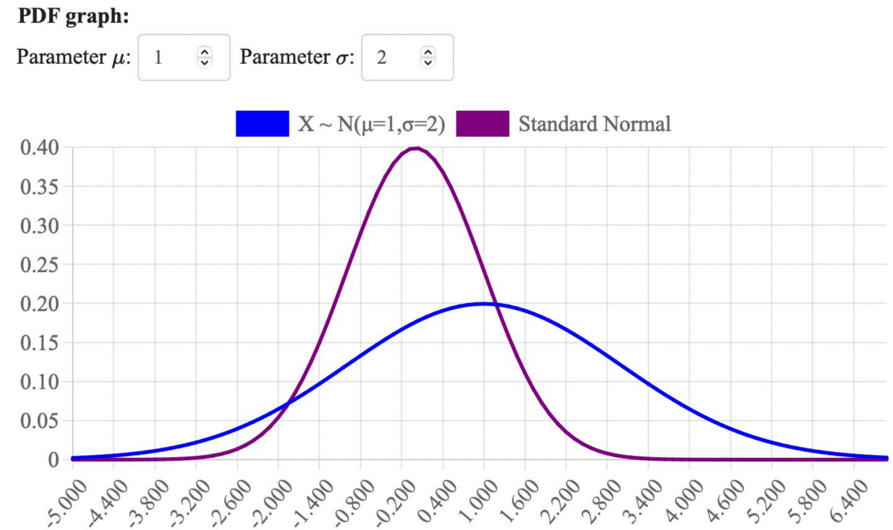
PDF

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

Moments

$$E[X] = \mu$$

$$Var[x] = \sigma^2$$



$$\sum_{i=1}^{n} X_i \sim \mathcal{N}(n\mu, n\sigma^2), \text{ as } n \to \infty$$

Uniform

Notation

$$X \sim Uni(a, b)$$

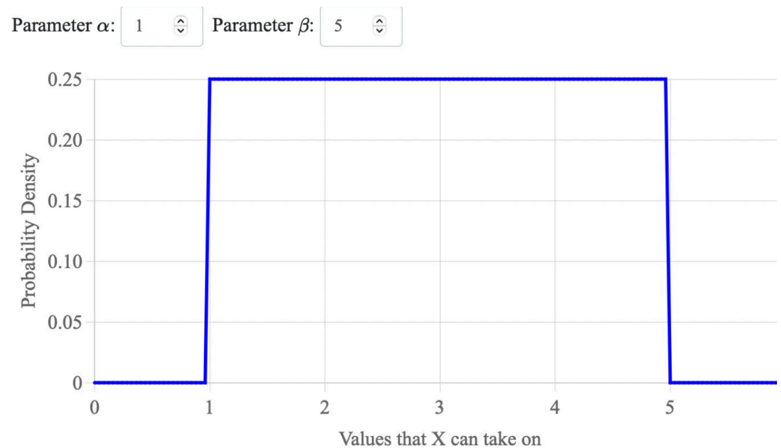
PDF

$$f(x) = \begin{cases} \dfrac{1}{\beta - \alpha}, a \leq x \leq b \\ 0 \end{cases}$$

Moments

$$E[X] = \frac{1}{2}(a + b)$$

$$Var(X) = \frac{1}{12}(b - a)^2$$



Consider a continuous random variable X, which exponential distribution parameterized by λ with density p(x|λ) = λ exp(-λx) with nonnegative support; compute the CDF of X.

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$1-\exp(-\lambda x)$

Additional Resources

- Chapter 2 of the Textbook: https://algorithmsbook.com/files/chapter-2.pdf
- CS109 Course reader: https://chrispiech.github.io/probabilityForComputerScientists/en/
- Brown's Probability Visualizations: https://seeing-theory.brown.edu
- Khan Academy probability: https://www.khanacademy.org/math/statistics-probability/probability-library

Thank You!
Questions?