ETH
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Formal Methods and Functional Programming

Session 8: Induction on the shape of derivation trees

Assignment

Here are repeated the Natural Deduction rules we consider:

$$\frac{}{\underline{\Gamma}, \underline{A} \vdash \underline{A}}\,(\mathrm{Ax}) \quad \frac{\underline{\Gamma} \vdash \underline{A} \quad \underline{\Gamma} \vdash \underline{B}}{\underline{\Gamma} \vdash \underline{A} \wedge \underline{B}}\,(\wedge\text{-I}) \quad \frac{\underline{\Gamma} \vdash \underline{A} \wedge \underline{B}}{\underline{\Gamma} \vdash \underline{A}}\,(\wedge\text{-EL})$$

The lemma to be proven is:

$$\begin{aligned}&\forall D_1, D_2, \Gamma, A, B \cdot \\ &\qquad \mathtt{root}(D_1) = (\Gamma, A \vdash B) \ \wedge \ \mathtt{root}(D_2) = (\Gamma \vdash A) \ \Rightarrow \ \exists D_3 \cdot \mathtt{root}(D_3) = (\Gamma \vdash B)\end{aligned}$$

We will prove this by induction on the shape of the derivation tree D_1.

Answer

Proof: Let $P(D_1) \equiv$

$$\begin{aligned}&\forall D_2, \Gamma, A, B \cdot \\ &\qquad \underbrace{\mathtt{root}(D_1) = (\Gamma, A \vdash B)}_{\textsc{Ant}_1} \wedge \underbrace{\mathtt{root}(D_2) = (\Gamma \vdash A)}_{\textsc{Ant}_2} \Rightarrow \exists D_3 \cdot \mathtt{root}(D_3) = (\Gamma \vdash B).\end{aligned}$$

We prove $\forall D_1 \cdot P(D_1)$ by induction on the shape of the derivation tree D_1. That way, we get the induction hypothesis $\forall D_1' \sqsubset D_1 \cdot P(D_1')$, that is, the proposition holds for all proper subtrees of D_1. For example, if the last applied rule of a derivation tree is $\wedge$-I, then the induction hypothesis holds for the two subtrees whose roots are the two premises of the rule, as well as for all their proper subtrees.

Let D_2, Γ, A, B be arbitrary, and assume that $\textsc{Ant}_1$ and $\textsc{Ant}_2$ hold. We need to show that the right-hand side of the implication holds. That is, $\exists D_3 \cdot \texttt{root}(D_3) = (\Gamma \vdash B)$. We proceed by a case analysis of the last rule applied in the derivation.

Case $\textsc{Ax}$:

Recall: the definition of the rule is:

$$\frac{}{\underline{\Gamma}, \underline{A} \vdash \underline{A}} \text{(Ax)}$$

and we know that $\texttt{root}(D_1) = (\Gamma, A \vdash B)$ and so, by unifying (or pattern matching) $(\Gamma, A \vdash B)$ and $(\underline{\Gamma}, \underline{A} \vdash \underline{A})$ we get $\underline{A} \equiv B$ and $(\underline{\Gamma}, \underline{A}) = (\Gamma', B) = (\Gamma, A)$ for some Γ'.

The use of equality in $(\Gamma', B) = (\Gamma, A)$ – and the fact that Γ, A denotes $\Gamma \cup \{A\}$ – also explains why we can neither conclude that $\Gamma = \Gamma'$, nor that $A \equiv B$. However, we do know $B \in (\Gamma, A)$.

The derivation is thus of the form:

$$\frac{}{\Gamma, A \vdash B} \text{(Ax)}$$

To show: $\exists D_3 \cdot \texttt{root}(D_3) = (\Gamma \vdash B)$

We perform a case distinction on whether or not $A \equiv B$:

Case $A \equiv B$:

In this case, we can conclude by choosing the required D_3 to be D_2, since, from $\textsc{Ant}_2$ we know that $\texttt{root}(D_2) = (\Gamma \vdash A)$.

Case $A \not\equiv B$:

From $(\Gamma', B) = (\Gamma, A)$ and $A \not\equiv B$ it follows that $B \in \Gamma$, and thus, for some Γ'', we have $\Gamma = (\Gamma'', B)$. Thus, we need a derivation tree with root $\Gamma'', B \vdash B$, which can be obtained by a single instance of the $\textsc{AX}$ rule.

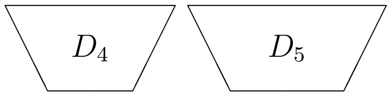
Case $\wedge$-I:

Recall: the definition of the rule is:

$$\frac{\underline{\Gamma} \vdash \underline{A} \quad \underline{\Gamma} \vdash \underline{B}}{\underline{\Gamma} \vdash \underline{A} \wedge \underline{B}} (\wedge\text{-I})$$

and we know that $\texttt{root}(D_1) = (\Gamma, A \vdash B)$ and so, by unifying (or pattern matching) $(\Gamma, A \vdash B)$ and $(\underline{\Gamma} \vdash \underline{A} \wedge \underline{B})$ we get $\underline{\Gamma} = (\Gamma, A)$ and $B \equiv (C \wedge D)$ for some C,D.

The derivation tree D_1 is then of the form



$$\frac{\Gamma \vdash C \qquad \Gamma, A \vdash D}{\Gamma, A \vdash C \wedge D}\ (\wedge\text{-I})$$

where D_4 and D_5 are derivation trees rooted on the left and right premises, respectively. That is, $\mathtt{root}(D_4) = (\Gamma \vdash C)$ and $\mathtt{root}(D_5) = (\Gamma, A \vdash D)$.

To show: $\exists D_3 \cdot \mathtt{root}(D_3) = (\Gamma \vdash C \wedge D)$

We can apply the induction hypothesis to the derivation trees D_4 and D_5 as they are proper subtrees of D_1. We rename quantified variables in order not to increase confusion and ambiguity.

By applying the induction hypothesis to D_4 we arrive at

$$\begin{aligned} P(D_4) \equiv\ & \forall D_2', \Gamma', A', B' \cdot \\ & \quad \mathtt{root}(D_4) = (\Gamma', A' \vdash B') \wedge \mathtt{root}(D_2') = (\Gamma' \vdash A') \\ & \quad \Rightarrow \exists D_3' \cdot \mathtt{root}(D_3') = (\Gamma' \vdash B') \end{aligned}$$

In order to get the conclusion of the above implication, we need to instantiate the quantified variables and show that the left-hand side of the implication holds.

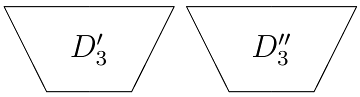
Let us consider the first antecedent, that is, $\mathtt{root}(D_4) = (\Gamma', A' \vdash B')$. We could instantiate the quantified variables that occur in it in any way we like, but because we already know that $\mathtt{root}(D_4) = (\Gamma, A \vdash C)$, the likely useful instantiation would be: $\Gamma' = \Gamma$, $A' \equiv A$ and $B' \equiv C$.

The second conjunct then becomes $\mathtt{root}(D_2') = (\Gamma \vdash A)$, which is $\textsc{Ant}_2$ if we instantiate $D_2 = D_2'$.

This gives us the desired conclusion of the implication, namely $\exists D_3' \cdot \mathtt{root}(D_3') = (\Gamma \vdash C)$.

By applying the induction hypothesis to D_5 and from instantiating the quantified variables in a similar way we then also obtain that there is some derivation D_3'' with $\mathtt{root}(D_3'') = (\Gamma \vdash D)$.

We conclude this case by choosing the required derivation D_3 to be:



$$\frac{\Gamma \vdash C \qquad \Gamma \vdash D}{\Gamma \vdash C \wedge D}\ (\wedge\text{-I})$$

Case $\wedge$-EL:

Recall: the definition of the rule is:

$$\frac{\underline{\Gamma} \vdash \underline{A} \wedge \underline{B}}{\underline{\Gamma} \vdash \underline{A}} \,(\wedge\text{-EL})$$

and we know that $\texttt{root}(D_1) = (\Gamma, A \vdash B)$ and so, by unifying (or pattern matching) $(\Gamma, A \vdash B)$ and $(\underline{\Gamma} \vdash \underline{A})$ we get $\underline{\Gamma} = (\Gamma, A)$ and $\underline{A} = B$. Then, for some C, the derivation tree D_1 is then of the form

$$\begin{array}{c} D_4 \\ \dfrac{\Gamma, A \vdash B \wedge C}{\Gamma, A \vdash B} \,(\wedge\text{-EL}) \end{array}$$

where D_4 is the derivation tree rooted at the premise with

$$\texttt{root}(D_4) = (\Gamma, A \vdash B \wedge C)$$

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To show: $\exists D_3 \cdot \texttt{root}(D_3) = \Gamma \vdash B$

We can apply the induction hypothesis to D_4 because it is a proper subtree of D_1. We rename quantified variables in order not to increase confusion and ambiguity, arriving to

$$\begin{aligned} &P(D_4) \equiv \forall D_2', \Gamma', A', B' \cdot \\ &\qquad \texttt{root}(D_4) = (\Gamma', A' \vdash B') \wedge \texttt{root}(D_2') = (\Gamma' \vdash A') \\ &\qquad \Rightarrow \exists D_3' \cdot \texttt{root}(D_3') = (\Gamma' \vdash B') \end{aligned}$$

Let us consider the first antecedent, that is, $\texttt{root}(D_4) = (\Gamma', A' \vdash B')$. We could instantiate the quantified variables that occur in it in any way we like, but because we already know that $\texttt{root}(D_4) = (\Gamma, A \vdash B \wedge C)$, the likely useful instantiation would be: $\Gamma' = \Gamma$, $A' \equiv A$ and $B' \equiv B \wedge C$.

The second conjunct then becomes $\texttt{root}(D_2') = (\Gamma \vdash A)$, which is $\textsc{Ant}_2$ if we instantiate $D_2 = D_2'$, allowing us to deduce:

$$\begin{aligned} &\texttt{root}(D_4) = (\Gamma, A \vdash B \wedge C) \wedge \texttt{root}(D_2) = (\Gamma \vdash A) \\ &\Rightarrow \exists D_3' \cdot \texttt{root}(D_3') = (\Gamma \vdash B \wedge C) \end{aligned}$$

that gives us the desired conclusion of the implication, namely, there is some D_3' with $\texttt{root}(D_3') = (\Gamma \vdash B \wedge C)$.

We conclude this case by choosing the required derivation D_3 to be:

$$\begin{array}{c} D_3' \\ \dfrac{\Gamma \vdash B \wedge C}{\Gamma \vdash B} \,(\wedge\text{-EL}) \end{array}$$