Multibody simulation

The Jacobian matrix (a tool for analysis)

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Main points covered

- system of linear equations
- null space of a matrix
- kinematic singularity
- redundancy
- statics
- inverse geometric model (numerical solution)

Geometric and kinematic models

So far, we have established a

geometric model

$$\begin{bmatrix} \boldsymbol{p} \\ \text{orientation} \end{bmatrix} = \boldsymbol{f}(\boldsymbol{q}),$$

it relates the **generalized position coordinates** $\boldsymbol{q}$ to the posture of a frame that could be fixed in any of the rigid bodies of the system.

kinematic model

$$\boldsymbol{\xi} = \begin{bmatrix} \boldsymbol{v} \\ \boldsymbol{\omega} \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

relates the **generalized velocity coordinates** $\dot{\boldsymbol{q}}$ to $(\boldsymbol{v}, \boldsymbol{\omega})$ of a frame that could be fixed in any of the rigid bodies of the system.

In Greek, "$\kappa\iota\nu\varepsilon\iota\nu$" (kinein) means to move. Kinematics describes the motion of objects without considering the cause for the motion ($i.e.$, the forces).

Even though we still have not formulated the equations of motion for a robotic manipulator, by inspecting the kinematics model, we are able to reveal many of the system's characteristics.

One of the most important quantities (for the purpose of analysis) is the Jacobian matrix $\boldsymbol{J}$. It reveals many of the properties of a system and can be used for

- the formulation of the equations of motion
- analysis of "special" system configurations
- static analysis
- motion planning, etc.

In the previous lecture, we discussed how to form the Jacobian matrix. Here, we will examine some of its important applications in robotics.

Inverse kinematics

We already noticed that

$$\boldsymbol{\xi} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

is a linear equation from $\dot{\boldsymbol{q}}$ to $\boldsymbol{\xi}$. In many cases, we are interested in finding what joint velocities $\dot{\boldsymbol{q}}$ result in given (desired) end-effector velocities. We call this: **inverse kinematics** problem. Solving it amounts to solving a system of linear equations.

Interpretation

- The row-size of $\boldsymbol{J}$ gives the number of equations. In our case these can be interpreted as the **number of tasks**.
- The column-size gives the number of unknowns (variables). In our case these are the **number of DoF** of the manipulator.

We will assume that $\boldsymbol{J} \in \mathbb{R}^{m \times n}$ with $m \leq n$, $i.e.$, the number of DoF is larger or equal to the number of tasks (constraints) to be satisfied.

Case 1: $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$ $(m = n)$

A square system of linear equations can have

- unique solution
- no solutions
- infinitely many solutions

If the columns of a matrix $\boldsymbol{J}$ form a basis for $\mathbb{R}^n$, we say that it is **nonsingular**. The following statements are equivalent

- the columns (and thus rows) of $\boldsymbol{J}$ form a basis for $\mathbb{R}^n$
- $\det(\boldsymbol{J}) \neq 0$
- $\mathcal{R}(\boldsymbol{J}) = \mathbb{R}^n$
- $\mathcal{N}(\boldsymbol{J}) = \{\boldsymbol{0}\}$
- $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$ has a unique solution $\dot{\boldsymbol{q}}$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$
- $\boldsymbol{J}$ has (left and right) inverse, denoted by $\boldsymbol{J}^{-1}$

$$\boldsymbol{J}\boldsymbol{J}^{-1} = \boldsymbol{J}^{-1}\boldsymbol{J} = \boldsymbol{I}.$$

Case 1: (singularity)

When the system has no solutions or infinitely many solutions, we say that the system is **singular**.

Question

Why do we put such different cases under the same label "singular"?

Answer

By only analyzing $\boldsymbol{J}$ we can **not** distinguish the cases with no, or infinitely many solutions. For that purpose we need the right-hand-side vector $\boldsymbol{b}$.

Example

$$\underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}}_{\boldsymbol{A}} \begin{bmatrix} x \\ y \end{bmatrix} = \boldsymbol{b}.$$

Clearly $\det(A) = 0$. For $\boldsymbol{b} = (3, 2)$ there is no solution, while for $\boldsymbol{b} = (3, 9)$, there are infinitely many solutions.

Case 2: $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$ $(m < n)$

A rectangular system of linear equations can have

- no solutions (when $\boldsymbol{\xi} \notin \mathcal{R}(\boldsymbol{J})$)
- infinitely many solutions

A solution (when it exists) can be generated as follows

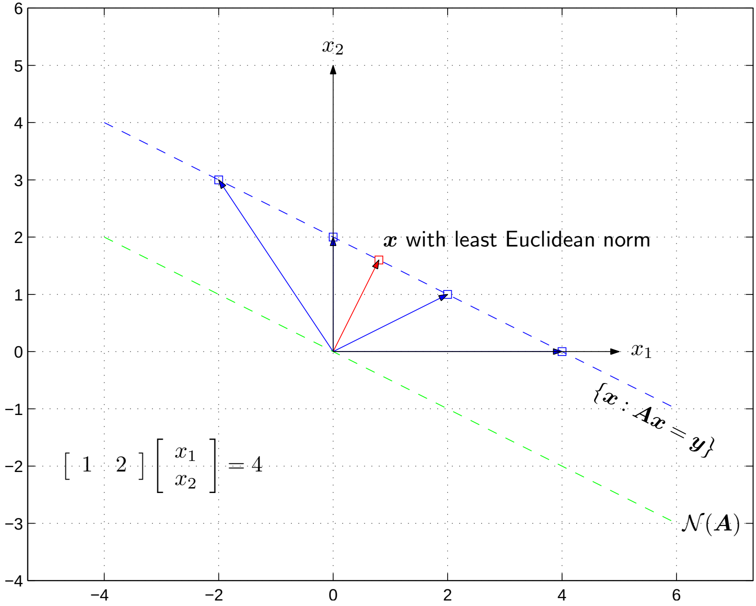
- solve $\boldsymbol{J}\dot{\boldsymbol{q}}_p = \boldsymbol{\xi}$ ($\dot{\boldsymbol{q}}_p$ is called a *particular solution*)
- solve $\boldsymbol{J}\dot{\boldsymbol{q}}_h = \boldsymbol{0}$ ($\dot{\boldsymbol{q}}_h$ is called a *homogeneous solution*)
- the *general solution* is given by $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_p + \dot{\boldsymbol{q}}_h$

Recall that the space of all solutions to $\boldsymbol{J}\dot{\boldsymbol{q}}_h = \boldsymbol{0}$ is called the **null space** of $\boldsymbol{J}$. The following equalities are satisfied

$$\boldsymbol{J}\dot{\boldsymbol{q}}_p = \boldsymbol{J}\dot{\boldsymbol{q}}_p + \boldsymbol{J}\dot{\boldsymbol{q}}_h = \boldsymbol{J}(\dot{\boldsymbol{q}}_p + \dot{\boldsymbol{q}}_h) = \boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}.$$

Note that if $\boldsymbol{J}$ is square and full rank, only the trivial solution $\dot{\boldsymbol{q}}_h = \boldsymbol{0}$ satisfies the *homogeneous equation*, $i.e.$, $\mathcal{N}(\boldsymbol{J}) = \{\boldsymbol{0}\}$.

Case 2 (example)



Kinematic singularity

Inverse kinematics

In many cases, we want to control directly the Cartesian velocities of an end-link. Hence, we need to compute what are the joint velocities that produce desired Cartesian velocities $\boldsymbol{\xi}$.

Question

Can we obtain an arbitrary $\boldsymbol{\xi}$ everywhere in the workspace of the manipulator?

Answering the above question requires to check whether $\boldsymbol{\xi} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$ has a solution for arbitrary $\boldsymbol{q}$ and $\boldsymbol{\xi}$. If $\text{rank}(\boldsymbol{J}) = m$, we are guaranteed to have a solution

- if $m = n$ the solution will be unique
- if $m < n$ there will be infinitely many solutions

Hence, we need to check whether there are manipulator configurations for which $\text{rank}(\boldsymbol{A}) < m$.

Example (2 DoF planar manipulator)

We already know that the Jacobian matrix of a 2 DoF planar manipulator (with revolute joints) is given by

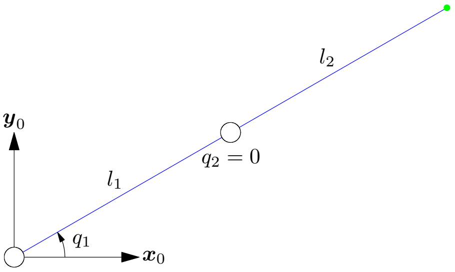
$$\boldsymbol{J} = \begin{bmatrix} -l_1 \sin q_1 - l_2 \sin(q_1 + q_2) & -l_2 \sin(q_1 + q_2) \\ l_1 \cos q_1 + l_2 \cos(q_1 + q_2) & l_2 \cos(q_1 + q_2) \end{bmatrix}$$

$\det(J) = l_1 l_2 \sin q_2$

When $q_2 = 0, \pi$, $\boldsymbol{J}$ loses rank. This means that we would not be able to produce an arbitrary velocity $\boldsymbol{\xi}$.

When $q_2 \neq 0, \pi$, the unique solution can be obtained using

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^{-1}\boldsymbol{\xi}.$$



Kinematics singularity (summary)

- A configuration where the Jacobian matrix looses rank is called a **singular configuration**. In such configurations the mobility of the system is reduced.
- In the neighborhood of a singularity, small Cartesian velocities $\boldsymbol{\xi}$ may cause large joint velocities $\dot{\boldsymbol{q}}$. For the above example, using $l_1 = l_2 = 1$, $q_1 = \pi/4$, $q_2 = 0.001$ leads to

$$J = \begin{bmatrix} -1.4149 & -0.7078 \\ 1.4135 & 0.7064 \end{bmatrix}$$

$$J^{-1} = \begin{bmatrix} 0.7064 & 0.7078 \\ -1.4135 & -1.4149 \end{bmatrix} 1000.$$

- Identifying singular configurations is important for the purposes of motion planning. Once we know regions where the mobility of the system is decreased, we can avoid them.

Kinematics redundancy

Consider the equation

$$\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}, \quad \boldsymbol{J} \in \mathbb{R}^{m \times n}.$$

In general, we say that a manipulator is redundant if $m < n$. Or in other words, the number of DoF of the robot is larger that the number of constraints (tasks) to be satisfied.

For example

if the tasks are to produce a desired linear and angular velocities ($i.e.$, 6 constraints), a manipulator with $n > 6$ is considered to be redundant

A word of caution

it is possible that a redundant manipulator is in a singular configuration, where $\boldsymbol{\xi} \notin \mathcal{R}(\boldsymbol{J})$ (can you give an example of such a case?)

Degree of redundancy

we call the dimension of $\mathcal{N}(\boldsymbol{J})$ **degree of redundancy**

The general solution of $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$ can be expressed as $\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}_p + \dot{\boldsymbol{q}}_h$, where $\dot{\boldsymbol{q}}_p$ and $\dot{\boldsymbol{q}}_h$ satisfy

- $\boldsymbol{J}\dot{\boldsymbol{q}}_p = \boldsymbol{\xi}$
- $\boldsymbol{J}\dot{\boldsymbol{q}}_h = \boldsymbol{0}$

A common choice for $\dot{\boldsymbol{q}}_p$ is

$$\dot{\boldsymbol{q}}_p = \boldsymbol{J}^\dagger \boldsymbol{\xi},$$

$\boldsymbol{J}^\dagger$ is the pseudoinverse of $\boldsymbol{J}$

Interpretation of $\boldsymbol{J}^\dagger \boldsymbol{\xi}$

$\boldsymbol{J}^\dagger \boldsymbol{\xi}$ is the vector with smallest Euclidean norm that satisfies $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$

General solution of $\boldsymbol{J}\dot{\boldsymbol{q}} = \boldsymbol{\xi}$ (again)

$$\dot{\boldsymbol{q}} = \boldsymbol{J}^\dagger \boldsymbol{\xi} + \boldsymbol{N}(\boldsymbol{q})\boldsymbol{\zeta},$$

where the columns of the matrix $\boldsymbol{N}(\boldsymbol{q})$ form a basis for $\mathcal{N}(\boldsymbol{J}(\boldsymbol{q}))$, and $\boldsymbol{\zeta}$ is an arbitrary vector (of appropriate dimensions)

Since $\boldsymbol{\zeta}$ is defined as an arbitrary vector, it can be utilized to satisfy additional criteria. For example, maximizing distance from mechanical joint limits (or obstacles) during the motion.

Example (self-motion of a 4 DoF planar system)

$l_i = 1$

$$\boldsymbol{q}_1 = \begin{bmatrix} 0.76 \\ 0.90 \\ 0.69 \\ 1.19 \end{bmatrix}$$

$$\boldsymbol{q}_9 = \begin{bmatrix} 1.81 \\ 2.06 \\ -1.70 \\ -1.16 \end{bmatrix}$$

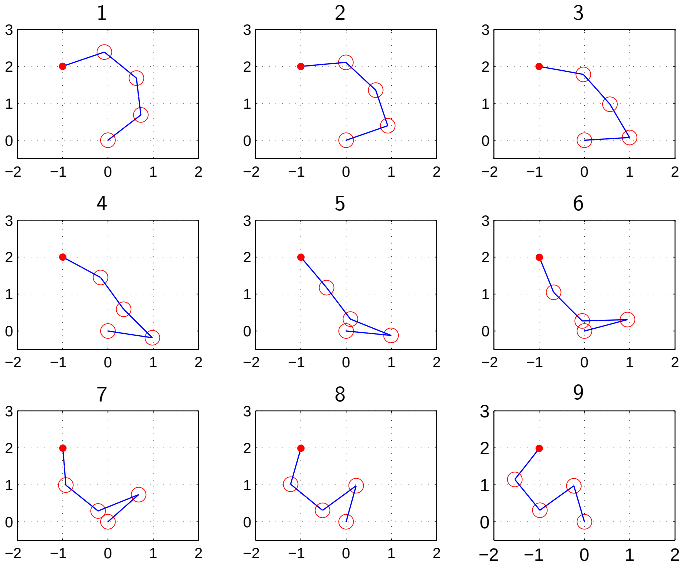
$\boldsymbol{p}_e = (-1, 2)$

$\boldsymbol{v}_e = (0, 0)$ desired

Joint velocities

$$\dot{\boldsymbol{q}} = \boldsymbol{N}(\boldsymbol{q})\boldsymbol{\zeta}$$

$$\boldsymbol{\zeta} = (-1, \dots, -1)$$



Static forces

Consider a manipulator system at rest ($\dot{\boldsymbol{q}} = \mathbf{0}$). Let $\boldsymbol{f}_e \in \mathbb{R}^3$ and $\boldsymbol{t}_e \in \mathbb{R}^3$ be **forces** and **torques** acting at the end-effector (possibly due to contact with the environment).

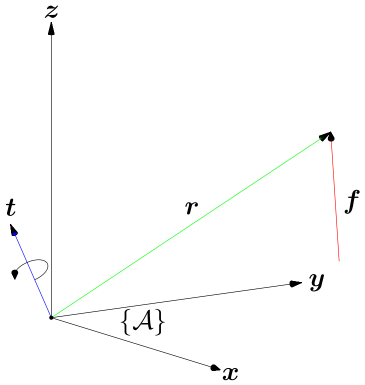
The goal of *statics* is to determine the relation between $\boldsymbol{f}_e$, $\boldsymbol{t}_e$ and the forces/torques $\boldsymbol{\tau}$ that have to be applied in the manipulator joints (by motors) so that the manipulator remains stationary ($\dot{\boldsymbol{q}} = \mathbf{0}$).

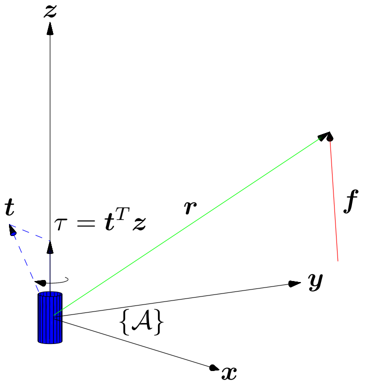
What is torque?

Loosely speaking, torque is a tendency of a force to rotate an object. We can think of torque as a "turning force".

Given a force $\boldsymbol{f}$ applied at a point $\boldsymbol{r}$ (both expressed in a common frame $\{\mathcal{A}\}$), the torque $\boldsymbol{t}$ about the origin of $\{\mathcal{A}\}$ is given by

$$\boldsymbol{t} = \boldsymbol{r} \times \boldsymbol{f}.$$





$t = r \times f$

torque t (about the origin of $\{\mathcal{A}\}$) as a result of a force f acting at point r

$\tau = t^T z = z \cdot (r \times f)$

torque τ at a joint with (unit) axis of rotation z as a result of a force f acting at point r

Given (for $i = 1, \ldots, 3$)

- $\boldsymbol{k}_i$ - axis of rotation of J_i
- $\boldsymbol{r}_i$ - vector from J_i to $\boldsymbol{p}_e$
- $(\boldsymbol{f}_e, \boldsymbol{t}_e)$ - force & torque acting at the end-effector

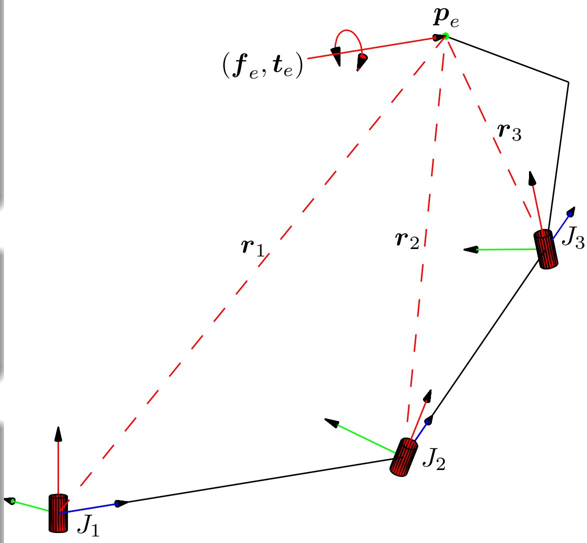
all vectors are expressed in the world frame (not depicted)

Problem

Find the joint torques $\boldsymbol{\tau}$ that would produce $(-\boldsymbol{f}_e, -\boldsymbol{t}_e)$ at the end-effector?

the torque at the i^{th} joint as a result of $(\boldsymbol{f}_e, \boldsymbol{t}_e)$ is given by

$$\tau_i = \boldsymbol{k}_i \cdot (\boldsymbol{r}_i \times \boldsymbol{f}_e) + \boldsymbol{k}_i \cdot \boldsymbol{t}_e$$



By using the "triple product rule" $\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c}$, we can represent $\tau_i = \boldsymbol{k}_i \cdot (\boldsymbol{r}_i \times \boldsymbol{f}_e) + \boldsymbol{k}_i \cdot \boldsymbol{t}_e$ as

$$\tau_i = \begin{bmatrix} (\boldsymbol{k}_i \times \boldsymbol{r}_i)^T & \boldsymbol{k}_i^T \end{bmatrix} \begin{bmatrix} \boldsymbol{f}_e \\ \boldsymbol{t}_e \end{bmatrix}.$$

Note that $\begin{bmatrix} (\boldsymbol{k}_i \times \boldsymbol{r}_i)^T & \boldsymbol{k}_i^T \end{bmatrix}$ is the i^{th} column of $\boldsymbol{J}$ transposed

$$\boldsymbol{J}_i^T = \begin{bmatrix} (\boldsymbol{k}_i \times \boldsymbol{r}_i)^T & \boldsymbol{k}_i^T \end{bmatrix}.$$

Static relation

$$\boldsymbol{\tau} = \boldsymbol{J}(\boldsymbol{q})^T \begin{bmatrix} \boldsymbol{f}_e \\ \boldsymbol{t}_e \end{bmatrix}$$

Kinematic relation

$$\begin{bmatrix} \boldsymbol{v}_e \\ \boldsymbol{\omega}_e \end{bmatrix} = \boldsymbol{J}(\boldsymbol{q})\dot{\boldsymbol{q}}$$

Applying joint torques $-\boldsymbol{\tau}$ (computed from the static relation) would counterbalance the $(\boldsymbol{f}_e, \boldsymbol{t}_e)$ acting at the end-effector, and the manipulator would remain stationary ($\dot{\boldsymbol{q}} = \boldsymbol{0}$).

Inverse geometric model (IGM) - numerical solution

Given

- initial configuration $\boldsymbol{q}^{(0)}$
- desired end-effector position $\boldsymbol{p}_{\text{des}}$
- error tolerance ϵ, α, β, set $i \to 0$

Recursion (do while $\|\boldsymbol{c}\| > \epsilon$)

- solve the FGM $\boldsymbol{p}_e^{(i)} = \boldsymbol{f}(\boldsymbol{q}^{(i)})$
- compute $\boldsymbol{c} = \boldsymbol{p}_{\text{des}} - \boldsymbol{p}_e^{(i)}$
- solve
$$\boldsymbol{J}(\boldsymbol{q}^{(i)})\dot{\boldsymbol{q}} = \alpha \boldsymbol{c},$$
where $\alpha > 0$ is a scaling factor
- update the joint angles using
$$\boldsymbol{q}^{(i+1)} = \boldsymbol{q}^{(i)} + \beta \dot{\boldsymbol{q}},$$
where $\beta > 0$ is a scaling factor
- set $i \to i + 1$

IGM for Cartesian position