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Recent advances in geometric problems related to energy efficiency

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Contents

Preface

The present Thesis tries to resume the main research interests during my PhD adventure at the University of Naples Federico II. During the last three years, I focused on mathematical problems arising from optimal design issues. These problems share a common ground: optimizing a shape with respect to some energy.

Insulation, elasticity, and electrostatics, all of these environments present some challenge that requires relating the efficiency of a shape to its geometric properties. I will be neither precise nor detailed in the physical interpretation of the following issues. The discussion on these topics will focus on the technical and mathematical point of view.

The present Thesis has two aims: the first is to collect different approaches and standard results used when dealing with the problem of optimizing shapes; the second is to offer some new perspectives needed in cases in which the classical tools found some obstacles.

The discussion in this work focuses on four main questions: the existence of an optimal shape, cases in which the solution is explicit, approximation, and stability. This dissertation and its bibliography are far from exhaustive and many important classical tools in this context will be overlooked. For instance, we are overlooking many details regarding regularity, symmetry, convex analysis, homogenization, and different boundary conditions. The plan of the thesis is the following.

In chapter 1, two examples of shape optimization problems are given, Dirichlet eigenvalues of the Laplacian, and thermal insulation problems. In this chapter, I collected well-known classical results for both examples, trying to provide the reader with a wide overview of standard tools and proofs in *shape optimization*.

In chapter 2, a brief list of notations and technical tools is given. Those will be useful in the following chapters.

In chapter 3, we provide the proof of two different (but similar) existence results related to the context of thermal insulation. The former is a linear free boundary problem, while the latter is a free boundary problem involving p-Laplacian and nonlinear boundary conditions. Those two results are inspired by the thermal insulation example in section 1.2.1, but they needed some particular attention to be adapted.

In chapter 4, I show three cases in which it is possible to explicit the optimal solution. The first out of three is devoted to proving that two concentric balls optimize an energy related to the ones in section 1.2.2. The third optimizes a Robin eigenvalue with a negative boundary parameter on the sphere, while the Euclidean case is in section 1.2.2. Finally, the second out of three shows that it is possible to relax the notion of optimal solution to something that is not an open set. Some geometric inequalities are optimized by collapsing sets, and sometimes some ways of collapsing are more efficient than others (or not efficient at all).

In chapter 5, I give two results related to elliptic diffraction PDEs with thin layers and Robin boundary conditions. Both of them try to improve the asymptotic study carried out in section 1.2.3, managing to obtain a first-order development by Γ-convergence with respect to the thickness of the thin layer.

Finally, chapter 6 contains a quantitative result for some special cases of the Talenti inequality. The latter is a rearrangement inequality that may be used to prove the *Faber-Krahn* inequality stated in section 1.1.2. I show that also in this rearrangement context, it is possible to restrict ourselves to stability issues similar to the ones shown in section 1.1.4.

For the sake of readability, all the chapters are written to be self-contained (the only needed exception is Chapter 2), and some things will be repeated throughout the book.

Chapter 1

Introduction: classical techniques in shape optimization

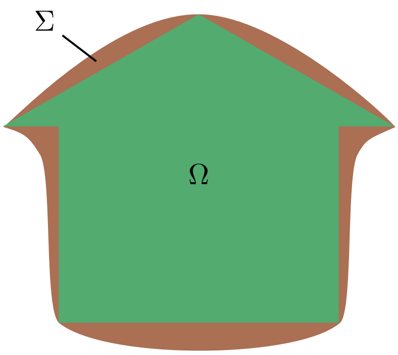
Let Ω be a house, and assume to have a fixed total mass m of insulating material (you can think of fiberglass, cellulose, rock wool, foams, etc.) to displace outside Ω. Assume to be able to build around your house a room Σ to be filled with your insulating material. If you aimed to avoid heat losses, how would you build Σ? It is undoubtedly impossible to answer with this amount of detail. However, this is a simple example of how a *shape optimization* problem is linked to real-life applications. Additionally, this example will allow us to discuss mathematical problems that have been addressed only in the last ten years and that require the employment of new techniques in the context of shape optimization.

The main aim of the present work is to show modern developments for some problems that have recently gained popularity. In particular, we will see how to *adapt classical techniques* and how to approach with *new perspectives* the challenging issue of designing optimal shapes to achieve energy efficiency.

A **shape optimization** problem consists of maximizing or minimizing a functional defined on shapes. Formally, let $\mathcal{O}$ be a class of subsets of $\mathbb{R}^n$ and consider a function

$$\mathcal{F} : \mathcal{O} \to \mathbb{R}$$

Figure 1.1: Insulation model.



usually called *shape functional.* We aim to find $\Omega_0 \in \mathcal{O}$ such that

$$\underset{(\geq)}{\mathcal{F}(\Omega_0) \leq \mathcal{F}(\Omega)} \qquad \forall \Omega \in \mathcal{O}$$

The importance of shape optimization problems goes back to ancient times, as proved by Virgil's narration of Queen Dido's famous solution to the isoperimetric problem. Other important shape optimization problems arise in:

- fluid mechanics, e.g. the designing of a car, or plane wings;
- electromagnetism, e.g. the designing of antennas or electronic components;
- thermal insulation and reinforcement, e.g. the introductory example of this section;
- spectral optimization problems, e.g. optimization of eigenvalues λ of the Laplacian with suitable boundary conditions
$$(\max) \min_{\Omega \in \mathcal{O}} \lambda(\Omega);$$
we will further inspect some eigenvalue examples and their real-life applications.

Far from being exhaustive, this list can help us to catch a glimpse of possible applications of the techniques developed by researchers in this context. For an overview of *shape optimization theory* and more detailed lists of examples and applications, we refer the reader to [12, 111, 113] and references therein.

Let us go back to our first example, where Ω is a house, and Σ is an insulating chamber. The question *"How would you build Σ?"* demands a deeper analysis, indeed various other questions are implied. First of all, *"Is there an optimal Σ?"*, and even if a solution exists, *"How many possible designs of Σ do minimize the heat loss?"*, *"Which is (or which are) the optimal shape?"*. When we are not able to answer the last question, we could try to approximate the solution and ask ourselves *"Which designs of Σ insulate Ω well enough?"*. And finally, *"Which is the necessary precision needed in the building process to optimize insulation?"*. These are only a few of the questions underlying the optimization problem (we are for instance overlooking regularity issues, geometric properties of the solutions, the choice of the constraints, and so on).

The questions listed above summarize the main interests of the present Thesis. In the following chapters, we are going to cover examples of shape optimization problems involving:

(a) **existence** of an optimal shape;

(b) explicit **computation** of the optimal shape;

(c) methods of **approximation** of an optimal shape;

(d) **quantitative** estimates of the optimal shape (if the functional is *nearly* at the minimum, then the shape is *nearly* optimal).

Among those examples, we will address issues *(a)*, *(b)*, and *(c)* for two mathematical problems related to the insulation model.

To better understand the differences and similarities between the problems that we are going to tackle and the, nowadays, *classical* techniques in shape optimization, we start by discussing important mathematical quantities that inspired mathematicians to pursue the ambitious goal of *hearing the shape of a drum.*

[1] Image realized with Desmos 3D, used with permission from Desmos Studio PBC.

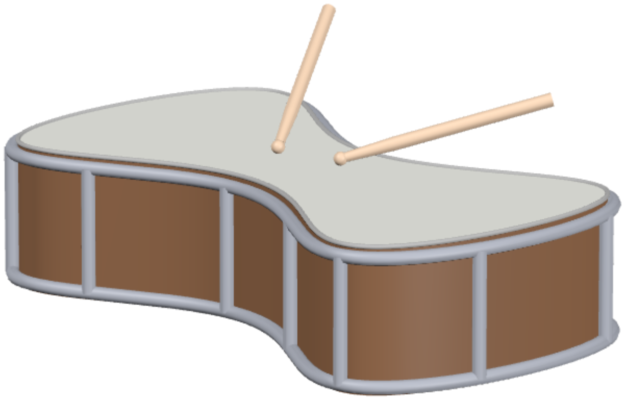


Figure 1.2: A drum with a generic horizontal section.[1]

1.1 The Dirichlet eigenvalues of the Laplacian

Let us build a drum, and let $\Omega \subset \mathbb{R}^2$ be the shape of the horizontal section of the instrument. We will assume that Ω is an open and bounded set. We may identify with 0 the height at which the membrane is fixed. For every $x \in \Omega$ and for every $t > 0$ we let $u(x,t) \in \mathbb{R}$ denote the height of the membrane in the point $x = (x_1, x_2)$ at time t. When the drum gets beaten then the membrane vibrates, and at every fixed time t the profile $u(x,t)$ solves the **wave equation**

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = c\Delta u, & \text{in } \Omega, \\ u(x,t) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where $c > 0$ is a constant depending on the material of the membrane, and Δ is the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2}.$$

From now on, for simplicity, we assume $c = 1$. Every solution u to (1.1.1) can be decomposed by separating variables. Let $\lambda > 0$, and let us consider functions

$$u(x,t) = u_\lambda(x)\cos(\lambda t) \qquad\qquad u(x,t) = u_\lambda(x)\sin(\lambda t)$$

with non-vanishing u_λ solution to the equation

$$-\Delta u_\lambda(x) = \lambda u_\lambda(x), \qquad x \in \Omega.$$

It is well known (see for instance [88, §6.5]) that there exists only a countable amount of such λ, i.e. for every $k \in \mathbb{N}$ there exist

$$0 < \lambda_1^{\mathrm{D}}(\Omega) \le \cdots \le \lambda_k^{\mathrm{D}}(\Omega) \le \lambda_{k+1}^{\mathrm{D}}(\Omega) \le \dots,$$

and non-vanishing functions u_k such that

$$\begin{cases} -\Delta u_k = \lambda_k^{\mathrm{D}}(\Omega) u_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1.2}$$

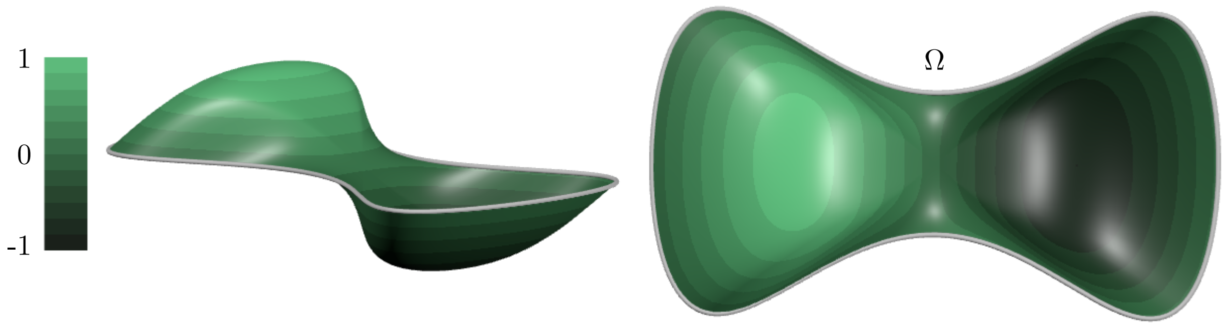


Figure 1.3: Elastic membrane example.[2]

and $\lim_k \lambda_k^{\mathrm{D}}(\Omega) = +\infty$. The Dirichlet boundary condition $u_k = 0$ motivates the choice to call $\lambda_k(\Omega)$ the k-th *Dirichlet eigenvalue* of the Laplacian.

The functions u_k are called *fundamental* (or *normal*) modes, and the numbers $\lambda_k^{\mathrm{D}}(\Omega)$ are also called *fundamental frequencies*, a concept which is not new to music experts. Indeed, these quantities are called *fundamental* because every possible vibration of the drum skin (which is uniquely determined by a function $u(x,t)$) can be decomposed in *fundamental modes*, exactly as every sound wave can be decomposed in *fundamental* waves. To be more precise, for every u solution to (1.1.1) there exist numbers α_k, β_k such that

$$u(x,t) = \sum_{k=1}^{+\infty} \alpha_k u_k(x) \cos(\lambda_k^{\mathrm{D}} t) + \sum_{k=1}^{+\infty} \beta_k u_k(x) \sin(\lambda_k^{\mathrm{D}} t).$$

The numbers $\lambda_k^{\mathrm{D}}(\Omega)$ are strictly related to the geometry of Ω. The relation of these quantities with geometry interested mathematicians for the last two centuries, to the point that in 1966 the question *"Can one hear the shape of a drum?"* gave the title to a famous paper by Mark Kac [116]. This formulation of the question is attributed to Lipman Bers, but the curiosity about the link between fundamental frequencies and geometry is way older than the paper. The same question was already formulated differently by Arthur Schuster in 1882. We refer the reader to [81] for a summary of the story.

The question is mathematically formulated in the following way: let Ω_1 and Ω_2 two open bounded sets in $\mathbb{R}^2$. If the spectra of Ω_1 and Ω_2 coincide, i.e.

$$\lambda_k^{\mathrm{D}}(\Omega_1) = \lambda_k^{\mathrm{D}}(\Omega_2) \qquad \forall k \in \mathbb{N},$$

can we infer that Ω_1 and Ω_2 are congruent? In other words, does it exist a rototranslation R such that $R(\Omega_1) = \Omega_2$? The answer to this question is "no", as proved by C. Gordon, D. L. Webb, and S. Wolpert in their paper of 1992 [106] entitled *"One cannot hear the shape of a drum"* (see the example in Figure 1.4).

Even though it is impossible in general to characterize the shape Ω in terms of the fundamental frequencies $\lambda_k^{\mathrm{D}}(\Omega)$, it is still possible to *hear the shape of a drum* in some particular cases. For instance, if Ω_1 is a set of volume $|\Omega_1| = m$ (here $|\cdot|$ represents the Lebesgue measure on $\mathbb{R}^2$) and Ω_2 is a ball such that $|\Omega_2| = m$, then we have that if

$$\lambda_1^{\mathrm{D}}(\Omega_1) = \lambda_1^{\mathrm{D}}(\Omega_2),$$

[2] Image realized with Desmos 3D, used with permission from Desmos Studio PBC.

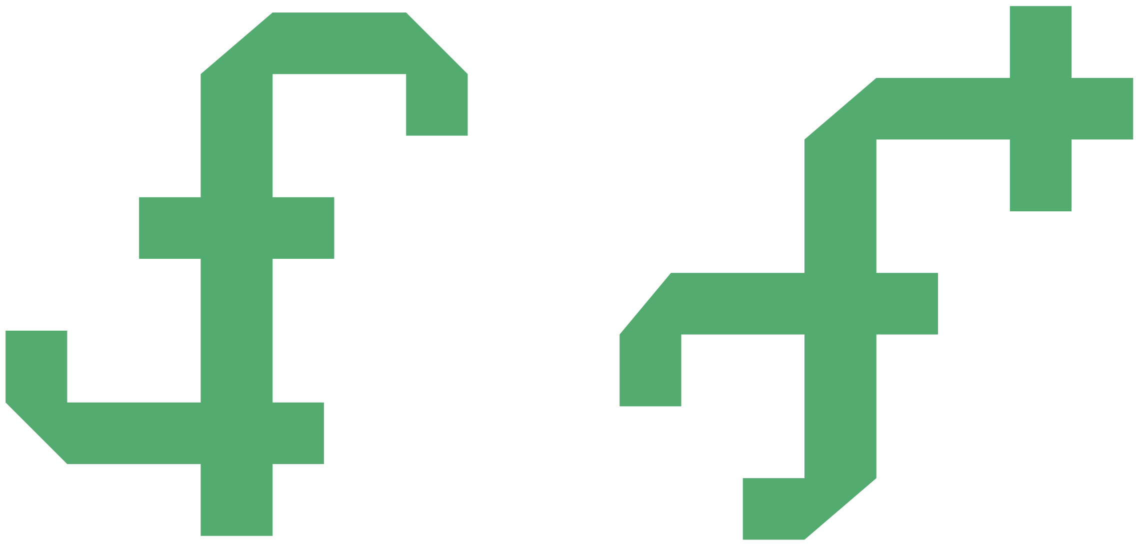


Figure 1.4: Two not congruent drums with the same spectrum. Example by [106].

then, up to translations, $\Omega_1 = \Omega_2$. This particular case, however, arises as a consequence of the study of a famous **shape optimization problem** that appeared in the book *The Theory of Sound* written by Lord Rayleigh in 1877 (see for instance [158]): if B is a ball of volume $m \in (0, +\infty)$, then B minimizes $\lambda_1(\Omega)$ among open sets Ω of volume m, i.e.

$$\lambda_1^{\mathrm{D}}(B) = \min \left\{ \lambda_1^{\mathrm{D}}(\Omega) \;\middle|\; \begin{array}{c} \Omega \subset \mathbb{R}^n \text{ open and bounded,} \\ |\Omega| = m. \end{array} \right\}$$

Lord Rayleigh's conjecture remained unproven till the 20s, when G. Faber [90] and E. Krahn [125] simultaneously proved the conjecture true.

Throughout this section, we will investigate the challenge of minimizing the k-th *Dirichlet eigenvalue* (i.e. the k-th fundamental frequency) with a volume constraint, and we will see the classical techniques developed in this context. In Section 1.1.2 we will focus on Rayleigh's conjecture and we will give a brief explanation of the ideas behind the celebrated *Faber-Krahn inequality*.

1.1.1 Existence of a minimizer of λ_k^{D}

In this paragraph we briefly discuss the existence of a minimizer for all the eigenvalues λ_k; we refer the reader to [111, Chapter 1] for a complete description of the problem.

Let us formalize the definition of λ_k. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set. We denote by $L^2(\Omega)$ the class of square summable functions, and by $H^1(\Omega) = W^{1,2}(\Omega)$ the Sobolev space of L^2 functions with distributional gradient in L^2, endowed with the classical H^1 scalar product

$$\langle u, v \rangle_{H^1(\Omega)} = \int_\Omega (uv + \nabla u \cdot \nabla v)\, dx.$$

We denote by $H_0^1(\Omega) = W_0^{1,2}(\Omega)$ the space of Sobolev functions with zero trace on the boundary $\partial\Omega$.

We define

$$\lambda_k^{\mathrm{D}}(\Omega) = \min_{V \in \mathcal{S}_k} \max_{v \in V \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla v|^2 \, dx}{\displaystyle\int_\Omega v^2 \, dx}, \tag{1.1.3}$$

where

$$\mathcal{S}_k = \left\{ V \subset H_0^1(\Omega) \;\middle|\; V \text{ is a space of dimension } k \right\}.$$

The *Courant-Fischer formula* (1.1.3) is equivalent to the definition of k-th *eigenvalue* of the Dirichlet Laplacian given at the beginning of the section. In particular, if we let $V_k \in \mathcal{S}_k$ denote a minimizer, then for every

$$u_k \in V_k \cap (V_{k-1} + \cdots + V_1)^\perp$$

we have that u_k solves the boundary value problem (1.1.2). We are interested in the following problem

$$\min_{\Omega \in \mathcal{A}_m(\mathbb{R}^n)} \lambda_k^{\mathrm{D}}(\Omega) \qquad\qquad \mathcal{A}_m(D) = \left\{ \Omega \subset \mathbb{R}^n \;\middle|\; \begin{array}{c} \Omega \text{ } quasi\text{-open}, \\ \Omega \subseteq D, \\ |\Omega| = m. \end{array} \right\}$$

We postpone the definition of *quasi-open* set to Definition 1.1.3, and we state here the main theorem of the paragraph, obtained, with different techniques, in [42, 140, 57].

Theorem 1.1.1. *For every $k \geq 3$ there exists $\Omega_0 \in \mathcal{A}_m(\mathbb{R}^n)$ such that*

$$\lambda_k(\Omega_0) \leq \lambda_k(\Omega) \qquad\qquad \forall \Omega \in \mathcal{A}_m(\mathbb{R}^n).$$

Four main tools have been employed to prove this result:

(A) existence theorem on $\mathcal{A}_m(D)$ with D bounded;

(B) *concentration-compactness* argument;

(C) *shape subsolutions*;

(D) *surgery* arguments.

The starting point was *(a)*, the existence theorem in the bounded case. This can be treated as a special case of the general theorem by Buttazzo and Dal Maso proved in [61], which applies to a wide class of *shape optimization* problems. To state this existence theorem (Theorem 1.1.5) we need a few definitions.

Definition 1.1.2. Let $\Omega \subseteq \mathbb{R}^n$. If Ω is compact, we define the *capacity* of Ω as

$$\mathrm{Cap}(\Omega) = \inf_{\substack{u \in C_0^\infty(\mathbb{R}^n) \\ u \geq 1 \text{ in } \Omega}} \int_{\mathbb{R}^n} |\nabla u|^2 \, dx.$$

If Ω is open we define

$$\mathrm{Cap}(\Omega) = \sup_{\substack{K \text{ compact} \\ K \subset \Omega}} \mathrm{Cap}(K).$$

If Ω is neither compact nor open, we define

$$\mathrm{Cap}(\Omega) = \inf_{\substack{U \text{ open} \\ \Omega \subset U}} \mathrm{Cap}(U).$$

Definition 1.1.3. Let $\Omega \subseteq \mathbb{R}^n$. We say that Ω is *quasi-open* if for every $\varepsilon > 0$ there exist an open set U_ε such that $\Omega \cup U_\varepsilon$ is open and $\mathrm{Cap}(U_\varepsilon) < \varepsilon$.

In the following for every $\Omega \subseteq \mathbb{R}^n$ we denote by $H^{-1}(\Omega)$ the dual space of $H^1_0(\Omega)$. Moreover, for every open set D, for every $f \in H^{-1}(D)$, and for every $\Omega \in \mathcal{A}_m(D)$, we denote by $u_f^\Omega \in H^1_0(\Omega)$ the unique solution to

$$\int_\Omega \nabla u_f^\Omega \cdot \nabla v \, dx = \langle f, v\rangle_{H^{-1}\times H^1_0} \qquad \forall v \in H^1_0(\Omega).$$

Sometimes we will consider $u_f^\Omega \in H^1_0(\mathbb{R}^n)$ by extending it to be zero outside of Ω.

Definition 1.1.4. Let $D \subseteq \mathbb{R}^n$ be an open set. For every $k \in \mathbb{N}$ let $\Omega_k, \Omega \subset D$ be quasi-open sets. We say that Ω_k *γ-converges* to Ω (relatively to D) and we write

$$\Omega_k \xrightarrow{\gamma} \Omega$$

if for every $f \in H^{-1}(D)$ we have that $u_f^{\Omega_k}$ strongly converges in $H^1_0(D)$ to u_f^Ω.

The definition of γ-convergence can be extended to measures on $\mathbb{R}^n$, but to avoid technicalities we refer the reader to [113, Definition 7.2.4].

Theorem 1.1.5 (Buttazzo-Dal Maso)**.** *Let $D \subset \mathbb{R}^n$ be an open and bounded set. Let*

$$\mathcal{F} : \{\, \Omega \subset D \mid \Omega \textit{ is quasi-open} \,\} \to \mathbb{R}$$

be a shape functional such that

(i) $\mathcal{F}$ is non-increasing with respect to the inclusion;

(ii) $\mathcal{F}$ is lower semi-continuous with respect to the γ-convergence topology.

Then for every $m \in (0, |D|)$ there exists $\Omega_0 \in \mathcal{A}_m(D)$ such that

$$\mathcal{F}(\Omega_0) \leq \mathcal{F}(\Omega) \qquad \forall \Omega \in \mathcal{A}_m(D).$$

From this theorem comes the choice of enlarging the class of admissible sets to *quasi-open* sets (instead of just open sets).

It is possible to prove that $\mathcal{F} = \lambda_k^{\mathrm{D}}$ satisfies the assumptions of Theorem 1.1.5 (see for instance [113, Corollary 4.7.4]).

The existence result for $\mathcal{A}_m(\mathbb{R}^n)$ requires more effort. The tools *(B)*, *(C)*, and *(D)* listed before are devoted to reducing the unbounded case to the bounded one.

The *concentration-compactness principle* is due to P.-L. Lions (see [133, 134]). D. Bucur adapted the concentration-compactness argument (see [43]) to apply it to the 3rd Dirichlet eigenvalue (see [54] by D.Bucur and A.Henrot) and later to the k-th Dirichlet eigenvalue (see [42] by D. Bucur). Before stating this adaptation to quasi-open sets, we fix the notation

$$d(\Omega_1, \Omega_2) := \inf_{\substack{x\in\Omega_1 \\ y\in\Omega_2}} |x - y|, \qquad \Omega_1, \Omega_2 \subseteq \mathbb{R}^n.$$

Theorem 1.1.6. *Let $\Omega_k \subseteq \mathbb{R}^n$ be a sequence of quasi-open sets equi-bounded in measure. There exists a subsequence (not relabelled) such that one of the following occurs:*

(Compactness) there exist points $y_k \in \mathbb{R}^n$ and a measure μ vanishing on zero capacity sets such that

$$y_k + \Omega_k \xrightarrow{\gamma} \mu;$$

(Dichotomy) there exist sets Ω_k^1, Ω_k^2 such that for every $f \in H^{-1}(\mathbb{R}^n)$

$$d(\Omega_k^1, \Omega_k^2) \xrightarrow{+} \infty, \qquad \|u_f^{\Omega_k} - u_f^{\Omega_k^1 \cup \Omega_k^2}\|_{H^1} \longrightarrow 0.$$

Even though the statement of the compactness ensures the existence of a γ-limit **measure**, in the case of the Dirichlet eigenvalues, in the minimization process, we may replace μ with a special quasi-open set A_μ (see for instance [113, Proof of Theorem 4.8.5]). On the other hand, the dichotomy reduces the problem to lower eigenvalues and their minimizers, and a boundedness property of those minimizers is required. This issue can be solved with either *shape subsolutions* or *surgery techniques*, and it is the reason why we stated Theorem 1.1.1 only for $k \geq 3$. The proof of the existence for $k \geq 3$ relies on the explicit *computation* of the minimizers for λ_1^{D} and λ_2^{D}. This explicit computation is the *Faber-Krahn inequality*, and it will be the main topic of Section 1.1.2.

The concept of *shape subsolution* was introduced in [42] to prove that minimizers to λ_k^{D} exist and they are bounded (and with finite perimeter). For every quasi-open set Ω and for every $f \in L^2(\Omega)$ let us call

$$\mathcal{T}_f(\Omega) = \min_{u \in H_0^1(\Omega)} \int_\Omega \Big(|\nabla u|^2 - 2fu\Big)\, dx$$

the *f-torsion* (or *Dirichlet energy*) of Ω. When $f = 1$ we call it *torsion* or *torsional rigidity* (we will explain this name later in Section 1.2.3).

Definition 1.1.7. Let $\Omega_1 \subseteq \mathbb{R}^n$ be a quasi-open set. We say that Ω_1 is a *shape subsolution* for $\mathcal{T}$ if there exists $\Lambda > 0$ such that

$$\mathcal{T}_1(\Omega_1) + \Lambda|\Omega_1| \leq \mathcal{T}_1(\Omega) + \Lambda|\Omega| \qquad \forall \Omega \subseteq \Omega_1 \text{ quasi-open.}$$

The main interest in this definition is the following.

Theorem 1.1.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a shape subsolution for $\mathcal{T}_1$. Then Ω is bounded (and it has finite perimeter).*

This result is one of the cases where *shape optimization* crossed the way of *free boundary problems*. Indeed, the proof of Theorem 1.1.8 uses Alt-Caffarelli estimates that can be found in [14].

Finally, the *surgery* argument was introduced by D. Mazzoleni and A. Pratelli in [140], and later combined with shape subsolutions in [57]. It is summarized in the following.

Theorem 1.1.9. *For every $M > 0$ there exists a positive constant $C = C(k, M, n)$ such that for every quasi-open set Ω if $\lambda_k^{\mathrm{D}}(\Omega) \leq M$, then there exists $\Omega_1 \subset \Omega$ quasi-open satisfying*

$$\mathrm{diam}(\Omega_1) \leq C, \qquad \lambda_i^{\mathrm{D}}(\Omega_1) \leq \lambda_i^{\mathrm{D}}(\Omega)$$

for every $i = 1, \dots, k$.

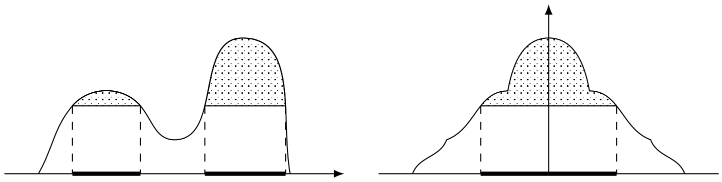


Figure 1.5: Schwarz rearrangement of a 1D function.[3]

We conclude this paragraph observing that this machinery is strong enough to deal with other energies and other constraints, such as Dirichlet energy, increasing functions of Dirichlet eigenvalues, and perimeter constraints (see [111, Chapter 2] for a more detailed discussion). On the other hand, the existence result in a bounded setting (Theorem 1.1.5) fails to solve analogous problems with other boundary conditions, such as Neumann or Robin boundary condition. We will address the latter in Section 1.2.

Some of the tools listed could still be adapted to the Robin boundary condition, but they have to be joined with a different theory, typical of *free boundary problems* or *free discontinuity problems*. This theory is the theory of *functions of bounded variation*, and it is a key tool in the proof of the existence of a minimizer for the perimeter under a volume constraint (see for instance [135, §14.2]). This same problem was one of the reasons for the development of symmetrization techniques, a central theme in the proof of the main theorem of the following paragraph.

1.1.2 Computation of the minimizer of λ_1^{D}

The long-standing conjecture of Lord Rayleigh (from 1877 to 1923) takes now the name of *Faber-Krahn inequality*, and it can be stated as follows.

Theorem 1.1.10. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let B be a ball having measure $|\Omega| = |B|$. Then*

$$\lambda_1^{\mathrm{D}}(B) \le \lambda_1^{\mathrm{D}}(\Omega).$$

Moreover, the equality holds if and only if, up to a zero measure set, Ω is a ball.

We will focus on the inequality without discussing the equality case, for which we refer the reader to [117, 76].

The proof of this theorem is due to two contemporary papers by G. Faber [90] and E. Krahn [125]. A modern version of the proof, which is not too far from the original ones of Faber and Krahn, involves rearrangements and an incredibly powerful tool called *Pólya-Szegő inequality*, due to G. Pólya and G. Szegő in [150].

The main tool of the proof is the Schwarz rearrangement that will be defined later on in Definition 2.4.6. For the purpose of this chapter, it is sufficient to imagine it as follows. Let Ω be a measurable set and let $\Omega^\sharp$ be the centered ball having the same measure as Ω. Let u be a measurable function (with respect to the Lebesgue measure) on Ω. The *Schwarz rearangement* of u is the unique measurable function $u^\sharp$ defined on $\Omega^\sharp$ such that for every $t \in \mathbb{R}$ we have that

$$\left\{ u^\sharp > t \right\} = \{ |u| > t \}^\sharp .$$

[3]Image kindly provided by Andrea Gentile.

By Cavalieri principle, this property implies that

$$\|u\|_{L^2(\Omega)} = \|u^\sharp\|_{L^2(\Omega^\sharp)}.$$

Since

$$\lambda_1^{\mathrm{D}}(\Omega) = \min_{u\in H_0^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla u|^2\,dx}{\displaystyle\int_\Omega u^2\,dx},$$

then the Faber-Krahn inequality can be easily seen as a consequence of the anticipated *Pólya-Szegő* inequality.

Theorem 1.1.11. *Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set, and let $u \in H_0^1(\Omega)$ be a non-negative function. Then*

$$\int_\Omega |\nabla u|^2\,dx \geq \int_\Omega |\nabla u^\sharp|^2\,dx.$$

As Krahn showed in [126], Theorem 1.1.10 gives as a corollary the minimizer for λ_2^{D}.

Corollary 1.1.12. *Let $\Omega \subseteq \mathbb{R}^n$ be a quasi-open set, and let B^1, B^2 be two disjoint balls having measure $|\Omega|/2$. Then*

$$\lambda_2^{\mathrm{D}}(B^1 \cup B^2) \leq \lambda_2^{\mathrm{D}}(\Omega).$$

Rearrangement techniques are an extremely powerful tool, and it should come as no surprise that they are used in a wide range of problems. We quote among others the *Saint-Venant inequality* (see [149]) which deals with the Dirichlet energy $\mathcal{T}_1$, and Talenti inequalities (see [159]) which we will inspect in more detail in Chapter 6. We refer to [118] for an overview of rearrangement techniques and their history.

However, Schwarz rearrangement fails to solve the minimization problem for eigenvalues when one tries to replace the Dirichlet boundary condition with a Robin boundary condition. The question of whether the first eigenvalue was minimized by balls remained open till 1986 when M.-H. Bossel [35] managed to solve the problem in 2 dimensions with the help of a tool known in the literature as H-function, and that will return in Section 4.1. The strategy was then adapted by D. Daners [85] in 2006 to every dimension.

Up to our knowledge, here ends the part where the minimizers of λ_k^{D} are explicit. The problem remains open for $k \geq 3$, but a lot of progress has been made in recent years trying to approximate the solution or, at least, guess reasonable candidates.

Sometimes it happens that the "minimizer" is not a proper open set. This is the case when, for instance, one tries to minimize more complex functionals involving eigenvalues of the Laplacian and geometric quantities under a convexity constraint. We will inspect a particular planar example in Section 4.2, where the minimizing sequences converge to a segment. As we will see, not all the collapsing sets are minimizing that functional: we have to develop a technique able to distinguish between collapsing rectangles and collapsing triangles.

1.1.3 Approximation of the minimizer of λ_k^{D}

In this paragraph, we aim to give a flavor of the ideas behind the numerical approximation of minimizers of λ_k^{D} in $\mathbb{R}^2$ under a volume constraint. We will not describe the technical details needed

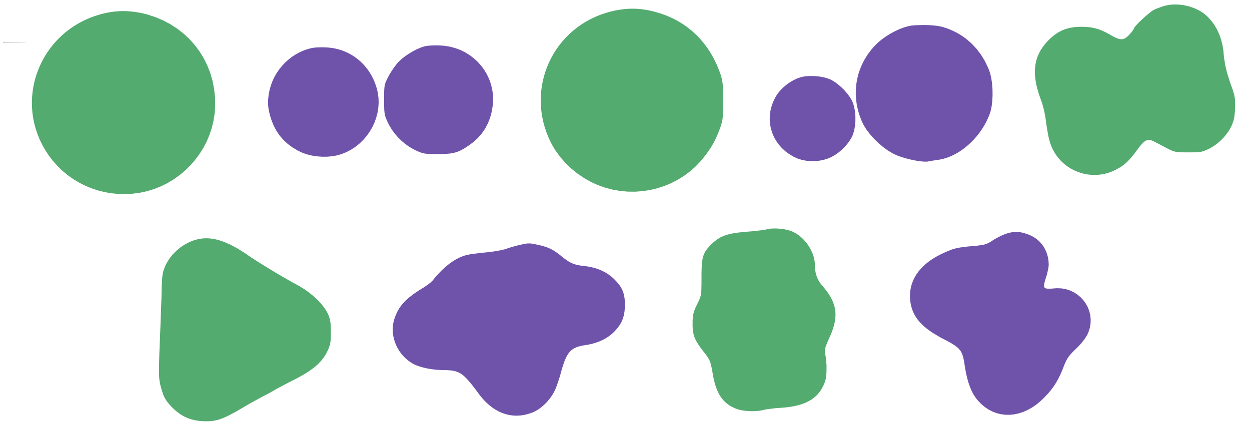


Figure 1.6: Numerical solutions minimizing λ_k^{D} under volume constraint with $k = 1, 2, 3, 4, 5$ above and $k = 6, 7, 8, 9$ below. Solutions from [31].

to improve the efficiency of the optimization algorithm. Instead, we will focus on the common strategy that inspires the results we are going to mention.

The main approximation results we are aware of in this context are [146, 19, 31, 169, 93]. All of them share a common ground, which is the minimization algorithm. Let $\mathcal{F}$ be a generic shape functional. The algorithm – inspired by gradient descent – reads as follows.

Step 1: choose a *starting shape* Ω_0;

Step 2: compute the **shape derivative** $\mathcal{F}'$ of the functional in Ω_0;

Step 3: deform Ω_0 to a new set Ω_1 obtained *"following the direction* $-\alpha_0\mathcal{F}'$*"* with some $\alpha_0 > 0$;

Step 4: *compute* the functional on Ω_1;

Step 5: repeat Step 1 with Ω_1 in place of Ω_0 until an *exit condition* is satisfied.

The aforementioned publications mainly differ in the strategies used to realize Steps 1,3,4. Nevertheless, the theoretical background behind the concept of *shape derivative* remains the same. Before describing what we mean by "shape derivative" and by "follow the direction -$\mathcal{F}'$", we refer the reader to [31, §6.2] for an insight into numerical known results for λ_k^{D} up to $k = 15$ (also in $\mathbb{R}^3$). It is important to highlight that the minimization algorithm is applied to the scaling invariant functional $\mathcal{F}(\cdot) = |\cdot|\lambda_K^{\mathrm{D}}(\cdot)$, so that the volume constraint can be neglected.

To our knowledge, the first gradient descent approach in shape optimization appeared in the paper [66] by J. Céa, J. Gioan, and J. Michel. The concept of *shape gradient*, instead, can be traced back to the publication of J. Hadamard [109] of 1908. Many authors contributed to the evolution of this concept, and we refer to [113, §5.1] for an extensive description of this bibliography.

The formal definition of shape derivative will be postponed to Section 2.6.1, but for the purposes of the present discussion, we may define it as follows. Let $\mathcal{O}$ be a class of sets, and let

$$\mathcal{F} : \mathcal{O} \to \mathbb{R}$$

be a functional. Let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a (small) deformation such that for every $t \in (0,1)$

$$(\mathrm{Id} + t\Phi)(\Omega) \in \mathcal{O} \qquad\qquad \forall \Omega \in \mathcal{O},$$

where Id denotes the identity map on $\mathbb{R}^n$. Then it is possible to define $\Omega_t = (\mathrm{Id} + t\Phi)(\Omega)$ and $\mathcal{F}(\Omega_t)$ is a function depending on one variable t. The shape derivative of $\mathcal{F}$ *in the direction* Φ is the limit, whenever it exists,

$$\mathcal{F}'(\Omega)[\Phi] = \lim_{t \to 0^+} \frac{\mathcal{F}(\Omega_t) - \mathcal{F}(\Omega)}{t}.$$

Let us now restrict to the case of λ_k^{D}. If the eigenvalue is simple (i.e. $\lambda_{k-1}^{\mathrm{D}} < \lambda_k^{\mathrm{D}} < \lambda_{k+1}^{\mathrm{D}}$), under the right regularity assumptions on Ω and Φ, we have that λ_k^{D} is *shape differentiable*. In particular, if we take $\mathcal{F}(\Omega) = |\Omega|\lambda_k^{\mathrm{D}}(\Omega)$, then

$$\mathcal{F}'(\Omega)[\Phi] = -\int_{\partial\Omega} \left(\lambda_k^{\mathrm{D}} - \left(\frac{\partial u_k}{\partial \nu} \right)^2 \right) (\Phi \cdot \nu)\, d\sigma,$$

where ν is the outer unit normal to $\partial\Omega$, and σ denotes the surface element of $\partial\Omega$ ($d\sigma$ can be interpreted as $d\mathcal{H}^{n-1}$ where $\mathcal{H}^{n-1}$ is the $(n-1)$-dimensional Hausdorff measure).

Let us look for the function Φ_0 which represents the best direction along which minimize $\mathcal{F}'(\Omega)[\cdot]$. Let us note that

$$\mathcal{F}'(\Omega)[\Phi] \geq -\int_{\partial\Omega} \left| \lambda_k^{\mathrm{D}} - \left(\frac{\partial u_k}{\partial \nu} \right)^2 \right| |\Phi|\, d\sigma,$$

and the equality is attained when we choose $\Phi = \alpha\Phi_0$ with $\alpha > 0$ and

$$\Phi_0(x) = \left(\lambda_k^{\mathrm{D}} - \left(\frac{\partial u_k}{\partial \nu} \right)^2 \right) \nu(x) \qquad x \in \partial\Omega.$$

This suggests that in some weak sense we could also see $\mathcal{F}'(\Omega)$ as a vector field:

$$\mathcal{F}'(\Omega)[\Phi] = -\int_{\partial\Omega} \Phi_0 \cdot \Phi\, d\sigma.$$

With this remark, we can understand the meaning of Step 3 as "define $\Omega_1 = (\mathrm{Id} + \alpha_0\Phi_0)(\Omega_0)$ with a *suitable* α_0". The choice of α_0 will depend on the specific implementation of the algorithm used to compute the approximation.

Shape derivative is a quite versatile tool, and it can be applied to a wide range of situations. Shape derivatives can deal with other boundary conditions, such as Neumann and Robin ones. Moreover, shape derivatives can provide necessary and sufficient conditions for minimality, just as first-order and second-order derivatives do for functions from $\mathbb{R}$ to $\mathbb{R}$. We are now interested in a further application of this tool: *sharp* stability properties of minimizers. Stability estimates for the minimizers of λ_1^{D} are our last stop in the study of Dirichlet eigenvalues.

1.1.4 Quantitative estimates of the minimizer of λ_1^{D}

We now discuss the problem of proving sharp stability inequalities for the first Dirichlet eigenvalue λ_1^{D}. The main result in this direction is the one proved by L. Brasco, G. De Philippis, and B. Velichkov in [38]. Before stating it, we define the *Fraenkel asymmetry* for bounded open sets Ω as

$$\mathcal{A}^{\mathrm{F}}(\Omega) = \min_{x_0 \in \mathbb{R}^n} \frac{|\Omega \Delta (x_0 + \Omega^\sharp)|}{|\Omega|}.$$

The Fraenkel asymmetry quantifies in some sense *"how far is Ω from being a ball"*.

Theorem 1.1.13. *There exists a positive constant $C = C(n)$ such that for every open set $\Omega \subseteq \mathbb{R}^n$ with finite measure and for every open ball B*

$$|\Omega|^{\frac{2}{n}} \lambda_1^{\mathrm{D}}(\Omega) - |B|^{\frac{2}{n}} \lambda_1^{\mathrm{D}}(B) \geq C \mathcal{A}^{\mathrm{F}}(\Omega)^2.$$

The exponent 2 on the Fraenkel asymmetry in Theorem 1.1.13 is sharp, in the sense that there exists a sequence Ω_n of sets such that

$$\lim_n \frac{|\Omega_n|^{\frac{2}{n}} \lambda_1^{\mathrm{D}}(\Omega_n) - |B|^{\frac{2}{n}} \lambda_1^{\mathrm{D}}(B)}{\mathcal{A}^{\mathrm{F}}(\Omega_n)^2} = C.$$

In particular, this implies that the exponent 2 on the asymmetry cannot be lowered. To understand the sketch of the proof, and to understand the origins of the tools needed, we briefly focus on the *quantitative isoperimetric inequality.*

As far as we know, the quantitative isoperimetric inequality by T. Bonnesen in 1921 [34] was the first attempt to obtain a *quantitative* inequality in shape optimization. Thereafter quantitative inequalities gained a lot of attention in the last century. The aim of [34] was to prove the isoperimetric inequality for planar convex sets Ω, retrieving from the computations some additional information in the case in which the perimeter of Ω was close to the one of the ball with the same area. Many mathematicians improved the *quantitative isoperimetric inequality*, and we refer to [99] for a survey on the topic. Letting P denote the perimeter (see Definition 2.3.5), the *sharp* quantitative version of the isoperimetric inequality reads as follows.

Theorem 1.1.14. *There exists a positive constant $C = C(n)$ such that for every set $\Omega \subset \mathbb{R}^n$ of finite perimeter*

$$\frac{P(\Omega) - P(\Omega^\sharp)}{P(\Omega^\sharp)} \geq C \mathcal{A}^{\mathrm{F}}(\Omega)^2.$$

Three main strategies have been employed to prove Theorem 1.1.14:

(A) direct symmetrization techniques (by N. Fusco, F. Maggi, and A. Pratelli in 2008 [101]);

(B) mass transportation (by A. Figalli, F. Maggi, and A. Pratelli in 2010 [95]);

(C) regularization (by M. Cicalese, and G. P. Leonardi in 2012 [77]).

We will focus on the approach given in [77], which turned out to be the most effective in proving a **sharp** quantitative Faber-Krahn inequality. Before getting into the details, we give a brief (non-exhaustive) summary of the history behind Theorem 1.1.13. We refer to [37, §1, §7] for a more detailed description.

To the best of our knowledge, the first investigation for a quantitative Faber-Krahn is due to W. Hansen and N. Nadirashvili in 1994 [110] and to A. D. Melas in 1992 [141]. In particular, the idea behind the proof in [110] relies on improving the classical proof of Faber-Krahn inequality using a quantitative version of the Pólya-Szegő principle (Theorem 1.1.11). The tools available at that time were not refined enough to obtain the best out of this technique, but to understand the power of their idea, we report here a result coming from an adaptation of the proof made in [37, §2.4]. As done by different authors [32, 100, 73, 26], thanks to the sharp quantitative isoperimetric inequality (Theorem 1.1.14) it is possible to prove a more refined version of the quantitative Pólya-Szegő principle than the one available at the time of [110]. Even using the consequences of the sharp quantitative isoperimetric inequality, the best result obtained so far with this strategy is the following.

Theorem 1.1.15. *There exists a positive constant $C = C(n)$ such that for every open set $\Omega \subseteq \mathbb{R}^n$ with finite measure and for every open ball B*

$$|\Omega|^{\frac{2}{n}}\lambda_1^{\mathrm{D}}(\Omega) - |B|^{\frac{2}{n}}\lambda_1^{\mathrm{D}}(B) \geq C\mathcal{A}^{\mathrm{F}}(\Omega)^3.$$

To improve the exponent on the Fraenkel asymmetry, the sharp quantitative isoperimetric inequality was probably not enough. This is the reason that led to a different strategy, the one developed in [38].

The strategy in [38] followed the outline of Cicalese and Leonardi's proof for the isoperimetric problem:

Step 1: local stability for regular small deformations of the ball;

Step 2: passing from local stability to global stability with a *selection principle*.

The first step draws inspiration from the computations made by B. Fuglede in 1989 [98] for the isoperimetric problem. The second step, instead, is inspired by the works of B. White in 1994 [167], and F. Morgan, and A. Ros in 2010 [142]. The selection principle requires a detailed study of the regularity properties of *"almost minimizers"* of the problem. This step is the most specific one, and the hardest to generalize to other problems. We now quickly look at both Steps, with a particular interest in Step 1.

Fuglede-type computations had a huge impact on many quantitative versions of geometric and spectral inequalities. The key notion here is the *nearly-spherical* sets.

Definition 1.1.16. Let $h \in C^{2,s}(\partial B_1)$ for some $s \in (0,1)$ with $\|h\|_\infty < 1/2$. We define the *nearly-spherical* set parametrized by h as the open, bounded set B_1^h such that

$$\partial B_1^h = \{\, (1+h(x))x \mid x \in \partial B_1 \,\}.$$

In the case of the Dirichlet-Laplacian, defining the $H^{1/2}(\partial B_1)$ norm as

$$\min_{u \in h + H_0^1(B_1)} \int_{B_1} |\nabla u|^2 \, dx,$$

Step 1 is resumed in the following.

Theorem 1.1.17. *Let $s \in (0,1]$. There exist a positive constant $\delta = \delta(n,s), C = C(n)$ such that if $\|h\|_{C^{2,s}} < \delta$ and $|B_1^h| = |B_1|$ with the barycenter in the origin, then*

$$\lambda_1^{\mathrm{D}}(B_1^h) - \lambda_1^{\mathrm{D}}(B_1) \geq C\|h\|_{H^{1/2}}^2.$$

It is important to notice that this result is not directly proven in [38]. Indeed, in the original paper, a Kohler-Jobin inequality (see [121, 122, 36]) is used to reduce the stability issue of the Faber-Krahn inequality to the stability issue of the Saint-Venant inequality, namely they prove

$$\mathcal{T}_1(B_1) - \mathcal{T}_1(B_1^h) \geq C\|h\|_{H^{1/2}}^2 \geq C_1\mathcal{A}^{\mathrm{F}}(B_1^h)^2,$$

with a smaller C_1. Even though the result is slightly different, as shown in [84, §5.1], the same computations can also be applied directly to the functional λ_1^{D}. In particular, the proof goes as follows. Let us consider $J(t) = \lambda_1^{\mathrm{D}}(B_1^{th})$ and $V(t) = |B_1^{th}|$. Since 0 is a minimum for $J(t)$ under the constraint $V(t) = |B_1|$, then by Lagrange multiplier

$$J'(0) + c_0 V'(0) = 0$$

for some $c_0 > 0$. We define the Lagrangian $L(t) = J(t) + c_0 V(t)$, and we observe that since $L'(0) = 0$, then

$$\lambda_1^{\mathrm{D}}(B_1^h) - \lambda_1^{\mathrm{D}}(B_1) = L(1) - L(0) = L''(\xi) + L''(0) - L''(0),$$

for some $\xi \in (0,1)$. At this point, one proves the following two facts:

- **Coercivity in 0:** there exists some positive constant C such that

$$L''(0) \geq C\|h\|_{H^{1/2}};$$

- **Improved continuity:** there exists a modulus of continuity ω (i.e. an increasing function with $\lim_{t\to 0}\omega(t) = 0$) such that

$$|L''(t) - L''(0)| \leq \omega(\|h\|_{C^{2,s}})\|h\|_{H^{1/2}}, \qquad \forall t \in (0,1).$$

From these two estimates, one obtains Theorem 1.1.17. We refer the reader to [84] for an abstract generalization of the above strategy that can be applied to a wide range of situations.

Finally, we summarize the *selection principle* for the Dirichlet eigenvalue in the following. Let us define another asymmetry index

$$\alpha(\Omega) = \int_{\Omega\Delta B_1(x_\Omega)} |1 - |x - x_\Omega||\, dx,$$

where x_Ω is the barycenter of Ω, and for some positive constant C we have $\alpha(\Omega) \geq C\mathcal{A}^{\mathrm{F}}(\Omega)^2$.

Theorem 1.1.18. *Let $R \geq 2$. There exist positive constants $\sigma = \sigma(n,R), C = C(n,R)$ such that if $\Omega_j \subseteq B_R$ is a sequence of sets such that $|\Omega_j| = |B_1|$, and*

$$\mathcal{T}_1(B_1) - \mathcal{T}_1(\Omega_j) \leq \sigma^4\alpha(\Omega_j),$$

then there exists a sequence of open, nearly-spherical sets $B_1^{h_j} \subseteq B_R$ of class C^∞ with h_j converging to 0 in C^k for every k, and

$$\limsup_{j\to+\infty} \frac{\mathcal{T}_1(B_1) - \mathcal{T}_1(B_1^{h_j})}{\alpha(B_1^{h_j})} \leq C\sigma.$$

The proof of Theorem 1.1.18 relies on regularity estimates for almost minimizers of a free boundary problem similar to the one by H. W. Alt and L. A. Caffarelli [14]. In some sense, Theorem 1.1.18 can be used as follows: assume that there exists a sequence Ω_j disproving the sharp stability property of Faber-Krahn inequality; we use Theorem 1.1.18 to replace Ω_j with smooth nearly-spherical sets, and obtain a contradiction from the local stability proven in Theorem 1.1.17.

We want to point out that the proof of Theorem 1.1.13 requires more technicalities and details to be written down, but we think that Theorem 1.1.17 and Theorem 1.1.18 embody the essence of a nowadays classical technique to approach quantitative geometric and spectral inequalities.

1.2 Thermal insulation

We finally go back to the initial example. Let us build a n-dimensional house, for instance, a bounded, open set $\Omega \subset \mathbb{R}^n$. Let $\Sigma \subset \mathbb{R}^n$ be disjoint from Ω representing an insulating material around Ω, and let us assume that the thermal exchange with the environment is given by convection (for instance, in the case in which the insulating material is in direct contact with air). For simplicity, let $A = \Omega \cup \Sigma$. If $f : \Omega \to \mathbb{R}^+$ is a heat source defined in Ω, then the steady-state temperature $u(x)$ (i.e., the temperature at the equilibrium, or the values of the temperature at infinite time),

$$\begin{cases} -k\Delta u = f, & \text{in } \Omega, \\ k\dfrac{\partial u^-}{\partial \nu} = \dfrac{\partial u^+}{\partial \nu} & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial A, \end{cases}$$

where $k, \beta > 0$ are constants depending on the thermal diffusivity of the environment and the efficiency of the insulating material, Δ is the Laplacian

$$\Delta u = \frac{\partial^2 u}{\partial x_1^2} + \cdots + \frac{\partial^2 u}{\partial x_n^2},$$

and u^+ and u^- denote the restrictions of u in $A \setminus \Omega$ and in Ω respectively. For simplicity, we will assume $k = 1$. The boundary condition

$$\frac{\partial u}{\partial \nu} + \beta u = 0$$

is the so-called *Robin boundary condition*, and it will be the main focus of this section. We saw in Section 1.1 different perspectives on shape optimization problems involving Dirichlet boundary conditions, and we will see that the Robin boundary condition will require different techniques to be approached.

This section focuses on two main problems related to thermal insulation. In particular, we study the two PDEs

Heat source

$$\begin{cases} -\Delta u = f, & \text{in } \Omega, \\ \dfrac{\partial u^+}{\partial \nu} = \dfrac{\partial u^-}{\partial \nu} & \text{on } \partial\Omega, \\ \Delta u = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial A, \end{cases}$$

Constant temperature

$$\begin{cases} v = 1, & \text{in } \Omega, \\ \Delta v = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial v}{\partial \nu} + \beta v = 0 & \text{on } \partial A, \end{cases}$$

and the respective energies

Heat source:

$$\mathcal{T}_f^\beta(\Omega, A) = \min_{\varphi \in H^1(A)} \int_A |\nabla \varphi|^2 \, dx - 2\int_\Omega f\varphi \, dx + \beta \int_{\partial A} \varphi^2 \, d\mathcal{H}^{n-1}.$$

Constant temperature:

$$\mathrm{Cap}^\beta(\Omega, A) = \min_{\substack{\varphi\in H^1(A)\\ \varphi=1 \text{in } \Omega}} \int_{A\setminus\Omega} |\nabla\varphi|^2\,dx + \beta\int_{\partial A}\varphi^2\,d\mathcal{H}^{n-1}.$$

Our aim is to minimize the two energy functionals when Ω is fixed, and $\Sigma = A\setminus\Omega$ is free to vary. When we take u and v to be the solutions to the equations above, then we have

$$\mathcal{T}_f^\beta(\Omega, A) = -\int_\Omega fu\,dx, \qquad \mathrm{Cap}^\beta(\Omega, A) = \int_{\partial\Omega}\beta v\,d\mathcal{H}^{n-1}.$$

From a physical point of view, minimizing $\mathcal{T}_f^\beta$ translates to maximizing the mean temperature u inside Ω (recall also that $f > 0$), while minimizing Cap^β translates to minimizing the heat flux across the surface ∂A. In both cases, the minimization is linked to the challenge of optimizing heat loss through the design of an optimal shape, the insulating layer. In this section, we are going to address three out of the four questions (a)-(d) raised in Chapter 1.

1.2.1 A free boundary problem in thermal insulation: existence

At the beginning of the present Thesis, we suggested to inspect the question of "finding the optimal design for Σ", and we had in mind the following minimization problem: let Ω be an open bounded set, let

$$\mathcal{A}(\Omega) = \left\{ A\subseteq\mathbb{R}^n \;\middle|\; \begin{array}{c} A \text{ open and bounded,}\\ P(A) < +\infty,\\ \Omega\subseteq A, \end{array}\right\},$$

where we recall that P denotes the perimeter in the Caccioppoli-De Giorgi sense (see Definition 2.3.5); for every $A\in\mathcal{A}(\Omega)$ we define

$$\mathrm{Cap}^\beta(\Omega, A) = \min_{\substack{\varphi\in H^1(A)\\ \varphi=1 \text{in } \Omega}} \int_{A\setminus\Omega} |\nabla\varphi|^2\,dx + \beta\int_{\partial^* A}\varphi^2\,d\mathcal{H}^{n-1},$$

with $\partial^* A$ denoting the essential boundary of A (see Definition 2.3.4); we are trying to solve

$$\min_{\substack{A\in\mathcal{A}(\Omega)\\ |A\setminus\Omega|\le m}} \mathrm{Cap}^\beta(\Omega, A). \tag{1.2.1}$$

The presence in the functional of a trace term (the integral on the boundary ∂A) gives to the problem (1.2.1) a peculiar behavior. As already pointed out by L. A. Caffarelli and D. Kriventsov in 2015 in [65], we heuristically expect that a minimizer A does not exist for some choices of Ω. In some sense, if we formulate the physical issue through (1.2.1), we are ignoring some possible optimal displacement of the insulator $A\setminus\Omega$: this model avoids competitors A with **internal cracks** (see Figure 1.7).

Following [65], we "relax" the problem (1.2.1) to a weaker setting where cracks are taken into account, SBV functions. We refer the reader to Section 2.3 for the definition of *special functions of bounded variation* SBV, but we can imagine them as "Sobolev functions which are allowed to jump". We transform (1.2.1) as follows:

- replace A with $\mathrm{supp}(\varphi) = \{\varphi > 0\}$;

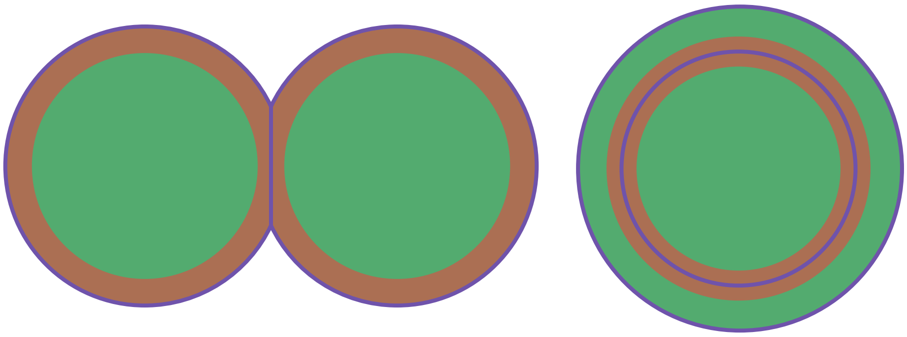


Figure 1.7: The set A (union of green and brown part) is made of two connected components sharing boundaries (the purple lines)

- replace $\varphi \in H^1(A)$ with $\varphi \in \mathrm{SBV}(\mathbb{R}^n)$;
- replace $\partial^* A$ with the set J_φ where φ is allowed to jump, which is *almost* an oriented hypersurface (see Section 2.3);
- replace $\int_{\partial^* A} \varphi^2 \, d\mathcal{H}^{n-1}$ with $\int_{J_u} \left(\underline{\varphi}^2 + \overline{\varphi}^2\right) d\mathcal{H}^{n-1}$ where $\underline{\varphi}$ and $\overline{\varphi}$ are respectively the lower trace and the upper trace from one side or the other of J_φ;
- replace the mass constraint $|A \setminus \Omega| \leq m$ with the volume penalization $C_0|\{\varphi > 0\} \setminus \Omega|$ for some cost $C_0 > 0$.

Then, letting

$$\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, d\mathcal{H}^{n-1} + C_0|\{\, v > 0 \,\} \setminus \Omega|,$$

our problem reads

$$\min_{\substack{\varphi \in \mathrm{SBV}(\mathbb{R}^n) \\ \varphi = 1 \text{ in } \Omega}} \mathcal{F}(\varphi). \tag{1.2.2}$$

We can also read the previous formulation (1.2.1) as

$$\min_{\substack{A \in \mathcal{A}(\Omega) \\ |A \setminus \Omega| \leq m}} \mathrm{Cap}^\beta(\Omega, A) = \min_{\substack{A \in \mathcal{A}(\Omega) \\ |A \setminus \Omega| \leq m}} \min_{\substack{\varphi \in H^1(A) \\ \varphi = 1 \text{ in } \Omega}} \mathcal{F}(\varphi \chi_A) - C_0 m.$$

The same problem of the existence and regularity of a minimizer to (1.2.2) was handled by D. Bucur and S. Luckhaus almost contemporarily in 2014 [56], with a similar yet different approach, obtaining a direct existence result employing compactness in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ (see Theorem 2.3.17). We focus on the methods of [65] that will be the inspiration to obtain further existence results in Chapter 3, but we do believe that the methods of [56] effectively adapt to those cases as well. For the purposes of this dissertation, the main result of [65] we are interested in is the following.

Theorem 1.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Then there exists $u \in \mathrm{SBV}(\mathbb{R}^n)$ such that*

$$\mathcal{F}(u) = \min_{\substack{\varphi \in \mathrm{SBV}(\mathbb{R}^n) \\ \varphi = 1 \textit{ in } \Omega}} \mathcal{F}(\varphi).$$

Moreover, $\{u > 0\}$ is bounded, and there exists a positive $\delta = \delta(\Omega)$ such that $u \geq \delta$ in $\{u > 0\}$.

The proof of the theorem relies on a concept very close to the notion of **shape subsolution**: we say that $v \in \mathrm{SBV}(\mathbb{R}^n)$ is an *inward minimizer* of $\mathcal{F}$ if for every A of finite perimeter such that $A \supset \Omega$ we have

$$\mathcal{F}(v) \le \mathcal{F}(v\chi_A).$$

The key point of the proof is the following

Proposition 1.2.2. *Let $\Omega \subset \mathbb{R}^n$ be open and contained in a ball B_R. Then there exists $\delta = \delta(\Omega, R) > 0$ such that for every inward minimizer v with $\mathcal{F}(v) \le \mathcal{F}(B_R)$ we have*

$$v(x) \ge \delta \qquad \forall x \in \{v > 0\}, \qquad\qquad \{v > 0\} \subseteq B_{1/\delta}.$$

Once Proposition 1.2.2 is proven, a compactness theorem in SBV joint with the semicontinuity of the functional $\mathcal{F}$ gives Theorem 1.2.1. It is worth mentioning that [65] also provides powerful tools to inspect the regularity of a minimizer u. For the discussion about regularity, we also refer the reader to [127].

Particular attention should be turned to the 2010 paper by D. Bucur and A. Giacomini [48], which is, up to our knowledge, the first paper proposing a variational approach to deal with the problem of optimizing the first eigenvalue of the Robin Laplacian. This approach inspired the techniques in [56], and it was the starting point to make steps towards the solution of the **open problem** of the existence of a minimizer for the k-th eigenvalue of the Robin Laplacian (see [50, 144]). Other results of existence close to the ones listed here are [51, 62].

Other approaches have been employed to tackle similar problems with internal cracks: we are aware of a direct strategy avoiding the generalization to SBV in [108] and another formulation taking into account cracks as a variable of the functional in [53, 78].

For problem (1.2.1), explicit solutions seem to be out of reach so far. We will get back in Section 1.2.3 to some attempts to approximate the problem and better understand its behavior. Before going in the direction of approximation, we inspect some cases in which minimizers are explicit: Rayleigh's conjecture for the Robin boundary condition, and the *double shape optimization* problem

$$\min_{\substack{|\Omega|=m_0,\\ |A\setminus\Omega|\le m}} \mathrm{Cap}^\beta(\Omega, A),$$

for some fixed $m_0 > 0$.

1.2.2 Computation of the minimizer: three examples

As done for the Dirichlet boundary condition, we define the first Robin eigenvalue of the Laplacian as follows: let

$$\mathcal{A} = \{\, \Omega \subseteq \mathbb{R}^n \mid \Omega \text{ open, bounded and with finite perimeter, } \}$$

and let $\beta \in \mathbb{R}$, then for every $\Omega \in \mathcal{A}$ we set

$$\lambda_1^\beta(\Omega) = \min_{\varphi \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla\varphi|^2\,dx + \beta\int_{\partial\Omega} \varphi^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\,dx}.$$

It is possible to define an increasing and diverging sequence of eigenvalues $\lambda_k^\beta(\Omega)$, and, as in the Dirichlet case, these eigenvalues are related to some physical interpretation in thermal insulation. In particular, if $t \in \mathbb{R}$ denotes the time and for $x \in \Omega$ the function $u(x,t)$ is the temperature inside the body Ω (we can still think of our house), then in the absence of heat sources the temperature (at fixed time) is ruled by the **heat equation**

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta_x u(x,t), & \text{in } \Omega \\ \dfrac{\partial u}{\partial \nu}(x,t) + \beta u(x,t) = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

where Δ_x denotes the spatial Laplacian with respect to $x = (x_1, \dots, x_n)$ and ν is the outer unit normal to Ω. If we impose the Robin boundary condition on $\partial\Omega$, and we take $u_k = u_k(x)$ eigenfunctions (constant in time) such that $-\Delta_x u_k = \lambda_k^\beta(\Omega) u_k$, then every solution to (1.2.3) is of the form

$$u(x,t) = \sum_{k=1}^{+\infty} \alpha_k e^{-\lambda_k^\beta(\Omega)t} u_k(x),$$

with suitable constants α_k. With this remark, it becomes clear that $\lambda_1^\beta(\Omega)$ is an indicator of how fast the temperature decays for the body Ω. Therefore, it could be interesting to understand which is the best shape Ω minimizing heat dispersion, i.e.

$$\min_{\substack{\Omega \in \mathcal{A} \\ |\Omega| = m}} \lambda_1^\beta(\Omega).$$

The answer to this question remained open for almost 30 years (the first appearance of the question could be due to M. Bareket in 1977 [27]) solved for Lipschitz sets by D. Daners in 2006 [85], who extended to every dimension the proof of M.-H. Bossel of 1986 [35] that solved the case $n = 2$. *Bossel-Daners inequality* reads as follows (we recall that, given Ω, the set $\Omega^\sharp$ denotes the centered ball of the same measure).

Theorem 1.2.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary, and let $\beta > 0$. Then*

$$\lambda_1^\beta(\Omega^\sharp) \le \lambda_1^\beta(\Omega).$$

The proof relies on the so-called H-function and a *derearranging technique*, tools that seem to be specific to the first Robin eigenvalue problem. Up to our knowledge, very few generalizations have been made of the H-function (see [46, 82, 58, 71, 4]), and it would be interesting to understand if it were possible to use this instrument for other shape optimization problems with the same boundary condition (for instance we could not find in literature any adaptation of the technique to the functional $\mathcal{T}_1^\beta(\Omega) := \mathcal{T}_1^\beta(\Omega, \Omega)$).

One of the examples where this generalization is possible (with some restrictions) is the *double shape optimization problem*

$$\min_{\substack{|\Omega| = m_0, \\ |A \setminus \Omega| \le m}} \mathrm{Cap}^\beta(\Omega, A).$$

The result proved in [58] is the following.

Theorem 1.2.4. *Let $\beta > 0$, and let $K \subset \mathbb{R}^n$ be a compact with $|K| = |B_1|$. Let $A \supset \Omega$ be an open set with volume $|\Omega| \le |B_R|$. Then*

$$\min\{\mathrm{Cap}^\beta(B_1, B_1). \mathrm{Cap}^\beta(B_1, B_R)\} \le \mathrm{Cap}^\beta(K, A).$$

The proof uses an adaptation of the Bossel' H-function, but it only works for the case $\beta > n-2$. For the general case, instead, a delicate symmetrization process is carried out, but the details of this part of the proof are beyond the scope of this Thesis. We oversimplified the results by D. Bucur, M. Nahon, C. Nitsch, and C. Trombetti, as their symmetrization technique applies to a wide class of boundary conditions.

The Bossel-Daners inequality with $\beta > 0$ has been proved also with a variational approach (with $\mathrm{SBV}^{1/2}$ functions), as in [48], followed up by generalizations for the torsional rigidity $\mathcal{T}_1^\beta$ and the eigenvalues of the Robin p-Laplacian [49, 52]. The same framework, joint with the H-function and the properties of inward minimizers, allowed D. Bucur, V. Ferone, C. Nitsch, and C. Trombetti [47] to obtain a sharp quantitative estimate for the first Robin eigenvalue, without shape derivatives (unlike [38]).

While for $\beta = 0$ the eigenvalue trivializes to $\lambda_1^\beta = 0$, for $\beta < 0$ the minimization question becomes a bit more delicate. First of all we have $\lambda_1^\beta < 0$ for $\beta < 0$. In this case, the minimization problem is ill-posed. Indeed, we have that

$$\inf_{\Omega \in \mathcal{A}} \lambda_1^\beta(\Omega) = -\infty,$$

one could prove this by computing the eigenvalue of small balls. We could then focus on another problem:

$$\sup_{\substack{\Omega \in \mathcal{A} \\ |\Omega| = m}} \lambda_1^\beta(\Omega) = - \inf_{\substack{\Omega \in \mathcal{A} \\ |\Omega| = m}} |\lambda_1^\beta(\Omega)|.$$

Bareket, in 1977 [27], conjectured that the supremum is attained and that the maximizer is the ball. Her conjecture was disproved by P. Freitas and D. Krejčiřík in 2015 [96]. On the other hand, by changing the constraint, other interesting inequalities can be obtained. Indeed, P. R. S. Antunes, P. Freitas, and D. Krejčiřík later in 2016 [20] proved that if $\Omega \subset \mathbb{R}^2$ is bounded, open and with boundary of class C^2, then if B is a ball with the same perimeter

$$\lambda_1^\beta(\Omega) \leq \lambda_1^\beta(B).$$

In 2019, a similar result was obtained in $\mathbb{R}^n$ under the convexity constraint: Bucur, Ferone, Nitsch, and Trombetti [47] proved the following.

Theorem 1.2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open convex set, let B be a ball having perimeter $P(B) = P(\Omega)$, and let $\beta < 0$. Then*

$$\lambda_1^\beta(\Omega) \leq \lambda_1^\beta(B).$$

The question of whether the ball is the maximizer among all open sets remains still open. The proof of [47] relies on *web functions* (or *parallel coordinates*, see [148] and [80]) and a derearrangement technique. We postpone the description of the proof to Section 4.3, where we will adapt these techniques to the Riemannian setting of the sphere. In the same paragraph, we will also discuss in detail the history of Bareket's conjecture.

1.2.3 Approximation with thin layers

As we anticipated in Section 1.2.1, finding the optimal set A to minimize $\mathrm{Cap}^\beta(\Omega, A)$ is tough. Attempts in this direction were partially made in [129], where C. Labourie and E. Milakis find sufficient conditions on Ω to have $\mathrm{Cap}^\beta(\Omega, \Omega) \leq \mathrm{Cap}^\beta(\Omega, A)$ for every A. Concerning numerical simulations, the only references we are aware of are [33, 161]. In [33] a gradient descent approach is

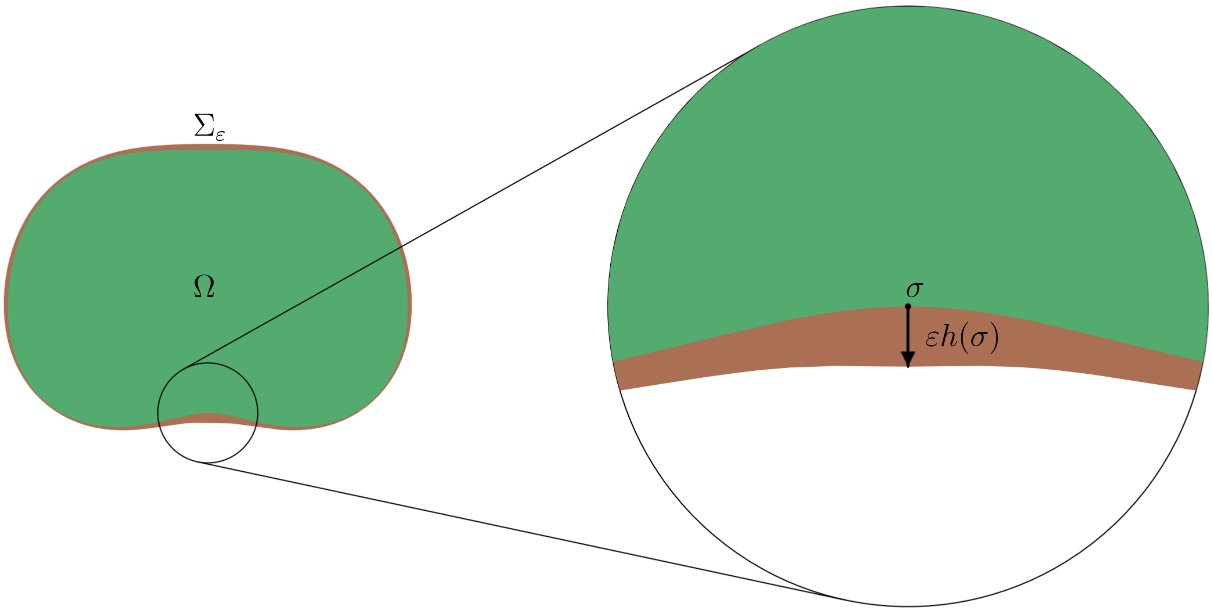


Figure 1.8: Description of Σ_ε using the *thickness* function h.[4]

employed, similar to the one described in Section 1.1.3. As of now, many properties of a minimizer remain unknown.

One idea to obtain finer information about this kind of problem is to restrict our case to sets A such that $A \setminus \Omega$ is very *"thin"*. Let $\Omega \subset \mathbb{R}^n$ a bounded open set with $C^{1,1}$ boundary. For a Lipschitz function $h : \partial\Omega \to \mathbb{R}^+$ and for $\varepsilon > 0$ we define

$$\Sigma_\varepsilon = \Sigma_\Omega(\varepsilon h) = \left\{ \sigma + t\nu(\sigma) \;\middle|\; \begin{array}{c} \sigma \in \partial\Omega, \\ 0 < t < \varepsilon h(\sigma), \end{array} \right\}$$

where $\nu(\sigma)$ denotes the outer unit normal to $\partial\Omega$ in σ (see Figure 1.8). If we now consider the problem

$$\lim_{\varepsilon\to 0^+} \mathrm{Cap}^\beta(\Omega, \Omega \cup \Sigma_\varepsilon) = \mathrm{Cap}^\beta(\Omega, \Omega) = \beta P(\Omega).$$

To better understand the behavior of the functional Cap^β in dependence of h, we modify it to make the limit non-trivial. Let us define $\Omega_\varepsilon = \Omega \cup \Sigma_\varepsilon$ and let for every $v \in H^1(\Omega_\varepsilon)$

$$\mathcal{E}_\varepsilon(v, h) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1}.$$

The limit functional will be non-trivial, and depend on h. In particular, we have the following

Theorem 1.2.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set with $C^{1,1}$ boundary, and let $h \in C^{0,1}(\partial\Omega)$. Then*

$$\lim_{\varepsilon\to 0^+} \min_{\substack{v \in H^1\Omega_\varepsilon \\ v=1 \text{ in } \Omega}} \mathcal{E}_\varepsilon(v, h) = \mathcal{E}_0(h),$$

[4] Image realized with Desmos Graphic Calculator, used with permission from Desmos Studio PBC.

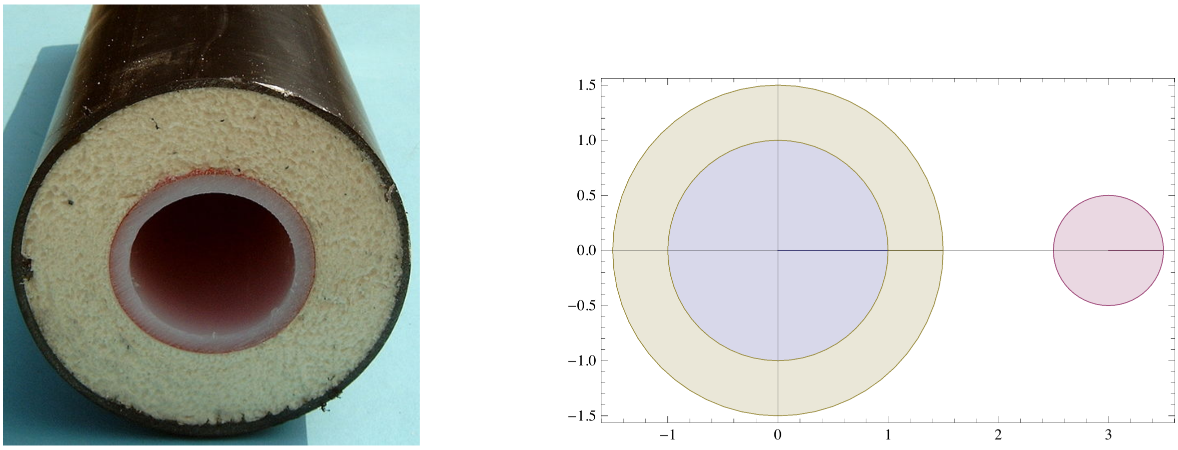


Figure 1.9: Examples in which the best thickness is non-constant: the inner layer is thicker in the case of a pipe; when Ω is the union of two balls with different radii, then, if the mass is small enough, the optimal choice is to leave the smaller ball uncovered. Images from [45].

where

$$\mathcal{E}_0(h) = \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}.$$

This result is proved in [7], but it is a direct consequence of the techniques used in [86] for a functional related to $\mathcal{T}^\beta(\Omega, \Omega_\varepsilon)$. Before discussing the proof of this result and the history of *thin layers* and *reinforcement problems*, we briefly discuss the consequences of Theorem 1.2.6.

Even though it seems an artificial constraint, the choice of putting an ε in front of the Dirichlet energy $\int |\nabla v|^2$ can be interpreted physically as putting around Ω a "very thin" chamber Σ_ε with a "very efficient" insulator. Indeed, if $u_{\varepsilon,h}$ is the minimizer of $\mathcal{E}_\varepsilon$, then it solves the boundary value problem

$$\begin{cases} \Delta u_{\varepsilon,h} = 0 & \text{in } \Sigma_\varepsilon, \\ u_{\varepsilon,h} = 1 & \text{in } \Omega, \\ \varepsilon \dfrac{\partial u_{\varepsilon,h}}{\partial \nu_\varepsilon} + \beta u_{\varepsilon,h} = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega. \end{cases}$$

Here ε represents the effectiveness of the insulating chamber. The function h represents the shape of the outer layer. For this reason, minimizing $\mathcal{E}_0(\cdot)$ with respect to h can give interesting suggestions on how to optimize $\mathrm{Cap}^\beta(\Omega, \Sigma)$ for a general thin Σ. Analogously to the mass constraint $|\Sigma_\varepsilon| = \varepsilon m$, we minimize $\mathcal{E}_0$ with the constraint $\|h\|_{1,\partial\Omega} = m$. By a convexity argument, it is possible to prove (see Proposition 5.1.4) that

$$\mathcal{E}_0\left(\frac{m}{P(\Omega)}\right) \le \mathcal{E}_0(h), \qquad \begin{aligned} & h \ge 0 \\ & \int_{\partial\Omega} h\, d\mathcal{H}^{n-1} = m. \end{aligned}$$

This result would suggest that the best insulation is achieved with layers of uniform thickness. However, real-life examples (see Figure 1.9) go in a different direction. This inspired [7] and later [5] to perform a first-order development of the functional $\min_v \mathcal{E}_\varepsilon(v, h)$ with respect to ε. This aspect will be treated in Chapter 5 for $\mathcal{E}_\varepsilon$ and for a functional related to $\mathcal{T}_f^\beta$, discovering the role of **mean curvature** in thermal insulation.

The proof of Theorem 1.2.6 goes back to the formulation introduced by E. Acerbi and G. Buttazzo in 1986 [10]. *Reinforcement problems* had been already investigated by different authors: as far as we know, the first appearance of a (internal, lens-shaped) thin layer was in 1970 by F. Sanchez-Palencia [153], followed up by [154] by the same author four years after; an external thin layer problem was then studied by L. A. Caffarelli and A. Friedman in 1978-80 [63, 64] in the 2-dimensional case, and by H. Brezis, L. A. Caffarelli, and A. Friedman in 1980 [40] in the n-dimensional case. The approach in [40] relied on some a priori H^2 estimates of the solutions to the PDE associated with the objective energy. Finally, in 1986 a different approach based on Γ-convergence was proposed by Acerbi and Buttazzo in [10] for the reinforcement problem. Referring the reader to Section 2.7 for the notion of Γ-convergence, we summarize the results obtained by using both the approaches (limiting ourselves to the particular case of the *torsional rigidity*) in the following. Let $\varepsilon > 0$, let $\delta(\varepsilon)$ be a function of ε, let

$$\mathcal{T}_{1,\varepsilon}(\Omega, h) := \min_{v \in H_0^1(\Omega_\varepsilon)} \int_\Omega \Big(|\nabla v|^2 - 2u \Big)\, dx + \delta(\varepsilon) \int_{\Sigma_\varepsilon} |\nabla u|^2\, dx,$$

and let $u_{\varepsilon,h} \in H_0^1(\Omega_\varepsilon)$ be the respective minimizer. Then we have that $u_{\varepsilon,h}$ solves the boundary value problem

$$\begin{cases} -\Delta u_{\varepsilon,h} = \chi_\Omega & \text{in } \Omega_\varepsilon, \\[2mm] \dfrac{\partial u_{\varepsilon,h}^-}{\partial \nu} = \delta(\varepsilon) \dfrac{\partial u_{\varepsilon,h}^+}{\partial \nu} & \text{on } \partial\Omega, \\[2mm] u_{\varepsilon,h} = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \end{cases}$$

where u^+ denotes the restriction of u to Σ_ε, and u^- denotes the restriction of u to Ω.

We called $\mathcal{T}_{1,\varepsilon}(\Omega, h)$ torsional rigidity because in $\mathbb{R}^2$ with $h = 0$ this functional is linked to the resistance to torsion of a cylinder with section Ω. This also motivates the name **"reinforcement"** for the problem of optimizing $\mathcal{T}_{1,\varepsilon}(\Omega, h)$ with respect to h. Passing to the limit for ε that goes to 0 we obtain the following.

Theorem 1.2.7. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $h : \partial\Omega \to \mathbb{R}^+$ be a Lipschitz function, and let*

$$L = \lim_{\varepsilon \to 0^+} \frac{\varepsilon}{\delta(\varepsilon)}.$$

Then the following hold:

(i) if $L = 0$, then

$$\lim_{\varepsilon \to 0^+} \mathcal{T}_{1,\varepsilon}(\Omega, h) = \min_{v \in H^1(\Omega)} \int_\Omega \Big(|\nabla u|^2 - 2u \Big)\, dx, \tag{1.2.4}$$

and u_ε converges weakly in $H^1(\Omega)$ to the minimizer u_0^{N} of (1.2.4);

(ii) if $L \in (0, +\infty)$, then

$$\lim_{\varepsilon \to 0^+} \mathcal{T}_{1,\varepsilon}(\Omega, h) = \min_{v \in H^1(\Omega)} \int_\Omega \Big(|\nabla u|^2 - 2u \Big)\, dx + L \int_{\partial\Omega} \frac{1}{h} u^2\, d\mathcal{H}^{n-1}, \tag{1.2.5}$$

and u_ε converges weakly in $H^1(\Omega)$ to the minimizer u_0^h of (1.2.5);

(iii) if $L = +\infty$, then

$$\lim_{\varepsilon \to 0^+} \mathcal{T}_{1,\varepsilon}(\Omega, h) = \min_{v \in H_0^1(\Omega)} \int_\Omega \Big(|\nabla u|^2 - 2u \Big)\, dx, \tag{1.2.6}$$

and u_ε converges weakly in $H^1(\Omega)$ to the minimizer u_0^h of (1.2.6).

We notice that u_0^{N} solves a Neumann boundary condition $\partial_\nu u = 0$ on $\partial\Omega$, u_0^h solves a Robin boundary condition with non-constant parameter $\partial_\nu u + (L/h)u = 0$, and u_0^{D} solves a Dirichlet boundary condition $u = 0$ on $\partial\Omega$. Even though we stated the result for the torsional rigidity $\mathcal{T}_{1,\varepsilon}$, it is possible to notice that this behavior is way more general, and applies to other functionals as well.

Equation (1.2.5) is one of the reasons for the interest in the Robin boundary condition. Indeed, this interest becomes clear when we interpret $\mathcal{T}_{1,\varepsilon}(\Omega, h)$ in terms of thermal insulation: $\mathcal{T}_{1,\varepsilon}(\Omega, h)$ is the *heat content* of a uniformly heated body Ω (heat source $f = 1$) and insulated by a thin layer Σ_ε and the temperature u_ε on the outer boundary $\partial\Omega_\varepsilon$ is equal to the one of the exterior environment. As the limit of this configuration, the Robin boundary condition on $\partial\Omega$ can also be seen as the mathematical representation of a thin film of insulating material around Ω. An interesting discussion of this fact can be found in the survey [45].

The functionals obtained with this approximation and limiting process led to surprising consequences. In 2017, D. Bucur, G. Buttazzo, and C. Nitsch [44] showed that if one considers a principal eigenvalue problem arising in thermal insulation, then an unexpected symmetry-breaking phenomenon occurs. We refer to the original paper for the details. This phenomenon was then confirmed by numerical evidence in [28, 29].

The same eigenvalue problem was initially investigated by A. Friedman in 1980 [97] and later by S. J. Cox., B. Kawohl, and P. X. Uhlig in 1999 [79]. The results in [79] are obtained under a uniform lower bound $h \geq \alpha > 0$, which, it turns out, prevented the symmetry breaking from happening.

Other insulation problems in this limit context were addressed by G. Buttazzo in 1988 [60] and by F. Della Pietra, R. Scala, C. Nitsch, and C. Trombetti in [86]. In both cases, an insulation problem with thin layers was studied, with the Dirichlet boundary condition in the former, and the Robin boundary condition in the latter. The results are similar to Theorem 1.2.6, and the geometry of the set Ω does not play an explicit role in these cases as well. The methods of [5] using a *first order development by Γ-convergence* allowed the authors to gather more information about how the problem is influenced by the curvature of the set Ω. This will be discussed in Section 5.2, where we inspect a technique that we expect to be adaptable to more general problems involving thin layers.

Chapter 2

Preliminaries

We denote by $\mathcal{L}^n$ the Lebesgue measure on $\mathbb{R}^n$, and by $\mathcal{H}^k$ the k-dimesnional Hausdorff measure. Given an open set $\Omega \subseteq \mathbb{R}^n$, we denote by $L^p(\Omega)$ the space of p-summable functions on Ω, and by $W^{k,p}(\Omega)$ the Sobolev space of k-differentiable functions with derivatives in $L^p(\Omega)$ respectively. Sometimes we use the notation $H^1(\Omega)$ to denote $W^{1,2}(\Omega)$.

We denote by $\mathbf{e}_i$ the canonical basis of $\mathbb{R}^n$, and for every $v, w \in \mathbb{R}^n$ we let the *tensorial product* $v \otimes w \in \mathbb{R}^{n\times n}$ be the unique linear operator on $\mathbb{R}^n$ such that, for every $z \in \mathbb{R}^n$,

$$(v \otimes w)z = (w \cdot z)v.$$

In particular, if

$$A = \sum_{i,j=1}^{n} a_{ij}\mathbf{e}_i \otimes \mathbf{e}_j,$$

and $v = \sum_j v_j\mathbf{e}_j$, then

$$Av = \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}v_j\,\mathbf{e}_i.$$

We denote by $I_n \in \mathbb{R}^{n\times n}$ the identity matrix on $\mathbb{R}^n$ and sometimes we will denote by Id the identity map on $\mathbb{R}^n$, so that

$$D\,\mathrm{Id}(x) = I_n, \qquad \forall x \in \mathbb{R}^n.$$

For every $A \in \mathbb{R}^{n\times n}$ we denote by A^T the transpose of A and, if A is invertible, we denote by A^{-T} the transpose of the inverse A^{-1}.

2.1 Distance function and curvatures

Definition 2.1.1 (Metric projection)**.** Let (X, d) be a metric space, and let $\Omega \subset X$ be a precompact set. We denote by

$$d_\Omega(x) = d(x, \Omega) = \min_{y\in\bar{\Omega}}|x - y|$$

the distance function from Ω. When the minimum is unique, we denote it by $\pi_\Omega(x)$ and we call it the *metric projection* of x onto Ω, i.e.

$$d_\Omega(x) = d(x, \pi_\Omega(x)).$$

When $X = \mathbb{R}^n$, we let $d(x, y) = |x - y|$. When possible, we will drop the dependence on Ω and we will only write $d = d_\Omega$.

Definition 2.1.2. Let (X, d) be a metric space, and let $K \subset X$. For every $t \geq 0$, we define the *inner parallel set*

$$(K)_t = \{\, p \in K \mid d(p, \partial K) \geq t \,\},$$

and the *outer parallel* set

$$(K)^t = \{\, p \in X \mid d(p, K) \leq t \,\}.$$

Definition 2.1.3 (Hausdorff distance)**.** Let (X, d) be a metric space, and let $K_1, K_2 \subset X$ be two compact sets. We define the Hausdorff distance as

$$d^{\mathrm{H}}(K_1, K_2) = \inf \left\{ t \geq 0 \;\middle|\; \begin{array}{l} K_1 \subset (K_2)^t \\ K_2 \subset (K_1)^t \end{array} \right\}$$

Proposition 2.1.4. *Let Ω be an open bounded set with $C^{1,1}$ boundary, then there exist a constant $C = C(\Omega) > 0$ and a $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that*

$$|\Omega \setminus (\Omega)_\varepsilon| \leq C\varepsilon \qquad\qquad \forall \varepsilon < \varepsilon_0.$$

Proof. Let $\Omega_\varepsilon = (\Omega)_\varepsilon$. Computing the first variation with respect to ε (see for instance [135, Theorem 17.5]) we have that there exist constants $C = C(\Omega)$ and $\varepsilon_0 = \varepsilon_0(\Omega) > 0$ such that

$$P(\Omega_\varepsilon) = P(\Omega) + C(\Omega)\,\varepsilon + O(\varepsilon^2),$$

for every $0 < \varepsilon < \varepsilon_0$. Let $d_{\partial\Omega}(x) = d(x, \partial\Omega)$ be the distance from the boundary of Ω. By coarea formula we have

$$|\Omega \setminus \Omega_\varepsilon| = \int_{\{\, 0 < d_{\partial\Omega} < \varepsilon \,\}} d\mathcal{L}^n = \int_0^\varepsilon P(\Omega_t)\; dt \leq C(\Omega)\varepsilon.$$

$\square$

Definition 2.1.5. Let $\Omega \subset \mathbb{R}^n$, and let $k : \mathbb{R}^n \to [0, +\infty)$. We define the *outer layer of thickness* k the set

$$\Sigma_\Omega(k) := \{\, x \in \mathbb{R}^n \mid d(x) < k(x) \,\} \setminus \bar{\Omega}.$$

Notice that when $k(x) = t > 0$ for every x, then

$$\Sigma_\Omega(t) = (\Omega)^t \setminus \bar{\Omega}.$$

We refer to [102, Section 14.6] for the following result.

Proposition 2.1.6. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary. Then Ω satisfies an uniform exterior ball condition, and in particular there exists a positive real number $d_0 = d_0(\Omega)$ such that*

$$d_\Omega \in C^{1,1}(\Sigma_\Omega(d_0)).$$

Moreover, the metric projection is well defined and $\pi_\Omega \in C^{0,1}(\Sigma_\Omega(d_0))$. We will denote by $\nu_\Omega = \nabla d_\Omega$, i.e. the unique extension of the outer unit normal to $\partial\Omega$ such that for every $z \in \Sigma_\Omega(d_0)$ we have $\nu_\Omega(z) = \nu_\Omega(\pi_\Omega(z))$.

For every $x \in \Sigma_\Omega(d_0)$ we can write

$$x = \pi_\Omega(x) + d(x)\nu_\Omega(x).$$

For $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ we can consider the Hessian matrix $D^2 d_\Omega(\sigma)$, and, since $|\nabla d_\Omega| = 1$, we have that $\nu_\Omega(\sigma)$ is an eigenvector with corresponding zero eigenvalue.

Definition 2.1.7. Let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary. Let $\sigma \in \partial\Omega$, and let $\{\, \tau_1(\sigma), \dots, \tau_{n-1}(\sigma), \nu_\Omega(\sigma) \,\}$ be an ordered system of normalized eigenvectors for $D^2 d_\Omega(\sigma)$. We define the *principal curvatures* $\kappa_1(\sigma), \dots, \kappa_{n-1}(\sigma)$ of $\partial\Omega$ at the point σ as the eigenvalues of the matrix $D^2 d_\Omega(\sigma)$ corresponding to the eigenvectors $\tau_1(\sigma), \dots, \tau_{n-1}(\sigma)$.

Let $t \in (0, d_0)$ and consider

$$\Gamma_t = \partial(\Omega)^t = \{\, x \in \mathbb{R}^n \mid d(x) = t \,\}.$$

When Ω has $C^{1,1}$ boundary, noticing that $\nabla d(x) = \nu_\Omega(x)$ is orthogonal to $\Gamma_{d(x)}$, then the matrix $D^2 d_\Omega(x)$ is symmetric and it represents the second fundamental form of the hypersurface $\Gamma_{d(x)}$. Moreover, by direct computation (see for instance [102, Lemma 14.17]) we have that $\tau_i(\pi_\Omega(x))$ are eigenvectors for $D^2 d_\Omega(x)$ with eigenvalues computed in the following definition.

Definition 2.1.8. Let $x \in \Gamma_{d_0}$. For every $i = 1, \dots, n-1$ we denote by

$$\tau_i(x) := \tau_i(\pi_\Omega(x)),$$

and we denote by $\kappa_1(x), \dots, \kappa_{n-1}(x)$ the corresponding sequence of eigenvalues of $D^2 d(x)$

$$\kappa_i(x) := \frac{\kappa_i(\pi_\Omega(x))}{1 + d(x)\kappa_i(\pi_\Omega(x))},$$

Remark 2.1.9. Since $\pi_\Omega(x) = d(x)\nu_\Omega(x) - x$, then we also have

$$D\pi_\Omega(x)\, \tau_i(x) = \frac{1}{1 + d(x)\kappa_i(\pi_\Omega(x))}\tau_i(x).$$

2.2 Calculus on hypersurfaces

We refer to [135] for the notions in this section. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with C^1 boundary and let ν_0 be the outer unit normal to its boundary. For every $\sigma \in \partial\Omega$ we denote by

$$T_\sigma\partial\Omega = \{\, y \in \mathbb{R}^n \mid \nu_0(\sigma) \cdot y = 0 \,\},$$

the *tangent space to* $\partial\Omega$ *at* σ. Let $\tau_1(\sigma), \dots, \tau_{n-1}(\sigma)$ be an orthonormal basis for $T_\sigma\partial\Omega$.

Definition 2.2.1 (Tangential gradient)**.** Let Ω be a bounded open set of class C^1, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U; \mathbb{R}^n)$. We define for $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ the *tangential gradient of* ϕ at σ as the the linear map

$$D^{\partial\Omega}\phi(\sigma) \colon T_\sigma\partial\Omega \to \mathbb{R}^n$$

defined as

$$\begin{aligned} D^{\partial\Omega}\phi(\sigma) =& \sum_{h=1}^{n-1} (\nabla\phi(\sigma)\tau_h) \otimes \tau_h \\ =& \nabla\phi(\sigma) - (\nabla\phi(\sigma)\nu_0) \otimes \nu_0. \end{aligned}$$

Notice that the definition of tangential gradient does not depend on the choice of the orthonormal basis $\tau_1, \dots, \tau_{n-1}$.

Definition 2.2.2 (Tangential Jacobian)**.** Let Ω be an open bounded set with C^1 boundary, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U;\mathbb{R}^n)$. We define *the tangential Jacobian of* ϕ as

$$\operatorname{Jac}^{\partial\Omega} \phi = \sqrt{\det\left((D^{\partial\Omega}\phi)^T(D^{\partial\Omega}\phi)\right)},$$

where the determinant has to be intended in the space $T_\sigma\partial\Omega \otimes T_\sigma\partial\Omega$.

Remark 2.2.3. It will be useful to notice that

$$\operatorname{Jac}^{\partial\Omega} \phi = |D\phi^{-T}\nu_\Omega| \operatorname{Jac} \phi.$$

Theorem 2.2.4 (Area formula on surfaces)**.** *Let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, let $\phi \in C^{0,1}(U;\mathbb{R}^n)$, and let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive Borel function. We have that*

$$\int_{\partial\Omega} g(\phi(\sigma)) \operatorname{Jac}^{\partial\Omega} \phi \, d\mathcal{H}^{n-1} = \int_{\phi(\partial\Omega)} g(\sigma)\, d\mathcal{H}^{n-1}.$$

Definition 2.2.5 (Tangential divergence)**.** Let Ω be a bounded open set of class C^1, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$ and let $\phi \in C^{0,1}(U;\mathbb{R}^n)$. We define the *tangential divergence of* ϕ as

$$\operatorname{div}^{\partial\Omega} \phi = \sum_{j=1}^{n-1} (D\phi\, \tau_j) \cdot \tau_j$$

Definition 2.2.6 (Mean Curvature)**.** Let Ω be a bounded open set with $C^{1,1}$ boundary and let

$$\nu : \partial\Omega \to \mathbb{S}^{n-1}$$

be the outer unit normal to its boundary. Let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$ and let $X \in C^{0,1}(U)$ be an extension of ν. For $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ we define the *mean curvature of* $\partial\Omega$ as

$$H_\Omega(\sigma) = \operatorname{div}^{\partial\Omega} X = \sum_{i=1}^{n-1} \kappa_i(\sigma).$$

Remark 2.2.7. Let Ω be a bounded open set of class $C^{1,1}$, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, let $X \in C^{0,1}(U;\mathbb{R}^n)$, and let $\phi(x) = x + tX(x)$. By direct computations, we have that

$$\operatorname{Jac}^{\partial\Omega} \phi(\sigma) = 1 + t \operatorname{div}^{\partial\Omega} X(\sigma) + t^2 R(t,\sigma)$$

where the remainder R is a bounded function. In particular, if X is an extension of ν_Ω, we have

$$\operatorname{Jac}^{\partial\Omega} \phi(\sigma) = 1 + tH_\Omega(\sigma) + t^2 R(t,\sigma).$$

2.3 Geometric measure theory and functions of bounded variation

In this section we recall some definitions in the context of geometric measure theory and proprieties of the spaces BV, SBV, and $\mathrm{SBV}^{\frac{1}{2}}$. We refer to [18], [48], [89] for a deep study of the properties of these functions.

Theorem 2.3.1 (Coarea formula)**.** *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a Lipschitz function, let $g\colon \mathbb{R}^n \to \mathbb{R}$ be an $L^1(\mathbb{R}^n)$ function and let $U \subset \mathbb{R}^n$ be an open set, then*

$$\int_U g(x)|\nabla f(x)|\,dx = \int_{\mathbb{R}} \int_{U\cap\{\, f=t\,\}} g(y)\,d\mathcal{H}^{n-1}(y)\,dt.$$

Remark 2.3.2. If Ω is of class $C^{1,1}$, then in particular it is of finite perimeter and it has generalized mean curvature with $H_\Omega \in L^\infty(\partial\Omega)$.

In the following, given an open set $\Omega \subseteq \mathbb{R}^n$ and $1 \le p \le \infty$, we will denote the $L^p(\Omega)$ norm of a function $v \in L^p(\Omega)$ as $\|v\|_{p,\Omega}$, in particular when $\Omega = \mathbb{R}^n$ we will simply write $\|v\|_p = \|v\|_{p,\mathbb{R}^n}$.

Definition 2.3.3 (BV)**.** Let $u \in L^1(\mathbb{R}^n)$. We say that u is a function of *bounded variation* in $\mathbb{R}^n$ and we write $u \in \mathrm{BV}(\mathbb{R}^n)$ if its distributional derivative is a Radon measure, namely

$$\int_\Omega u\,\frac{\partial\varphi}{\partial x_i} = \int_\Omega \varphi\,dD_i u \qquad \forall\varphi \in C_c^\infty(\mathbb{R}^n),$$

with Du a $\mathbb{R}^n$-valued measure in $\mathbb{R}^n$. We denote with $|Du|$ the total variation of the measure Du. The space $\mathrm{BV}(\mathbb{R}^n)$ is a Banach space equipped with the norm

$$\|u\|_{\mathrm{BV}(\mathbb{R}^n)} = \|u\|_1 + |Du|(\mathbb{R}^n).$$

Definition 2.3.4. Let $E \subseteq \mathbb{R}^n$ be a measurable set. We define the *set of points of density 1 for E* as

$$E^{(1)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r\to 0^+} \frac{|B_r(x)\cap E|}{|B_r(x)|} = 1 \right\},$$

and the *set of points of density 0 for E* as

$$E^{(0)} = \left\{ x \in \mathbb{R}^n \;\middle|\; \lim_{r\to 0^+} \frac{|B_r(x)\cap E|}{|B_r(x)|} = 0 \right\}.$$

Moreover, we define the *essential boundary* of E as

$$\partial^* E = \mathbb{R}^n \setminus (E^{(0)} \cup E^{(1)}).$$

2.3.1 Isoperimetric inequalities

Definition 2.3.5. Let $E \subseteq \mathbb{R}^n$ be a measurable set and let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the *relative perimeter* of E inside Ω as

$$P(E;\Omega) = \sup\left\{ \int_E \operatorname{div}\varphi\,d\mathcal{L}^n \;\middle|\; \begin{array}{c} \varphi \in C_c^1(\Omega,\mathbb{R}^n) \\ |\varphi| \le 1 \end{array} \right\}.$$

When $\Omega = \mathbb{R}^n$ we write $P(E) := P(E;\mathbb{R}^n)$. If $P(E) < +\infty$ we say that E is a *set of finite perimeter*.

Definition 2.3.6 (Generalized mean curvature)**.** Let $\Omega \subset \mathbb{R}^n$ be a set of finite perimeter, and let $p \in [1, +\infty]$. We say that Ω has *generalized mean curvature in* L^p if there exists $H_\Omega \in L^p(\partial\Omega)$ such that

$$\int_{\partial^*\Omega} \operatorname{div}^{\partial\Omega} F \, d\mathcal{H}^{n-1} = \int_{\partial^*\Omega} H_\Omega F \cdot \nu \, d\mathcal{H}^{n-1},$$

for any $F \in C_c^\infty(A; \mathbb{R}^n)$ with A open set containing Ω.

Here we recall the classical isoperimetric inequality (see for instance [135]).

Theorem 2.3.7 (Isoperimetric inequality)**.** *Let E be a measurable set, and let B be the ball such that $|B| = |E|$. Then*

$$\mathcal{H}^{n-1}(\partial^* E) \geq \mathcal{H}^{n-1}(\partial B),$$

and the equality holds if and only if $E = B$ up to a set of measure 0.

Here we state a relative isoperimetric inequality that can be found in [138].

Theorem 2.3.8 (Relative Isoperimetric Inequality)**.** *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a positive constants $C = C(\Omega)$ such that*

$$\min\{\, |\Omega \cap E|, |\Omega \setminus E| \,\}^{\frac{n-1}{n}} \leq C P(E; \Omega),$$

for every set E of finite perimeter.

Finally, we state a comparison between some notions of "internal" and "external" perimeter whose proof can be found in [75, Theorem 2.3] (notice that if Ω is a Lipschitz set, then it is an admissible set in the sense defined in [75], see [170, Remark 5.10.2]).

Theorem 2.3.9. *Let Ω be an open, bounded, connected set with Lipschitz boundary. Then there exists a constant $C = C(\Omega) > 0$ such that*

$$\mathcal{H}^{n-1}(\partial^* E \cap \partial\Omega) \leq C \mathcal{H}^{n-1}(\partial^* E \cap \Omega)$$

for every set of finite perimeter $E \subset \Omega$ with $0 < |E| \leq |\Omega|/2$.

2.3.2 Properties of BV and SBV functions

Definition 2.3.10 (Approximate upper and lower limits)**.** Let $u \colon \mathbb{R}^n \to \mathbb{R}$ be a measurable function. We define the *approximate upper and lower limits* of u, respectively, as

$$\overline{u}(x) = \inf\left\{\, t \in \mathbb{R} \;\middle|\; \limsup_{r \to 0^+} \frac{|B_r(x) \cap \{\, u > t \,\}|}{|B_r(x)|} = 0 \,\right\},$$

and

$$\underline{u}(x) = \sup\left\{\, t \in \mathbb{R} \;\middle|\; \limsup_{r \to 0^+} \frac{|B_r(x) \cap \{\, u < t \,\}|}{|B_r(x)|} = 0 \,\right\}.$$

We define the *jump set* of u as

$$J_u = \{\, x \in \mathbb{R}^n \mid \underline{u}(x) < \overline{u}(x) \,\}.$$

We denote by K_u the closure of the jump set

$$K_u = \overline{J_u}.$$

If $\overline{u}(x) = \underline{u}(x) = l$, we say that l is the approximate limit of u as y tends to x, and we have that, for any $\varepsilon > 0$,

$$\limsup_{r\to 0^+} \frac{|B_r(x) \cap \{\,|u-l| \geq \varepsilon\,\}|}{|B_r(x)|} = 0.$$

If $u \in \mathrm{BV}(\mathbb{R}^n)$, the jump set J_u is a $(n-1)$-rectifiable set, i.e. $J_u \subseteq \bigcup_{i\in\mathbb{N}} M_i$, up to a $\mathcal{H}^{n-1}$-negligible set, with M_i a C^1-hypersurface in $\mathbb{R}^n$ for every i. We can then define $\mathcal{H}^{n-1}$-almost everywhere on J_u a normal ν_u coinciding with the normal to the hypersurfaces M_i. Futhermore, the direction of $\nu_u(x)$ is chosen in such a way that the approximate upper and lower limits of u coincide with the approximate limit of u on the half-planes

$$H^+_{\nu_u} = \{\, y \in \mathbb{R}^n \mid \nu_u(x)\cdot(y-x) \geq 0\,\}$$

and

$$H^-_{\nu_u} = \{\, y \in \mathbb{R}^n \mid \nu_u(x)\cdot(y-x) \leq 0\,\}$$

respectively.

Theorem 2.3.11 (Decomposition of BV functions)**.** *Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then we have*

$$dDu = \nabla u \; d\mathcal{L}^n + |\overline{u} - \underline{u}|\nu_u \; d\mathcal{H}^{n-1}\lfloor_{J_u} + dD^c u,$$

where ∇u is the density of Du with respect to the Lebesgue measure, ν_u is the normal to the jump set J_u and $D^c u$ is the Cantor part *of the measure Du. The measure $D^c u$ is singular with respect to the Lebesgue measure and concentrated out of J_u.*

Definition 2.3.12. Let $v \in \mathrm{BV}(\mathbb{R}^n)$, let $\Gamma \subseteq \mathbb{R}^n$ be a $\mathcal{H}^{n-1}$-rectifiable set and let $\nu(x)$ be the generalized normal to Γ defined for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$. For $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$ we define the traces $\gamma^\pm_\Gamma(v)(x)$ of v on Γ by the following Lebesgue-type limit quotient relation

$$\lim_{r\to 0} \frac{1}{r^n} \int_{B^\pm_r(x)} |v(y) - \gamma^\pm_\Gamma(v)(x)| \; dy = 0,$$

where

$$B^+_r(x) = \{\, y \in B_r(x) \mid \nu(x)\cdot(y-x) > 0\,\},$$
$$B^-_r(x) = \{\, y \in B_r(x) \mid \nu(x)\cdot(y-x) < 0\,\}.$$

Remark 2.3.13. Notice that, by [18, Remark 3.79], for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma$, $(\gamma^+_\Gamma(v)(x), \gamma^-_\Gamma(v)(x))$ coincides with either $(\overline{v}(x), \underline{v}(x))$ or $(\underline{v}(x), \overline{v}(x))$, while, for $\mathcal{H}^{n-1}$-a.e. $x \in \Gamma \setminus J_v$, we have that $\gamma^+_\Gamma(v)(x) = \gamma^-_\Gamma(v)(x)$ and they coincide with the approximate limit of v in x. In particular, if $\Gamma = J_v$, we have

$$\gamma^+_{J_v}(v)(x) = \overline{v}(x) \qquad \gamma^-_{J_v}(v)(x) = \underline{v}(x)$$

for $\mathcal{H}^{n-1}$-a.e. $x \in J_v$.

We now focus our attention on the BV functions whose Cantor parts vanish.

Definition 2.3.14 (SBV)**.** Let $u \in \mathrm{BV}(\mathbb{R}^n)$. We say that u is a *special function of bounded variation* and we write $u \in \mathrm{SBV}(\mathbb{R}^n)$ if $D^c u = 0$.

For SBV functions we have the following.

Theorem 2.3.15 (Chain rule)**.** *Let $g\colon \mathbb{R} \to \mathbb{R}$ be a differentiable function. Then if $u \in \mathrm{SBV}(\mathbb{R}^n)$, we have*

$$\nabla g(u) = g'(u)\nabla u.$$

Furthermore, if g is increasing,

$$\overline{g(u)} = g(\overline{u}), \quad \underline{g(u)} = g(\underline{u})$$

while, if g is decreasing,

$$\overline{g(u)} = g(\underline{u}), \quad \underline{g(u)} = g(\overline{u}).$$

We now give the definition of the following class of functions.

Definition 2.3.16 ($\mathrm{SBV}^{1/2}$)**.** Let $u \in L^2(\mathbb{R}^n)$ be a non-negative function. We say that $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ if $u^2 \in \mathrm{SBV}(\mathbb{R}^n)$. In addition, we define

$$J_u := J_{u^2} \qquad \overline{u} := \sqrt{\overline{u^2}} \qquad \underline{u} := \sqrt{\underline{u^2}}$$

$$\nabla u := \frac{1}{2u}\nabla(u^2)\chi_{\{\, u>0 \,\}}$$

Notice that this definition extends the validity of the Chain Rule to the functions in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$. We refer to [48, Lemma 3.2] for the coherence of this definition.

Theorem 2.3.17 (Compactness in $\mathrm{SBV}^{1/2}$)**.** *Let u_k be a sequence in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ and let $C > 0$ be such that for every $k \in \mathbb{N}$*

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \, d\mathcal{L}^n + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2\right) \, d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \, d\mathcal{L}^n < C$$

Then there exists $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ and a subsequence u_{k_j} such that

- Compactness:

$$u_{k_j} \xrightarrow{L^2_{\mathrm{loc}}(\mathbb{R}^n)} u$$

- Lower semicontinuity: *for every open set Ω we have*

$$\int_\Omega |\nabla u|^2 \, d\mathcal{L}^n \leq \liminf_{j\to+\infty} \int_\Omega |\nabla u_{k_j}|^2 \, d\mathcal{L}^n$$

and

$$\int_{J_u \cap \Omega} \left(\overline{u}^2 + \underline{u}^2\right) \, d\mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{J_{u_{k_j}} \cap \Omega} \left(\overline{u}_{k_j}^2 + \underline{u}_{k_j}^2\right) \, d\mathcal{H}^{n-1}$$

Theorem 2.3.18 (Compactness in SBV)**.** *Let u_k be a sequence in $\mathrm{SBV}(\mathbb{R}^n)$. Let $p, q > 1$, and let $C > 0$ such that for every $k \in \mathbb{N}$*

$$\int_{\mathbb{R}^n} |\nabla u_k|^p \, d\mathcal{L}^n + \|u_k\|_\infty + \mathcal{H}^{n-1}(J_{u_k}) < C.$$

Then there exists $u \in \mathrm{SBV}(\mathbb{R}^n)$ and a subsequence u_{k_j} such that

- Compactness:

$$u_{k_j} \xrightarrow{L^1_{\mathrm{loc}}(\mathbb{R}^n)} u$$

- Lower semicontinuity: *for every open set* A *we have*

$$\int_A |\nabla u|^p \, d\mathcal{L}^n \leq \liminf_{j \to +\infty} \int_A |\nabla u_{k_j}|^p \, d\mathcal{L}^n$$

and

$$\int_{J_u \cap A} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} \leq \liminf_{j \to +\infty} \int_{J_{u_{k_j}} \cap A} \left(\overline{u}^q_{k_j} + \underline{u}^q_{k_j} \right) d\mathcal{H}^{n-1}$$

We refer to [18, Theorem 4.7, Theorem 4.8, Theorem 5.22] for the proof of the previous theorem. We now conclude this section with the following proposition whose proof can be found in [65, Lemma 3.1].

Proposition 2.3.19. *Let* $u \in \mathrm{BV}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$*. Then*

$$\int_0^1 P(\{\, u > s \,\}; \mathbb{R}^n \setminus J_u) \, ds = |Du|(\mathbb{R}^n \setminus J_u).$$

2.4 Rearrangements and Talenti's inequality

We refer the reader to [160, 118] for an overview about definitions and properties of rearrangements.

Definition 2.4.1. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *distribution function* $\mu_u : [0, +\infty[\to [0, +\infty[$ of u as the function

$$\mu_u(t) = |\{\, x \in \Omega \,:\, |u(x)| > t \,\}|$$

Definition 2.4.2. Let Ω an open set of finite measure, and let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *decreasing rearrangement* u^* of u as

$$u^*(s) = \inf \{\, t > 0 \mid \mu_u(t) \leq s \,\}.$$

Definition 2.4.3. Let Ω an open set of finite measure, and let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *increasing rearrangement* u_* of u as

$$u_*(s) = \inf \{\, t > 0 \mid \mu_u(t) \leq |\Omega| - s \,\}.$$

Remark 2.4.4. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $u : \Omega \to \mathbb{R}$ be a measurable function. Then u, its decreasing rearrangement u^*, and its increasing rearrangement u_* are equi-measurable namely

$$\mu_u = \mu_{u_*} = \mu_{u^*}.$$

In particular, for every $p \in [1, +\infty)$,

$$\|u\|_{L^p(\Omega)} = \|u^*\|_{L^p(0,|\Omega|)} = \|u_*\|_{L^p(0,|\Omega|)}.$$

Remark 2.4.5. When Ω is connected and $u \in W^{1,1}(\Omega)$, the function μ_u is strictly decreasing (in particular it is injective). Moreover, let $m \in (0, |\Omega|)$. If in addition u has no plateau at level $u^*(m)$, i.e.

$$|\{|\nabla u| = 0\} \cap \{u = u^*(m)\}| = 0,$$

then $\{u > u^*(m)\}$ is the unique superlevel set of u having measure m. See [41, Lemma 2.3] for a reference.

Definition 2.4.6. Let $u : \Omega \to \mathbb{R}$ be a measurable function. We define the *Schwarz rearrangement* of $u^\sharp$ of u as

$$u^\sharp(x) = u^*(\omega_n |x|^n) \qquad\qquad x \in \Omega^\sharp,$$

where $\Omega^\sharp$ denotes the centered ball having the same volume as Ω.

For the following, see [159, Theorem I].

Theorem 2.4.7 (Talenti comparison)**.** *Let Ω be a measurable set, let $f \in L^q(\Omega)$ with*

$$q \in \left(\frac{2n}{n+2}, +\infty\right),$$

and let $v_f \in W_0^{1,2}(\Omega)$ and $v_{f^\sharp} \in W_0^{1,2}(\Omega^\sharp)$ be the solutions to

$$\begin{cases} -\Delta v_f = f & \text{in } \Omega, \\ v_f = 0 & \text{on } \partial\Omega, \end{cases} \qquad\qquad \begin{cases} -\Delta v_{f^\sharp} = f^\sharp & \text{in } \Omega^\sharp, \\ v_{f^\sharp} = 0 & \text{on } \partial\Omega^\sharp. \end{cases}$$

Then

$$v_f^\sharp \leq v_{f^\sharp}.$$

In particular, for every set E, letting $u_E = v_{\chi_E}$ and $u_{E^\sharp} = v_{\chi_{E^\sharp}}$, we have

$$u_E^\sharp \leq u_{E^\sharp}.$$

For the following rigidity result we refer to [15, Theorem 1].

Theorem 2.4.8. *Let $\Omega \subseteq \mathbb{R}^n$ be a measurable set, and let $f \in L^q(\Omega)$ with*

$$q \in \left(\frac{2n}{n+2}, +\infty\right).$$

If $v_f^\sharp = v_{f^\sharp}$ almost everywhere, then there exists $x_0 \in \mathbb{R}^n$ such that up to negligible sets $\Omega = x_0 + \Omega^\sharp$, $f(\cdot) = f^\sharp(\cdot + x_0)$, and $v_f(\cdot) = v_{f^\sharp}(\cdot + x_0)$ almost everywhere.

Remark 2.4.9. Let us notice that the condition $v_f^\sharp = v_{f^\sharp}$ in Theorem 2.4.7 is implied by

$$\mu_{v_f}(t) = \mu_{v_{f^\sharp}}(t) \qquad\qquad \text{for a.e. } t \in \mathbb{R}. \tag{2.4.1}$$

This is a consequence of the definition of $v_f^\sharp$, the fact that $v_{f^\sharp} = (v_{f^\sharp})^\sharp$, and the monotonicity of the distribution function. Since v_f and $v_f^\sharp$ are equimeasurable, then we also have that $v_f^\sharp = v_{f^\sharp}$ and (2.4.1) are equivalent.

2.5 Convexity in the Euclidean setting

Here we define standard quantities for convex sets and the formal definition of thin domain. This definition passes through the ones of support function and minimal width (or thickness).

We refer to [112] for the proof of the lemmas in this section.

Definition 2.5.1. Let $\Omega \subset \mathbb{R}^N$ be a bounded, open, and convex set. We define the *support function* of Ω as

$$h_\Omega(y) = \sup_{x\in\Omega}(x \cdot y), \qquad y \in \mathbb{R}^n.$$

Definition 2.5.2. Let $\Omega \subset \mathbb{R}^N$ be a bounded, open and convex set, and let $y \in \mathbb{R}^n$. We define the *width* of Ω in the direction y as

$$\omega_\Omega(y) = h_\Omega(y) + h_\Omega(-y)$$

and we define the *minimal width* of Ω as

$$w_\Omega = \min\{\omega_\Omega(y) \mid y \in \mathbb{S}^{n-1}\}.$$

Hence, if $\mathrm{diam}(\Omega)$ denotes the diameter of Ω, then we have

Definition 2.5.3. Let $\Omega_\delta \subset \mathbb{R}^n$ be a family of non-empty, bounded, open, and convex sets. We say that Ω_δ is a family of *thinning domains* if

$$\lim_{\delta\to 0} \frac{w_{\Omega_\delta}}{\mathrm{diam}(\Omega_\delta)} = 0.$$

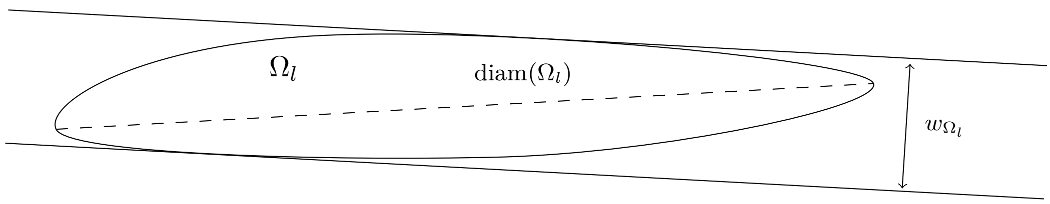


Figure 2.1: Minimal width and diameter of a convex set.

Let us now consider a particular family of thinning domains. Let

$$\mathcal{P} = \{\, h \in L^\infty(0,1) \colon\ h \text{ non negative, concave and not identically zero} \,\}$$

be the family of admissible *profile functions*, and let

$$\mathcal{P}_1 = \left\{ h \in \mathcal{P} \;\middle|\; \int_0^1 h(t)\,dt = 1 \right\}.$$

Definition 2.5.4. Let $h^+, h^- \in \mathcal{P}$ such that $h^+ + h^- \in \mathcal{P}_1$. We define

$$\Omega_\varepsilon = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; \begin{array}{c} 0 \le x \le 1, \\ -\varepsilon h_-(x) \le y \le \varepsilon h_+(x). \end{array} \right\}, \tag{2.5.1}$$

and we say that the family Ω_ε is *thinning with profile* (h^+, h^-).

We inspect some compactness and convexity properties of $\mathcal{P}_1$. The proof of the following lemma can be found in [112, Lemma 3.1].

Lemma 2.5.5. *Let $h_n \in \mathcal{P}_1$ be a sequence of functions, then there exists $h \in \mathcal{P}$ such that, up to a subsequence, we have:*

- *h_n converges to h in $L^2(0,1)$;*
- *h_n converges to h uniformly on every compact subset of $(0,1)$.*

Definition 2.5.6. Let V be a vector space, let $C \subset V$ be a convex set, and let $z \in C$. We say that z is an *extreme point* of C if it cannot be written as a convex combination of distinct elements of C. More precisely, if $z = (1-t)x + ty$, with $x, y \in C$ and $t \in [0,1]$, then $x = y = z$.

Since we will need to characterize the extreme points of $\mathcal{P}_1$, then we define the triangular profiles as follows.

Definition 2.5.7 (Triangular profiles)**.** Let $x_0 \in (0,1)$. We define the *triangular profile with vertex in x_0* as

$$T_{x_0}(x) := \begin{cases} \dfrac{2x}{x_0} & x \in [0, x_0), \\[2ex] \dfrac{2(1-x)}{1-x_0} & x \in [x_0, 1]. \end{cases}$$

If $x_0 = 0$ or $x_0 = 1$ we define

$$T_0(x) = 2(1-x), \qquad\qquad T_1(x) = 2x.$$

Proposition 2.5.8. *Let $h \in \mathcal{P}_1$. Then h is an extreme point for $\mathcal{P}_1$ if and only if there exists $x_0 \in [0,1]$ such that $h = T_{x_0}$.*

Proof. Let us start by proving that for every $x_0 \in [0,1]$ the triangle T_{x_0} is an extreme point of $\mathcal{P}_1$. Let $h \in \mathcal{P}_1$ and let x_M be a maximum point for h, then the concavity of h ensures

$$h \geq \frac{h(x_M)}{2} T_{x_M}.$$

Recalling that $\int_0^1 h\,dx = 1$, we get that

$$h(x_M) = \max_{[0,1]} h \leq 2, \tag{2.5.2}$$

and the equality holds if and only if $h = T_{x_M}$.

Let now $x_0 \in [0,1]$, and assume that

$$T_{x_0}(x) = (1-t)h_0(x) + t h_1(x) \qquad x \in [0,1],$$

with $h_0, h_1 \in \mathcal{P}_1$, and $t \in [0,1]$. Since

$$2 = T_{x_0}(x_0) \leq \max\{h_0(x_0), h_1(x_0)\},$$

and

$$2 = (1-t)h_0(x_0) + t h_1(x_0),$$

we get equality in (2.5.2) for both h_0 and h_1. Therefore, $h_0 = h_1 = T_{x_0}$, and we have proved that T_{x_0} is an extreme point of $\mathcal{P}_1$.

We now prove that the triangles are the only extreme points of $\mathcal{P}_1$. Let $h \in \mathcal{P}_1$ be such that $h \neq T_{x_0}$ for every $x_0 \in [0,1]$.

We begin by assuming that $h(1) > 0$. Notice that, in this setting, there exists $s \in (0,1)$, such that the function

$$h_s = \frac{h - sT_0}{1-s} \in \mathcal{P}_1.$$

In particular, we get

$$h = (1-s)h_s + sT_0,$$

that is, h is not an extreme point of $\mathcal{P}$. An analogous computation can be done when $h(0) > 0$.

Assume now that $h(0) = h(1) = 0$ and let ν be the positive Radon measure representing $-h''$. Since $h \neq T_{x_0}$ for every x_0, then there exists $y_0 \in (0,1)$ such that $\nu([0,y_0]) > 0$ and $\nu((y_0,1]) > 0$. Let

$$\nu_1 = \nu|_{[0,x_0]}, \qquad\qquad \nu_2 = \nu|_{(x_0,1]},$$

and let h_1, h_2 be the solutions to

$$\begin{cases} -h_1'' = \nu_1, \\ h_1(0) = h_1(1) = 0, \end{cases} \qquad\qquad \begin{cases} -h_2'' = \nu_2, \\ h_2(0) = h_2(1) = 0. \end{cases}$$

We have that $h_1, h_2 \in \mathcal{P}$ and $h = h_1 + h_2$, so that, letting

$$\tilde{h}_i = \frac{h_i}{\int_0^1 h_i\,dx}, \qquad i = 1,2,$$

we get $\tilde{h}_1, \tilde{h}_2 \in \mathcal{P}_1$, and

$$h = t\tilde{h}_1 + (1-t)\tilde{h}_2,$$

with $t \in (0,1)$. Hence, h is not an extreme point of $\mathcal{P}_1$. □

Finally, we conclude the section recalling the definition of quasiconcave function.

Definition 2.5.9. A function $f : \mathbb{R} \to \mathbb{R}$ is *quasiconcave* if for all $x, y \in \mathbb{R}$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \geq \min\{f(x), f(y)\}.$$

A function defined on an interval is quasiconcave if and only if it is monotone or 'increasing then decreasing', i.e. if there are two complementary intervals (one of which may be empty) such that it is increasing on the former and decreasing on the latter.

2.6 Shape functionals

2.6.1 Shape derivative

Definition 2.6.1 (Shape derivative)**.** Let $s \in (0,1)$, let $\mathcal{O}$ be a family of $C^{1,s}$ subsets of $\mathbb{R}^n$, and let

$$\mathcal{F} : \mathcal{O} \to X$$

with X a Banach space. Let $p > n$ big enough to have $W^{1,p}(B_1;\mathbb{R}^n) \hookrightarrow C^{1,s}(B_1;\mathbb{R}^n)$. For every set $E \in \mathcal{O}$, we define the *shape derivative* as the Fréchet derivative of the functional

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto \mathcal{F}((\operatorname{Id}+\Phi)(E)) \in X.$$

In particular, the first order *shape derivative* of $\mathcal{F}$ at E along Φ will be denoted by

$$\mathcal{F}'(E)[\Phi] := \lim_{t\to 0} \frac{\mathcal{F}((\operatorname{Id}+t\Phi)(E)) - \mathcal{F}(E)}{t},$$

whenever the limit exists.

The second order *shape derivative* of $\mathcal{F}$ at E along Φ will be denoted by

$$\mathcal{F}''(E)[\Phi,\Psi] := \lim_{t\to 0} \frac{\mathcal{F}'((\operatorname{Id}+t\Psi)(E))[\Phi] - \mathcal{F}'(E)[\Phi]}{t},$$

whenever the limit exists.

In the following, given a function $\Phi \in W^{2,p}(\mathbb{R}^n;\mathbb{R}^n)$ with p big enough and a $C^{1,s}$ set E, we denote by $E^{\Phi} = (\operatorname{Id}+\Phi)(E)$.

Remark 2.6.2. In our notations, when $X = \mathbb{R}$ and $F(t) = \mathcal{F}(E^{t\Phi})$, then

$$F'(t) = \mathcal{F}'(E^{t\Phi})[\tilde{\Phi}], \qquad\qquad F''(t) = \mathcal{F}''(E^{t\Phi})[\tilde{\Phi},\tilde{\Phi}] + \mathcal{F}'(E^{t\Phi})[-D\tilde{\Phi}\tilde{\Phi}],$$

with $\tilde{\Phi} = \Phi \circ (\operatorname{Id}+t\Phi)^{-1}$. This comes from the fact that when t_0 is fixed

$$(\operatorname{Id}+(t_0+s)\Phi)(E) = \Big(\operatorname{Id}+s\Big(\Phi\circ(\operatorname{Id}+t_0\Phi)^{-1}\Big)\Big)(E^{t_0\Phi}),$$

and

$$\partial_t\tilde{\Phi} = -D\tilde{\Phi}\tilde{\Phi}.$$

Remark 2.6.3 (Lagrange Multiplier)**.** Let $\mathcal{O}$ and p as in Definition 2.6.1, and let

$$\mathcal{F} : \mathcal{O} \to \mathbb{R}$$

be a shape functional. When we maximize (or minimize) $\mathcal{F}$ under a volume constraint, we may apply classical theory of Lagrange multipliers in Banach spaces to shape derivatives. In particular, let us denote by

$$\mathcal{L}_\tau(E) = \mathcal{F}(E) + \tau|E|.$$

If

$$\mathcal{F}(E_*) = \max_{|E|=m} \mathcal{F}(E),$$

then there exists $\tau \in \mathbb{R}$ such that $\mathcal{L}'_\tau(E_*)[\Phi] = 0$ for every $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$, and

$$\mathcal{L}''_\tau(E_*)[\Phi,\Phi] \le 0, \qquad\qquad \forall \Phi \in W^{2,p}(B_1;\mathbb{R}^n) \text{ such that } \int_{\partial E_*} \Phi\cdot\nu\, d\mathcal{H}^{n-1} = 0.$$

2.6.2 Robin eigenvalue

Definition 2.6.4 (Robin Eigenvalue)**.** Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with Lipschitz boundary, let $\beta > 0$. We define $\lambda^\beta(\Omega)$ as

$$\lambda^\beta(\Omega) = \inf \left\{ \frac{\displaystyle\int_\Omega |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 \, d\mathcal{L}^n} \;\middle|\; v \in W^{1,2}(\Omega) \setminus \{ 0 \} \right\}. \tag{2.6.1}$$

Standard tools of calculus of variation ensure that the infimum in (2.6.1) is achieved. We recall that the Robin eigenvalue does not satisfy an homogeneity with respect to rescaling. However, it is possible to prove the following scaling inequality.

Lemma 2.6.5. *For every $0 < r < R$, the following inequality holds*

$$\lambda^\beta(B_r) \leq \left(\frac{|B_R|}{|B_r|} \right)^{\frac{2}{n}} \lambda^\beta(B_R),$$

where B_R and B_r are balls with radii R and r respectively.

Proof. Let φ be a minimum of (2.6.1) for $\Omega = B_R$ and with $\|\varphi\|_{2,B_R} = 1$. We define

$$w(x) = \varphi\left(\frac{R}{r} x \right) \qquad \forall x \in B_r.$$

Therefore,

$$\begin{aligned}\lambda^\beta(B_r) &\leq \frac{\displaystyle\int_{B_r} |\nabla w(x)|^2 \, dx + \int_{\partial B_r} w(x)^2 \, d\mathcal{H}^{n-1}(x)}{\displaystyle\int_{B_r} w(x)^2 \, dx} \\ &= \frac{\left(\frac{r}{R} \right)^{n-2} \displaystyle\int_{B_R} |\nabla \varphi(y)|^2 \, dy + \left(\frac{r}{R} \right)^{n-1} \int_{\partial B_R} \varphi(y)^2 \, d\mathcal{H}^{n-1}(y)}{\left(\frac{r}{R} \right)^n}.\end{aligned}$$

Since $r/R < 1$, by minimality of φ, we get

$$\lambda^\beta(B_r) \leq \frac{\left(\frac{r}{R} \right)^{n-2}}{\left(\frac{r}{R} \right)^n} \lambda^\beta(B_R) = \left(\frac{|B_r|}{|B_R|} \right)^{-\frac{2}{n}} \lambda^\beta(B_R).$$

□

Let $\beta, m > 0$, and let us denote by

$$\Lambda_{\beta,m} = \inf \left\{ \frac{\displaystyle\int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^2 + \overline{v}^2 \right) \, d\mathcal{H}^{n-1}}{\displaystyle\int_{\mathbb{R}^n} v^2 \, d\mathcal{L}^n} \;\middle|\; \begin{array}{l} v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \setminus \{0\} \\ |\{ v > 0 \}| \leq m \end{array} \right\}.$$

Here we state a theorem, referring to [48, Theorem 5] for the proof.

Theorem 2.6.6. *Let $B \subseteq \mathbb{R}^n$ be a ball of volume m. Then*

$$\Lambda_{\beta,m} = \lambda^\beta(B).$$

2.6.3 Neumann and Steklov eigenvalues

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, connected and Lipschitz set. We define the Neumann and Steklov eigenvalues as follows: find positive constants μ, σ such that there exist non-zero solutions to the boundary value problems

$$\begin{cases} -\Delta u = \lambda^{\mathrm{N}} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda^{\mathrm{S}} v & \text{on } \partial\Omega. \end{cases}$$

The regularity assumption we made on Ω ensures that we can find two increasing and divergent sequences of eigenvalues

$$0 = \lambda_0^{\mathrm{N}}(\Omega) < \lambda_1^{\mathrm{N}}(\Omega) \le \lambda_2^{\mathrm{N}}(\Omega) \le \dots \le \lambda_k^{\mathrm{N}}(\Omega) \le \dots,$$
$$0 = \lambda_0^{\mathrm{S}}(\Omega) < \lambda_1^{\mathrm{S}}(\Omega) \le \lambda_2^{\mathrm{S}}(\Omega) \le \dots \le \lambda_k^{\mathrm{S}}(\Omega) \le \dots,$$

which are the spectrum of the Neumann laplacian and the spectrum of the Dirichlet-to-Neumann map respectively. We recall the variational characterization of the eigenvalues, for $k \ge 0$:

$$\lambda_k^{\mathrm{N}}(\Omega) = \inf_{E \in \mathcal{S}_{k+1}(\Omega)} \sup_{w \in E\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_\Omega w^2\,dx},$$

$$\lambda_k^{\mathrm{S}}(\Omega) = \inf_{E \in \mathcal{S}_{k+1}(\Omega)} \sup_{w \in E\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1}},$$

where $\mathcal{S}_{k+1}(\Omega)$ is the family of all linear subspaces of $H^1(\Omega)$ of dimension $k+1$. In particular, we are interested in the principal eigenvalues, i.e. $k = 1$, namely

$$\lambda_1^{\mathrm{N}}(\Omega) = \inf_{\substack{w \in H^1(\Omega)\setminus\{0\} \\ \int_\Omega w = 0}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_\Omega w^2\,dx}, \qquad \lambda_1^{\mathrm{S}}(\Omega) = \inf_{\substack{w \in H^1(\Omega)\setminus\{0\} \\ \int_{\partial\Omega} w = 0}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1}}.$$

If we take Ω_ε a family of sets thinning with profile (h^+, h^-), then we have that both the principal eigenvalues of the Neumann and Steklov problems converge to the principal eigenvalues of the Sturm-Liouville problems (4.2.3) and (4.2.4) respectively. More precisely, if we define

$$\lambda_1^{\mathrm{N}}(h) = \inf_{\substack{u \in H^1(0,1) \\ \int_0^1 uh\,dx = 0}} \frac{\displaystyle\int_0^1 (u')^2 h\,dx}{\displaystyle\int_0^1 u^2 h\,dx}, \qquad (2.6.2) \qquad \lambda_1^{\mathrm{S}}(h) = \inf_{\substack{v \in H^1(0,1) \\ \int_0^1 v\,dx = 0}} \frac{\displaystyle\int_0^1 (v')^2 h\,dx}{\displaystyle\int_0^1 v^2\,dx}, \qquad (2.6.3)$$

we have the following lemmas (see [112, Lemma 3.2, Lemma 3.5])

Lemma 2.6.7. *Let $\{\Omega_\varepsilon\}$ be family of thinning domains as in (2.5.1) and let $h = h_- + h_+$. Then*

$$\lambda_1^{\mathrm{N}}(\Omega_\varepsilon) = \lambda_1^{\mathrm{N}}(h) + o(1) \ \text{ as } \varepsilon \to 0,$$
$$\lambda_1^{\mathrm{S}}(\Omega_\varepsilon) = \frac{\lambda_1^{\mathrm{S}}(h)}{2}\varepsilon + o(\varepsilon) \ \text{ as } \varepsilon \to 0.$$

We also recall a continuity property of the eigenvalues $\lambda_1^{\mathrm{N}}(h)$ and $\lambda_1^{\mathrm{S}}(h)$ (proved in [112, Lemma 3.4, Lemma 3.6]).

Lemma 2.6.8. *Let $h_n, h \in \mathcal{P}$ be a sequence such that h_n converges in $L^2(0,1)$ to h. Then we have*

$$\lim_n \lambda_1^{\mathrm{N}}(h_n) = \lambda_1^{\mathrm{N}}(h),$$
$$\lim_n \lambda_1^{\mathrm{S}}(h_n) = \lambda_1^{\mathrm{S}}(h).$$

2.7 Γ-convergence

In this section, we recall some basic properties of the Γ-convergence and the asymptotic development by Γ-convergence. We refer for instance to [83] and [21] for the following notions.

Definition 2.7.1. Let X be a metric space and, for any $\varepsilon > 0$, let us consider the functionals $\mathcal{F}_\varepsilon, \mathcal{F}_0 : X \to \mathbb{R} \cup \{+\infty\}$. We will say that $\mathcal{F}_\varepsilon$ *Γ-converges, with respect to the strong topology in X, as $\varepsilon \to 0^+$* to $\mathcal{F}_0$ if for every $x \in X$ the following conditions hold:

- for every sequence $\{x_\varepsilon\} \subset X$ converging to x,
$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \geq \mathcal{F}_0(x);$$
- there exists a sequence $\{x_\varepsilon\} \subset X$ converging to x such that
$$\limsup_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) \leq \mathcal{F}_0(x).$$

In particular, from the definition, if $\mathcal{F}_\varepsilon$ Γ-converges to $\mathcal{F}_0$, for every $x \in X$ there exists a recovery sequence $\{x_\varepsilon\} \subset X$, converging to x, such that

$$\lim_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(x_\varepsilon) = \mathcal{F}_0(x).$$

We have the following

Proposition 2.7.2. *Let X be a metric space and, for any $\varepsilon > 0$, let us consider the functionals $\mathcal{F}_\varepsilon, \mathcal{F}_0 : X \to \mathbb{R} \cup \{+\infty\}$ such that $\mathcal{F}_\varepsilon$ Γ-converges, with respect to the strong topology in X as $\varepsilon \to 0^+$ to $\mathcal{F}_0$. Let $\{x_\varepsilon\}$ be a sequence in X such that*

$$\mathcal{F}_\varepsilon(x_\varepsilon) = \min_X \mathcal{F}_\varepsilon.$$

If there exists $\overline{x} \in X$ such that x_ε converges to $\overline{x}$, then

$$\mathcal{F}_0(\overline{x}) = \min_X \mathcal{F}_0 = \lim_{\varepsilon \to 0^+} \min_X \mathcal{F}_\varepsilon.$$

Let
$$m_0 = \inf_X \mathcal{F}_0,$$
and, for every $x \in X$, let
$$\delta\mathcal{F}_\varepsilon(x) = \frac{\mathcal{F}_\varepsilon(x) - m_0}{\varepsilon}$$

Definition 2.7.3. If there exists a functional $\mathcal{F}^{(1)} \colon X \to \mathbb{R} \cup \{ +\infty \}$ such that $\delta\mathcal{F}_\varepsilon$ Γ-converges, with respect to the strong topology in X, as $\varepsilon \to 0^+$ to $\mathcal{F}^{(1)}$, we say that $\mathcal{F}^{(1)}$ is the *first-order asymptotic development by* Γ*-convergence* for the functional $\mathcal{F}_\varepsilon$.

Let
$$\mathcal{U}_0 = \{ x \in X \mid \mathcal{F}_0(x) = m_0 \},$$
the interest in the previous definition is justified by the following

Remark 2.7.4. Let $\{ x_\varepsilon \}$ be a sequence in X such that
$$\mathcal{F}_\varepsilon(x_\varepsilon) = \min_X \mathcal{F}_\varepsilon,$$
and assume that there exists $\overline{x} \in X$ such that x_ε converges to $\overline{x}$; then, by Proposition 2.7.2, we have that $\overline{x} \in \mathcal{U}_0$ and
$$\mathcal{F}^{(1)}(\overline{x}) = \min_X \mathcal{F}^{(1)} = \lim_{\varepsilon \to 0^+} \frac{\mathcal{F}_\varepsilon(x_\varepsilon) - m_0}{\varepsilon}.$$
In particular, we have
$$\mathcal{F}_\varepsilon(x_\varepsilon) = m_0 + \varepsilon\mathcal{F}^{(1)}(\bar{x}) + o(\varepsilon).$$

Chapter 3

Existence

3.1 A free boundary problem in thermal insulation with a prescribed heat source

The results of this section are contained in the paper [6].

Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set with smooth boundary, let $f \in L^2(\Omega)$ be a positive function and let β, C_0 be positive constants. We consider the following energy functional

$$\mathcal{F}(A, v) = \int_A |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, d\mathcal{L}^n + C_0 |A \setminus \Omega|, \tag{3.1.1}$$

and the variational problem

$$\inf \left\{ \mathcal{F}(A, v) \, \middle| \, \begin{array}{l} A \supseteq \Omega \text{ open, bounded and Lipschitz} \\ v \in W^{1,2}(A), \; v \geq 0 \text{ in } A \end{array} \right\}. \tag{3.1.2}$$

As anticipated in Section 1.2, this problem is related to a thermal insulation problem, let us recall it: for a given heat source f distributed in a conductor Ω, find the best possible configuration of insulating material surrounding Ω. We also recall that similar problems have been studied in [44] and [86] for a thin insulating layer, and in [65], [56] and [58] for a prescribed temperature in Ω.

For a fixed open set A with Lipschitz boundary, we have, via the direct methods of the calculus of variations, that there exists $u_A \in W^{1,2}(A)$ such that

$$\mathcal{F}(A, u_A) \leq \mathcal{F}(A, v),$$

for all $v \in W^{1,2}(A)$, with $v \geq 0$ in A. Furthermore u_A solves the following stationary problem, with Robin boundary condition on ∂A. Precisely

$$\begin{cases} -\Delta u_A = f & \text{in } \Omega, \\ \dfrac{\partial u_A^+}{\partial \nu} = \dfrac{\partial u_A^-}{\partial \nu} & \text{on } \partial\Omega, \\ \Delta u_A = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u_A}{\partial \nu} + \beta u_A = 0 & \text{on } \partial A, \end{cases}$$

where u_A^- and u_A^+ denote the traces of u_A on $\partial\Omega$ in Ω and in $A \setminus \Omega$ respectively. That is

$$\int_A \nabla u_A \cdot \nabla \varphi \, d\mathcal{L}^n + \beta \int_{\partial A} u_A \varphi \, d\mathcal{H}^{n-1} = \int_\Omega f\varphi \, d\mathcal{L}^n, \tag{3.1.3}$$

for all $\varphi \in W^{1,2}(A)$. The Robin boundary condition represents the case when the heat transfer with the environment is conveyed by convection.

If for any couple (A, v) with A an open bounded set with Lipschitz boundary containing Ω and $v \in W^{1,2}(A)$, $v \geq 0$ in A, we identify v with $v\chi_A$, where χ_A is the characteristic function of A, and the set A with the support of v, then the energy functional (3.1.1) becomes

$$\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) d\mathcal{H}^{n-1} - 2\int_\Omega fv \, d\mathcal{L}^n + C_0|\{ v > 0 \} \setminus \Omega|, \tag{3.1.4}$$

and the minimization problem (3.1.2) becomes

$$\inf \left\{ \mathcal{F}(v) \;\middle|\; v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \right\}, \tag{3.1.5}$$

where $\overline{v}$ and $\underline{v}$ are respectively the approximate upper and lower limits of v, J_v is the jump set and ∇v is the absolutely continuous part of the derivative of v. See Section 2.3.2 for the definitions.

We state the main results of this section in the two following theorems, referring to Definition 2.6.4 for the definition of Robin eigennvalue $\lambda^\beta(B)$.

Theorem 3.1.1. *Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $f \in L^2(\Omega)$, with $f > 0$ almost everywhere in Ω. Assume in addition that, if $n = 2$,*

$$\|f\|_{2,\Omega}^2 < C_0 \lambda^\beta(B)\mathcal{L}^2(\Omega), \tag{3.1.6}$$

where B is a ball having the same measure of Ω. Then problem (3.1.5) admits a solution. Moreover, if $p > n$ and $f \in L^p(\Omega)$, then there exists a positive constant $C = C(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (3.1.5) then

$$\|u\|_\infty \leq C.$$

Theorem 3.1.2. *Let $n \geq 2$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set with $C^{1,1}$ boundary, let $p > n$ and let $f \in L^p(\Omega)$, with $f > 0$ almost everywhere in Ω. Assume in addition that, if $n = 2$ condition (3.1.6) holds true. Then there exist positive constants $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$, $C = C(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (3.1.5) then*

$$u \geq \delta_0 \qquad \mathcal{L}^n\text{-a.e. in } \{ u > 0 \},$$

and the jump set J_u satisfies the density estimates

$$cr^{n-1} \leq \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq Cr^{n-1},$$

with $x \in \overline{J_u}$, and $0 < r < d(x, \partial\Omega)$. In particular, we have

$$\mathcal{H}^{n-1}(\overline{J_u} \setminus J_u) = 0.$$

Section 3.1.1 is devoted to the proof of Theorem 3.1.1, while Section 3.1.2 is devoted to the proof of Theorem 3.1.2.

We notice that the assumptions on the function f do not seem to be sharp. Indeed, it is well known that (see for instance [102, Theorem 8.15]), in the more regular case, the assumption $f \in L^p(\Omega)$ with $p > n/2$ ensures the boundedness of solutions to equation (3.1.3).

3.1.1 Existence of minimizers

In this section we prove Theorem 3.1.1: in Proposition 3.1.5 we prove the existence of a minimizer to problem (3.1.5); in Proposition 3.1.9 we prove the L^∞ estimate for a minimizer.

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, that $f \in L^2(\Omega)$ is a positive function and that β, C_0 are positive constants. We consider the energy functional $\mathcal{F}$ defined in (3.1.4).

Lemma 3.1.3. *Let $n \geq 2$ and assume that, if $n = 2$, condition (3.1.6) holds true. Then there exist two positive constants $c = c(\Omega, f, \beta, C_0)$ and $C = C(\Omega, f, \beta, C_0)$ such that if $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, with $\mathcal{F}(v) \leq 0$ and $\Omega \subseteq \{ v > 0 \}$, then*

$$|\{ v > 0 \}| \leq c, \tag{3.1.7}$$

$$\|v\|_2 \leq C. \tag{3.1.8}$$

Proof. Let B' be a ball with the same measure as $\{ v > 0 \}$. By Theorem 2.6.6

$$\begin{aligned} 0 \geq \mathcal{F}(v) \geq \lambda^\beta(B') \int_{\mathbb{R}^n} v^2 \, d\mathcal{L}^n - 2 \int_\Omega f v \, d\mathcal{L}^n \\ + C_0 |\{ v > 0 \} \setminus \Omega|. \end{aligned}$$

By Lemma 2.6.5 and Hölder inequality

$$\begin{aligned} 0 \geq &\lambda^\beta(B) \left(\frac{|\Omega|}{|\{ v > 0 \}|} \right)^{\frac{2}{n}} \|v\|_2^2 - 2\|f\|_{2,\Omega} \|v\|_2 \\ &+ C_0 |\{ v > 0 \} \setminus \Omega| \end{aligned} \tag{3.1.9}$$

where B is a ball with the same measure as Ω. Obviously (3.1.9) implies that

$$\|f\|_{2,\Omega}^2 - \lambda^\beta(B) \left(\frac{|\Omega|}{|\{ v > 0 \}|} \right)^{\frac{2}{n}} C_0 |\{ v > 0 \} \setminus \Omega| \geq 0.$$

Let $M = |\{ v > 0 \}|$, and notice that, since $\Omega \subseteq \{ v > 0 \}$,

$$|\{ v > 0 \} \setminus \Omega| = M - |\Omega|,$$

therefore

$$\|f\|_{2,\Omega}^2 \geq C_0 \lambda^\beta(B) |\Omega|^{\frac{2}{n}} \left(M^{1-\frac{2}{n}} - M^{-\frac{2}{n}} |\Omega| \right).$$

This implies (taking into account (3.1.6) if $n = 2$) that there exists $c = c(\Omega, f, \beta, C_0) > 0$ such that

$$|\{ v > 0 \}| < c.$$

Finally observe that by (3.1.9) it follows

$$\|v\|_2 \leq C(M), \tag{3.1.10}$$

where

$$C(M) = \frac{M^{\frac{2}{n}}\left(\|f\|_{2,\Omega} + \sqrt{\|f\|_{2,\Omega}^2 - C_0\lambda^\beta(B)\left(\dfrac{|\Omega|}{M}\right)^{\frac{2}{n}}(M-|\Omega|)}\right)}{\lambda^\beta(B)|\Omega|}$$
$$\leq \frac{2c^{\frac{2}{n}}\|f\|_{2,\Omega}}{\lambda^\beta(B)|\Omega|}$$

□

Remark 3.1.4. Let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, it is always possible to choose a function v_0 such that $v_0 = v$ in $\mathbb{R}^n \setminus \Omega$, $\mathcal{F}(v_0) \leq \mathcal{F}(v)$, and $\Omega \subseteq \{\, v_0 > 0 \,\}$. Indeed the function $v_0 \in W^{1,2}(\Omega)$, weak solution to the following boundary value problem

$$\begin{cases} -\Delta v_0 = f & \text{in } \Omega, \\ v_0 = \gamma^-_{\partial\Omega}(v) & \text{on } \partial\Omega, \end{cases} \tag{3.1.11}$$

satisfies

$$\int_\Omega \nabla v_0 \cdot \nabla\varphi \; d\mathcal{L}^n = \int_\Omega f\varphi \; d\mathcal{L}^n$$

for every $\varphi \in W^{1,2}_0(\Omega)$ and $v_0 = \gamma^-_{\partial\Omega}(v)$ on $\partial\Omega$ in the sense of the trace. Then, extending v_0 to be equal to v outside of Ω, we have that $\Omega \subset \{\, v_0 > 0 \,\}$ and $\mathcal{F}(v_0) \leq \mathcal{F}(v)$.

Proposition 3.1.5 (Existence)**.** *Let $n \geq 2$ and, if $n = 2$, assume that condition (3.1.6) holds true. Then there exists a solution to problem (3.1.5).*

Proof. Let $\{u_k\}$ be a minimizing sequence for problem (3.1.5). Without loss of generality we may always assume that, for all $k \in \mathbb{N}$, $\mathcal{F}(u_k) \leq \mathcal{F}(0) = 0$, and, by Remark 3.1.4, $\Omega \subseteq \{\, u_k > 0 \,\}$. Therefore we have

$$\begin{aligned} 0 \geq \mathcal{F}(u_k) &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \; d\mathcal{L}^n + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) d\mathcal{H}^{n-1} - 2\int_\Omega fv \; d\mathcal{L}^n \\ &\geq \int_{\mathbb{R}^n} |\nabla u_k|^2 \; d\mathcal{L}^n + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) d\mathcal{H}^{n-1} - 2\|f\|_{2,\Omega}\|u_k\|_{2,\Omega}\,, \end{aligned}$$

and by (3.1.8),

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \; d\mathcal{L}^n + \beta \int_{J_{u_k}} \left(\overline{u_k}^2 + \underline{u_k}^2\right) d\mathcal{H}^{n-1} \leq C\|f\|_{2,\Omega}\,.$$

Then we have that there exists a positive constant still denoted by C, independent on the sequence $\{u_k\}$, such that

$$\int_{\mathbb{R}^n} |\nabla u_k|^2 \; d\mathcal{L}^n + \int_{J_{u_k}} \left(\overline{u}_k^2 + \underline{u}_k^2\right) d\mathcal{H}^{n-1} + \int_{\mathbb{R}^n} u_k^2 \; d\mathcal{L}^n < C. \tag{3.1.12}$$

The compactness theorem in $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ (Theorem 2.3.17), ensures that there exists a subsequence $\{u_{k_j}\}$ and a function $u \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, such that u_{k_j} converges to u strongly in $L^2_{\mathrm{loc}}(\mathbb{R}^n)$, weakly in $W^{1,2}(\Omega)$, almost everywhere in $\mathbb{R}^n$ and

$$\int_{\mathbb{R}^n} |\nabla u|^2 \, d\mathcal{L}^n \leq \liminf_{j\to+\infty} \int_{\mathbb{R}^n} |\nabla u_{k_j}|^2 \, d\mathcal{L}^n,$$

$$\int_{J_u} \left(\overline{u}^2 + \underline{u}^2\right) \, d\mathcal{H}^{n-1} \leq \liminf_{j\to+\infty} \int_{J_{u_{k_j}}} \left(\overline{u}_{k_j}^2 + \underline{u}_{k_j}^2\right) \, d\mathcal{H}^{n-1},$$

$$|\{\, u > 0 \,\} \setminus \Omega| \leq \liminf_{j\to+\infty} |\{\, u_{k_j} > 0 \,\} \setminus \Omega|.$$

Finally we have

$$\mathcal{F}(u) \leq \liminf_{j\to+\infty} \mathcal{F}(u_{k_j}) = \inf \left\{ \mathcal{F}(v) \;\middle|\; v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega) \right\},$$

Therefore u is a minimizer to problem (3.1.5). □

Theorem 3.1.6 (Euler-Lagrange equation)**.** *Let u be a minimizer to problem (3.1.5), and let $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n)$ such that $J_v \subseteq J_u$. Assume that there exists $t > 0$ such that $\{\, v > 0 \,\} \subseteq \{\, u > t \,\}$ $\mathcal{L}^n$-a.e., and that*

$$\int_{J_u \setminus J_v} v^2 \, d\mathcal{H}^{n-1} < +\infty.$$

Then

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, d\mathcal{L}^n + \beta \int_{J_u} \left(\overline{u}\gamma^+(v) + \underline{u}\gamma^-(v)\right) \, d\mathcal{H}^{n-1} = \int_\Omega fv \, d\mathcal{L}^n, \tag{3.1.13}$$

where $\gamma^\pm = \gamma^\pm_{J_u}$.

Proof. Notice that since $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $J_v \subseteq J_u$ we have that $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$. Assume $v \in \mathrm{SBV}^{1/2}(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. If $s \in \mathbb{R}$, recalling that $\{\, v > 0 \,\} \subseteq \{\, u > t \,\}$ $\mathcal{L}^n$-a.e.,

$$u(x) + sv(x) = u(x) \geq 0 \qquad \mathcal{L}^n\text{-a.e. } \forall x \in \{\, u \leq t \,\},$$

while, for $|s|$ small enough,

$$u(x) + sv(x) \geq t - |s| \, \|v\|_\infty > 0 \qquad \forall x \in \{\, u > t \,\}.$$

Therefore we still have

$$u + sv \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n, \mathbb{R}^+).$$

Moreover by minimality of u we have, for every $|s| \leq s_0$

$$\begin{aligned} \mathcal{F}(u) &\leq \mathcal{F}(u + sv) \\ &= \int_{\mathbb{R}^n} |\nabla u + s\nabla v|^2 \, d\mathcal{L}^n + \\ &\quad + \int_{J_{u+sv}} \left[\left(\gamma^+(u) + s\gamma^+(v)\right)^2 + \left(\gamma^-(u) + s\gamma^-(v)\right)^2 \right] d\mathcal{H}^{n-1} + \\ &\quad - 2 \int_{\mathbb{R}^n} f(u + sv) \, d\mathcal{L}^n + C_0 |\{\, u > 0 \,\}|. \end{aligned}$$

Claim: the set

$$S := \left\{ s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(J_u \setminus J_{u+sv}) \neq 0 \right\}$$

is at most countable.

Let us define

$$D_0 = \left\{ x \in J_u \;\middle|\; \gamma^+(u)(x) \neq \gamma^-(u)(x) \right\},$$

$$D_s = \left\{ x \in J_u \;\middle|\; \gamma^+(u+sv)(x) \neq \gamma^-(u+sv)(x) \right\},$$

and notice that

$$\mathcal{H}^{n-1}(J_u \setminus D_0) = 0, \qquad \mathcal{H}^{n-1}(J_{u+sv} \setminus D_s) = 0.$$

Then we have to prove that

$$\left\{ s \in [-s_0, s_0] \;\middle|\; \mathcal{H}^{n-1}(D_0 \setminus D_s) \neq 0 \right\}$$

is at most countable. Observe that if $t \neq s$,

$$(D \setminus D_t) \cap (D \setminus D_s) = \emptyset.$$

Indeed if $x \in D \setminus D_s$

$$\gamma^+(u)(x) \neq \gamma^-(u)(x),$$

$$\gamma^+(u) + s\gamma^+(v)(x) = \gamma^-(u) + s\gamma^-(v)(x),$$

then

$$\gamma^+(v)(x) \neq \gamma^-(v)(x),$$

and so

$$s = \frac{\gamma^-(u)(x) - \gamma^+(u)(x)}{\gamma^+(v)(x) - \gamma^-(v)(x)}.$$

If $\mathcal{H}^0$ denotes the counting measure in $\mathbb{R}$, we can write

$$\int_{-s_0}^{s_0} \mathcal{H}^{n-1}(D_0 \setminus D_s)\, d\mathcal{H}^0 = \mathcal{H}^{n-1}\left(\bigcup_{(-s_0, s_0)} D_0 \setminus D_s \right) \leq \mathcal{H}^{n-1}(J_u) < +\infty,$$

then the claim is proved.

We are now able to differentiate in $s = 0$ the function $\mathcal{F}(u + sv)$, and observing that $0 \notin S$ is a minimum for $\mathcal{F}(u + sv)$, we get

$$\frac{1}{2}\mathcal{F}'(u)[v] = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \; d\mathcal{L}^n + \beta \int_{J_u} \left[\overline{u}\gamma^+(v) + \underline{u}\gamma^-(v) \right] \, d\mathcal{H}^{n-1} - \int_\Omega fv \; d\mathcal{L}^n = 0.$$

If $v \notin L^\infty(\mathbb{R}^n)$, we consider $v_h = \min\{ v, h \}$. Then

$$\mathcal{F}'(u)[v_h] = 0 \qquad \forall h > 0.$$

Observe that, since $\gamma^\pm(v_h) = \min\{ \gamma^\pm(v), h \}$,

$$\gamma^\pm(v_h) \to \gamma^\pm(v) \qquad \mathcal{H}^{n-1}\text{-a.e. in } J_u.$$

Therefore, passing to limit for $h \to +\infty$, by dominated convergence on the term

$$\int_{\mathbb{R}^n} \nabla u \cdot \nabla v_h \, d\mathcal{L}^n,$$

and by monotone convergence on the terms

$$\beta \int_{J_u} \left[\overline{u}\gamma^+(v_h) + \underline{u}\gamma^-(v_h)\right] \, d\mathcal{H}^{n-1}, \qquad \int_{\Omega} f v_h \, d\mathcal{L}^n,$$

we get

$$0 = \lim_h \mathcal{F}'(u)[v_h] = \mathcal{F}'(u)[v].$$

□

We now want to use the Euler-Lagrange equation (3.1.13) to prove that if f belongs to $L^p(\Omega)$ with $p > n$, and if u is a minimizer to problem (3.1.5) then u belongs to $L^\infty(\mathbb{R}^n)$. In order to prove this we need the following

Lemma 3.1.7. *Let m be a positive real number. There exists a positive constant $C = C(m, \beta, n)$ such that, for every function $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ with $|\{ v > 0 \}| \le m$,*

$$\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*} \, d\mathcal{L}^n\right)^{\frac{1}{1^*}} \le C\left[\int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, d\mathcal{H}^{n-1}\right],$$

where $1^* = \dfrac{n}{n-1}$ is the Sobolev conjugate of 1.

Proof. Classical Embedding of $BV(\mathbb{R}^n)$ in $L^{1^*}(\mathbb{R}^n)$ ensures that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*} \, d\mathcal{L}^n\right)^{\frac{1}{1^*}} \le& C(n)\left|Dv^2\right|(\mathbb{R}^n) \\ =& C(n)\left[\int_{\mathbb{R}^n} 2v|\nabla v| \, d\mathcal{L}^n + \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, d\mathcal{H}^{n-1}\right].\end{aligned}$$

For every $\varepsilon > 0$, using Young's and Hölder's inequalities, we have

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*} \, d\mathcal{L}^n\right)^{\frac{1}{1^*}} \le& \frac{C(n)}{\varepsilon}\int_{\mathbb{R}^n} v^2 \, d\mathcal{L}^n + \\ &+ C(n)\left[\varepsilon\int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, d\mathcal{H}^{n-1}\right] \\ \le& \frac{C(n)\, m^{\frac{1}{n}}}{\varepsilon}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*} \, d\mathcal{L}^n\right)^{\frac{1}{1^*}} + \\ &+ C(n)\left[\varepsilon\int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \int_{J_v} \left(\overline{v}^2 + \underline{v}^2\right) \, d\mathcal{H}^{n-1}\right].\end{aligned}$$

Setting $\varepsilon = 2C(n)m^{\frac{1}{n}}$, we can find two constants $C(m,n), C(m,\beta,n) > 0$ such that

$$\begin{aligned}\left(\int_{\mathbb{R}^n} v^{2\cdot 1^*}\, d\mathcal{L}^n\right)^{\frac{1}{1^*}} &\le C(m,n)\left[\int_{\mathbb{R}^n} |\nabla v|^2\, d\mathcal{L}^n + \int_{J_v}\left(\overline{v}^2 + \underline{v}^2\right)\, d\mathcal{H}^{n-1}\right] \\ &\le C(m,\beta,n)\left[\int_{\mathbb{R}^n} |\nabla v|^2\, d\mathcal{L}^n + \beta\int_{J_v}\left(\overline{v}^2 + \underline{v}^2\right)\, d\mathcal{H}^{n-1}\right].\end{aligned}$$

□

We refer to [143] for the following lemma.

Lemma 3.1.8. *Let $g : [0,+\infty) \to [0,+\infty)$ be a decreasing function and assume that there exist $C, \alpha > 0$ and $\theta > 1$ constants such that for every $h > k \ge 0$,*

$$g(h) \le C(h-k)^{-\alpha} g(k)^{\theta}.$$

Then there exists a constant $h_0 > 0$ such that

$$g(h) = 0 \qquad \forall h \ge h_0.$$

In particular we have

$$h_0 = C^{\frac{1}{\alpha}} g(0)^{\frac{\theta-1}{\alpha}} 2^{\theta(\theta-1)}.$$

Proposition 3.1.9 (L^∞ bound)**.** *Let $n \ge 2$ and assume that, if $n = 2$, condition (3.1.6) holds true. Let $f \in L^p(\Omega)$, with $p > n$. Then there exists a constant $C = C(\Omega, f, p, \beta, C_0) > 0$ such that if u is a minimizer to problem (3.1.5), then*

$$\|u\|_\infty \le C.$$

Proof. Let $\gamma^\pm = \gamma^\pm_{J_u}$. For every $\varphi, \psi \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n)$ satisfying $J_\varphi, J_\psi \subseteq J_u$, define

$$a(\varphi,\psi) = \int_{\mathbb{R}^n} \nabla\varphi \cdot \nabla\psi\, d\mathcal{L}^n + \beta\int_{J_u}\left[\gamma^+(\varphi)\gamma^+(\psi) + \gamma^-(\varphi)\gamma^-(\psi)\right]\, d\mathcal{H}^{n-1}.$$

For every v satisfying the assumptions of Theorem 3.1.6, it holds that

$$a(u,v) = \int_\Omega f v\, d\mathcal{L}^n.$$

In particular, let us fix $k \in \mathbb{R}^+$ and define

$$\varphi_k(x) = \begin{cases} u(x) - k & \text{if } u(x) \ge k, \\ 0 & \text{if } u(x) < k, \end{cases}$$

then

$$\gamma^+(\varphi_k)(x) = \begin{cases} \overline{u}(x) - k & \text{if } \overline{u}(x) \ge k, \\ 0 & \text{if } \overline{u}(x) < k, \end{cases}$$

and analogously for $\gamma^-(\varphi_k)$. Furthermore, let us define

$$\mu(k) = |\{\, u > k \,\}|.$$

We want to prove that $\mu(k) = 0$ for sufficiently large k. From Theorem 3.1.6, we have

$$a(u, \varphi_k) = \int_\Omega f\varphi_k \, d\mathcal{L}^n, \tag{3.1.14}$$

and we can observe that

$$\begin{aligned} a(u, \varphi_k) &= \int_{\{u>k\}} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{u>k\}} [\overline{u}(\overline{u} - k) + \underline{u}(\underline{u} - k)] \, d\mathcal{H}^{n-1} \\ &\geq \int_{\{u>k\}} |\nabla u|^2 \, d\mathcal{L}^n + \beta \int_{J_u \cap \{u>k\}} \left[(\overline{u} - k)^2 + (\underline{u} - k)^2\right] \, d\mathcal{H}^{n-1} \\ &= a(\varphi_k, \varphi_k). \end{aligned}$$

Moreover, by minimality, $\mathcal{F}(u) \leq \mathcal{F}(0) = 0$ and by Remark 3.1.4, $\Omega \subseteq \{u > 0\}$. Therefore, (3.1.7) holds true and we can apply Lemma 3.1.7, having that there exists $C = C(\Omega, f, \beta, C_0) > 0$ such that

$$\int_\Omega f\varphi_k \, d\mathcal{L}^n = a(u, \varphi_k) \geq a(\varphi_k, \varphi_k) \geq C\|\varphi_k\|^2_{2\cdot 1^*}. \tag{3.1.15}$$

On the other hand

$$\begin{aligned} \int_\Omega f\varphi_k \, d\mathcal{L}^n = \int_{\Omega\cap\{u>k\}} f(u-k) \, d\mathcal{L}^n &\leq \left(\int_{\Omega\cap\{u>k\}} f^{\frac{2n}{n+1}} \, d\mathcal{L}^n\right)^{\frac{n+1}{2n}} \|\varphi_k\|_{2\cdot 1^*} \\ &\leq \|f\|_{p,\Omega} \, \|\varphi_k\|_{2\cdot 1^*} \, \mu(k)^{\frac{n+1}{2n\sigma'}}, \end{aligned} \tag{3.1.16}$$

where

$$\sigma = \frac{p(n+1)}{2n} > 1,$$

since $p > n$. Joining (3.1.15) and (3.1.16), we have

$$\|\varphi_k\|_{2\cdot 1^*} \leq C\|f\|_{p,\Omega} \, \mu(k)^{\frac{n+1}{2n\sigma'}}. \tag{3.1.17}$$

Let $h > k$, then

$$\begin{aligned} (h-k)^{2\cdot 1^*} \mu(h) &= \int_{\{u>h\}} (h-k)^{2\cdot 1^*} \, d\mathcal{L}^n \\ &\leq \int_{\{u>h\}} (u-k)^{2\cdot 1^*} \, d\mathcal{L}^n \\ &\leq \int_{\{u>k\}} (u-k)^{2\cdot 1^*} \, d\mathcal{L}^n = \|\varphi_k\|^{2\cdot 1^*}_{2\cdot 1^*}. \end{aligned}$$

Using (3.1.17) and the previous inequality, we have

$$\mu(h) \leq C(h-k)^{-2\cdot 1^*} \mu(k)^{\frac{n+1}{(n-1)\sigma'}}.$$

Since $p > n$, then $\sigma' < (n+1)/(n-1)$. By Lemma 3.1.8, we have that $\mu(h) = 0$ for all $h \geq h_0$ with $h_0 = h_0(\Omega, f, \beta, C_0) > 0$, which implies

$$\|u\|_\infty \leq h_0.$$

□

Proof of Theorem 3.1.1. The result is obtained by joining Proposition 3.1.5 and Proposition 3.1.9. □

3.1.2 Density estimates for the jump set

In this section we prove Theorem 3.1.2: in Proposition 3.1.15 we prove the lower bound for minimizers to problem (3.1.5); in Proposition 3.1.17 and Proposition 3.1.18 we prove the density estimates for the jump set of a minimizer to problem (3.1.5).

In this section, we will assume that $\Omega \subseteq \mathbb{R}^n$ is an open bounded set with $C^{1,1}$ boundary, that $f \in L^p(\Omega)$, with $p > n$, is a positive function, and that β, C_0 are positive constants. We consider the energy functional $\mathcal{F}$ defined in (3.1.4).

In order to show that if u is a minimizer to problem (3.1.5) then u is bounded away from 0, we will first prove that there exists a positive constant δ such that $u > \delta$ almost everywhere in Ω, and then we will show that this implies the existence of a positive constant δ_0 such that $u > \delta_0$ almost everywhere in the set $\{\, u > 0 \,\}$. In the following we will denote by $U_t := \{\, u < t \,\} \cap \Omega$.

Remark 3.1.10. Let u be a minimizer to (3.1.5), by Remark 3.1.4, u is a solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u \geq 0 & \text{on } \partial\Omega \end{cases}$$

Let $u_0 \in W_0^{1,2}(\Omega)$ be the solution to the following boundary value problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.1.18}$$

Then, by maximum principle,

$$u \geq u_0 \quad \text{in } \Omega \subseteq \{\, u > 0 \,\} \quad \text{and} \quad \{\, u < t \,\} \cap \Omega = U_t \subseteq \{\, u_0 < t \,\} \cap \Omega.$$

Lemma 3.1.11. *There exist two positive constants $t_0 = t_0(\Omega, f)$ and $C = C(\Omega, f)$ such that if u is a minimizer to (3.1.5) then for every $t \in [0, t_0]$ it results*

$$|U_t| \leq C\, t. \tag{3.1.19}$$

Proof. Let u_0 be the solution to (3.1.18), fix $\varepsilon > 0$ such that the set

$$\Omega_\varepsilon = (\Omega)_\varepsilon = \{\, x \in \Omega \mid d(x, \partial\Omega) > \varepsilon \,\}$$

is not empty. Since u_0 is superharmonic and non-negative in Ω, by maximum principle we have that

$$\alpha = \inf_{\Omega_\varepsilon} u_0 > 0.$$

then u_0 solves

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ u_0 = 0 & \text{on } \partial\Omega, \\ u_0 \geq \alpha & \text{on } \partial\Omega_\varepsilon. \end{cases}$$

Therefore, if we consider the solution v to the following boundary value problem

$$\begin{cases} -\Delta v = 0 & \text{in } \Omega \setminus \bar{\Omega}_\varepsilon, \\ v = 0 & \text{on } \partial\Omega, \\ v = \alpha & \text{in } \bar{\Omega}_\varepsilon, \end{cases}$$

we have that $u \geq u_0 \geq v$ almost everywhere in Ω and

$$\{ u < t \} \cap \Omega = U_t \subseteq \{ u_0 < t \} \cap \Omega \subseteq \{ v < t \} \cap \Omega.$$

Hopf Lemma implies that
there exists a constant $\tau = \tau(\Omega) > 0$ such that

$$\frac{\partial v}{\partial \nu} < -\tau \qquad \text{on } \partial\Omega.$$

Let $x \in \bar{\Omega}$, and let x_0 be a projection of x onto the boundary $\partial\Omega$, then

$$|x - x_0| = d(x, \partial\Omega), \qquad \frac{x - x_0}{|x - x_0|} = -\nu_\Omega(x_0),$$

where ν_Ω denotes the exterior normal to $\partial\Omega$. We can write

$$\begin{aligned} v(x) &= \underbrace{v(x_0)}_{=0} + \nabla v(x_0) \cdot (x - x_0) + o(|x - x_0|) \\ &= -\frac{\partial v}{\partial \nu}(x_0)|x - x_0| + o(|x - x_0|) \\ &\geq \tau |x - x_0| + o(|x - x_0|) \\ &> \frac{\tau}{2}|x - x_0| = \frac{\tau}{2} d(x, \partial\Omega) \end{aligned} \tag{3.1.20}$$

for every x such that $d(x, \partial\Omega) < \sigma_0$ for a suitable $\sigma_0 = \sigma_0(\Omega, f) > 0$. Notice that if $\bar{x} \in \overline{\Omega}$ and $\lim_{x \to \bar{x}} v(x) = 0$ then necessarily $\bar{x} \in \partial\Omega$. Therefore, there exists a $t_0 = t_0(\Omega, f) > 0$ such that $v(x) < t_0$ implies $d(x, \partial\Omega) < \sigma_0$. Consequently, if $t < t_0$, we have that

$$\{ v < t \} \subseteq \{ d(x, \partial\Omega) < \sigma_0 \},$$

and by (3.1.20), we get

$$|U_t| \leq |\{ v < t \}| \leq \left| \left\{ x \in \Omega \;\middle|\; d(x, \partial\Omega) \leq \frac{2}{\tau} t \right\} \right|.$$

Since Ω is $C^{1,1}$, by Proposition 2.1.4, we conclude the proof.

□

Lemma 3.1.12. *Let $g : [0, t_1] \to [0, +\infty)$ be an increasing, absolutely continuous function such that*

$$g(t) \leq C t^\alpha (g'(t))^\sigma \qquad \forall t \in [0, t_1], \tag{3.1.21}$$

with $C > 0$ and $\alpha > \sigma > 1$. Then there exists $t_0 > 0$ such that

$$g(t) = 0 \qquad \forall t \leq t_0.$$

Precisely,

$$t_0 = \left(\frac{C(\alpha - \sigma)}{\sigma - 1} g(t_1)^{\frac{\sigma - 1}{\sigma}} + t_1^{\frac{\sigma - \alpha}{\sigma}} \right)^{\frac{\sigma}{\sigma - \alpha}}.$$

Proof. Assume by contradiction that $g(t) > 0$ for every $t > 0$. Inequality (3.1.21) implies

$$\frac{g'}{g^{\frac{1}{\sigma}}} \geq \frac{1}{C} t^{-\frac{\alpha}{\sigma}}.$$

Integrating between t_0 and t_1, we have

$$\frac{\sigma}{\sigma-1}\left(g(t_1)^{\frac{\sigma-1}{\sigma}} - g(t_0)^{\frac{\sigma-1}{\sigma}}\right) \geq \frac{1}{C}\frac{\sigma}{\sigma-\alpha}\left(t_1^{\frac{\sigma-\alpha}{\sigma}} - t_0^{\frac{\sigma-\alpha}{\sigma}}\right).$$

Since $\alpha > \sigma > 1$, we have

$$0 \leq g(t_0)^{\frac{\sigma-1}{\sigma}} \leq \frac{\sigma-1}{C(\alpha-\sigma)}\left(t_1^{\frac{\sigma-\alpha}{\sigma}} - t_0^{\frac{\sigma-\alpha}{\sigma}}\right) + g(t_1)^{\frac{\sigma-1}{\sigma}},$$

which is a contradiction if

$$t_0 \leq \left(\frac{C(\alpha-\sigma)}{\sigma-1} g(t_1)^{\frac{\sigma-1}{\sigma}} + t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

□

Remark 3.1.13. Let g be as in Lemma 3.1.12 and assume that $g(t_1) \leq K$, then $g(t) = 0$ for all $0 < t < \tilde{t}$ where

$$\tilde{t} = \left(\frac{C(\alpha-\sigma)}{\sigma-1} K^{\frac{\sigma-1}{\sigma}} + t_1^{\frac{\sigma-\alpha}{\sigma}}\right)^{\frac{\sigma}{\sigma-\alpha}}.$$

We now have the tools to prove the lower bound inside Ω.

Proposition 3.1.14. *There exists a positive constant $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that if u is a minimizer to problem (3.1.5) then*

$$u \geq \delta$$

almost everywhere in Ω.

Proof. Assume that Ω is connected and define the function

$$u_t(x) = \begin{cases} \max\{\, u, t\,\} & \text{in } \Omega, \\ u & \text{in } \mathbb{R}^n \setminus \Omega \end{cases}$$

Recalling that $U_t = \{\, u < t\,\} \cap \Omega$, we have

$$J_{u_t} \setminus \partial^* U_t = J_u \setminus \partial^* U_t,$$

and on this set $\underline{u_t} = \underline{u}$ and $\overline{u_t} = \overline{u}$.

Then we get by minimality of u, and using the fact that $J_{u_t} \cap \partial^* U_t \subseteq \partial\Omega$,

$$\begin{aligned}
0 &\geq \mathcal{F}(u) - \mathcal{F}(u_t) \\
&= \int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n - 2\int_{U_t} f(u-t)\, d\mathcal{L}^n + \beta\int_{\partial^* U_t \cap J_u} \left(\underline{u}^2 + \overline{u}^2\right) d\mathcal{H}^{n-1} + \\
&\quad - \beta \int_{J_{u_t}\cap\partial^* U_t \cap J_u} \left[t^2 + (\gamma^+_{\partial\Omega}(u))^2\right] d\mathcal{H}^{n-1} - \beta\int_{(J_{u_t}\cap \partial^* U_t)\setminus J_u} \left(t^2 + u^2\right) d\mathcal{H}^{n-1} \\
&\geq \int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n - 2\beta t^2 \mathcal{H}^{n-1}(\partial^* U_t \cap \partial\Omega)
\end{aligned}$$

where we ignored all the non-negative terms except the integral of $|\nabla u|^2$, and we used that $u \leq t$ in $\partial^* U_t \setminus J_u$. By Lemma 3.1.11, we can choose t small enough to have $|U_t| \leq |\Omega|/2$, then applying the isoperimetric inequality in Theorem 2.3.9 to the set $E = U_t$, we get

$$\int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n \leq 2\beta\, C\, t^2\, P(U_t;\Omega). \tag{3.1.22}$$

Let us define

$$p(t) = P(U_t;\Omega),$$

and consider the absolutely continuous function

$$g(t) = \int_{U_t} u\,|\nabla u| \, d\mathcal{L}^n = \int_0^t s\,p(s)\, ds.$$

By minimality of u we can apply the a priori estimates (3.1.12) to prove the equiboundedness of g, i.e. there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $g(t) \leq K$ for all $t > 0$. Using the Hölder inequality and the estimate (3.1.22) we have

$$g(t) \leq \left(\int_{U_t} u^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} \left(\int_{U_t} |\nabla u|^2 \, d\mathcal{L}^n \right)^{\frac{1}{2}} \leq \sqrt{2\beta C}\, t\, |U_t|^{\frac{1}{2}} (t^2 p(t))^{\frac{1}{2}}.$$

Fix $1 > \varepsilon > 0$. Then we can write $|U_t| = |U_t|^{\varepsilon} |U_t|^{1-\varepsilon}$, and by Lemma 3.1.11 there exists a constant $C = C(\Omega, f, \beta) > 0$ such that

$$g(t) \leq C\, t^{2+\frac{1-\varepsilon}{2}}\, |U_t|^{\frac{\varepsilon}{2}}\, p(t)^{\frac{1}{2}}.$$

By the relative isoperimetric inequality in Theorem 2.3.8, we can estimate

$$|U_t|^{\frac{\varepsilon}{2}} \leq C(\Omega, n) p(t)^{\frac{\varepsilon n}{2(n-1)}},$$

and, noticing that $p(t) = g'(t)/t$, we get

$$g(t) \leq C t^{\alpha} (g'(t))^{\sigma},$$

where

$$\alpha = 2 - \frac{\varepsilon}{2}\left(1 + \frac{n}{n-1}\right), \qquad \sigma = \frac{1}{2} + \frac{\varepsilon}{2}\frac{n}{n-1}.$$

In particular, if we choose

$$\varepsilon \in \left(\frac{n-1}{n}, \frac{3n-3}{3n-1} \right),$$

we have that $\alpha > \sigma > 1$, and then, using Lemma 3.1.12 and Remark 3.1.13, there exists a $\delta = \delta(\Omega, f, p, \beta, C_0) > 0$ such that $g(t) = 0$ for every $t < \delta$. Then $|\{\, u < t \,\} \cap \Omega| = 0$ for every $t < \delta$, hence

$$u \geq \delta$$

almost everywhere in Ω.

When Ω is not connected, then

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_N,$$

with Ω_i pairwise disjoint connected open sets. Using u_t as the function u truncated inside a single Ω_i, we find constants $\delta_i > 0$ such that

$$u(x) \geq \delta_i$$

almost everywhere in Ω_i. Therefore choosing $\delta = \min\{\, \delta_1, \ldots, \delta_N \,\}$ we have $u(x) > \delta$ almost everywhere in Ω. ∎

Finally, following the approach in [65], we have

Proposition 3.1.15 (Lower Bound)**.** *There exists a positive constant $\delta_0 = \delta_0(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (3.1.5) then*

$$u \geq \delta_0$$

almost everywhere in $\{ u > 0 \}$.

Proof. Let δ be the constant in Proposition 3.1.14. For every $0 < t \leq \delta$ let us define the absolutely continuous function

$$h(t) = \int_{\{ u \leq t \} \setminus J_u} u|\nabla u| \; d\mathcal{L}^n = \int_0^t sP(\{ u > s \} ; \mathbb{R}^n \setminus J_u) \; ds.$$

By minimality of u we can apply the a priori estimates (3.1.12) to prove the equiboundedness of h, i.e. there exists $K = K(\Omega, f, \beta, C_0) > 0$ such that $h(t) \leq K$ for all $t > 0$.

We will show that h satisfies a differential inequality. For any $0 < t < \delta$, let us consider $u^t = u\chi_{\{ u>t \}}$, where $\chi_{\{ u>t \}}$ is the characteristic funtion of the set $\{ u > t \}$, as a competitor for u. We observe that, by Proposition 3.1.14, $\Omega \subseteq \{ u > t \}$, so we have that

$$\begin{aligned}
0 \geq& \mathcal{F}(u) - \mathcal{F}(u^t) \\
=& \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \; d\mathcal{L}^n + \beta \int_{J_u \cap \{ u>t \}^{(0)}} \left(\underline{u}^2 + \overline{u}^2 \right) \; d\mathcal{H}^{n-1} + \\
&+ \beta \int_{J_u \cap \partial^* \{ u>t \}} \underline{u}^2 \; d\mathcal{H}^{n-1} - \beta \int_{\partial^* \{ u>t \} \setminus J_u} u^2 \; d\mathcal{H}^{n-1} + \\
&+ C_0 |\{ 0 < u \leq t \}|.
\end{aligned}$$

Rearranging the terms,

$$\begin{aligned}
&\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^2 \; d\mathcal{L}^n + \beta \int_{J_u \cap \{ u>t \}^{(0)}} \left(\underline{u}^2 + \overline{u}^2 \right) \; d\mathcal{H}^{n-1} + \\
&+ \beta \int_{J_u \cap \partial^* \{ u>t \}} \underline{u}^2 \; d\mathcal{H}^{n-1} + C_0 |\{ 0 < u \leq t \}| \\
&\leq \beta t^2 P(\{ u > t \} ; \mathbb{R}^n \setminus J_u) = \beta t h'(t).
\end{aligned} \tag{3.1.23}$$

On the other hand using Hölder's inequality, we have

$$h(t) \leq \left(\int_{\{ u \leq t \}} |\nabla u|^2 \; d\mathcal{L}^n \right)^{\frac{1}{2}} |\{ 0 < u \leq t \}|^{\frac{1}{2n}} \left(\int_{\{ u \leq t \}} u^{2 \cdot 1^*} \; d\mathcal{L}^n \right)^{\frac{1}{2 \cdot 1^*}}. \tag{3.1.24}$$

Classical Embedding of BV in L^{1^*} applied to $u^2 \chi_{\{ u \leq t \}}$, ensures

$$\left(\int_{\{ u \leq t \}} u^{2 \cdot 1^*} \; d\mathcal{L}^n \right)^{\frac{1}{1^*}} \leq C(n) \left| D(u^2 \chi_{\{ u \leq t \}}) \right| (\mathbb{R}^n),$$

and, using (3.1.23),

$$\begin{aligned}\Big|D(u^2\chi_{\{u\le t\}})\Big|(\mathbb{R}^n) =& 2\int_{\{u\le t\}} u|\nabla u|\; d\mathcal{L}^n + \int_{J_u\cap\{u>t\}^{(0)}}\Big(\underline{u}^2+\overline{u}^2\Big)\; d\mathcal{H}^{n-1}+\\ &+\int_{J_u\cap\partial^*\{u>t\}} \underline{u}^2\; d\mathcal{H}^{n-1} + \int_{\partial^*\{u>t\}\setminus J_u} u^2\; d\mathcal{H}^{n-1}\\ \le& 2t\bigg(|\{0<u\le t\}|\int_{\{u\le t\}\setminus J_u}|\nabla u|^2\; d\mathcal{L}^n\bigg)^{\frac{1}{2}} + 3th'(t)\\ \le&\Big(2\frac{\delta\beta}{\sqrt{C_0}}+3\Big)th'(t).\end{aligned} \tag{3.1.25}$$

Therefore, joining (3.1.24), (3.1.23), and (3.1.25), we have

$$h(t)\le C_3(th'(t))^{1+\frac{1}{2n}}, \tag{3.1.26}$$

where

$$C_3=\beta^{\frac12}\Big(\frac{\beta}{C_0}\Big)^{\frac{1}{2n}}C(n)^{\frac12}\Big(2\frac{\delta\beta}{\sqrt{C_0}}+3\Big)^{\frac12}.$$

By (3.1.26) we now want to show that there exists $\delta_0=\delta_0(\Omega,f,p,\beta,C_0)>0$ such that $h(t)=0$ for every $0\le t<\delta_0$. Indeed assume by contradiction that $h(t)>0$ for every $0<t\le\delta$. We have

$$\frac{h'(t)}{h(t)^{\frac{2n}{2n+1}}}\ge\frac{C_3^{-\frac{2n}{2n+1}}}{t}.$$

Integrating from $t_0>0$ to δ, we get

$$\Big(h(\delta)^{\frac{1}{2n+1}}-h(t_0)^{\frac{1}{2n+1}}\Big)\ge C_4\log\Big(\frac{\delta}{t_0}\Big),$$

where

$$C_4=\frac{C_3^{-\frac{2n}{2n+1}}}{2n+1}.$$

Then

$$h(t_0)^{\frac{1}{2n+1}}\le h(\delta)^{\frac{1}{2n+1}}+C_4\log\Big(\frac{t_0}{\delta}\Big).$$

Finally, for any

$$0<t_0\le\tilde{\delta}=\delta\exp\Big(-h(\delta)^{\frac{1}{2n+1}}/C_4\Big),$$

we have $h(t_0)<0$, which is a contradiction. Then, setting $\delta_0=\delta\exp\Big(-K^{\frac{1}{2n+1}}/C_4\Big)\le\tilde{\delta}$, we conclude that $h(t)=0$ for any $0<t<\delta_0$, from which we have

$$u\ge\delta_0$$

almost everywhere in $\{u>0\}$. □

Remark 3.1.16. From Proposition 3.1.15, if u is a minimizer to problem (3.1.5), we have that

$$\partial^* \{ u > 0 \} \subseteq J_u \subseteq K_u. \tag{3.1.27}$$

Indeed, on $\partial^* \{ u > 0 \}$ we have that, by definition, $\underline{u} = 0$ and that, since $u \geq \delta_0$ $\mathcal{L}^n$-a.e. in $\{ u > 0 \}$, $\overline{u} \geq \delta_0$.

In the following we denote by $\{u > 0\}^{(1)}$ set of points with Lebesgue density equal to 1.

Proposition 3.1.17 (Density Estimates)**.** *There exist positive constants* $C = C(\Omega, f, p, \beta, C_0)$, $c = c(\Omega, f, p, \beta, C_0)$ *and* $\delta_1 = \delta_1(\Omega, f, p, \beta, C_0)$ *such that if* u *is a minimizer to problem* (3.1.5) *then for every* $B_r(x)$ *such that* $B_r(x) \cap \Omega = \emptyset$*, we have:*

(a) For every $x \in \mathbb{R}^n \setminus \Omega$*,*
$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C r^{n-1}; \tag{3.1.28}$$

(b) For every $x \in K_u$*,*
$$|B_r(x) \cap \{ u > 0 \}| \geq c r^n; \tag{3.1.29}$$

(c) The function u *has bounded support, namely*
$$\{ u > 0 \} \subseteq B_{1/\delta_1}.$$

Proof. This theorem is a consequence of Proposition 3.1.15, since we immediately have

$$\int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2 \right) \, d\mathcal{H}^{n-1} \geq \delta_0^2 \mathcal{H}^{n-1}(J_u \cap B_r(x)),$$

and by minimality of u we have

$$\begin{aligned}
0 &\geq \mathcal{F}(u) - \mathcal{F}(u \chi_{\mathbb{R}^n \setminus B_r(x)}) \\
&\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2 \right) \, d\mathcal{H}^{n-1} - \int_{\partial B_r(x) \setminus J_u} u^2 \, d\mathcal{H}^{n-1} \\
&\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2 \right) \, d\mathcal{H}^{n-1} - \int_{\partial B_r(x) \cap \{ u > 0 \}^{(1)}} \left(\underline{u}^2 + \overline{u}^2 \right) \, d\mathcal{H}^{n-1} \\
&\geq \int_{J_u \cap B_r(x)} \left(\underline{u}^2 + \overline{u}^2 \right) \, d\mathcal{H}^{n-1} - 2\|u\|_\infty^2 \mathcal{H}^{n-1}\left(\partial B_r(x) \cap \{ u > 0 \}^{(1)} \right),
\end{aligned}$$

where, in the second inequality, we have used (3.1.27). Thus we have

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \frac{2\|u\|_\infty^2}{\delta_0^2} \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)}) \leq C r^{n-1}, \tag{3.1.30}$$

where $C = C(\Omega, f, p, \beta, C_0) > 0$, which proves *(a)*.

We now prove *(b)* using the estimate (3.1.28) together with the relative isoperimetric inequality in order to get a differential inequality for the volume of $B_r(x) \cap \{ u > 0 \}^{(1)}$. Let $x \in K_u$, then for almost every r we have

$$\begin{aligned}
0 < V(r) := |B_r(x) \cap \{ u > 0 \}^{(1)}| &\leq k \, P(B_r(x) \cap \{ u > 0 \}^{(1)})^{\frac{n}{n-1}} \\
&\leq k \, \mathcal{H}^{n-1}(\partial B_r(x) \cap \{ u > 0 \}^{(1)})^{\frac{n}{n-1}},
\end{aligned}$$

where $k = k(\Omega, f, p, \beta, C_0) > 0$, and in the last inequality we used that (3.1.27) and (3.1.30) imply

$$P\Big(B_r(x) \cap \{ u > 0 \}^{(1)}\Big) \leq \mathcal{H}^{n-1}\Big(\partial B_r(x) \cap \{ u > 0 \}^{(1)}\Big) + \mathcal{H}^{n-1}(J_u \cap B_r(x))$$
$$\leq \left(1 + \frac{2\|u\|_\infty^2}{\delta_0^2}\right) P(B_r(x); \{ u > 0 \}^{(1)}).$$

Then we have

$$\frac{V'(r)}{V(r)^{\frac{n-1}{n}}} \geq \frac{1}{k},$$

which implies

$$|B_r(x) \cap \{ u > 0 \}^{(1)}| \geq c\, r^n.$$

Finally, to prove *(c)*, let $x \in K_u$ such that $d(x, \partial\Omega) \geq 1/\delta_1$. From (3.1.29), noticing that $\mathcal{F}(u) \leq \mathcal{F}(0) = 0$, we have that

$$c\, \delta_1^{-n} \leq |\{ u > 0 \} \setminus \Omega| \leq \frac{2\|u\|_\infty}{C_0} \int_\Omega f \, d\mathcal{L}^n,$$

which is a contradiction if δ_1 is sufficiently small. Then the thesis is given by (3.1.27). □

Finally, we have

Proposition 3.1.18 (Lower Density Estimate)**.** *There exists a positive constant $c = c(\Omega, f, p, \beta, C_0)$ such that if u is a minimizer to problem (3.1.5) then*

1. *For any $x \in K_u$ and $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$,*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \geq c r^{n-1};$$

2. *J_u is essentially closed, namely*

$$\mathcal{H}^{n-1}(K_u \setminus J_u) = 0;$$

The proof of Proposition 3.1.18 relies on classical techniques used in [103] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford-Shah functional. We refer to [65, Theorem 5.1] and [65, Corollary 5.4] for the details of the proof.

Proof of Theorem 3.1.2. The result is obtained by joining Proposition 3.1.15, Proposition 3.1.17, and Proposition 3.1.18. □

Remark 3.1.19. Given the summability assumption on the function f and the lower bound given in Proposition 3.1.15, we have that minimizers to (3.1.2) are almost-quasi-minimizers of the functional $\mathcal{G}$, defined on $\mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$ as

$$\mathcal{G}(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v),$$

that is, there exists $C(\Omega, f, p, \beta, C_0) > 0$, $\Lambda(\Omega, f, p, \beta, C_0) \geq \lambda$ and $\alpha(n, p) > n - 1$ such that, if $B_r(x)$ is a ball of radius $r \leq 1$, and $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, then

$$\mathcal{G}_\lambda(u; B_r(x)) \leq \mathcal{G}_\Lambda(v; B_r(x)) + C r^\alpha,$$

where

$$\mathcal{G}_\lambda(v; B_r(x)) := \int_{B_r(x)} |\nabla v|^2 \, d\mathcal{L}^n + \lambda \mathcal{H}^{n-1}(J_v \cap B_r(x)).$$

Indeed, let u be a minimizer to (3.1.2), let $B_r(x)$ be a ball of radius $r \leq 1$, and let $v \in \mathrm{SBV}^{\frac{1}{2}}(\mathbb{R}^n) \cap W^{1,2}(\Omega)$, with $\{ u \neq v \} \subset B_r(x)$, and let

$$w = \min \{ \max \{ v, 0 \}, \|u\|_\infty \}.$$

By minimality of u we have that

$$\mathcal{G}_\lambda(u; B_r(x)) \leq \mathcal{G}_\Lambda(v; B_r(x)) + Cr^n,$$

where $\lambda = \beta \delta_0^2$ and $\Lambda = 2\beta \|u\|_\infty^2$. Moreover,

$$\int_{\Omega \cap B_r(x)} fu \, d\mathcal{L}^n \leq \|f\|_{p,\Omega} \|u\|_\infty |B_r|^{1/p'} = C(\Omega, f, p, \beta, C_0) r^\alpha,$$

where

$$n > \alpha = \frac{n}{p'} > n - 1.$$

Finally, we have

$$\mathcal{G}_\lambda(u; B_r(x)) \leq \mathcal{G}_\Lambda(v; B_r(x)) + Cr^\alpha.$$

Such a minimality property can be used to prove that the lower density estimate in Proposition 3.1.18 still holds even when $B_r(x) \cap \Omega$ is non-empty. Indeed, the above density estimate is a consequence of the following decay lemma

Lemma 3.1.20 (Decay lemma). *Let $1 > \gamma > n - \alpha$. There exists $\tau_0 = \tau_0(n, \Omega, \gamma, \lambda) > 0$ such that for every $\tau_0 > \tau > 0$ there exist $r_0 = r_0(\tau, \Omega)$, $\varepsilon_0 = \varepsilon_0(\tau, \Omega) > 0$ such that, if $x_0 \in \partial\Omega$, $r_0 > r > 0$, and u is a almost-quasi minimizer on $B_r = B_r(x_0)$ for the functional $\mathcal{G}$ such that*

$$\mathcal{H}^{n-1}(J_u \cap B_r) \leq \varepsilon_0 r^{n-1},$$

then we have that either

$$\mathcal{G}_\lambda(u; B_r) \leq r^{n-\gamma},$$

or

$$\mathcal{G}_\lambda(u; B_{\tau r}) \leq \tau^{n-\gamma} \mathcal{G}_\lambda(u; B_r).$$

Proof. The proof of the decay lemma is similar to the one in [65, Lemma 5.3], [51, Section 4], [103, Lemma 4.9]; the main difference is in the construction of the blow-up sequence of almost-quasi minimizers.

Let u_k be a sequence of almost-quasi minimizer on B_{r_k} contradicting the lemma, with $\lim_k r_k = 0$. To reach a contradiction one usually constructs a sequence of functions $\tilde{v}_k$ on the unit ball, related to the sequence u_k, that converges to an harmonic function v. In order to prove that v is harmonic we construct a sequence of admissible test functions ψ_k on B_{r_k} and use the minimality property of u_k. If $d(x_0, \Omega) > 0$, then the test function are only required to be in $\mathrm{SBV}(B_{r_k})$, while, if $x_0 \in \partial\Omega$ the additional constraint $\psi_k \in \mathrm{SBV}(B_{r_k}) \cap W^{1,2}(\Omega \cap B_{r_k})$ should be treated with more carefulness.

Without loss of generality let $x_0 = 0$ and let $E_k = r_k^{2-n}\mathcal{G}_\lambda(u_k; B_{r_k})$, and define

$$v_k(x) = \frac{1}{E_k^{1/2}} u_k(r_k x).$$

For any k, we extend $u_k \in W^{1,2}(\Omega \cap B_{r_k})$ to $Lu_k \in W^{1,2}(B_{r_k})$, which is a function such that $u_k - Lu_k \equiv 0$ in Ω. Let us define, with a slight abuse of notation,

$$Lv_k(x) = \frac{1}{E_k^{1/2}} Lu_k(r_k x), \qquad\qquad w_k = v_k - Lv_k,$$

so that, by construction, and by properties of the blow-up,

$$\liminf_k |\{w_k = 0\}| \geq \liminf_k r_k^{-n} |\Omega \cap B_{r_k}| > 0. \tag{3.1.31}$$

This is the key property: by Poincaré inequality in SBV, there exist $\tilde{w}_k$ truncated functions, such that

$$\lim_k |\{w_k \neq \tilde{w}_k\}| = 0, \tag{3.1.32}$$

and, up to subtracting medians, w_k converge in L^2 to some Sobolev function. By (3.1.31) and (3.1.32), and considering that $\tilde{w}_k$ is a truncation of w_k, then for big enough k, up to $\mathcal{L}^n$-negligible sets,

$$\{w_k = 0\} \subseteq \{w_k = \tilde{w}_k\}.$$

This means that if we define $\tilde{v}_k = \tilde{w}_k + Lv_k$, then the scaled back functions

$$\tilde{u}_k(x) := E_k^{1/2} \tilde{v}_k\left(\frac{x}{r_k}\right)$$

respect the property $\tilde{u}_k \equiv u_k$ in $\Omega \cap B_{r_k}$. Moreover, it is possible to choose an extension L (see Lemma 3.1.21) such that, combining the Poincaré inequality in SBV and the Poincaré inequality in $W^{1,2}$, then there exist constants c_k such that $\tilde{v}_k - c_k$ converge in L^2 to a function $v \in W^{1,2}(B_1)$. This ensures that, if we take $\rho < \rho'$ small enough, η cut-off functions between B_ρ and $B_{\rho'}$, and $\varphi \in W^{1,2}(B_1)$, then the test functions $\psi_k = E_k^{1/2}\varphi_k(x/r_k)$, with

$$\varphi_k = \Big(\eta(\varphi + c_k) + (1-\eta)\tilde{v}_k\Big)\chi_{B_{\rho'}} + v_k \chi_{B_1 \setminus B_{\rho'}},$$

are admissible test functions for any $\varphi \in W^{1,2}(B_1)$, leading to similar computations that can be found in the aforementioned papers. □

Lemma 3.1.21. *Let Ω be an open set with Lipschitz boundary, and let $x_0 \in \partial\Omega$. There exist positive constants $\rho_0 = \rho_0(\Omega, x_0)$, $C = C(\Omega, x_0)$, $\delta = \delta(\Omega, x_0) > 1$, and an extension operator*

$$L : W^{1,2}(\Omega) \to W^{1,2}(B_{\rho_0}(x_0))$$

such that, for any $u \in W^{1,2}(\Omega)$, and for any $r < \rho_0$, we have that $Lu \equiv u$ in $\Omega \cap B_{\rho_0}(x_0)$ and

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, d\mathcal{L}^n \leq C \int_{\Omega \cap B_{\delta r}(x_0)} |\nabla u|^2 \, d\mathcal{L}^n. \tag{3.1.33}$$

Proof. We can assume without loss of generality that $x_0 = 0$, and, if s is small enough, we have that, up to rotations,

$$\Omega \cap B_s = \{ (x', x_n) \in B_s \mid \gamma(x') < x_n \},$$

for a suitable Lipschitz function γ, with $\gamma(0) = 0$. We denote by Φ the diffeomorphism that flattens the boundary $\partial\Omega$, namely

$$\Phi(x', x_n) = (x', x_n - \gamma(x')), \qquad \Phi^{-1}(y', y_n) = (y', y_n + \gamma(y')).$$

Let $M = \|\nabla\gamma\|_\infty$, we claim that for any $r < (1+M)^{-2}s$ we have

$$\Phi(B_r) \subset B_{(1+M)r} \subset \Phi(B_{(1+M)^2 r}). \tag{3.1.34}$$

Indeed, let $x \in B_r$, then

$$|\Phi(x)|^2 \le |x|^2 + 2|x_n\gamma(x)| + |\gamma(x)|^2,$$

so that, we have

$$|\gamma(x)| \le |x| \|\nabla\gamma\|_\infty,$$

and then

$$|\Phi(x)| \le (1+M)r.$$

In a similar way, we have that for any $x \in B_{(1+M)r}$,

$$|\Phi^{-1}(x)| \le (1+M)^2 r,$$

thus the claim is proved.

Let us take a ball B_t such that $\Phi^{-1}(B_t) \subset B_s$, which we can find thanks to (3.1.34), and let us reflect the function $v(x) = u(\Phi(x))$ as follows: for any $x \in B_t$, we define

$$Lv(x) = \begin{cases} v(x) & \text{if } x_n < 0, \\ -3v(x', -x_n) + 4v(x', -\frac{x_n}{2}) & \text{if } x_n > 0, \end{cases}$$

which is still a Sobolev function in B_t. Moreover, we have

$$\int_{B_t} |\nabla Lv|^2 \, d\mathcal{L}^n \le C \int_{B_t \cap \{x_n < 0\}} |\nabla v|^2 \, d\mathcal{L}^n,$$

where C is independent of Ω. We put $Lu(x) = Lv(\Phi^{-1}(x))$, and by change of variables, we get

$$\int_{\Phi^{-1}(B_t)} |\nabla Lu|^2 \, d\mathcal{L}^n \le C(\Omega) \int_{\Omega \cap \Phi^{-1}(B_t)} |\nabla u|^2 \, d\mathcal{L}^n. \tag{3.1.35}$$

Finally, taking $\rho_0 = (1+M)^{-2}s$, and $t_0 = (1+M)^{-1}s$, we have $B_{\rho_0} \subset \Phi^{-1}(B_{t_0})$. Therefore, denoting by $\delta = (1+M)^2$, by (3.1.34) and (3.1.35), we get, for $r < \rho_0$,

$$\int_{B_r(x_0)} |\nabla Lu|^2 \, d\mathcal{L}^n \le \int_{\Phi^{-1}(B_{\sqrt{\delta} r})} |\nabla Lu|^2 \, d\mathcal{L}^n \le C \int_{\Omega \cap \Phi^{-1}(B_{\sqrt{\delta} r})} |\nabla u|^2 \, d\mathcal{L}^n \le C \int_{\Omega \cap B_{\delta r}} |\nabla u|^2 \, d\mathcal{L}^n.$$

□

Remark 3.1.22. Notice that if Ω is bounded, the constants in Lemma 3.1.21 can be chosen independent of the point x_0.

Remark 3.1.23. Let u be a minimizer to (3.1.5) and let $A = \{ \overline{u} > 0 \} \setminus K_u$, then the boundary of A is equal to K_u: in first place, assume by contradiction that there exists an $x \in (\partial A) \setminus K_u$, then u is superharmonic in a small ball centered in x with radius r. Therefore, being

$$\{ u > 0 \} \cap B_r(x) \neq \emptyset,$$

it is necessary that $u > 0$ in the entire ball, and then $x \notin \partial A$, which is a contradiction. In other words,

$$\partial A \subseteq K_u$$

By the same argument we also have that A is open, and moreover $J_u \subseteq \partial A$, then

$$K_u \subseteq \partial A.$$

In particular, the pair (A, u) is a minimizer for the functional

$$\mathcal{F}(E, v) = \int_E |\nabla v|^2 \, d\mathcal{L}^n - 2 \int_\Omega f v \, d\mathcal{L}^n + \int_{\partial E} \left(\underline{v}^2 + \overline{v}^2 \right) \, d\mathcal{H}^{n-1} + C_0 |E \setminus \Omega|$$

over all pairs (E, v) with E open set of finite perimeter containing Ω and $v \in W^{1,2}(E)$.

3.2 A free boundary problem for the p-Laplacian with non-linear boundary conditions

The results of this section are contained in the paper [3].

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with smooth boundary, and let A be a set containing Ω. Consider the functional

$$F(A, v) = \int_A |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{\partial A} v^2 \, d\mathcal{H}^{n-1} + C_0 |A|, \tag{3.2.1}$$

with $v \in H^1(A)$, $v = 1$ in Ω and $\beta, C_0 > 0$ fixed positive constants. As we've seen, the problem of minimizing this functional arises in the environment of thermal insulation: F represents the energy of a heat configuration v when the temperature is maintained constant inside the body Ω and there's a bulk layer $A \setminus \Omega$ of insulating material whose cost is represented by C_0 and the heat transfer with the external environment is conveyed by convection. For simplicity's sake in the following we will set $C_0 = 1$. The variational formulation in (3.2.1) leads to an Euler-Lagrange equation, which is the weak form of the following problem:

$$\begin{cases} \Delta u = 0 & \text{in } A \setminus \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial A, \\ u = 1 & \text{in } \Omega, \end{cases} \tag{3.2.2}$$

The problems we are interested in concern the existence of a solution and its regularity. In this sense, one could be interested in studying a more general setting in which it is possible to consider possibly irregular sets A. Specifically, we could generalize the problem into the context of SBV functions, aiming to minimize the functional

$$F(v) = \int_{\mathbb{R}^n} |\nabla v|^2 \, d\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^2 + \overline{v}^2 \right) d\mathcal{H}^{n-1} + |\{ v > 0 \} \setminus \Omega|$$

with $v \in \mathrm{SBV}(\mathbb{R}^n)$ and $v = 1$ in Ω. This problem has been studied in [65], where the authors have proved the existence of a solution u for the problem and the regularity of its jump set. Similar two-phase problems in the linear case can be found in [6], and [108]. With regards to the non-linear context, analogous versions of the problem have been addressed in [51], and in [62] with a boundedness constraint.

In this section, our main aim is to generalize the problem and techniques employed in [65] to a nonlinear formulation. In detail, for $p, q > 1$ fixed, we consider the functional

$$\mathcal{F}(v) = \int_{\mathbb{R}^n} |\nabla v|^p \, d\mathcal{L}^n + \beta \int_{J_v} \left(\underline{v}^q + \overline{v}^q \right) d\mathcal{H}^{n-1} + |\{ v > 0 \} \setminus \Omega|, \tag{3.2.3}$$

and in the following we are going to study the problem

$$\inf \left\{ \mathcal{F}(v) \;\middle|\; \begin{array}{l} v \in \mathrm{SBV}(\mathbb{R}^n) \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}.$$

Notice that if $v \in \mathrm{SBV}(\mathbb{R}^n)$ with $v = 1$ a.e. in Ω, letting $v_0 = \max \{ 0, \min \{ v, 1 \} \}$ we have that $v_0 \in \mathrm{SBV}(\mathbb{R}^n)$ with $v_0 = 1$ a.e. in Ω and $\mathcal{F}(v_0) \leq \mathcal{F}(v)$ so it suffices to consider the problem

$$\inf \left\{ \mathcal{F}(v) \;\middle|\; \begin{array}{c} v \in \mathrm{SBV}(\mathbb{R}^n), \\ v(x) \in [0,1] \; \mathcal{L}^n\text{-a.e.}, \\ v(x) = 1 \text{ in } \Omega \end{array} \right\}. \tag{3.2.4}$$

In a more regular setting, problem (3.2.4) can be seen as a PDE. Let us fix Ω, A sufficiently smooth open sets, $u \in W^{1,p}(A)$ with $u = 1$ on Ω, and let us define the functional

$$F(A, u) = \int_{\Omega} |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{\partial\Omega} |u|^q \, d\mathcal{H}^{n-1} + |A \setminus \Omega|. \tag{3.2.5}$$

minimizers u to (3.2.5) solve the following boundary value problem

$$\begin{cases} \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right) = 0 & \text{in } A \setminus \Omega, \\[2ex] |\nabla u|^{p-2} \dfrac{\partial u}{\partial \nu} + \beta \dfrac{q}{p} |u|^{q-2} u = 0 & \text{on } \partial A, \\[2ex] u = 1 & \text{in } \Omega. \end{cases} \tag{3.2.6}$$

We will prove the existence of a minimizer u of (3.2.4), under a prescribed condition on p and q. Finally, we will prove density estimates for the jump set J_u.

We resume in the following theorems the main results of this paragraph.

Theorem 3.2.1. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying one of the following conditions:*

- $1 < p < n$, *and* $1 < q < \dfrac{p(n-1)}{n-p} := p_*$;
- $n \leq p < \infty$, *and* $1 < q < \infty$.

Then there exists a solution u to problem (3.2.4) and there exists a constant $\delta_0 = \delta_0(\Omega, \beta, p, q) > 0$ such that

$$u > \delta_0 \tag{3.2.7}$$

$\mathcal{L}^n$-almost everywhere in $\{ u > 0 \}$, and there exists $\rho(\delta_0) > 0$ such that

$$\operatorname{supp} u \subseteq B_{\rho(\delta_0)}.$$

Theorem 3.2.2. *Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set, and let $p, q > 1$ be exponents satisfying the assumptions of Theorem 3.2.1. Then there exist positive constants $C(\Omega, \beta, p, q)$, $c(\Omega, \beta, p, q)$, $C_1(\Omega, \beta, p, q)$ such that if u is a minimizer to problem (3.2.4), then*

$$c\, r^{n-1} \leq \mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C\, r^{n-1},$$

and

$$|B_r(x) \cap \{ u > 0 \}| \geq C_1\, r^n,$$

for every $x \in \overline{J_u}$ with $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$.

In particular, this implies the essential closedness of the jump set J_u outside of Ω, namely

$$\mathcal{H}^{n-1}((\overline{J_u} \setminus J_u) \setminus \bar{\Omega}) = 0.$$

In Section 3.2.1 we prove that the a priori estimate (3.2.7) holds for inward minimizers (see Definition 3.2.4), such an estimate will be crucial in the proof of Theorem 3.2.1 in Section 3.2.2. Finally, in Section 3.2.3 we prove Theorem 3.2.2.

Remark 3.2.3. Notice that the condition on the exponents is undoubtedly verified when $p \geq q > 1$. Furthermore, if Ω is a set with Lipschitz boundary, the exponent p_* is the optimal exponent such that

$$W^{1,p}(\Omega) \subset\subset L^q(\partial\Omega) \qquad \forall q \in [1, p_*).$$

3.2.1 Lower Bound

In the following, we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set and that p and q are two positive real numbers such that

$$\frac{q'}{p'} > 1 - \frac{1}{n} \tag{3.2.8}$$

where p' and q' are the Hölder conjugates of p and q respectively.

Definition 3.2.4. Let $v \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $v = 1$ a.e. in Ω. We say that v is an *inward minimizer* if

$$\mathcal{F}(v) \leq \mathcal{F}(v\chi_A),$$

for every set of finite perimeter A containing Ω, where χ_A is the characteristic function of set A.

Let $A \subset \mathbb{R}^n$ be a set of finite perimeter such that $\Omega \subset A$, and let $v \in \mathrm{SBV}(\mathbb{R}^n)$. We will make use of the following expression

$$\begin{aligned}\mathcal{F}(v\chi_A) = &\int_A |\nabla v|^p \, d\mathcal{L}^n + \beta \int_{J_v \cap A^{(1)}} (\underline{v}^q + \overline{v}^{\,q}) \, d\mathcal{H}^{n-1} + \beta \int_{\partial^* A \setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &+ \beta \int_{J_v \cap \partial^* A} \gamma^-_{\partial A}(v)^q \, d\mathcal{H}^{n-1} + |(\{ v > 0 \} \cap A) \setminus \Omega|,\end{aligned} \tag{3.2.9}$$

Let B be a ball containing Ω, then $\chi_B \in \mathrm{SBV}(\mathbb{R}^n)$ and $\chi_B = 1$ in Ω, we will denote $\mathcal{F}(\chi_B)$ by $\tilde{\mathcal{F}}$.

Theorem 3.2.5. *There exists a positive constant $\delta = \delta(\Omega, \beta, p, q)$ such that if u is an inward minimizer with $\mathcal{F}(u) \leq 2\tilde{\mathcal{F}}$, then*

$$u > \delta$$

$\mathcal{L}^n$-almost everywhere in $\{ u > 0 \}$.

Proof. Let $0 < t < 1$ and

$$f(t) = \int_{\{ u \leq t \} \setminus J_u} u^{q-1} |\nabla u| \, d\mathcal{L}^n = \int_0^t s^{q-1} P(\{ u > s \} ; \mathbb{R}^n \setminus J_u) \, ds.$$

For every such t, we have

$$f(t) \leq \left(\int_{\{ u \leq t \}} u^{(q-1)p'} \, d\mathcal{L}^n \right)^{\frac{1}{p'}} \left(\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \leq \mathcal{F}(u) \leq 2\tilde{\mathcal{F}}. \tag{3.2.10}$$

Let $u_t = u\chi_{\{ u > t \}}$. Using (3.2.9) we have

$$\begin{aligned} 0 \leq &\mathcal{F}(u_t) - \mathcal{F}(u) \\ = &\beta \int_{\partial^* \{ u > t \} \setminus J_u} \overline{u}^q \, d\mathcal{H}^{n-1} - \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n - \beta \int_{J_u \cap \partial^* \{ u > t \}} \underline{u}^q \, d\mathcal{H}^{n-1} + \\ &- \beta \int_{J_u \cap \{ u > t \}^{(0)}} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} - |\{ 0 < u \leq t \}|, \end{aligned}$$

and rearranging the terms,

$$\begin{aligned} \int_{\{ u \leq t \} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{J_u \cap \partial^* \{ u > t \}} \underline{u}^q \, d\mathcal{H}^{n-1} + \beta \int_{J_u \cap \{ u > t \}^{(0)}} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} + \\ + |\{ 0 < u \leq t \}| \leq \beta t^q P(\{ u > t \} ; \mathbb{R}^n \setminus J_u) = \beta t f'(t). \end{aligned} \tag{3.2.11}$$

On the other hand,

$$\begin{aligned} f(t) = &\int_{\{ u \leq t \} \setminus J_u} u^{q-1} |\nabla u| \, d\mathcal{L}^n \\ \leq &\left(\int_{\{ u \leq t \}} u^{(q-1)p'} \, d\mathcal{L}^n \right)^{\frac{1}{p'}} \left(\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}} \\ \leq &|\{ 0 < u \leq t \}|^{\frac{1}{p'\gamma'}} \left(\int_{\{ u \leq t \}} u^{q1^*} \, d\mathcal{L}^n \right)^{\frac{1}{q'1^*}} \left(\int_{\{ u \leq t \} \setminus J_u} |\nabla u|^p \, d\mathcal{L}^n \right)^{\frac{1}{p}}, \end{aligned}$$

where we used

$$1^* = \frac{n}{n-1}, \qquad \text{and} \qquad \gamma = \frac{q1^*}{(q-1)p'},$$

and $\gamma > 1$ by (3.2.8). By classical BV embedding in L^{1^*} applied to the function $(u\chi_{\{u\leq t\}})^q$ and the estimate (3.2.11), we have

$$f(t) \leq C(n,\beta)\Big(tf'(t)\Big)^{1-\frac{n-1}{q'n}}\left(\int_{\mathbb{R}^n} d\Big|D(u\chi_{\{u\leq t\}})^q\Big|\right)^{\frac{1}{q'}}.$$

We can compute

$$\begin{aligned}\int_{\mathbb{R}^n} d\Big|D(u\chi_{\{u\leq t\}})^q\Big| \leq q|\{0<u\leq t\}|^{\frac{1}{p'}}\left(\int_{\{u\leq t\}\setminus J_u}|\nabla u|^p\,d\mathcal{L}^n\right)^{\frac{1}{p}}+\\ +\int_{J_u\cap\{u>t\}^{(0)}}(\overline{u}^q+\underline{u}^q)\,d\mathcal{H}^{n-1}+\int_{J_u\cap\partial^*\{u>t\}}\underline{u}^q\,d\mathcal{H}^{n-1}+\\ +t^qP(\{u>t\};\mathbb{R}^n\setminus J_u)\leq(2+q\beta)tf'(t).\end{aligned}$$

We therefore get

$$f(t) \leq C(n,\beta,q)\big(tf'(t)\big)^{1+\frac{1}{nq'}}.$$

Let $0<t_0<1$ such that $f(t_0)>0$, then for every $t_0<t<1$, we have $f(t)>0$ and

$$\frac{f'(t)}{f(t)^{\frac{nq}{q(n+1)-1}}} \geq \frac{C(n,\beta,q)}{t},$$

integrating from t_0 to 1, we have

$$f(1)^{\frac{q-1}{q(n+1)-1}} - f(t_0)^{\frac{q-1}{q(n+1)-1}} \geq C(n,\beta,q)\log\frac{1}{t_0},$$

so that, using (3.2.10),

$$f(t_0)^{\frac{q-1}{q(n+1)-1}} \leq (2\tilde{\mathcal{F}})^{\frac{q-1}{q(n+1)-1}} + C(n,\beta,q)\log t_0.$$

Let

$$\delta = \exp\left(-\frac{(2\tilde{\mathcal{F}})^{\frac{q-1}{q(n+1)-1}}}{C(n,\beta,q)}\right),$$

for every $t_0<\delta$ we would have $f(t_0)<0$, which is a contradiction. Therefore $f(t)=0$ for every $t<\delta$, from which $u>\delta$ $\mathcal{L}^n$-almost everywhere on $\{u>0\}$. □

Remark 3.2.6. From Theorem 3.2.5, if u is an inward minimizer with $\mathcal{F}(u)\leq 2\tilde{\mathcal{F}}$, we have that

$$\partial^*\{u>0\}\subseteq J_u\subseteq K_u.$$

Indeed, on $\partial^*\{u>0\}$ we have that, by definition, $\underline{u}=0$ and that, since $u\geq\delta$ $\mathcal{L}^n$-almost everywhere in $\{u>0\}$, $\overline{u}\geq\delta$.

Proposition 3.2.7. *There exists a positive constant $\delta_0 = \delta_0(\Omega, \beta, p, q) < \delta$ such that if u is an inward minimizer with $\mathcal{F}(u) \leq 2\tilde{\mathcal{F}}$, then u is supported on $B_{\rho(\delta_0)}$, where $\rho(\delta_0) = \delta_0^{1-q}$ and $B_{\rho(\delta_0)}$ is the ball centered at the origin with radius $\rho(\delta_0)$. Moreover there exist positive constants $C(\Omega, \beta, p, q), C_1(\Omega, \beta, p, q)$ such that, for any $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$ we have*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq C(\Omega, p, q) r^{n-1}, \tag{3.2.12}$$

and if $x \in K_u$, then

$$|B_r(x) \cap \{ u > 0 \}| \geq C_1(\Omega, p, q) r^n. \tag{3.2.13}$$

Proof. By Theorem 3.2.5, if u is an inward minimizer, we have

$$\int_{J_u \cap B_r(x)} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \geq \delta^q \mathcal{H}^{n-1}(J_u \cap B_r(x)),$$

on the other hand, using $u\chi_{\mathbb{R}^n \setminus B_r(x)}$ as a competitor for u, we have

$$\int_{J_u \cap B_r(x)} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \leq \int_{\partial B_r(x) \cap \{ u>0 \}^{(1)}} (\overline{u}^q + \underline{u}^q)\, d\mathcal{H}^{n-1} \leq C(n) r^{n-1}.$$

Let now $x \in K_u$ and consider $\mu(r) = \left| B_r(x) \cap \{ u > 0 \}^{(1)} \right|$. Using the isoperimetric inequality and inequality (3.2.12), we have that for almost every $r \in (0, d(x, \Omega))$

$$\begin{aligned} 0 < \mu(r) &\leq K(n)\, P\Big(B_r(x) \cap \{ u > 0 \}^{(1)}\Big)^{\frac{n}{n-1}} \\ &\leq K(\Omega, \beta, p, q)\, P\Big(B_r(x); \{ u > 0 \}^{(1)}\Big)^{\frac{n}{n-1}}. \end{aligned}$$

Notice that we used Remark 3.2.6 in the last inequality. We have

$$\mu(r) \leq K \mu'(r)^{\frac{n}{n-1}}.$$

Integrating the differential inequality, we obtain

$$|B_r(x) \cap \{ u > 0 \}| \geq C_1(\Omega, \beta, p, q) r^n.$$

Finally, let $\delta_0 > 0$ and $x \in K_u$ such that $d(x, \Omega) > \rho(\delta_0) = \delta_0^{1-q}$. By (3.2.13)

$$C_1(\Omega, \beta, p, q) \rho(\delta_0)^n \leq |\{ u > 0 \} \cap \Omega| \leq 2\tilde{\mathcal{F}},$$

which leads to a contradiction if δ_0 is too small, hence there exists a positive value $\delta_0(\Omega, \beta, p, q)$ such that $\{ u > 0 \} \subset B_{\rho(\delta_0)}$. □

3.2.2 Existence

In this section, we are going to prove the existence of a solution u to the problem (3.2.4). Let us denote

$$\mathcal{K}_a = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{c} u(x) = 1 \text{ in } \Omega \\ u(x) \in \{ 0 \} \cup [a, 1] \ \mathcal{L}^n\text{-a.e.} \\ \operatorname{supp} u \subseteq B_{\frac{1}{a^{q-1}}} \end{array} \right\}.$$

We also denote by $\mathcal{K}_0$ the set

$$\mathcal{K}_0 = \left\{ u \in \mathrm{SBV}(\mathbb{R}^n) \;\middle|\; \begin{array}{c} u(x) = 1 \text{ in } \Omega \\ u(x) \in [0,1] \ \mathcal{L}^n\text{-a.e.} \end{array} \right\}.$$

Notice that if $u \in \mathcal{K}_0$ is an inward minimizer, by Theorem 3.2.5 and Proposition 3.2.7, then $u \in \mathcal{K}_{\delta_0}$.

Proposition 3.2.8. *Let $u \in \mathcal{K}_0$. Then u is a minimizer for the functional (3.2.3) on $\mathcal{K}_0$ if and only if $u \in \mathcal{K}_{\delta_0}$ and*

$$\mathcal{F}(u) \le \mathcal{F}(v) \qquad \forall v \in \mathcal{K}_{\delta_0}.$$

Proof. As we observed before, if u is a minimizer over $\mathcal{K}_0$ then u is in $\mathcal{K}_{\delta_0}$, hence it is a minimizer over $\mathcal{K}_{\delta_0}$. Conversely, let us take $u \in \mathcal{K}_{\delta_0}$ a minimizer over $\mathcal{K}_{\delta_0}$, and let us consider in addition $v \in \mathcal{K}_0$. Without loss of generality assume $\mathcal{F}(v) \le 2\tilde{\mathcal{F}}$. We will prove that there exists a sequence w_k of inward minimizers such that

$$\mathcal{F}(w_k) \le \mathcal{F}(v) + \frac{C}{k^{q-1}}.$$

We first construct a family of functions $v_a \in \mathcal{K}_a$ such that

$$\mathcal{F}(v_a) \le \mathcal{F}(v) + r(a),$$

with $\lim_{a\to 0} r(a) = 0$. Let $0 < a < 1$, and let $v_a = v\chi_{\{v \ge a\} \cap B_{\rho(a)}}$, where $\rho(a) = a^{1-q}$, we have

$$\begin{aligned} \mathcal{F}(v_a) - \mathcal{F}(v) &\le \beta \int_{\partial^*(\{v\ge a\}\cap B_{\rho(a)})\setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &\le \beta a^q P(\{v \ge a\}) + \beta \int_{(\partial B_{\rho(a)} \cap \{v \ge a\})\setminus J_v} v^q \, d\mathcal{H}^{n-1} \\ &\le \beta a^q \left(P(\{v \ge a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v\ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \right). \end{aligned} \tag{3.2.14}$$

In order to estimate the right-hand side, fix $t \in (0,1)$, and observe that by the coarea formula

$$\int_0^t P(\{v \ge a\}) \, da \le |Dv|(\mathbb{R}^n), \tag{3.2.15}$$

while, with a change of variables,

$$\int_0^t \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v\ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \, da \le (q-1) \int_0^{+\infty} \int_{\partial B_r \setminus J_v} v \, d\mathcal{H}^{n-1} \, dr = (q-1)\|v\|_{L^1(\mathbb{R}^n)}.$$

$$\int_0^t \left(P(\{v \ge a\}) + \frac{1}{a^q} \int_{(\partial B_{\rho(a)} \cap \{v\ge a\})\setminus J_v} v \, d\mathcal{H}^{n-1} \right) da \le q\|v\|_{\mathrm{BV}}.$$

By mean value theorem, for every $k \in \mathbb{N}$ we can find $a_k \le 1/k$ such that

$$P(\{v \ge a_k\}) + \frac{1}{a_k^q} \int_{(\partial B_{\rho(a_k)} \cap \{v \ge a_k\})\setminus J_v} v \, d\mathcal{H}^{n-1} \le \frac{q\|v\|_{\mathrm{BV}}}{a_k},$$

and in (3.2.14) we get

$$\mathcal{F}(v_{a_k}) \leq \mathcal{F}(v) + q\beta a_k^{q-1}\|v\|_{\mathrm{BV}} \leq \mathcal{F}(v) + q\beta\frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

We now construct the aforementioned sequence of inward minimizers: let us consider the functional

$$\mathcal{G}_k(A) = \mathcal{F}(v_{a_k}\chi_A),$$

with A containing Ω and contained in $\{ v_{a_k} > 0 \}$. If we consider A_j a minimizing sequence for $\mathcal{G}_k$, then they are certainly equibounded. Moreover,

$$\begin{aligned}\mathcal{G}_k(A_j) &\geq |A_j \setminus \Omega| + \beta\int_{J_{\chi_{A_j}v_{a_k}}} \left(\underline{\chi_{A_j}v_{a_k}}^q + \overline{\chi_{A_j}v_{a_k}}^q\right) d\mathcal{H}^{n-1} \\ &\geq |A_j| + \beta a_k^q\mathcal{H}^{n-1}(J_{\chi_{A_j}v_{a_k}}) - |\Omega|.\end{aligned}$$

Notice in addition that since $v_{a_k} \geq a_k$ on its support, then the jump set $J_{\chi_{A_j}v_{a_k}}$ clearly contains $\partial^* A_j$. We now have that χ_{A_j} satisfies the conditions of Theorem 2.3.18, and eventually extracting a subsequence we can suppose that

$$A_j \xrightarrow{L^1} A^{(k)},$$

with a suitable $A^{(k)}$, and moreover, letting $w_k = \chi_{A^{(k)}}v_{a_k}$, we have

$$\mathcal{F}(w_k) \leq \inf_{\Omega\subseteq A\subseteq\{ v_{a_k}>0 \}} \mathcal{G}_k(A) \leq \mathcal{F}(v_{a_k}) \leq \mathcal{F}(v) + q\beta\frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

By construction w_k is an inward minimizer, therefore we have $w_k \in \mathcal{K}_{\delta_0}$, and consequently, we can compare it with u, obtaining

$$\mathcal{F}(u) \leq \mathcal{F}(w_k) \leq \mathcal{F}(v) + q\beta\frac{\|v\|_{\mathrm{BV}}}{k^{q-1}}.$$

Letting k go to infinity we get the thesis. □

Proposition 3.2.9. *There exists a minimizer for problem* (3.2.4).

Proof. By Proposition 3.2.8 and Theorem 3.2.5 it is enough to find a minimizer in $\mathcal{K}_{\delta_0}$. Let u_k be a minimizing sequence in $\mathcal{K}_{\delta_0}$, then, for k large enough, we have

$$\beta\delta_0^q\mathcal{H}^{n-1}(J_{u_k}) + \int_{\mathbb{R}^n} |\nabla u_k|^p \, d\mathcal{L}^n \leq \mathcal{F}(u_k) \leq 2\tilde{\mathcal{F}}.$$

From Theorem 2.3.18 we have that there exists $u \in \mathcal{K}_{\delta_0}$ such that, up to a subsequence, u_k converges to u in L^1_{loc} and

$$\mathcal{F}(u) \leq \liminf_k \mathcal{F}(u_k),$$

therefore u is a solution. □

Proof of Theorem 3.2.1. The result is obtained by joining Proposition 3.2.9 and Theorem 3.2.5. □

3.2.3 Density estimates

In this section, we prove the density estimates in Theorem 3.2.2 by adapting techniques used in [65] analogous to classical ones used in [103] to prove density estimates for the jump set of almost-quasi minimizers of the Mumford-Shah functional.

Definition 3.2.10. Let $u \in \mathrm{SBV}(\mathbb{R}^n)$ be a function such that $u = 1$ a.e. in Ω. We say that u is a *local minimizer* for $\mathcal{F}$ on a set of finite perimeter $E \subset \mathbb{R}^n \setminus \Omega$, if

$$\mathcal{F}(u) \leq \mathcal{F}(v),$$

for every $v \in \mathrm{SBV}(\mathbb{R}^n)$ such that $u - v$ has support in E.

Let E be a set of finite perimeter. We introduce the notation

$$\mathcal{F}(u;E) = \int_E |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{J_u \cap E} (\overline{u}^q + \underline{u}^q) \, d\mathcal{H}^{n-1} + |\{ u > 0 \} \cap E|.$$

To prove Theorem 3.2.2 we will use the following Poincaré-Wirtinger type inequality whose proof can be found in [103, Theorem 3.1 and Remark 3.3]. Let γ_n be the isoperimetric constant relative to the balls of $\mathbb{R}^n$, i.e.

$$\min \left\{ |E \cap B_r|^{\frac{n-1}{n}}, |E \setminus B_r|^{\frac{n-1}{n}} \right\} \leq \gamma_n P(E; B_r),$$

for every Borel set E, then we have the following.

Proposition 3.2.11. *Let $p \geq 1$ and let $u \in \mathrm{SBV}(B_r)$ such that*

$$\Big(2\gamma_n \mathcal{H}^{n-1}(J_u \cap B_r)\Big)^{\frac{n}{n-1}} < \frac{|B_r|}{2}. \tag{3.2.16}$$

Then there exist numbers $-\infty < \tau^- \leq m \leq \tau^+ < +\infty$ such that the function

$$\tilde{u} = \max \left\{ \min \left\{ u, \tau^+ \right\}, \tau^- \right\},$$

satisfies

$$\|\tilde{u} - m\|_{L^p} \leq C \|\nabla u\|_{L^p}$$

and

$$|\{ u \neq \tilde{u} \}| \leq C \Big(\mathcal{H}^{n-1}(J_u \cap B_r)\Big)^{\frac{n}{n-1}},$$

where the constants depend only on n, p, and r.

This allows us to prove the following decay lemma.

Lemma 3.2.12. *Let $u \in \mathcal{K}_s$ be a local minimizer on $B_r(x)$ in the sense of definition Definition 3.2.10. For sufficiently small values of τ, there exist values r_0, ε_0 depending only on n, τ, β, p, q and s such that, if $r < r_0$,*

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \leq \varepsilon_0 r^{n-1},$$

and

$$\mathcal{F}(u; B_r(x)) \geq r^{n-\frac{1}{2}},$$

then

$$\mathcal{F}(u; B_{\tau r}(x)) \leq \tau^{n-\frac{1}{2}} \mathcal{F}(u; B_r(x)).$$

Proof. Without loss of generality, assume $x = 0$. Assume by contradiction that the conclusion fails, then for every $\tau > 0$ there exists a sequence $u_k \in \mathcal{K}_s$ of local minimizers on B_{r_k}, with $\lim_k r_k = 0$, such that

$$\frac{\mathcal{H}^{n-1}(J_{u_k} \cap B_{r_k})}{r_k^{n-1}} = \varepsilon_k,$$

with $\lim_k \varepsilon_k = 0$,

$$\mathcal{F}(u_k; B_{r_k}) \geq r_k^{n-\frac{1}{2}}, \tag{3.2.17}$$

and yet

$$\mathcal{F}(u_k; B_{\tau r_k}) > \tau^{n-\frac{1}{2}} \mathcal{F}(u_k; B_{r_k}). \tag{3.2.18}$$

For every $t \in [0,1]$, we define the sequence of monotone functions

$$\alpha_k(t) = \frac{\mathcal{F}(u_k; B_{t r_k})}{\mathcal{F}(u_k, B_{r_k})} \leq 1.$$

By compactness of $\mathrm{BV}([0,1])$ in $L^1([0,1])$, we can assume that, up to a subsequence, α_k converges $\mathcal{L}^1$-almost everywhere to a monotone function α. Moreover, notice that, by (3.2.18), for every k

$$\alpha_k(\tau) > \tau^{n-\frac{1}{2}}. \tag{3.2.19}$$

Our final aim is to prove that there exists a p-harmonic function $v \in W^{1,p}(B_1)$ such that for every t

$$\lim_{k \to +\infty} \alpha_k(t) = \alpha(t) = \int_{B_t} |\nabla v|^p \, d\mathcal{L}^n.$$

Let us transform the sequence v_k in such a way to obtain a limit v in $W^{1,p}$. Let

$$E_k = r_k^{p-n} \mathcal{F}(u_k; B_{r_k}), \qquad v_k(x) = \frac{u_k(r_k x)}{E_k^{1/p}}.$$

Then $v_k \in \mathrm{SBV}(B_1)$, and

$$\int_{B_1} |\nabla v_k|^p \, d\mathcal{L}^n \leq 1, \qquad \mathcal{H}^{n-1}(J_{v_k} \cap B_1) = \varepsilon_k.$$

Thus, applying the Poincaré-Wirtinger type inequality in Proposition 3.2.11 to functions v_k we obtain truncated functions $\tilde{v}_k$ and values m_k, such that

$$\int_{B_1} |\tilde{v}_k - m_k|^p \, d\mathcal{L}^n \leq C$$

and

$$|\{ v_k \neq \tilde{v}_k \}| \leq C \Big(\mathcal{H}^{n-1}(J_{v_k} \cap B_1) \Big)^{\frac{n}{n-1}} \leq C \varepsilon_k^{\frac{n}{n-1}}. \tag{3.2.20}$$

We prove that there exists $v \in W^{1,p}(B_1)$ such that

$$\tilde{v}_k - m_k \xrightarrow{L^p(B_1)} v,$$

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \alpha(\rho), \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1, \tag{3.2.21}$$

and

$$\lim_k \frac{r_k^{p-1}}{E_k}\mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_\rho) = 0, \qquad \text{for } \mathcal{L}^1\text{-a.e. } \rho < 1. \tag{3.2.22}$$

Notice that

$$\int_{B_1} |\nabla(\tilde{v}_k - m_k)|^p \, d\mathcal{L}^n \leq \int_{B_1} |\nabla v_k|^p \, d\mathcal{L}^n \leq 1,$$

since $\tilde{v}_k$ is a truncation of v. From compactness theorems in SBV (see for instance [103, Theorem 3.5]), we have that $\tilde{v}_k - m_k$ converges in $L^p(B_1)$ and $\mathcal{L}^n$-almost everywhere to a function $v \in W^{1,p}(B_1)$, since $\mathcal{H}^{n-1}(J_{\tilde{v}_k})$ goes to 0 as $k \to +\infty$. Moreover, for every $\rho < 1$,

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n,$$

and

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \leq \liminf_k \int_{B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n \leq \liminf_k \alpha_k(\rho) = \alpha(\rho),$$

since by definition

$$\int_{B_\rho} |\nabla v_k|^p \, d\mathcal{L}^n = \frac{r_k^{p-n}}{E_k} \int_{B_{\rho r_k}} |\nabla u_k|^p \, d\mathcal{L}^n \leq \frac{r_k^{p-n}}{E_k} \mathcal{F}(u_k; B_{\rho r_k}) \leq \alpha_k(\rho).$$

Finally, up to a subsequence,

$$\lim_k \frac{r_k^{p-1}}{E_k} |\{ v_k \neq \tilde{v}_k \}| = 0.$$

Indeed, by (3.2.20),

$$\frac{r_k^{p-1}}{E_k} |\{ v_k \neq \tilde{v}_k \}| \leq C \frac{r_k^{p-1}}{E_k} \varepsilon_k^{\frac{n}{n-1}},$$

which tends to zero as long as r_k^{p-1}/E_k is bounded. On the other hand, if r_k^{p-1}/E_k diverges, we could use the fact that $\varepsilon_k \leq s^{-q} \mathcal{F}(u_k; B_{r_k}) r_k^{1-n}$ and get

$$\frac{r_k^{p-1}}{E_k} |\{ v_k \neq \tilde{v}_k \}| \leq C \frac{r_k^{p-1}}{E_k} \left(\frac{E_k}{r_k^{p-1}} \right)^{\frac{n}{n-1}}$$

which goes to zero. Using Fubini's theorem we have (3.2.22).

Let $\tilde{u}_k(x) = E_k^{1/p} \tilde{v}_k(\frac{x}{r_k})$, and for every $t \in [0,1]$ we define

$$\tilde{\alpha}_k(t) = \frac{\mathcal{F}(\tilde{u}_k; B_{t r_k})}{\mathcal{F}(u_k, B_{r_k})}.$$

The $\tilde{\alpha}_k$ functions are also monotone and bounded: the jump set of $\tilde{u}_k$ is contained in J_{u_k}, therefore we can write

$$\tilde{\alpha}_k(t) \leq \alpha_k(t) + \frac{2\beta \mathcal{H}^{n-1}(J_{u_k} \cap B_{t r_k})}{\mathcal{F}(u_k; B_{r_k})} \leq \left(1 + \frac{2}{s^q} \right) \alpha_k(t),$$

using the fact that $u_k \in \mathcal{K}_s$. As done for α_k, we can assume that $\tilde{\alpha}_k$ converges $\mathcal{L}^1$-almost everywhere to a function $\tilde{\alpha}$.

Let $I \subset [0,1]$ be the set of values ρ for which (3.2.22) holds, α_k and $\tilde{\alpha}_k$ converge and α and $\tilde{\alpha}$ are continuous. Notice that $\mathcal{L}^1(I) = 1$. Fix $\rho, \rho' \in I$ with $\rho < \rho' < 1$ and let

$$\mathcal{I}_k(\xi) = \beta E_k^{q/p-1} r_k^{p-1} \int_{J_\xi \cap (B_{\rho'} \setminus B_\rho)} \left(\overline{\xi}^q + \underline{\xi}^q \right) d\mathcal{H}^{n-1},$$

with $\xi \in \mathrm{SBV}(B_1)$. Let $w \in W^{1,p}(B_1)$ and consider η a smooth cutoff function supported on $B_{\rho'}$ and identically equal to 1 in B_ρ. Let

$$\varphi_k = ((w + m_k)\eta + \tilde{v}_k(1-\eta))\chi_{B_{\rho'}} + v_k \chi_{B_1 \setminus B_{\rho'}}.$$

We want to prove that

$$\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) \geq \int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + \mathcal{I}_k(\tilde{v}_k), \tag{3.2.23}$$

and

$$\alpha_k(\rho') \leq R_k + \int_{B_{\rho'}} |\nabla \varphi_k|^p \, d\mathcal{L}^n + \mathcal{I}_k(\varphi_k), \tag{3.2.24}$$

where R_k goes to zero as k goes to infinity. We immediately compute

$$\begin{aligned} \tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho) =& \mathcal{F}(u_k; B_{r_k})^{-1} \left[\int_{B_{\rho' r_k} \cap B_{\rho r_k}} |\nabla \tilde{u}_k|^p \, d\mathcal{L}^n + \beta \int_{J_{\tilde{u}_k} \cap (B_{\rho' r_k} \setminus B_{\rho r_k})} (\overline{\tilde{u}_k}^q + \underline{\tilde{u}_k}^q) \, d\mathcal{H}^{n-1} \right] \\ &+ \mathcal{F}(u_k; B_{r_k})^{-1} |\{ \tilde{u}_k > 0 \} \cap (B_{\rho' r_k} \setminus B_{\rho r_k})| \\ \geq& \int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + E_k^{q/p-1} r_k^{p-1} \beta \int_{J_{\tilde{v}_k} \cap (B_{\rho'} \setminus B_\rho)} (\overline{\tilde{v}_k}^q + \underline{\tilde{v}_k}^q) \, d\mathcal{H}^{n-1}, \end{aligned}$$

and then we have (3.2.23). Now let $\psi_k = E_k^{1/p} \varphi_k(x/r_k)$ and observe that ψ_k coincides with u_k outside $B_{\rho' r_k}$. We get from the local minimality of u_k that

$$\begin{aligned} \mathcal{F}(u_k; B_{r_k}) \leq \mathcal{F}(\psi_k; B_{r_k}) &= \mathcal{F}(\psi_k; B_{\rho' r_k}) + \beta \int_{\{ u_k \neq \tilde{u}_k \} \cap \partial B_{\rho' r_k}} \left(\underline{\psi_k}^q + \overline{\psi_k}^q \right) d\mathcal{H}^{n-1} \\ &\quad + \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho' r_k}}) \\ &\leq \mathcal{F}(\psi_k; B_{\rho' r_k}) + 2\beta r_k^{n-1} \mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_{\rho'}) \\ &\quad + \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho' r_k}}). \end{aligned} \tag{3.2.25}$$

So, in particular, we have

$$\begin{aligned} \mathcal{F}(u_k; B_{\rho' r_k}) =& \mathcal{F}(u_k; B_{r_k}) - \mathcal{F}(u_k; B_{r_k} \setminus \overline{B_{\rho'_k}}) - \beta \int_{J_{u_k} \cap \partial B_{\rho' r_k}} (\overline{u_k}^q + \underline{u_k}^q) \, d\mathcal{H}^{n-1} \\ \leq& 2\beta r_k^{n-1} \mathcal{H}^{n-1}(\{ v_k \neq \tilde{v}_k \} \cap \partial B_{\rho'}) + \mathcal{F}(\psi_k; B_{\rho' r_k}). \end{aligned}$$

Dividing by $\mathcal{F}(u_k; B_{r_k})$ and using (3.2.22) we get

$$\alpha_k(\rho') \leq R_k + r_k^{p-n} E_k^{-1} \mathcal{F}(\psi_k; B_{\rho' r_k}).$$

With appropriate rescalings we have

$$r_k^{p-n} E_k^{-1} \mathcal{F}(\psi_k; B_{\rho' r_k}) = \int_{B_{\rho'}} |\nabla \varphi_k|^p \, d\mathcal{L}^n + \mathcal{I}_k(\varphi_k) + r_k^p E_k^{-1} |\{ \varphi_k > 0 \} \cap B_{\rho'}|.$$

From (3.2.17) and the definition of E_k, we have

$$r_k^p E_k^{-1} |\{ \varphi_k > 0 \} \cap B_{\rho'}| \leq \omega_n r_k^{1/2},$$

and then we get (3.2.24).

We now want to prove that for every $\varphi \in W^{1,p}(B_1)$ such that $v - \varphi$ is supported on B_ρ, we have

$$\alpha(\rho') \leq \int_{B_\rho} |\nabla \varphi|^p \, d\mathcal{L}^n + C[\tilde{\alpha}(\rho') - \tilde{\alpha}(\rho)] + C \int_{B_{\rho'} \setminus B_\rho} |\nabla \varphi|^p \, d\mathcal{L}^n, \tag{3.2.26}$$

where C does not depend on either ρ or ρ'. From the definition of φ_k, we have that on B_ρ

$$\nabla \varphi_k = \nabla w$$

and on $B_{\rho'} \setminus B_\rho$

$$\nabla \varphi_k = \eta \nabla w + (w + m_k - \tilde{v}_k) \nabla \eta + \nabla \tilde{v}_k (1 - \eta),$$

so that

$$\begin{aligned} \int_{B_{\rho'}} |\nabla \varphi_k|^p \, d\mathcal{L}^n &\leq \int_{B_\rho} |\nabla w|^p \, d\mathcal{L}^n \\ &\quad + C \left[\int_{B_{\rho'} \setminus B_\rho} |\nabla \tilde{v}_k|^p \, d\mathcal{L}^n + \int_{B_{\rho'} \setminus B_\rho} (|\nabla w|^p + |w + m_k - \tilde{v}_k|^p |\nabla \eta|^p) \, d\mathcal{L}^n \right]. \end{aligned} \tag{3.2.27}$$

We split the proof into two cases: either

$$\limsup_k E_k > 0 \tag{3.2.28}$$

or

$$\lim_k E_k = 0. \tag{3.2.29}$$

Assume (3.2.28) occurs. Notice that $s \leq u_k \leq 1$ for every k, then by definition we have that, for every k, $s \leq E_k^{1/p} \tilde{v}_k \leq 1$ and, since m_k is a median of v_k, $0 \leq E_k^{1/p} m_k \leq 1$. In particular we have that

$$|\tilde{v}_k - m_k| \leq \frac{2}{E_k^{1/p}},$$

passing to the limit when k goes to infinity we have that

$$\|v\|_\infty \leq \liminf_k \frac{2}{E_k^{1/p}} < +\infty \quad \mathcal{L}^n\text{-a.e.}$$

then v belongs to $L^\infty(B_1)$ and there exists a positive constant C independent of k, and a natural number $\overline{k}$ such that

$$|v + m_k - \tilde{v}_k| \leq \frac{C}{E_k^{1/p}} \leq \frac{C}{s} \tilde{v}_k \quad \mathcal{L}^n\text{-a.e.}$$

for all $k > \overline{k}$. Let $\varphi \in W^{1,p}(B_1)$ with $v - \varphi$ supported on B_ρ, and let $w = \varphi$ in the definition of φ_k, then, for every $k > \overline{k}$, we have

$$|\varphi_k| = |\tilde{v}_k + (v + m_k - \tilde{v}_k)\eta| \le C\tilde{v}_k \tag{3.2.30}$$

$\mathcal{L}^n$-a.e. on $B_{\rho'} \setminus B_\rho$. From (3.2.30) we have that

$$\mathcal{I}_k(\varphi_k) \le C\mathcal{I}_k(\tilde{v}_k). \tag{3.2.31}$$

Notice, in addition, that (3.2.27) reads as

$$\begin{aligned}\int_{B_{\rho'}} |\nabla\varphi_k|^p \, d\mathcal{L}^n \le \int_{B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n \\ + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\tilde{v}_k|^p \, d\mathcal{L}^n + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n + R_k.\end{aligned} \tag{3.2.32}$$

finally joining (3.2.24), (3.2.32), (3.2.31), and (3.2.23), we have

$$\alpha_k(\rho') \le \int_{B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n + C[\tilde{\alpha}_k(\rho') - \tilde{\alpha}_k(\rho)] + C\int_{B_{\rho'}\setminus B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n + R_k.$$

Letting k go to infinity we get (3.2.26).

Suppose now that (3.2.29) occurs. The functions $|\tilde{v}_k - m_k|^p$, $|v|^p$ are uniformly integrable, namely for every $\varepsilon > 0$ there exists a $\sigma = \sigma_\varepsilon < \varepsilon$ such that if A is a measurable set with $|A| < \sigma$, then

$$\int_A |\tilde{v}_k - m_k|^p \, d\mathcal{L}^n + \int_A |v|^p \, d\mathcal{L}^n < \varepsilon. \tag{3.2.33}$$

Since $v \in L^p(B_1)$, we can find $M > 1/\varepsilon$ such that

$$|\{\, |v| > M \,\}| < \sigma. \tag{3.2.34}$$

Setting $w = \varphi_M = \max\{\, -M, \min\{\, \varphi, M \,\}\,\}$, then (3.2.27) reads as

$$\begin{aligned}\int_{B_{\rho'}} |\nabla\varphi_k|^p \, d\mathcal{L}^n \le \int_{B_\rho \cap \{\,|\varphi| \le M\,\}} |\nabla\varphi|^p \, d\mathcal{L}^n + C\int_{(B_{\rho'}\setminus B_\rho)\cap\{\,|\varphi|\le M\,\}} |\nabla\varphi|^p \, d\mathcal{L}^n \\ + C\left[\int_{B_{\rho'}\setminus B_\rho} |\nabla\tilde{v}_k|^p \, d\mathcal{L}^n + \int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla\eta|^p \, d\mathcal{L}^n\right].\end{aligned} \tag{3.2.35}$$

We can estimate the last integral as follows

$$\begin{aligned}\int_{B_{\rho'}\setminus B_\rho} |\varphi_M + m_k - \tilde{v}_k|^p |\nabla\eta|^p \, d\mathcal{L}^n &\le C\varepsilon + \int_{(B_{\rho'}\setminus B_\rho)\cap\{\,|v|\le M\,\}} |v + m_k - \tilde{v}_k|^p |\nabla\eta|^p \, d\mathcal{L}^n]. \\ &= C\varepsilon + R_k,\end{aligned} \tag{3.2.36}$$

where we used (3.2.34) and (3.2.33), and C only depends on ρ and ρ'. Furthermore, we have

$$\mathcal{I}_k(\varphi_k) \le R_k + C\mathcal{I}_k(\tilde{v}_k). \tag{3.2.37}$$

Indeed, as before, $|\tilde{v}_k - m_k| \le C\tilde{v}_k$, while

$$\begin{aligned} E_k^{q/p-1} r_k^{p-1} \int_{J_{\tilde{v}_k}\cap(B_{\rho'}\setminus B_\rho)} |\varphi_M|^q \, d\mathcal{H}^{n-1} &\le M^q E_k^{q/p-1} r_k^{p-1} \mathcal{H}^{n-1}(J_{\tilde{v}_k} \cap (B_{\rho'} \setminus B_\rho)) \\ &\le M^q E_k^{\frac{q}{p}} \frac{r_k^{p-1}\varepsilon_k}{E_k} \\ &\le \frac{M^q}{s^q} E_k^{\frac{q}{p}}, \end{aligned}$$

which goes to 0 as $k \to \infty$. Finally, joining (3.2.24), (3.2.35), (3.2.36), (3.2.37), and (3.2.23), we have

$$\alpha_k(\rho') \le R_k + \int_{B_\rho \cap \{ |\varphi| \le M \}} |\nabla\varphi|^p + C[\tilde{\alpha}(\rho') - \tilde{\alpha}(\rho)] + C \int_{(B_{\rho'}\setminus B_\rho)\cap\{ |\varphi|\le M \}} |\nabla\varphi|^p \, d\mathcal{L}^n + C\varepsilon.$$

Taking the limit as k tends to infinity, and then the limit as ε tends to 0, we get (3.2.26).

We are now in a position to prove that v is p-harmonic: taking the limit as ρ' tends to ρ in (3.2.26), we have that if $\varphi \in W^{1,p}(B_1)$, with $v - \varphi$ supported on B_ρ,

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n \le \alpha(\rho) \le \int_{B_\rho} |\nabla\varphi|^p \, d\mathcal{L}^n,$$

for every $\rho \in I$, therefore v is p-harmonic in B_1. Notice that this implies that v is a locally Lipschitz function (see [18, Theorem 7.12]) . Moreover, if $\varphi = v$, we have

$$\int_{B_\rho} |\nabla v|^p \, d\mathcal{L}^n = \alpha(\rho)$$

for every $\rho \in I$, so that α is continuous on the whole interval $[0, 1]$, $\alpha(1) = 1$ and $\alpha(\tau) = \lim_k \alpha_k(\tau) \ge \tau^{n-1/2}$. Nevertheless, if τ is sufficiently small this contradicts the fact that v is locally Lipschitz, since

$$\tau^{n-\frac{1}{2}} \le \int_{B_\tau} |\nabla v|^p \, d\mathcal{L}^n \le C\,\tau^n,$$

where C is a positive constant depending only on n and p. □

Proof of Theorem 3.2.2. Let u be a minimizer for the problem (3.2.4). By Proposition 3.2.7 there exist two positive constants $C(\Omega, \beta, p, q), C_1(\Omega, \beta, p, q)$ such that if $B_r(x) \subseteq \mathbb{R}^n \setminus \Omega$, then

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \le C(\Omega, \beta, p, q) r^{n-1},$$

and if $x \in K_u$

$$|B_r(x) \cap \{ u > 0 \}| \ge C_1(\Omega, \beta, p, q) r^n.$$

We now prove that there exists a positive constant $c = c(\Omega, \beta, p, q)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \ge c(\Omega, \beta, p, q) r^{n-1} \tag{3.2.38}$$

for every $x \in K_u$ and $B_r(x) \subset \mathbb{R}^n \setminus \Omega$. Assume by contradiction that there exists $x \in J_u$ such that, for $r > 0$ small enough,

$$\mathcal{H}^{n-1}(J_u \cap B_r(x)) \le \varepsilon_0 r^{n-1},$$

where ε_0 is the one in Lemma 3.2.12. Iterating Lemma 3.2.12 it can be proven (see [65, Theorem 5.1]) that

$$\lim_{r\to 0^+} r^{1-n}\mathcal{F}(u;B_r) = 0,$$

which, in particular, implies

$$\lim_{r\to 0^+} r^{1-n}\left[\int_{B_r(x)} |\nabla u|^p \, d\mathcal{L}^n + \mathcal{H}^{n-1}(J_u \cap B_r(x))\right] = 0. \tag{3.2.39}$$

By [103, Theorem 3.6], (3.2.39) implies that $x \notin J_u$, which is a contradiction. Finally, if $x \in K_u$ and

$$\mathcal{H}^{n-1}(J_u \cap B_{2r}(x)) \le \varepsilon_0 r^{n-1},$$

there exists $y \in J_u \cap B_r(x)$ such that

$$\mathcal{H}^{n-1}(J_u \cap B_r(y)) \le \varepsilon_0 r^{n-1}$$

which, again, is a contradiction. Then the assertion is proved. The density estimate (3.2.38) implies in particular that

$$K_u \setminus \bar{\Omega} \subset \left\{ x \in \mathbb{R}^n \;\middle|\; \limsup_{r\to 0^+} r^{1-n}\left[\int_{B_r(x)} |\nabla u|^p \, d\mathcal{L}^n + \mathcal{H}^{n-1}(J_u \cap B_r(x))\right] > 0 \right\},$$

hence $\mathcal{H}^{n-1}((K_u \setminus J_u) \setminus \bar{\Omega}) = 0$ (see for instance [103, Lemma 2.6]). □

Remark 3.2.13. Let u be a minimizer for problem (3.2.4), then from Theorem 3.2.5 we have that the function $u_0 = (\beta\delta^q)^{-1/p}u$ is an almost-quasi minimizer for the Mumford-Shah functional

$$\mathrm{MS}_p(v) = \int_{\mathbb{R}^n} |\nabla v|^p \, d\mathcal{L}^n + \mathcal{H}^{n-1}(J_v)$$

with the Dirichlet condition $u_0 = (\beta\delta^q)^{-1/p}$ on Ω. If Ω is sufficiently smooth we can apply the results in [51] to have that the density estimate for the jump set of minimizers holds up to the boundary of Ω.

Chapter 4

Cases with explicit optimizers

4.1 Double shape optimization problem realted to p-capacity

The results of this section are contained in the paper [4].

Let $p > 1$, $\beta > 0$ be real numbers. For every open bounded set $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, and every compact set $K \subseteq \overline{\Omega}$ with Lipschitz boundary, we define

$$\mathrm{Cap}_p^\beta(K,\Omega) = \inf_{\substack{v\in W^{1,p}(\Omega)\\ v=1 \text{ in } K}} \left(\int_\Omega |\nabla v|^p \, dx + \beta \int_{\partial\Omega} |v|^p \, d\mathcal{H}^{n-1} \right). \tag{4.1.1}$$

We notice that it is sufficient to minimize among all functions $v \in H^1(\Omega)$ with $v = 1$ in K and $0 \le v \le 1$ a.e. Moreover, if K, Ω are sufficiently smooth, a minimizer u satisfies

$$\begin{cases} u = 1 & \text{in } K, \\ \Delta_p u = 0 & \text{in } \Omega \setminus K, \\ |\nabla u|^{p-2}\dfrac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = 0 & \text{on } \partial\Omega \setminus \partial K, \end{cases} \tag{4.1.2}$$

where $\Delta_p u = \mathrm{div}\left(|\nabla u|^{p-2}\nabla u\right)$ is the p-Laplacian of u and ν is the outer unit normal to $\partial\Omega$. If $\mathring{K} = \Omega$, equation (4.1.2) has to be understood as $u = 1$ in Ω, and the energy is

$$\mathrm{Cap}_p^\beta(\Omega,\Omega) = \beta \mathcal{H}^{n-1}(\partial\Omega).$$

In general, equation (4.1.2) has to be interpreted in the weak sense, that is: for every $\varphi \in W^{1,p}(\Omega)$ such that $\varphi \equiv 0$ in K,

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi \, d\mathcal{L}^n + \beta \int_{\partial\Omega} u^{p-1}\varphi \, d\mathcal{H}^{n-1} = 0. \tag{4.1.3}$$

In particular, if u is a minimizer, letting $\varphi = u - 1$, we have that

$$\mathrm{Cap}_p^\beta(K,\Omega) = \int_\Omega |\nabla u|^p \, dx + \beta \int_{\partial\Omega} u^p \, d\mathcal{H}^{n-1} = \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}.$$

Moreover, from the strict convexity of the functional, the minimizer is the unique solution to (4.1.3).

This problem is related to the so-called *relative p-capacity of K with respect to Ω*, defined as

$$\operatorname{Cap}_p(K,\Omega) := \inf_{\substack{v\in W_0^{1,p}(\Omega)\\ v=1 \text{ in } K}} \left(\int_\Omega |\nabla v|^p \, dx \right).$$

In the case $p=2$ it represents the electrostatic capacity of an annular condenser consisting of a conducting surface $\partial\Omega$, and a conductor K, where the electrostatic potential is prescribed to be 1 inside K and 0 outside Ω. Let ω_n be the measure of the unit sphere in $\mathbb{R}^n$, and let $M>\omega_n$, then it is well known that there exists some $r\geq 1$ such that

$$\min_{\substack{|K|=\omega_n\\ |\Omega|\leq M}} \operatorname{Cap}_p(K,\Omega) = \operatorname{Cap}_p(B_1,B_r).$$

This is an immediate consequence of the Pólya-Szegő inequality for the Schwarz rearrangement (see for instance [150, 118]). We are interested in studying the same problem for the energy defined in (4.1.1), which corresponds to changing the Dirichlet boundary condition on $\partial\Omega$ into a Robin boundary condition, namely, we consider the following problem

$$\inf_{\substack{|K|=\omega_n\\ |\Omega|\leq M}} \operatorname{Cap}_p^\beta(K,\Omega). \tag{4.1.4}$$

In this case, the previous symmetrization techniques cannot be employed anymore.

Problem (4.1.4) has been studied in the linear case $p=2$ in [58], with more general boundary conditions on $\partial\Omega$, namely

$$\frac{\partial u}{\partial \nu} + \frac{1}{2}\Theta'(u) = 0,$$

where Θ is a suitable increasing function vanishing at 0. This kind of problem has also been addressed in the context of thermal insulation (see for instance [65, 3, 6]). Our main result reads as follows.

Theorem 4.1.1. *Let $\beta>0$ such that*

$$\beta^{\frac{1}{p-1}} > \frac{n-p}{p-1}.$$

Then, for every $M>\omega_n$ the solution to problem (4.1.4) is given by two concentric balls (B_1,B_r), that is

$$\min_{\substack{|K|=\omega_n\\ |\Omega|\leq M}} \operatorname{Cap}_p^\beta(K,\Omega) = \operatorname{Cap}_p^\beta(B_1,B_r),$$

in particular we have that either $r=1$ or $M=\omega_n r^n$.

Moreover, if $K_0\subseteq\overline{\Omega_0}$ are such that

$$\operatorname{Cap}_p^\beta(K_0,\Omega_0) = \min_{\substack{|K|=\omega_n\\ |\Omega|\leq M}} \operatorname{Cap}_p^\beta(K,\Omega),$$

and u is the minimizer of $\operatorname{Cap}_p^\beta(K_0,\Omega_0)$, then the sets $\{\, u=1 \,\}$ and $\{\, u>0 \,\}$ coincide with two concentric balls up to a $\mathcal{H}^{n-1}$-negligible set.

Remark 4.1.2. In the case
$$\beta^{\frac{1}{p-1}} \leq \frac{n-p}{p-1},$$
adapting the symmetrization techniques used in [58], it can be proved that a solution to problem (4.1.4) is always given by the pair (B_1, B_1).

We point out that the proof of the theorem relies on the techniques involving the H-function introduced in [35, 85].

The case in which Ω is the Minkowski sum $\Omega = K + B_r(0)$, the energy $\mathrm{Cap}_p^\beta(K, \Omega)$, has been studied in [25] under suitable geometrical constraints (see also [87]).

4.1.1 Proof of the theorem

To prove Theorem 4.1.1, we start by studying the function

$$R \mapsto \mathrm{Cap}_p^\beta(B_1, B_R).$$

A similar study of the previous function can also be found in [25]. Let

$$\Phi_{p,n}(\rho) = \begin{cases} \log(\rho) & \text{if } p = n, \\ -\dfrac{p-1}{n-p}\dfrac{1}{\rho^{\frac{n-p}{p-1}}} & \text{if } p \neq n. \end{cases}$$

For every $R > 1$, consider

$$u_R(x) = 1 - \frac{\beta^{\frac{1}{p-1}}\left(\Phi_{p,n}(|x|) - \Phi_{p,n}(1)\right)_+}{\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}}\left(\Phi_{p,n}(R) - \Phi_{p,n}(1)\right)}, \tag{4.1.5}$$

the solution to
$$\begin{cases} u_R = 1 & \text{in } B_1, \\ \Delta_p u_R = 0 & \text{in } B_R \setminus B_1, \\ |\nabla u_R|^{p-2}\dfrac{\partial u_R}{\partial \nu} + \beta|u_R|^{p-2}u_R = 0 & \text{on } \partial B_R. \end{cases}$$
We have that

$$\begin{aligned} \mathrm{Cap}_p^\beta(B_1, B_R) &= \int_{B_R} |\nabla u_R|^p \, dx + \beta \int_{\partial B_R} |u_R|^p \, d\mathcal{H}^{n-1} \\ &= \frac{n\omega_n\beta}{\left[\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}}\left(\Phi_{p,n}(R) - \Phi_{p,n}(1)\right)\right]^{p-1}}. \end{aligned} \tag{4.1.6}$$

Notice that $\mathrm{Cap}_p^\beta(B_1, B_R)$ is decreasing in $R > 0$ if and only if

$$\frac{d}{dR}\Big(\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}}\Phi_{p,n}(R)\Big) \geq 0$$

that is, if and only if

$$R \geq \frac{n-1}{p-1} \frac{1}{\beta^{\frac{1}{p-1}}} =: \alpha_{\beta,p}.$$

Moreover

$$\mathrm{Cap}_p^\beta(B_1, B_1) = n\omega_n \beta,$$

$$\lim_{R\to\infty} \mathrm{Cap}_p^\beta(B_1, B_R) = \begin{cases} n\omega_n \left(\dfrac{n-p}{p-1}\right)^{p-1} & \text{if } p < n, \\ 0 & \text{if } p \geq n. \end{cases}$$

Therefore, there are three cases:

(i) if

$$\beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1},$$

$R \in [1, +\infty) \mapsto \mathrm{Cap}_p^\beta(B_1, B_R)$ is decreasing;

(ii) if

$$\frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1},$$

$R \in [1, +\infty) \mapsto \mathrm{Cap}_p^\beta(B_1, B_R)$ increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, with the existence of a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that $\mathrm{Cap}_p^\beta(B_1, B_{R_{\beta,p}}) = \mathrm{Cap}_p^\beta(B_1, B_1)$;

(iii) if

$$\beta^{\frac{1}{p-1}} \leq \frac{n-p}{p-1},$$

$R \in [1, +\infty) \mapsto \mathrm{Cap}_p^\beta(B_1, B_R)$ reaches its minimum at $R = 1$.

See for instance Figure 4.1, where

$$\beta_1 = \left(\frac{n-p}{p-1}\right)^{p-1}, \qquad \beta_2 = \left(\frac{n-1}{p-1}\right)^{p-1}, \qquad p = 2.5, \qquad n = 3.$$

In the following, we will need

Lemma 4.1.3. *Let $R > 1$,$\beta > 0$ and let u_R be the solution of the problem on (B_1, B_R). Then*

$$\frac{|\nabla u_R|}{u_R} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$, if and only if

$$\mathrm{Cap}_p^\beta(B_1, B_\rho) \geq \mathrm{Cap}_p^\beta(B_1, B_R)$$

for every $\rho \in [1, R]$.

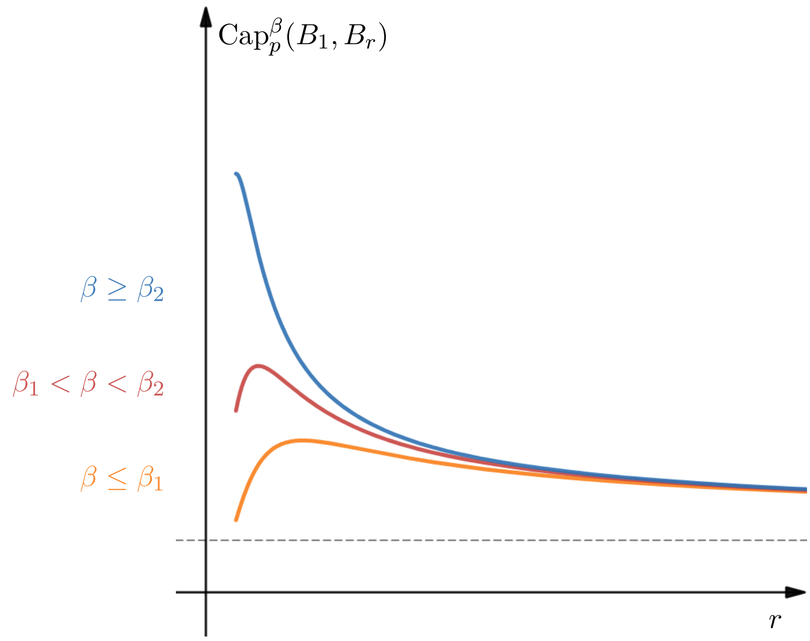


Figure 4.1: $\mathrm{Cap}_p^\beta(B_1, B_r)$ depending on the value of β

Proof. Recalling the expressions of u_R in (4.1.5), by straightforward computations we have that

$$\frac{|\nabla u_R|}{u_R} \leq \beta^{\frac{1}{p-1}}$$

in $B_R \setminus B_1$ if and only if

$$\Phi'_{p,n}(R) + \beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(R) - \Phi_{p,n}(1)\big) \geq \Phi'_{p,n}(\rho) + \beta^{\frac{1}{p-1}}\big(\Phi_{p,n}(\rho) - \Phi_{p,n}(1)\big) \tag{4.1.7}$$

for every $\rho \in [1, R]$, using the expression of $\mathrm{Cap}_p^\beta(B_1, B_\rho)$ in (4.1.6), (4.1.7) is equivalent to

$$\mathrm{Cap}_p^\beta(B_1, B_\rho) \geq \mathrm{Cap}_p^\beta(B_1, B_R)$$

for every $\rho \in [1, R]$. $\square$

Definition 4.1.4. Let $\Omega \subseteq \mathbb{R}^n$ be an open set, and let $U \subseteq \Omega$ be another set. We define the *internal boundary* of U as

$$\partial_{\mathrm{i}} U = \partial U \cap \Omega,$$

and the *external boundary* of U as

$$\partial_{\mathrm{e}} U = \partial U \cap \partial\Omega.$$

Let $K \subseteq \overline{\Omega} \subseteq \mathbb{R}^n$ be open bounded sets, and let u be the minimizer of $E_{\beta.p}(K, \Omega)$. In the following, we denote by

$$U_t = \{\, x \in \Omega \mid u(x) > t \,\}\,.$$

Definition 4.1.5 (H-function)**.** Let $\varphi \in W^{1,p}(\Omega)$. We define

$$H(t, \varphi) = \int_{\partial_{\mathrm{i}} U_t} |\varphi|^{p-1}\, d\mathcal{H}^{n-1} - (p-1)\int_{U_t} |\varphi|^p\, d\mathcal{L}^n + \beta\mathcal{H}^{n-1}(\partial_{\mathrm{e}} U_t).$$

Notice that this definition is slightly different from the one given in [46].

Lemma 4.1.6. *Let $K \subseteq \Omega \subseteq \mathbb{R}^n$ be open, bounded sets, and let u be the minimizer of $\mathrm{Cap}_p^\beta(K,\Omega)$. Then for a.e. $t \in (0,1)$ we have*

$$H\left(t, \frac{|\nabla u|}{u}\right) = \mathrm{Cap}_p^\beta(K,\Omega).$$

Proof. Recall that

$$\mathrm{Cap}_p^\beta(K,\Omega) = \int_\Omega |\nabla u|^p \, d\mathcal{L}^n + \beta \int_{\partial\Omega} u^p = \beta \int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}. \tag{4.1.8}$$

Let $t \in (0,1)$, we construct the following test functions: let $\varepsilon > 0$, and let

$$\varphi_\varepsilon(x) = \begin{cases} -1 & \text{if } u(x) \le t, \\ \dfrac{u(x)-t}{\varepsilon u(x)^{p-1}} - 1 & \text{if } t < u(x) \le t+\varepsilon, \\ \dfrac{1}{u(x)^{p-1}} - 1 & \text{if } u(x) > t+\varepsilon, \end{cases}$$

so that φ_ε is an approximation the function $(u^{1-p}\chi_{U_t} - 1)$, and

$$\nabla\varphi_\varepsilon(x) = \begin{cases} 0 & \text{if } u(x) \le t, \\ \dfrac{1}{\varepsilon}\left(\dfrac{\nabla u(x)}{u(x)^{p-1}} - (p-1)\dfrac{\nabla u(x)(u(x)-t)}{u(x)^p}\right) & \text{if } t < u(x) \le t+\varepsilon, \\ -(p-1)\dfrac{\nabla u(x)}{u(x)^p} & \text{if } u(x) > t+\varepsilon. \end{cases}$$

We have that φ_ε is an admissible test function for the Euler-Lagrange equation (4.1.3), which entails

$$\begin{aligned} 0 =& \frac{1}{\varepsilon}\int_{\{t<u\le t+\varepsilon\}\cap\Omega} \frac{|\nabla u|^{p-1}}{u^{p-1}} |\nabla u| \, d\mathcal{L}^n - (p-1)\int_{\{t<u\le t+\varepsilon\}\cap\Omega} \frac{|\nabla u|^p}{u^p}\frac{u-t}{\varepsilon} \, d\mathcal{L}^n \\ &- (p-1)\int_{\{u>t+\varepsilon\}\cap\Omega} \frac{|\nabla u|^p}{u^p} \, d\mathcal{L}^n + \beta\int_{\{t<u\le t+\varepsilon\}\cap\partial\Omega} \frac{u-t}{\varepsilon} \, d\mathcal{H}^{n-1} \\ &+ \beta\mathcal{H}^{n-1}(\partial\Omega \cap \{u > t+\varepsilon\}) - \beta\int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1}. \end{aligned}$$

Letting now ε go to 0, by coarea formula we get that for a.e. $t \in (0,1)$

$$\begin{aligned} \beta\int_{\partial\Omega} u^{p-1} \, d\mathcal{H}^{n-1} =& \int_{\partial_{\mathrm{i}} U_t} \left(\frac{|\nabla u|}{u}\right)^{p-1} d\mathcal{H}^{n-1} - (p-1)\int_{U_t}\left(\frac{|\nabla u|}{u}\right)^p d\mathcal{L}^n \\ &+ \beta\mathcal{H}^{n-1}(\partial_{\mathrm{e}} U_t). \end{aligned} \tag{4.1.9}$$

Joining (4.1.8) and (4.1.9), the lemma is proven. □

Remark 4.1.7. Notice that if K, and Ω are two concentric balls, the minimizer u is the one written in (4.1.5), for which the statement of the above Lemma holds for every $t \in (0,1)$.

Lemma 4.1.8. *Let $\varphi \in L^\infty(\Omega)$. Then there exists $t \in (0,1)$ such that*

$$H(t,\varphi) \leq \mathrm{Cap}_p^\beta(K,\Omega).$$

Proof. Let

$$w = |\varphi|^{p-1} - \left(\frac{|\nabla u|}{u}\right)^{p-1},$$

then we evaluate

$$\begin{aligned} H(t,\varphi) - H\Big(t, \frac{|\nabla u|}{u}\Big) &= \int_{\partial_\mathrm{i} U_t} w\, d\mathcal{H}^{n-1} - (p-1)\int_{U_t} \left(|\varphi|^p - \left(\frac{|\nabla u|}{u}\right)^p\right) d\mathcal{L}^n \\ &\leq \int_{\partial_\mathrm{i} U_t} w\, d\mathcal{H}^{n-1} - p\int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n \\ &= -\frac{1}{t^{p-1}}\frac{d}{dt}\left(t^p \int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n\right), \end{aligned}$$

where we used the inequality

$$a^p - b^p \leq \frac{p}{p-1} a\left(a^{p-1} - b^{p-1}\right) \qquad \forall a,b \geq 0. \tag{4.1.10}$$

Multiplying by t^{p-1} and integrating, we get

$$\int_0^1 t^{p-1}\left(H(t,\varphi) - H\Big(t, \frac{|\nabla u|}{u}\Big)\right) dt \leq -\left[t^p \int_{U_t} \frac{|\nabla u|}{u} w\, d\mathcal{L}^n\right]_0^1 = 0, \tag{4.1.11}$$

from which we obtain the conclusion of the proof. □

Remark 4.1.9. Notice that the inequality (4.1.10) holds as equality if and only if $a = b$. Therefore, if $\varphi \neq \frac{|\nabla u|}{u}$ on a set of positive measure, then the inequality in (4.1.11) is strict, since

$$\left|\left\{\, \varphi \neq \frac{\nabla u}{u} \,\right\} \cap U_t\right| > 0$$

for small enough t. Therefore, there exists $S \subset (0,1)$ such that $\mathcal{L}^1(S) > 0$ and for every $t \in S$

$$H(t,\varphi) < \mathrm{Cap}_p^\beta(K,\Omega).$$

In the following, we fix a radius R such that $|B_R| \geq |\Omega|$, u_R the minimizer of $\mathrm{Cap}_p^\beta(B_1, B_R)$, and

$$\begin{aligned} H^*(t,\varphi) = &\int_{\partial\{\, u_R > t \,\}\cap B_R} |\varphi|^{p-1}\, d\mathcal{H}^{n-1} - (p-1)\int_{\{\, u_R>t \,\}} |\varphi|^p\, d\mathcal{L}^n \\ &+ \beta\mathcal{H}^{n-1}(\partial\{\, u_R < t \,\} \cap \partial B_R). \end{aligned}$$

Proposition 4.1.10. *Let $\beta > 0$. Assume that*

$$\frac{|\nabla u_R|}{u_R} \leq \beta^{\frac{1}{p-1}}. \tag{4.1.12}$$

Then we have that

$$\mathrm{Cap}_p^\beta(K,\Omega) \geq \mathrm{Cap}_p^\beta(B_1, B_R).$$

Proof. In the following, if v is a radial function on B_R and $r \in (0, R)$, we denote with abuse of notation

$$v(r) = v(x),$$

where x is any point on ∂B_r. By Lemma 4.1.6 we know that for every $t \in (0,1)$

$$H^*\Big(t, \frac{|\nabla u_R|}{u_R}\Big) = \mathrm{Cap}_p^\beta(B_1, B_R), \tag{4.1.13}$$

while by Lemma 4.1.8, for every $\varphi \in L^\infty(\Omega)$ there exists a $t \in (0,1)$ such that

$$\mathrm{Cap}_p^\beta(K, \Omega) \geq H(t, \varphi). \tag{4.1.14}$$

We aim to find a suitable φ such that, for some t,

$$H(t, \varphi) \geq H^*\Big(t, \frac{|\nabla u_R|}{u_R}\Big), \tag{4.1.15}$$

so that combining (4.1.14), (4.1.15), and (4.1.13) we conclude the proof. In order to construct φ, for every $t \in (0,1)$ we define

$$r(t) = \left(\frac{|U_t|}{\omega_n}\right)^{\frac{1}{n}}, \tag{4.1.16}$$

then we set, for every $x \in \Omega$,

$$\varphi(x) = \frac{|\nabla u_R|}{u_R}(r(u(x))).$$

Claim: the functions $\varphi \chi_{U_t}$ and $\frac{|\nabla u_R|}{u_R}\chi_{B_{r(t)}}$ are equi-measurable, in particular

$$\int_{U_t} \varphi^p \, d\mathcal{L}^n = \int_{B_{r(t)}} \left(\frac{|\nabla u_R|}{u_R}\right)^p d\mathcal{L}^n. \tag{4.1.17}$$

Indeed, let $g(r) = \frac{|\nabla u_R|}{u_R}(r)$, and by coarea formula,

$$\begin{aligned}
|U_t \cap \{\, \varphi > s \,\}| &= \int_{U_t \cap \{\, g(r(u(x)))>s \,\}} d\mathcal{L}^n \\
&= \int_t^{+\infty} \int_{\Omega \cap \partial^* U_\tau \cap \{\, g(r(\tau))>s \,\}} \frac{1}{|\nabla u(x)|} \, d\mathcal{H}^{n-1}(x) \, d\tau \\
&= \int_0^{r(t)} \int_{\Omega \cap \partial^* U_{r^{-1}(\sigma)}} \frac{1}{|\nabla u(x)||r'(r^{-1}(\sigma))|} \chi_{\{\, g(\sigma)>s \,\}} \, d\mathcal{H}^{n-1}(x) \, d\sigma.
\end{aligned} \tag{4.1.18}$$

Notice now that, since

$$\omega_n r(\tau)^n = |U_\tau|,$$

then

$$r'(\tau) = -\frac{1}{n\omega_n r(\tau)^{n-1}} \int_{\Omega \cap \partial^* U_\tau} \frac{1}{|\nabla u(x)|} \, d\mathcal{H}^{n-1}(x). \tag{4.1.19}$$

Therefore, substituting in (4.1.18), we get

$$|U_t \cap \{\,\varphi > s\,\}| = \int_0^{r(t)} n\omega_n \sigma^{n-1} \chi_{\{\,g(\sigma)>s\,\}}\, d\sigma = \left| B_{r(t)} \cap \left\{ \frac{|\nabla u_R|}{u_R} > s \right\} \right|;$$

where we have used polar coordinates to get the last equality. Thus, the claim is proved.

Recalling the definition of φ, (4.1.12) reads

$$\beta \geq \varphi^{p-1},$$

then using (4.1.17) and the definition of H (see Definition 4.1.5), we have

$$\begin{aligned} H(t,\varphi) &= \beta \mathcal{H}^{n-1}(\partial_{\mathrm{e}} U_t) + \int_{\partial_{\mathrm{i}} U_t} \varphi^{p-1}\, d\mathcal{H}^{n-1} - (p-1)\int_{U_t} \varphi^p \, d\mathcal{L}^n \\ &\geq \int_{\partial U_t} \varphi^{p-1}\, d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u_R|}{u_R}\right)^p d\mathcal{L}^n \\ &\geq \int_{\partial B_{r(t)}} \left(\frac{|\nabla u_R|}{u_R}\right)^{p-1} d\mathcal{H}^{n-1} - (p-1) \int_{B_{r(t)}} \left(\frac{|\nabla u_R|}{u_R}\right)^p d\mathcal{L}^n \\ &= H^*\left(u_R(r(t)), \frac{|\nabla u_R|}{u_R}\right) \\ &= \mathrm{Cap}_p^\beta(B_1, B_R), \end{aligned} \tag{4.1.20}$$

where in the last inequality we have used the isoperimetric inequality and the fact that φ is constant on ∂U_t. □

Remark 4.1.11. By Remark 4.1.9, we have that if K and Ω are such that

$$\mathrm{Cap}_p^\beta(K,\Omega) = \mathrm{Cap}_p^\beta(B_1, B_R),$$

then

$$\varphi = \frac{|\nabla u|}{u} \qquad \text{for a. e. } x \in \Omega,$$

so that, by Lemma 4.1.6, we have equality in (4.1.20) for a.e. $t \in (0,1)$. Thus, by the rigidity of the isoperimetric inequality, we get that U_t coincides with a ball up to a $\mathcal{H}^{n-1}$-negligible set for a.e. $t \in (0,1)$. In particular, $\{\,u > 0\,\} = \bigcup_t U_t$ and $\{\,u = 1\,\} = \bigcap_t U_t$ coincide with two balls up to a $\mathcal{H}^{n-1}$-negligible set.

Proof of Theorem 4.1.1. Fix $M = \omega_n R^n$ with $R > 1$. We divide the proof of the minimality of balls into two cases, and subsequently, we study the equality case.

Let us assume that

$$\beta^{\frac{1}{p-1}} \geq \frac{n-1}{p-1},$$

and recall that in this case the function

$$\rho \in [1, +\infty) \mapsto \mathrm{Cap}_p^\beta(B_1, B_\rho)$$

is decreasing. Let u_R be the minimizer of $\mathrm{Cap}_p^\beta(B_1, B_R)$, by Lemma 4.1.3 condition (4.1.12) holds and, by Proposition 4.1.10, we have that a solution to (4.1.4) is given by the concentric balls (B_1, B_R).

Assume now that
$$\frac{n-p}{p-1} < \beta^{\frac{1}{p-1}} < \frac{n-1}{p-1},$$
then, in this case, letting
$$\alpha_{\beta,p} = \frac{(n-1)}{(p-1)\beta^{\frac{1}{p-1}}},$$
the function
$$\rho \in [1, +\infty) \mapsto \mathrm{Cap}_p^\beta(B_1, B_\rho)$$
increases on $[1, \alpha_{\beta,p}]$ and decreases on $[\alpha_{\beta,p}, +\infty)$, and there exist a unique $R_{\beta,p} > \alpha_{\beta,p}$ such that $\mathrm{Cap}_p^\beta(B_1, B_{R_{\beta,p}}) = \mathrm{Cap}_p^\beta(B_1, B_1)$. If $R \geq R_{\beta,p}$ the function u_R, minimizer of $\mathrm{Cap}_p^\beta(B_1, B_R)$, still satisfies condition (4.1.12) and, as in the previous case, a solution to (4.1.4) is given by the concentric balls (B_1, B_R). On the other hand, if $R < R_{\beta,p}$, we can consider $u_{\beta,p}$ the minimizer of $\mathrm{Cap}_p^\beta(B_1, B_{R_{\beta,p}})$. By Lemma 4.1.3 we have that, for the function $u_{\beta,p}$, condition (4.1.12) holds and, by Proposition 4.1.10, we have that if K and Ω are open bounded Lipschitz sets with $K \subseteq \Omega$, $|K| = \omega_n$, and $|\Omega| \leq M$, then
$$\mathrm{Cap}_p^\beta(K, \Omega) \geq \mathrm{Cap}_p^\beta(B_1, B_{R_{\beta,p}}) = \mathrm{Cap}_p^\beta(B_1, B_1)$$
and a solution to (4.1.4) is given by the pair (B_1, B_1).

For what concerns the equality case, we will follow the outline of the rigidity problem given in [137, Section 3] (see also [22, Section 2]). Let $K_0 \subseteq \overline{\Omega_0}$ be such that
$$\mathrm{Cap}_p^\beta(K_0, \Omega_0) = \min_{\substack{|K|=\omega_n \\ |\Omega|\leq M}} \mathrm{Cap}_p^\beta(K, \Omega),$$
let u be the minimizer of $\mathrm{Cap}_p^\beta(K_0, \Omega_0)$. If $\mathring{K}_0 = \Omega_0$, then $|\Omega_0| = |B_1|$ and isoperimetric inequality yields
$$\mathcal{H}^{n-1}(\partial\Omega_0) \geq \mathcal{H}^{n-1}(\partial B_1),$$
while, from the minimality of (K_0, Ω_0) we have that
$$\mathrm{Cap}_p^\beta(K_0, \Omega_0) = \beta\mathcal{H}^{n-1}(\partial\Omega_0) \leq \mathrm{Cap}_p^\beta(B_1, B_1) = \beta\mathcal{H}^{n-1}(\partial B_1),$$
so that $\mathcal{H}^{n-1}(\Omega_0) = \mathcal{H}^{n-1}(\partial B_1)$. Hence, by the rigidity of the isoperimetric inequality we have that $\mathring{K}_0 = \Omega_0$ are balls of radius 1. On the other hand, if $\mathring{K}_0 \neq \Omega_0$, from the first part of the proof, there exists $R_0 > 1$ such that $|B_{R_0}| \geq M$ and
$$\mathrm{Cap}_p^\beta(K_0, \Omega_0) = \mathrm{Cap}_p^\beta(B_1, B_{R_0}).$$
Therefore, by Remark 4.1.11, we have that for a.e. $t \in (0,1)$, the superlevel sets U_t coincide with balls up to $\mathcal{H}^{n-1}$-negligible sets, and $\{ u = 1 \}$ and $\{ u > 0 \}$ coincide with balls, up to $\mathcal{H}^{n-1}$-negligible sets, as well. We only have to show that $\{ u = 1 \}$ and $\{ u > 0 \}$ are concentric balls. To this aim, let us denote by $x(t)$ the center of the ball U_t and by $r(t)$ the radius of U_t, as already done in (4.1.16). In addition, we also have that
$$\frac{|\nabla u_R|}{u_R}\Big(r\big(u(x)\big)\Big) = \varphi(x) = \frac{|\nabla u|}{u}(x),$$

so that, if $u(x) = t$, then $|\nabla u(x)| = C_t > 0$. This ensures that we can write

$$\begin{aligned} x(t) &= \frac{1}{|U_t|} \int_{U_t} x \, dx \\ &= \frac{1}{|U_t|} \left(\int_t^1 \int_{\partial U_s} \frac{x}{|\nabla u(x)|} \, d\mathcal{H}^{n-1}(x) \, ds + \int_K x \, dx \right), \end{aligned}$$

and we can infer that $x(t)$ is an absolutely continuous function, since $|\nabla u| > 0$ implies that $|U_t|$ is an absolutely continuous function as well. Moreover, on ∂U_t we have that for every $\nu \in \mathbb{S}^{n-1}$,

$$u(x(t) + r(t)\nu) = t, \tag{4.1.21}$$

from which

$$\nabla u(x(t) + r(t)\nu) = -C_t \nu. \tag{4.1.22}$$

Differentiating (4.1.21), and using (4.1.22), we obtain

$$-C_t \, x'(t) \cdot \nu - C_t \, r'(t) = 1. \tag{4.1.23}$$

Finally, joining (4.1.23) and (4.1.19), and the fact that $|\nabla u| = C_t$ on ∂U_t, we get

$$x'(t) \cdot \nu = 0$$

for every $\nu \in \mathbb{S}^{n-1}$, so that $x(t)$ is constant and U_t are concentric balls for a.e. $t \in (0, 1)$. In particular, $\{ u = 1 \} = \bigcap_t U_t$ and $\{ u > 0 \} = \bigcup_t U_t$ share the same center. □

4.2 A case in which the optimal set is a segment

The results of this section are contained in the paper [1].

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open, connected and Lipschitz set. We define the Neumann and Steklov eigenvalues as follows: find positive constants μ, σ such that there exist non-zero solutions to the boundary value problems

$$\begin{cases} -\Delta u = \lambda^{\mathrm{N}} u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} \Delta v = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial \nu} = \lambda^{\mathrm{S}} v & \text{on } \partial\Omega. \end{cases}$$

The regularity assumption we made on Ω ensures that we can find two increasing and divergent sequences of eigenvalues

$$0 = \lambda_0^N(\Omega) < \lambda_1^{\mathrm{N}}(\Omega) \le \lambda_2^{\mathrm{N}}(\Omega) \le \cdots \le \lambda_k^{\mathrm{N}}(\Omega) \le \dots,$$

$$0 = \lambda_0^{\mathrm{S}}(\Omega) < \lambda_1^{\mathrm{S}}(\Omega) \le \lambda_2^{\mathrm{S}}(\Omega) \le \cdots \le \lambda_k^{\mathrm{S}}(\Omega) \le \dots,$$

which are the spectrum of the Neumann laplacian and the spectrum of the Dirichlet-to-Neumann map respectively. We recall the variational characterization of the eigenvalues, for $k \ge 0$:

$$\lambda_k^{\mathrm{N}}(\Omega) = \inf_{E \in \mathcal{S}_{k+1}(\Omega)} \sup_{w \in E \setminus \{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2 \, dx}{\displaystyle\int_\Omega w^2 \, dx},$$

$$\lambda_k^{\mathrm{S}}(\Omega) = \inf_{E\in\mathcal{S}_{k+1}(\Omega)} \sup_{w\in E\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1}},$$

where $\mathcal{S}_{k+1}(\Omega)$ is the family of all linear subspaces of $H^1(\Omega)$ of dimension $k+1$. In particular, we are interested in the principal eigenvalues, i.e. $k=1$, namely

$$\lambda_1^{\mathrm{N}}(\Omega) = \inf_{\substack{w\in H^1(\Omega)\setminus\{0\}\\ \int_\Omega w=0}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_\Omega w^2\,dx}, \qquad \lambda_1^{\mathrm{S}}(\Omega) = \inf_{\substack{w\in H^1(\Omega)\setminus\{0\}\\ \int_{\partial\Omega} w=0}} \frac{\displaystyle\int_\Omega |\nabla w|^2\,dx}{\displaystyle\int_{\partial\Omega} w^2\,d\mathcal{H}^{n-1}}.$$

Many authors in the literature identified remarkable similarities between the two families of eigenvalues. Moreover, an underlying relationship holds between the two quantities. For instance, Steklov eigenvalues can be seen as limits of weighted Neumann eigenvalues, while Neumann eigenvalues can be obtained as limits of Steklov eigenvalues by suitably perforating the set Ω. We refer, for instance, to [130], and [104] for these results. We want to explore the same relationship between the two eigenvalues, from the shape optimization point of view.

Namely, we could be interested in the scale invariant ratio

$$F(\Omega) = \frac{|\Omega|\lambda_1^{\mathrm{N}}(\Omega)}{P(\Omega)\lambda_1^{\mathrm{S}}(\Omega)},$$

and, consequently, in the two problems

$$\min_{\Omega\in\mathcal{K}} F(\Omega), \qquad \max_{\Omega\in\mathcal{K}} F(\Omega), \tag{4.2.1}$$

where $\mathcal{K}$ is a suitable class of sets, $|\cdot|$ denotes the area, and $P(\cdot)$ denotes the perimeter. Unfortunately, the choice

$$\mathcal{K} = \left\{ \Omega\subset\mathbb{R}^2 \;\middle|\; \Omega \text{ bounded, open and Lipschitz} \right\}$$

causes the problems in (4.2.1) to be ill-posed, in the sense that

$$\inf_{\mathcal{K}} F(\Omega) = 0, \qquad \sup_{\mathcal{K}} F(\Omega) = +\infty,$$

as shown in [112], [55], and [105].

In order to obtain some comparison between Neumann and Steklov eigenvalues, we address the problems in (4.2.1) restricting the class of admissible sets to

$$\mathcal{K}_c = \left\{ \Omega\subset\mathbb{R}^2 \;\middle|\; \Omega \text{ bounded, open and convex} \right\}. \tag{4.2.2}$$

This choice of $\mathcal{K}_c$ avoids shapes that could make F degenerate, and precisely it could be shown, as in [112], that there exist two constants $c, C>0$ such that

$$c \le F(\Omega) \le C \qquad \forall\,\Omega\in\mathcal{K}_c.$$

Additionally, numerical simulations lead the authors to state the following

Conjecture 4.2.1 (Henrot, Michetti [112]). *Let $\mathcal{K}_c$ be as in (4.2.2), then*

$$1 < F(\Omega) < 2 \qquad \forall\, \Omega \in \mathcal{K}_c.$$

Moreover, the inequalities are sharp in the following sense: there exists a sequence R_n of thinning rectangles, and a sequence T_n of thinning triangles such that

$$\lim_n F(R_n) = 1, \qquad \lim_n F(T_n) = 2.$$

The aim of this section is to take steps towards proving the conjecture; however, we do not provide an exhaustive solution.

The numerical simulations which support Conjecture 4.2.1 also suggest that the infimum and the supremum of $F(\Omega)$, in the class $\mathcal{K}_c$, are asymptotically achieved by particular sequences of thinning domains. Therefore, we focus on the limits of $F(\Omega_\varepsilon)$, where Ω_ε is a family of thinning domains of the type (2.5.1). Indeed, following in the footsteps of [112], for such a family, there exists a non-negative concave function $h : [0,1] \to \mathbb{R}$ such that

$$\lim_{\varepsilon\to 0} \lambda_1^{\mathrm{N}}(\Omega_\varepsilon) = \lambda_1^{\mathrm{N}}(h) \qquad \lim_{\varepsilon\to 0} \frac{P(\Omega_\varepsilon)\lambda_1^{\mathrm{S}}(\Omega_\varepsilon)}{|\Omega_\varepsilon|} = \lambda_1^{\mathrm{S}}(h)\left(\int_0^1 h(t)\,dt\right)^{-1},$$

where $\lambda_1^{\mathrm{N}}(h)$ is the first eigenvalue of the Sturm-Liouville problem

$$\begin{cases} -\dfrac{d}{dx}\left(h(x)\dfrac{dv}{dx}(x)\right) = \lambda_1^{\mathrm{N}}(h)h(x)v(x) & x \in (0,1), \\[2ex] h(0)\dfrac{dv}{dx}(0) = h(1)\dfrac{dv}{dx}(1) = 0, \end{cases} \tag{4.2.3}$$

while $\lambda_1^{\mathrm{S}}(h)$ is the first eigenvalue of the Sturm-Liouville problem

$$\begin{cases} -\dfrac{d}{dx}\left(h(x)\dfrac{dv}{dx}(x)\right) = \lambda_1^{\mathrm{S}}(h)v(x) & x \in (0,1), \\[2ex] h(0)\dfrac{dv}{dx}(0) = h(1)\dfrac{dv}{dx}(1) = 0. \end{cases} \tag{4.2.4}$$

The function h, in some sense, represents the profile of the thinning sets Ω_ε, and, in particular, we have that $h \equiv 1$ represents the limit of a family of thinning rectangles. On the other hand, for every $x_0 \in (0,1)$, we let

$$T_{x_0}(x) := \begin{cases} \dfrac{2x}{x_0} & x \in [0, x_0), \\[2ex] \dfrac{2(1-x)}{1-x_0} & x \in [x_0, 1], \end{cases}$$

and

$$T_0(x) = 2(1-x), \qquad T_1(x) = 2x,$$

which represents the limit of a family of thinning triangles. Consequently, familiarizing oneself with the properties of $\lambda_1^{\mathrm{N}}(h)$ and $\lambda_1^{\mathrm{S}}(h)$ can offer advantages when it comes to analyzing the eigenvalues $\lambda_1^{\mathrm{N}}(\Omega)$ and $\lambda_1^{\mathrm{S}}(\Omega)$. It is worth mentioning that the quantities $\lambda_1^{\mathrm{N}}(h)$ and $\lambda_1^{\mathrm{S}}(h)$ are in a way related to a weighted Hardy constant (see [128], [145], [147], and Proposition 4.2.10).

Following this path, we refer to [147], [112] and [162] for the proof of the subsequent properties: let

$$\mathcal{P} = \{\, h \in L^\infty(0,1) \colon\ h \text{ non negative, concave and not identically zero} \,\},$$

and

$$\mathcal{P}_1 = \left\{ h \in \mathcal{P} \;\middle|\; \int_0^1 h(t)\,dt = 1 \right\},$$

then for every $h \in \mathcal{P}_1$, we have that

$$\pi^2 = \lambda_1^{\mathrm{N}}(1) \le \lambda_1^{\mathrm{N}}(h) \le \lambda_1^{\mathrm{N}}(T_{1/2})$$

$$\lambda_1^{\mathrm{S}}(h) \le \lambda_1^{\mathrm{S}}(p) = 12,$$

where p is the arc of parabola $p(x) = 6x(1-x)$.

Here we state the main results of this section

Theorem 4.2.2. *The minimum problem*

$$\min_{h \in \mathcal{P}_1} \lambda_1^{\mathrm{S}}(h) \tag{4.2.5}$$

admits the functions T_0 and T_1 as unique solutions.

We prove the theorem following two distinct approaches. Section 4.2.1 is devoted to the former, while Section 4.2.2 is devoted to the latter, which relies on a rearrangement method that, up to our knowledge, appears to be new. Finally, in Section 4.2.3 we establish a relationship between $\lambda_1^{\mathrm{N}}(h)$ and $\lambda_1^{\mathrm{S}}(h)$.

Theorem 4.2.3. *There exists an invertible operator*

$$\mathcal{G} : \mathcal{P} \to \mathcal{P}$$

such that, for every $h, k \in \mathcal{P}$, we have

$$\left(\int_0^1 h(t)\,dt \right)^2 \lambda_1^{\mathrm{N}}(h) = \lambda_1^{\mathrm{S}}(\mathcal{G}(h)), \tag{4.2.6}$$

and

$$\left(\int_0^1 \frac{1}{\sqrt{k(t)}}\,dt \right)^2 \lambda_1^{\mathrm{S}}(k) = \lambda_1^{\mathrm{N}}(\mathcal{G}^{-1}(k)). \tag{4.2.7}$$

It may help to solve problems obtained by studying (4.2.1) among thinning domains, namely

$$\min_{h \in \mathcal{P}} \frac{\lambda_1^{\mathrm{N}}(h) \displaystyle\int_0^1 h(t)\,dt}{\lambda_1^{\mathrm{S}}(h)}, \qquad \max_{h \in \mathcal{P}} \frac{\lambda_1^{\mathrm{N}}(h) \displaystyle\int_0^1 h(t)\,dt}{\lambda_1^{\mathrm{S}}(h)}.$$

In particular, we can fully solve the maximizing problem, and partially solve the minimizing problem. We summarize these results in the following theorem.

Theorem 4.2.4. *Let $h \in \mathcal{P}_1$. Then*

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \le 2,$$

and the equality holds if and only if $h = T_{x_0}$ for some $x_0 \in [0,1]$. If, in addition, $h(x) = h(1-x)$ for every $x \in [0,1]$, then

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \ge 1.$$

4.2.1 Minimization of the Steklov eigenvalue

For every $h \in \mathcal{P}_1$ we consider the Sturm-Liouville eigenvalue $\lambda_1^{\mathrm{S}}(h)$ defined in (2.6.3)

Lemmas Lemma 2.5.5 and Lemma 2.6.8 prove that the problems

$$\max \left\{ \lambda_1^{\mathrm{S}}(h) \colon h \in \mathcal{P}_1 \right\}$$

$$\min \left\{ \lambda_1^{\mathrm{S}}(h) \colon h \in \mathcal{P}_1 \right\}$$

admit solutions. In particular, the solution to the maximization problem (see for instance [162]) is given by the parabola $p(x) = 6x(1-x)$, with corresponding eigenvalue $\lambda_1^{\mathrm{S}}(p) = 12$. In this section, we aim to prove Theorem 4.2.2, namely that the problem

$$\min \left\{ \lambda_1^{\mathrm{S}}(h) \colon h \in \mathcal{P}_1 \right\},$$

admits as unique solutions the functions $T_0(x) = 2(1-x)$ and $T_1(x) = 2x$ with corresponding eigenvalue

$$\lambda_1^{\mathrm{S}}(T_0) = \lambda_1^{\mathrm{S}}(T_1) = (j'_{0,1})^2/2,$$

where $j'_{0,1}$ is the first positive zero of the first derivative of the Bessel function J_0.

Remark 4.2.5. The function

$$h \in \mathcal{P} \longmapsto \lambda_1^{\mathrm{S}}(h),$$

satisfies the following properties:

- **monotonicity**: for every $h_0, h_1 \in \mathcal{P}$, if $h_0 \leq h_1$ then
$$\lambda_1^{\mathrm{S}}(h_0) \leq \lambda_1^{\mathrm{S}}(h_1);$$
- **homogeneity**: for every $h \in \mathcal{P}$ and for every $\alpha > 0$,
$$\lambda_1^{\mathrm{S}}(\alpha h) = \alpha \lambda_1^{\mathrm{S}}(h);$$
- **concavity**: for every $h_0, h_1 \in \mathcal{P}$ and for every $t \in [0,1]$, letting $h_t = (1-t)h_0 + th_1$, we have that
$$\lambda_1^{\mathrm{S}}(h_t) \geq (1-t)\lambda_1^{\mathrm{S}}(h_0) + t\,\lambda_1^{\mathrm{S}}(h_1);$$
- **symmetry**: let $h \in \mathcal{P}$, and let $k(x) = h(1-x)$, then
$$\lambda_1^{\mathrm{S}}(k) = \lambda_1^{\mathrm{S}}(h). \tag{4.2.8}$$

Proposition 4.2.6. *Let $h \in \mathcal{P}_1$ be a solution to problem (4.2.5), then h is an extreme point of $\mathcal{P}_1$.*

Proof. Let $h \in \mathcal{P}_1$ be a solution to problem (4.2.5). By contradiction assume that h is not an extreme point of $\mathcal{P}_1$. Let $h_0, h_1 \in \mathcal{P}_1 \setminus \{ h \}$ and $t \in (0,1)$ such that

$$h = (1-t)h_0 + th_1.$$

Let $v \in H^1(0,1)$ be an eigenfunction for $\lambda_1^{\mathrm{S}}(h)$ with

$$\int_0^1 v^2\,dx = 1,$$

then

$$\begin{aligned}\lambda_1^{\mathrm{S}}(h) = \int_0^1 (v')^2 h\,dx &= (1-t)\int_0^1 (v')^2 h_0\,dx + t\int_0^1 (v')^2 h_1\,dx\\ &\geq (1-t)\lambda_1^{\mathrm{S}}(h_0) + t\lambda_1^{\mathrm{S}}(h_1).\end{aligned}$$

On the other hand, by the minimality of $\lambda_1^{\mathrm{S}}(h)$, we have

$$\lambda_1^{\mathrm{S}}(h_0) = \int_0^1 (v')^2 h_0\,dx, \qquad\qquad \lambda_1^{\mathrm{S}}(h_1) = \int_0^1 (v')^2 h_1\,dx.$$

Therefore, v is also an eigenfunction for $\lambda_1^{\mathrm{S}}(h_0)$ and $\lambda_1^{\mathrm{S}}(h_1)$. Let us now prove that $h_0 = h$, thus reaching a contradiction. From the weak form of equation (4.2.4), we have that for every $\varphi \in H^1(0,1)$

$$\begin{aligned}\int_0^1 v'\varphi' h\,dx &= \lambda_1^{\mathrm{S}}(h)\int_0^1 v\varphi\,dx\\ &= \lambda_1^{\mathrm{S}}(h_0)\int_0^1 v\varphi\,dx = \int_0^1 v'\varphi' h_0\,dx,\end{aligned}$$

that is

$$\int_0^1 (h-h_0)v'\varphi'\,dx = 0$$

for every $\varphi \in H^1(0,1)$, which yields $h = h_0$, since, for every $\psi \in L^2(0,1)$, we can choose

$$\varphi(x) = \int_0^x \psi(t)\,dt.$$

$\square$

In order to study the minimum problem (4.2.5), we need to evaluate λ_1^{S} on triangles, and we will need the following result, whose proof can be found in [112].

Lemma 4.2.7. *Let $x_0 \in [0,1]$. Then $\lambda_1^{\mathrm{S}}(T_{x_0})$ is the first non-zero root σ of the equation*

$$J_0\Big(\sqrt{2\sigma}x_0\Big)J_0'\Big(\sqrt{2\sigma}(1-x_0)\Big) + J_0\Big(\sqrt{2\sigma}(1-x_0)\Big)J_0'\Big(\sqrt{2\sigma}x_0\Big) = 0. \tag{4.2.9}$$

In addition, here we summarize the properties of the Bessel function J_0 which we will use.

Proposition 4.2.8. *Let J_0 be the Bessel function of the first kind of order 0, and let $j_{0,1}$ and $j_{0,1}'$ be the first zero of J_0 and J_0' respectively. Then*

$$0 < j_{0,1} < j_{0,1}',$$

and

$$\begin{aligned} J_0(x) &\geq 0 && \forall\, x \in (0, j_{0,1}),\\ J_0'(x) &\leq 0 && \forall\, x \in (0, j_{0,1}'),\\ J_0(x) &< 0 && \forall\, x \in (j_{0,1}, j_{0,1}'). \end{aligned}$$

Proof of Theorem 4.2.2. By Lemma 2.5.5 and Lemma 2.6.8 we have that the minimum problem (4.2.5) admits a solution. On the other hand, by Proposition 4.2.6 and Proposition 2.5.8 we have that a solution to (4.2.5) has to be a triangle T_{x_0} for some $x_0 \in [0,1]$. By the symmetry of λ_1^{S} stated in (4.2.8), we notice that to prove the theorem it is sufficient to show that the function

$$x_0 \in \left[0, \frac{1}{2}\right] \mapsto \lambda_1^{\mathrm{S}}(T_{x_0}),$$

attains its minimum for $x_0 = 0$.

Let $j_{0,1}$ and $j'_{0,1}$ be the first positive roots of J_0 and J'_0 respectively. For every $x \in [0, 1/2]$, and $s \in [0, +\infty)$, let

$$F(x,s) = J_0(sx)J'_0(s(1-x)) + J_0(s(1-x))J'_0(sx),$$

which is the function defined in Lemma 4.2.7 that determines the value $\lambda_1^{\mathrm{S}}(T_{x_0})$. Let $x_0 \in (0, 1/2)$ and let $s(x_0)$ be the smallest non-zero root of the equation

$$F(x_0, s) = 0. \tag{4.2.10}$$

We claim that

$$s(x_0) \in I_{x_0} = \left(\frac{j_{0,1}}{(1-x_0)}, \min\left\{ \frac{j_{0,1}}{x_0}, \frac{j'_{0,1}}{1-x_0} \right\} \right). \tag{4.2.11}$$

Indeed, since J_0 and $-J'_0$ are positive in $(0, j_{0,1})$, and $x_0 < 1 - x_0$, then

$$F(x_0, s) < 0 \qquad \forall\, s \in \left(0, \frac{j_{0,1}}{1-x_0}\right].$$

On the other hand, using again the properties in Proposition 4.2.8, a direct computation gives

$$F\left(x_0, \min\left\{ \frac{j_{0,1}}{x_0}, \frac{j'_{0,1}}{1-x_0} \right\} \right) > 0,$$

thus proving the claim. Notice that (4.2.11) gives

$$\begin{aligned} &J_0(s(x_0)x_0) > 0, \qquad && J_0(s(x_0)(1-x_0)) < 0, \\ &J'_0(s(x_0)x_0) < 0, \qquad && J'_0(s(x_0)(1-x_0)) < 0. \end{aligned} \tag{4.2.12}$$

Since J_0 solves the equation

$$J''_0(t) + \frac{J'_0(t)}{t} + J_0(t) = 0, \tag{4.2.13}$$

then we have

$$\begin{aligned} \partial_s F(x_0, s) =& J'_0(sx_0)J'_0(s(1-x_0)) - J_0(sx_0)J_0(s(1-x_0)) \\ &- \frac{1}{s}\big(J_0(sx_0)J'_0(s(1-x_0)) + J_0(s(1-x_0))J'_0(sx_0)\big). \end{aligned}$$

In particular, (4.2.10) and (4.2.12) ensure that

$$\partial_s F(x_0, s(x_0)) > 0. \tag{4.2.14}$$

By the implicit function theorem, the function $x_0 \mapsto s(x_0)$ is continuous, differentiable and

$$s'(x_0)\partial_s F(x_0, s(x_0)) + \partial_x F(x_0, s(x_0)) = 0.$$

Using (4.2.13), direct computations give

$$\partial_x F(x_0, s(x_0)) = -\frac{J_0(s(x_0)(1-x_0))J_0'(s(x_0)x_0)}{x_0} + \frac{J(s(x_0)x_0)J_0'(s(x_0)(1-x_0))}{1-x_0}.$$

As before, (4.2.12) ensure that

$$\partial_x F(x_0, s(x_0)) < 0. \tag{4.2.15}$$

Joining (4.2.14) and (4.2.15), we have that $s'(x_0) > 0$ and $x_0 \mapsto s(x_0)$ is increasing. Finally,

$$\lambda_1^{\mathrm{S}}(T_{x_0}) = s^2(x_0)/2$$

is increasing for $x_0 \in (0, 1/2)$, and the minimum is achieved when $x_0 = 0$. $\square$

Remark 4.2.9. Equation (4.2.9) for $x_0 = 0$ reduces to

$$J_0'\left(\sqrt{2\sigma}\right) = 0$$

that is, $\lambda_1^{\mathrm{S}}(2x) = \lambda_1^{\mathrm{S}}(T_0) = (j_{0,1}')^2/2$.

4.2.2 An alternative proof for the minimum of $\lambda_1^{\mathrm{S}}(h)$

In this section, we minimize $\lambda_1^{\mathrm{S}}(h)$ using an alternative approach that avoids the explicit computation of the eigenvalue. In particular, our aim is to define a particular kind of symmetrization that allows us to prove that solutions to (4.2.5) have to be monotone. Before defining the aforementioned symmetrization we prove an equivalent formulation for the eigenvalue $\lambda_1^{\mathrm{S}}(h)$, referring to the ideas for the proof in [145, Lemma 4.2]

Proposition 4.2.10. *Let $h \in \mathcal{P}_1$, then*

$$\lambda_1^{\mathrm{S}}(h) = \min\left\{ \frac{\displaystyle\int_0^1 (\varphi')^2\,dx}{\displaystyle\int_0^1 \frac{\varphi^2}{h}\,dx} \;\middle|\; \begin{array}{l} \varphi \in H_0^1(0,1), \\ \displaystyle\int_0^1 \frac{\varphi^2}{h}\,dx < \infty \end{array} \right\}.$$

Proof. Let $v \in H^1(0,1)$ be a weak solution to (4.2.4) and let $w = hv'$. Then, since $w' = -\lambda_1^{\mathrm{S}}(h)v$, we have that $w \in H^2(0,1) \cap H_0^1(0,1)$ and that is a solution to

$$\begin{cases} -w''(x) = \dfrac{\lambda_1^{\mathrm{S}}(h)}{h(x)} w(x) & x \in (0,1) \\ w(0) = w(1) = 0, \end{cases} \tag{4.2.16}$$

and

$$\frac{\displaystyle\int_0^1 (w')^2\,dx}{\displaystyle\int_0^1 \frac{w^2}{h}\,dx} = \lambda_1^{\mathrm{S}}(h). \tag{4.2.17}$$

Following classical arguments (see for instance [94]) we have that v vanishes in one and only one point $x_0 \in (0,1)$, so that w' vanishes only in x_0. Without loss of generality, we can assume that w' is positive in $[0, x_0)$ and it is negative in $(x_0, 1]$. Let now $\varphi \in H^1_0(0,1)$ be such that

$$\int_0^1 \frac{\varphi^2}{h}\,dx < +\infty.$$

Then, for every $0 < x < x_0$, we have that

$$\begin{aligned}\frac{1}{h(x)}\varphi^2(x) &\le \frac{1}{h(x)}\left(\int_0^x \frac{(\varphi'(t))^2}{w'(t)}\,dt\right)\left(\int_0^x w'(t)\,dt\right)\\ &= \frac{w(x)}{h(x)}\int_0^x \frac{(\varphi'(t))^2}{w'(t)}\,dt.\end{aligned}$$

Then, integrating from 0 to $x_1 < x_0$, we get

$$\begin{aligned}\int_0^{x_1}\frac{\varphi^2(x)}{h(x)}\,dx &\le \int_0^{x_1}\frac{w(x)}{h(x)}\int_0^x \frac{(\varphi'(t))^2}{w'(t)}\,dt\,dx\\ &= \int_0^{x_1}\frac{(\varphi'(t))^2}{w'(t)}\int_t^{x_1}\frac{w(x)}{h(x)}\,dx\,dt.\end{aligned}$$

Using (4.2.16), and the fact that $w'(x_1) > 0$, then we have

$$\int_0^{x_1}\frac{\varphi^2(x)}{h(x)}\,dx \le \frac{1}{\lambda_1^{\mathrm{S}}(h)}\int_0^{x_1}(\varphi'(t))^2\,dt.$$

Letting x_1 go to x_0 we have

$$\int_0^{x_0}\frac{\varphi^2(x)}{h(x)}\,dx \le \frac{1}{\lambda_1^{\mathrm{S}}(h)}\int_0^{x_0}(\varphi'(x))^2\,dx. \tag{4.2.18}$$

Similar computations can be done in the case $x > x_0$, so that we have

$$\int_{x_0}^{1}\frac{\varphi^2(x)}{h(x)}\,dx \le \frac{1}{\lambda_1^{\mathrm{S}}(h)}\int_{x_0}^{1}(\varphi'(x))^2\,dx. \tag{4.2.19}$$

Joining (4.2.18) and (4.2.19) we have

$$\int_0^1\frac{\varphi^2(x)}{h(x)}\,dx \le \frac{1}{\lambda_1^{\mathrm{S}}(h)}\int_0^1(\varphi'(x))^2\,dx,$$

that is

$$\frac{\displaystyle\int_0^1 (\varphi')^2\,dx}{\displaystyle\int_0^1 \frac{\varphi^2}{h}\,dx} \ge \lambda_1^{\mathrm{S}}(h). \tag{4.2.20}$$

Since w is an admissible function, the assertion follows from (4.2.17) and (4.2.20). $\square$

We now define the rearrangement mentioned above. Let

$$w : [0,1] \to \mathbb{R}$$

be a quasi-concave piecewise C^1 function such that

$$|\{w' = 0\}| = 0, \qquad w(0) = w(1) = 0,$$

and let us denote by

$$w_M = \max_{[0,1]} w,$$

and by x_M the maximum point of w. We aim to rearrange w in such a way that the derivative of the rearranged function w° concentrates at the left of the new maximum point x_M^*.

For every $t \in (0, w_M)$, we define $(x_t, y_t) := \{w(x) > t\}$, and the distribution functions

$$\begin{aligned} \eta_1(t) &= x_M - x_t = |\{\, w > t \,\} \cap (0, x_M)|, \\ \eta_2(t) &= y_t - x_M = |\{\, w > t \,\} \cap (x_M, 1)|. \end{aligned} \tag{4.2.21}$$

Notice that

$$\eta_1 : (0, w_M) \to (0, x_M), \qquad \eta_2 : (0, w_M) \to (0, 1 - x_M)$$

are both decreasing, invertible, absolutely continuous functions, and that, for a.e. $t \in [0,1]$,

$$\eta_1'(t) = -\frac{1}{|w'(x_t)|}, \qquad \eta_2'(t) = -\frac{1}{|w'(y_t)|}.$$

Let us now define the rearranged distribution functions in such a way that, for a.e. $t \in [0,1]$,

$$\begin{aligned} \eta_{*,1}'(t) &= \max\{\eta_1'(t), \eta_2'(t)\}, \\ \eta_{*,2}'(t) &= \min\{\eta_1'(t), \eta_2'(t)\}, \end{aligned} \tag{4.2.22}$$

namely,

$$\begin{aligned} \eta_{*,1}(t) &:= -\int_t^{w_M} \max\{\eta_1'(s), \eta_2'(s)\}\, ds, \\ \eta_{*,2}(t) &:= -\int_t^{w_M} \min\{\eta_1'(s), \eta_2'(s)\}\, ds. \end{aligned} \tag{4.2.23}$$

Remark 4.2.11. Here we emphasize some properties of these distribution functions:

- for every $t \in (0, w_M)$, we have

$$\eta_1(t) + \eta_2(t) = \eta_{*,1}(t) + \eta_{*,2}(t) = |\{w > t\}|;$$

- by (4.2.22), we have that, for a.e. $t \in (0, w_M)$,

$$\begin{aligned} \frac{1}{|\eta_{*,1}'(t)|} &= \max\left\{\frac{1}{|\eta_1'(t)|}, \frac{1}{|\eta_2'(t)|}\right\} \\ &= \max\{|w'(x_t)|, |w'(y_t)|\}, \end{aligned}$$

  and

$$\begin{aligned} \frac{1}{|\eta_{*,2}'(t)|} &= \min\left\{\frac{1}{|\eta_1'(t)|}, \frac{1}{|\eta_2'(t)|}\right\} \\ &= \min\{|w'(x_t)|, |w'(y_t)|\}. \end{aligned}$$

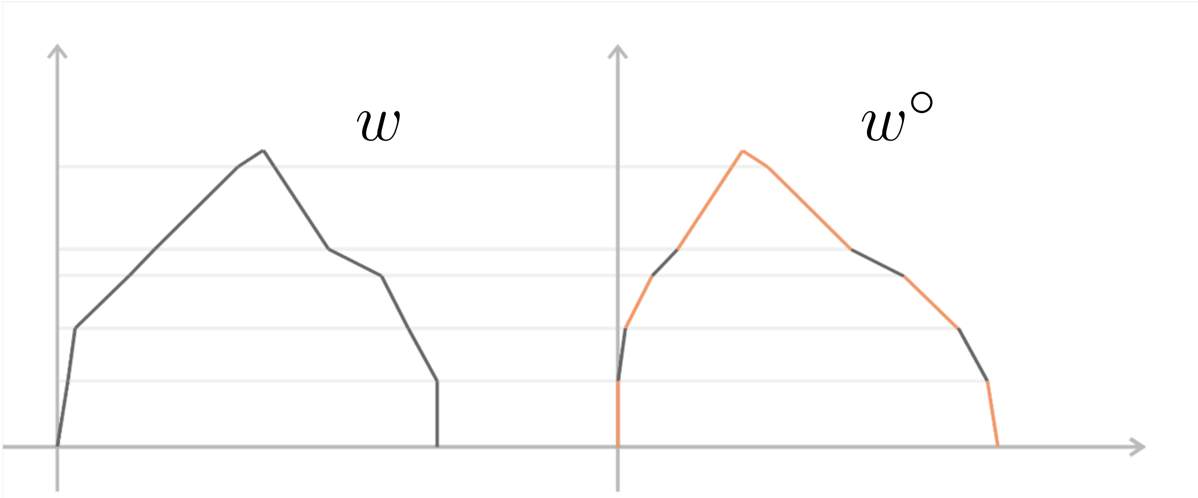


Figure 4.2: Function w° when w is a quasi-concave affine function

- By (4.2.22), we have

$$\frac{1}{|\eta'_{*,1}(t)|^\alpha} + \frac{1}{|\eta'_{*,2}(t)|^\alpha} = \frac{1}{|\eta'_1(t)|^\alpha} + \frac{1}{|\eta'_2(t)|^\alpha} \tag{4.2.24}$$

  for every $\alpha \in \mathbb{R}$.

- for $t = 0$, we denote by

$$x^*_M := \eta_{*,1}(0) = 1 - \eta_{*,2}(0),$$

  this will play the role of the maximum point for the rearranged function.

- the functions

$$\eta_{*,1} : (0, w_M) \to (0, x^*_M), \qquad \eta_{*,2} : (0, w_M) \to (0, 1 - x^*_M)$$

  are decreasing, invertible, absolutely continuous functions.

We now define the rearrangement w° as follows:

Definition 4.2.12. Let w be a quasi-concave piecewise C^1 function such that

$$|\{w' = 0\}| = 0, \qquad w(0) = w(1) = 0,$$

and let $\eta_1, \eta_2, \eta_{*,1}$ and $\eta_{*,2}$ be the functions defined in (4.2.21), and (4.2.23). We define the competitor w° as

$$w^\circ(x) = \begin{cases} w_m - \eta_{*,1}^{-1}(x) & \text{if } x \le x^*_M, \\ w_m - \eta_{*,2}^{-1}(1 - x) & \text{if } x > x^*_M. \end{cases}$$

Remark 4.2.13. From the definition we have that w° is increasing in $[0, x_M^*)$ and decreasing in $(x_M^*, 0]$, so that w° is quasi-concave. Moreover, we have that w° and w are equi-measurable, i.e.

$$\|w^\circ\|_{L^p(0,1)} = \|w\|_{L^p(0,1)}$$

for every $p \in [1, +\infty]$.

We now prove some useful properties of this rearrangement.

Lemma 4.2.14. *Let w be a quasi-concave piecewise C^1 function such that*

$$|\{w' = 0\}| = 0, \qquad w(0) = w(1) = 0,$$

and let w° be the competitor defined in Definition 4.2.12. Then

$$w^\circ(x) = (w(1-x))^\circ.$$

Proof. Let us set $v(x) = w(1-x)$ and $\nu_1, \nu_2, \nu_{*,1}$ and $\nu_{*,2}$ the equivalent quantities defined for v. Then we have

$$\nu_1'(t) = \eta_2'(t), \qquad \nu_2'(t) = \eta_1'(t),$$

and, in particular,

$$\nu_{*,1}'(t) = \eta_{*,1}'(t), \qquad \nu_{*,2}'(t) = \eta_{*,2}'(t).$$

□

Lemma 4.2.15. *Let w be a quasi-concave piecewise C^1 function such that*

$$|\{w' = 0\}| = 0, \qquad w(0) = w(1) = 0,$$

and let w° be its competitor defined in Definition 4.2.12. Then,

$$\|(w^\circ)'\|_{L^p(0,1)} = \|w'\|_{L^p(0,1)} \qquad \forall\, p \geq 1. \tag{4.2.25}$$

Proof. Let us compute separately the norms: by the coarea formula (see Theorem 2.3.1), we get

$$\begin{aligned}
\int_0^1 |w'(x)|^p\, dx &= \int_0^{w_M} \int_{\{w=t\}} |w'(x)|^{p-1}\, d\mathcal{H}^0(x)\, dt \\
&= \int_0^{w_M} \left(|w'(x_t)|^{p-1} + |w'(y_t)|^{p-1}\right) dt \\
&= \int_0^{w_M} \left(\frac{1}{|\eta_1'(t)|^{p-1}} + \frac{1}{|\eta_2'(t)|^{p-1}}\right) dt.
\end{aligned} \tag{4.2.26}$$

Analogously,

$$\int_0^1 |(w^\circ)'(x)|^p\, dx = \int_0^{w_M} \frac{1}{|\eta_{*,1}'(t)|^{p-1}} + \frac{1}{|\eta_{*,2}'(t)|^{p-1}}\, dt. \tag{4.2.27}$$

Joining (4.2.26), (4.2.27), and (4.2.24), we get (4.2.25). □

We now state the property of w° that will be crucial in the proof of Theorem 4.2.2.

Lemma 4.2.16. *Let w be a quasi-concave piecewise C^1 function such that*

$$|\{w' = 0\}| = 0, \qquad w(0) = w(1) = 0,$$

and let w° be its competitor defined in Definition 4.2.12. Assume that

$$h : (0,1) \to [0, +\infty)$$

is a concave function, and let h_* be its increasing rearrangement. Then

$$\int_0^1 \frac{w^2}{h}\,dx \le \int_0^1 \frac{(w^\circ)^2}{h_*}\,dx.$$

Proof. By Fubini's theorem, we can write

$$\int_0^1 \frac{w^2(x)}{h(x)}\,dx = \int_0^1 w^2(x) \int_0^{\frac{1}{h(x)}} dt\,dx = \int_0^\infty \int_{\left\{\frac{1}{h(x)} > t\right\}} w^2(x)\,dx\,dt.$$

The same computation leads to

$$\int_0^1 \frac{(w^\circ)^2(x)}{h_*(x)}\,dx = \int_0^\infty \int_{\left\{\frac{1}{h_*(x)} > t\right\}} (w^\circ)^2(x)\,dx\,dt.$$

Hence, to prove the lemma it is sufficient to prove that for a.e. $t > 0$

$$\int_{\left\{\frac{1}{h(x)} > t\right\}} w^2(x)\,dx \le \int_{\left\{\frac{1}{h_*(x)} > t\right\}} (w^\circ)^2(x)\,dx.$$

For every $t \in (0, \|1/h\|_\infty)$, let us define

$$D_t := \left\{\frac{1}{h(x)} > t\right\} = (0, \tilde{x}_t) \cup (\tilde{y}_t, 1),$$

for some $\tilde{x}_t, \tilde{y}_t \in (0,1)$. In an analogous way, by the definition of increasing rearrangement (see Definition 2.4.3), we have

$$D_t^* = \left\{\frac{1}{h_*(x)} > t\right\} = (0, \tilde{x}_t + 1 - \tilde{y}_t).$$

Let $m = \min\{w(\tilde{x}_t), w(\tilde{y}_t)\}$, and let us define the following auxiliary functions

$$f(x) = \min\{w(x), m\}^2, \qquad g(x) = (w^2 - m^2)_+,$$

so that

$$\int_{D_t} w^2\,dx = \int_{D_t} f\,dx + \int_{D_t} g\,dx.$$

Similarly, we define

$$f_0(x) = \min\{w^\circ(x), m\}^2, \qquad g_0(x) = ((w^\circ)^2 - m^2)_+,$$

so that

$$\int_{D_t^*} (w^\circ)^2\,dx = \int_{D_t^*} f_0\,dx + \int_{D_t^*} g_0\,dx.$$

We now evaluate separately the two terms:

1. By the definition of m, we have that

$$w(x) > m \qquad \forall\, x \in (0,1) \setminus D_t.$$

Therefore, since f and f_0 are equi-measurable, we get

$$\begin{aligned}\int_{D_t} f(x)\,dx &= \int_0^1 f(x)\,dx - (1 - |D_t|)m^2 \\ &= \int_0^1 f_0(x)\,dx - \int_{(0,1)\setminus D_t^*} m^2\,dx \\ &\le \int_0^1 f_0(x)\,dx - \int_{(0,1)\setminus D_t^*} f_0(x)\,dx \\ &= \int_{D_t^*} f_0(x)\,dx,\end{aligned} \tag{4.2.28}$$

where we have used that $|D_t| = |D_t^*|$, and that $m \ge f_0$;

2. Lemma 4.2.14 allows us to assume without loss of generality that $w(\tilde{y}_t) = m$. Therefore, the quasi-concavity of w ensures that

$$w(x) \le m \qquad \forall\, x \in (\tilde{y}_t, 1),$$

and we can write

$$\int_{D_t} g(x)\,dx = \int_0^{\tilde{x}_t} (w^2(x) - m^2)_+\,dx = \int_m^{w_M} 2r|\{w > r\} \cap (0, \tilde{x}_t)|\,dr. \tag{4.2.29}$$

On the other hand,

$$\begin{aligned}\int_{D_t^*} g_0(x)\,dx &= \int_0^{\tilde{x}_t + 1 - \tilde{y}_t} g_0(x)\,dx \\ &\ge \int_0^{\tilde{x}_t} g_0(x)\,dx \\ &= \int_m^{w_M} 2r|\{w^\circ > r\} \cap (0, \tilde{x}_t)|\,dr.\end{aligned} \tag{4.2.30}$$

We now claim that

$$|\{w^\circ > r\} \cap (0, \tilde{x}_t)| \ge |\{w > r\} \cap (0, \tilde{x}_t)|. \tag{4.2.31}$$

Indeed, if we let

$$\{w > r\} = (x_r, y_r), \qquad\qquad \{w^\circ > r\} = (x_r^*, y_r^*),$$

then (4.2.22) gives

$$x_r^* = -\int_0^r \eta'_{*,1}(s)\,ds \le -\int_0^r \eta'_1(s)\,ds = x_r,$$

while the equi-misurability of w and w° gives

$$y_r^* = (y_r^* - x_r^*) + x_r^* = (y_r - x_r) + x_r^* \leq y_r.$$

Therefore we get

$$\begin{aligned} |\{w^\circ > r\} \cap (0, \tilde{x}_t)| &= |\{w > r\} \cap (0, \tilde{x}_t)| \qquad &\text{if } y_r \leq \tilde{x}_t, \\ |\{w^\circ > r\} \cap (0, \tilde{x}_t)| &> |\{w > r\} \cap (0, \tilde{x}_t)| \qquad &\text{if } y_r > \tilde{x}_t, \end{aligned}$$

thus the claim is proved. Finally, joining (4.2.29), (4.2.30), and (4.2.31), we have that

$$\int_{D_t^*} g_0(x)\, dx \geq \int_{D_t} g(x)\, dx, \tag{4.2.32}$$

and the result follows from (4.2.32), and (4.2.28). □

We now turn our attention to the eigenvalue problem.

Alternative proof of Theorem 4.2.2. Let $h \in \mathcal{P}_1$, by Proposition 4.2.10 we have that

$$\lambda_1^{\mathrm{S}}(h) = \min \left\{ \frac{\displaystyle\int_0^1 (\varphi')^2\, dx}{\displaystyle\int_0^1 \frac{\varphi^2}{h}\, dx} : \varphi \in H_0^1(0,1) \right\}. \tag{4.2.33}$$

Let w be a minimizer in (4.2.33), then by Lemma 4.2.15, and Lemma 4.2.16, we have

$$\lambda_1^{\mathrm{S}}(h) = \frac{\displaystyle\int_0^1 (w')^2}{\displaystyle\int_0^1 \frac{w^2}{h}} \geq \frac{\displaystyle\int_0^1 ((w^\circ)')^2}{\displaystyle\int_0^1 \frac{(w^\circ)^2}{h_*}} \geq \lambda_1^{\mathrm{S}}(h_*). \tag{4.2.34}$$

By Proposition 4.2.6, and Proposition 2.5.8, we have that the minimum of λ_1^{S} is a triangle T_{x_0} for some $x_0 \in [0,1]$. Let $h = T_{x_0}$, then $h_* = T_1$ and, from (4.2.34), we have

$$\lambda_1^{\mathrm{S}}(T_{x_0}) \geq \lambda_1^{\mathrm{S}}(T_1),$$

which concludes the proof. □

4.2.3 Ratio μ/σ

In this section, we prove Theorem 4.2.3 and Theorem 4.2.4. We begin by defining an operator $\mathcal{G}$ on $\mathcal{P}$ as follows: let $h \in \mathcal{P}$, and let

$$H(x) = \frac{1}{\int_0^1 h(t)\, dt} \int_0^x h(t)\, dt; \tag{4.2.35}$$

we notice that H is a strictly increasing function such that $H(0) = 0$ and $H(1) = 1$. We then define

$$\mathcal{G}(h)(x) = h^2(H^{-1}(x)).$$

Lemma 4.2.17. *Let $h \in \mathcal{P}$. Then $\mathcal{G}(h) \in \mathcal{P}$, and the map*

$$\mathcal{G} : \mathcal{P} \to \mathcal{P}$$

is invertible.

Proof. Since $h \in \mathcal{P}$, then h' is defined a.e. in $[0,1]$, and h' is decreasing. We also have that H^{-1} is a locally Lipschitz function and

$$\frac{d}{dx} H^{-1}(x) = \frac{1}{h(H^{-1}(x))} \int_0^1 h(t)\, dt. \tag{4.2.36}$$

Therefore, $\mathcal{G}(h)$ is a.e. differentiable and

$$\frac{d}{dx}\mathcal{G}(h)(x) = 2\alpha h'(H^{-1}(x)),$$

where

$$\alpha = \int_0^1 h(t)\, dt.$$

Since H^{-1} is an increasing function and h' is decreasing, then $\mathcal{G}(h)$ is a concave function, and $\mathcal{G}(h) \in \mathcal{P}$.

Let $k \in \mathcal{P}$ and define

$$K(x) = \frac{1}{\int_0^1 \frac{1}{\sqrt{k(t)}}\, dt} \int_0^x \frac{1}{\sqrt{k(t)}}\, dt, \tag{4.2.37}$$

then we want to prove that

$$\sqrt{k(K^{-1}(x))} = \mathcal{G}^{-1}(k)(x). \tag{4.2.38}$$

First we prove that $\sqrt{k \circ K^{-1}} \in \mathcal{P}$. By direct computation,

$$\frac{d}{dx}\sqrt{k(K^{-1}(x))} = \frac{\beta k'(K^{-1}(x))}{2k(K^{-1}(x))},$$

where

$$\beta = \int_0^1 \frac{1}{\sqrt{k(t)}}\, dt.$$

This proves that $\sqrt{k \circ K^{-1}}$ is concave, since K^{-1} is increasing and h'/h is decreasing because of the concavity of h. On the other hand, to prove (4.2.38), we observe that with a change of variables we get

$$\int_0^x \sqrt{k(K^{-1}(t))}\, dt = \frac{K(x)}{\int_0^1 \frac{1}{\sqrt{k(t)}}\, dt},$$

and by definition of $\mathcal{G}$ we get

$$\mathcal{G}\Big(\sqrt{k \circ K^{-1}}\Big)(x) = k(x).$$

$\square$

We now prove that $\mathcal{G}$ is the operator in Theorem 4.2.3.

Proof of Theorem 4.2.3. Let $v \in H^1(0,1)$ be a function such that

$$\int_0^1 v(t)h(t)\,dt = 0,$$

and let H denote the integral function defined in (4.2.35). The change of variables $H(t) = s$ yields

$$\left(\int_0^1 h(t)\,dt\right)\int_0^1 v(t)\,h(t)\,dt = \int_0^1 v(H^{-1}(s))\,ds,$$

$$\left(\int_0^1 h(t)\,dt\right)\int_0^1 (v')^2(t)\,h(t)\,dt = \int_0^1 (v')^2(H^{-1}(s))\,ds,$$

and

$$\left(\int_0^1 h(t)\,dt\right)\int_0^1 v^2(t)\,h(t)\,dt = \int_0^1 v^2(H^{-1}(s))\,ds.$$

Let $w(x) = v(H^{-1}(x))$, then by (4.2.36),

$$w'(x) = \left(\int_0^1 h(t)\,dt\right) v'(H^{-1}(x))\,(\mathcal{G}(h)(x))^{-\frac{1}{2}}.$$

Hence,

$$\frac{\int_0^1 (v')^2(t)\,h(t)\,dt}{\int_0^1 v^2(t)\,h(t)\,dt} = \left(\int_0^1 h(t)\,dt\right)^{-2}\frac{\int_0^1 (w')^2(t)\,\mathcal{G}(h)(t)\,dt}{\int_0^1 w^2(t)\,dt}.$$

Choosing $v = v_\mu$ to be the eigenfunction of $\lambda_1^{\mathrm{N}}(h)$, then we get

$$\lambda_1^{\mathrm{N}}(h) \geq \left(\int_0^1 h(t)\,dt\right)^{-2}\lambda_1^{\mathrm{S}}(h).$$

On the other hand, choosing $w = w_\sigma$ to be the eigenfunction of $\lambda_1^{\mathrm{S}}(\mathcal{G}(h))$, we get

$$\lambda_1^{\mathrm{N}}(h) \leq \left(\int_0^1 h(t)\,dt\right)^{-2}\lambda_1^{\mathrm{S}}(\mathcal{G}(h)),$$

which gives (4.2.6).

Let $k \in \mathcal{P}$ and let K be the integral function defined in (4.2.37). If we evaluate the integral on the right-hand side by means of the change of variables $t = K(s)$, we finally get

$$\int_0^1 h(t)\,dt = \left(\int_0^1 \frac{1}{\sqrt{k(t)}}\,dt\right)^{-1},$$

which gives (4.2.7). □

The following punctual estimate will be crucial.

Proposition 4.2.18. *Let $h \in \mathcal{P}$. Then*

$$\left(\int_0^1 h(t)\,dt\right)^{-1} \mathcal{G}(h)(x) \le 2h(x).$$

Proof. Up to rescaling h, we can assume without loss of generality that $h \in \mathcal{P}_1$. Notice, in addition, that if $h \equiv 1$, then the proof is trivial. Therefore, let $h \in \mathcal{P}_1$ and $h \ne 1$, and define

$$H(x) = \int_0^x h(t)\,dt.$$

We claim that there exists a unique $\bar{x} \in [0,1]$ such that

$$\begin{aligned} H(x) &\le x \qquad && x \in [0,\bar{x}], \\ H(x) &\ge x \qquad && x \in [\bar{x},1]. \end{aligned} \tag{4.2.39}$$

Indeed, if we denote by

$$f(x) = H(x) - x,$$

then, by the concavity of h and the integral constraint, we have that the equation $h = 1$ admits at most two solutions (h cannot be equal to 1 in an entire interval, otherwise the concavity would give $\|h\|_1 < 1$). Therefore, we have that there exist two points $x_1 \in [0,1)$, and $x_2 \in (0,1]$ such that

$$\begin{aligned} f'(x) &< 0 \qquad && x \in [0,x_1) \cup (x_2,1], \\ f'(x) &> 0 \qquad && x \in (x_1,x_2). \end{aligned}$$

Finally, noticing that $f(0) = f(1) = 0$, then we have that there exists a unique zero $\bar{x}$ of f in the interval $[x_1,x_2]$, thus the claim is proved. In particular, we have that

$$\begin{aligned} H^{-1}(x) &\ge x \qquad && x \in [0,\bar{x}], \\ H^{-1}(x) &\le x \qquad && x \in [\bar{x},1], \end{aligned}$$

and $h(\bar{x}) > 1$.

These estimates allow us to compare the derivatives of $\mathcal{G}(h)$ and h. Denoting by $g(x) = \mathcal{G}(h)(x)$, we have that

$$g'(x) = 2h'(H^{-1}(x)) \le 2h'(x) \qquad x \in [0,\bar{x}], \tag{4.2.40}$$

$$g'(x) = 2h'(H^{-1}(x)) \ge 2h'(x) \qquad x \in [\bar{x},1]. \tag{4.2.41}$$

We recall that, as in (2.5.2), the concavity of h ensures that

$$\|h\|_\infty \le 2.$$

Therefore, we get

$$g(0) = h^2(0) \le 2h(0),$$

and by (4.2.40),

$$g(x) \le 2h(x) \qquad x \in [0,\bar{x}].$$

Analogously,

$$g(1) = h^2(1) \le 2h(1),$$

and, by (4.2.41),

$$g(x) \le 2h(x) \qquad x \in [\bar{x},1].$$

□

Proposition 4.2.19. *Let $h \in \mathcal{P}_1$.*

$$\mathcal{G}(h) = 2h$$

if and only if $h = T_{x_0}$ for some $x_0 \in [0,1]$.

Proof. By direct computation, one can prove that if $h = T_{x_0}$ for some $x_0 \in [0,1]$, then

$$h^2(x) = 2h(H(x)).$$

Let us now assume that $\mathcal{G}(h) = 2h$. Notice that, if $y \in [0,1]$ is a fixed point of the integral function H, then

$$h^2(y) = h^2(H^{-1}(y)) = \mathcal{G}(h)(y) = 2h(y), \tag{4.2.42}$$

so that either $h(y) = 0$ or $h(y) = 2$. In particular, if $h(y) = 2$, then by the concavity of h, we have that

$$h(x) = T_y(x) \qquad \forall\, x \in [0,1].$$

Since 0 and 1 are always fixed points of H, if either $h(0) = 2$ or $h(1) = 2$ the assertion is proved. Therefore, let us assume that

$$h(0) = 0 = h(1),$$

then the equation $h = 1$ admits at least two distinct solutions $0 < x_1 < x_2 < 1$ and, arguing as in the proof of Proposition 4.2.18, we have that there exists a fixed point $\bar{x} \in [x_1, x_2]$ for the function H and, by (4.2.42), necessarily $h(\bar{x}) = 2$ and $h = T_{\bar{x}}$. □

Proposition 4.2.20. *Let $h \in \mathcal{P}_1$, then*

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \leq 2 \tag{4.2.43}$$

and the equality holds if and only if $h = T_{x_0}$ for some $x_0 \in [0,1]$.

Proof. Let w be an eigenfunction for $\lambda_1^{\mathrm{S}}(h)$. Using Theorem Theorem 4.2.3, Proposition 4.2.18, and the variational characterization of $\lambda_1^{\mathrm{S}}(\mathcal{G}(h))$, we obtain

$$\lambda_1^{\mathrm{N}}(h) = \lambda_1^{\mathrm{S}}(\mathcal{G}(h)) \leq \frac{\displaystyle\int_0^1 (w')^2 \mathcal{G}(h)\, dx}{\displaystyle\int_0^1 w^2\, dx} \leq \frac{2\displaystyle\int_0^1 (w')^2 h\, dx}{\displaystyle\int_0^1 w^2\, dx} = 2\lambda_1^{\mathrm{S}}(h), \tag{4.2.44}$$

thus proving (4.2.43). Assume now that for some $h \in \mathcal{P}_1$ equality holds, then by (4.2.44) we have

$$\int_0^1 (w')^2 (\mathcal{G}(h) - 2h)\, dx = 0. \tag{4.2.45}$$

Since $\mathcal{G}(h) \leq 2h$, then (4.2.45) yields $\mathcal{G}(h) = 2h$, and Proposition 4.2.19 ensures that $h = T_{x_0}$ for some $x_0 \in [0,1]$. □

Remark 4.2.21. Since it is not possible in general to have that $\mathcal{G}(h) \geq h$, then the same argument cannot be used for the lower bound

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \geq 1.$$

For instance, let

$$h(x) = \frac{1}{2} + x,$$

then $\mathcal{G}(h)(0) = h^2(0) < h(0)$, while $\mathcal{G}(h)(1) = h^2(1) > h(1)$.

Here we prove the lower bound in Theorem 4.2.4 in the symmetric case.

Proposition 4.2.22. *Let $h \in \mathcal{P}_1$ such that $h(1-x) = h(x)$ for all $x \in [0,1]$. Then*

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \geq 1. \tag{4.2.46}$$

Proof. Let $g = \mathcal{G}(h)$. By the variational characterization (4.2.33) of λ_1^{S}, and Theorem 4.2.3, we can find a function $w \in H^2(0,1)$, symmetric with respect to $x = 1/2$, such that

$$\lambda_1^{\mathrm{N}}(h) = \lambda_1^{\mathrm{S}}(g) = \frac{\displaystyle\int_0^1 (w')^2(x)\,dx}{\displaystyle\int_0^1 \frac{w^2(x)}{g(x)}\,dx}, \tag{4.2.47}$$

and w solves the problem

$$\begin{cases} -w''(x) = \dfrac{\lambda_1^{\mathrm{S}}(h)}{h(x)} w(x) & x \in (0,1), \\ w(0) = w(1) = 0. \end{cases}$$

We can choose w to be positive and concave, so that

$$\begin{aligned} w'(x) &\geq 0 && \text{in } \left(0, \frac{1}{2}\right), \\ w'(x) &\leq 0 && \text{in } \left(\frac{1}{2}, 1\right). \end{aligned} \tag{4.2.48}$$

Moreover, by the variational characterization (4.2.33), we get

$$\lambda_1^{\mathrm{S}}(h) \leq \frac{\displaystyle\int_0^1 (w')^2(x)\,dx}{\displaystyle\int_0^1 \frac{w^2(x)}{h(x)}\,dx}, \tag{4.2.49}$$

and then, joining (4.2.47) and (4.2.49), we get

$$\frac{\lambda_1^{\mathrm{N}}(h)}{\lambda_1^{\mathrm{S}}(h)} \geq \frac{\displaystyle\int_0^1 \frac{w^2(x)}{h(x)}\,dx}{\displaystyle\int_0^1 \frac{w^2(x)}{g(x)}\,dx}.$$

To prove (4.2.46) it is sufficient to prove that

$$\int_0^1 \frac{w^2(x)}{g(x)}\,dx \leq \int_0^1 \frac{w^2(x)}{h(x)}\,dx.$$

We now compute the left-hand side by means of the change of variables $x = H(y)$, where

$$H(y) = \int_0^y h(t)\,dt,$$

so that
$$\int_0^1 \frac{w^2(x)}{g(x)}\,dx = \int_0^1 \frac{w^2(H(y))}{h(y)}\,dy.$$
We now notice that the symmetry of h gives (4.2.39) with $\bar{x} = 1/2$, namely
$$\begin{aligned} &H(y) \le y && x \in \left[0, \frac{1}{2}\right], \\ &H(y) \ge y && x \in \left[\frac{1}{2}, 1\right]. \end{aligned} \tag{4.2.50}$$
Finally, joining (4.2.50), and (4.2.48), we have
$$\int_0^1 \frac{w^2(x)}{g(x)}\,dx = \int_0^1 \frac{w^2(H(y))}{h(y)}\,dy \le \int_0^1 \frac{w^2(y)}{h(y)}\,dy,$$
which concludes the proof. □

Proof of Theorem 4.2.4. The result follows from Proposition 4.2.20 and Proposition 4.2.22 □

4.3 A spectral isoperimetric inequality on the n-sphere

The results of this section are contained in the paper [2].

Let (M, g) be a complete Riemannian n-manifold, and let $\Omega \subset M$ be a bounded open set with smooth boundary. For every $\beta \in \mathbb{R}$ consider the Robin-Laplacian eigenvalue problem on Ω, that is
$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.3.1}$$
where Δ is the Laplace-Beltrami operator on M and ν is the unit outer normal to the boundary of Ω. (4.3.1) admits an increasing sequence of eigenvalues diverging to infinity. Moreover, if Ω is connected, any first eigenfunction has a sign, so that, by linearity, the first eigenvalue $\lambda^\beta(\Omega)$ is simple (see for example [132]).

Let $\lambda^\beta(\Omega)$ be the smallest eigenvalue for (4.3.1), then the following variational characterization holds
$$\lambda^\beta(\Omega) = \inf_{v \in H^1(\Omega)} \frac{\displaystyle\int_\Omega |\nabla v|^2\,d\mu + \beta \int_{\partial\Omega} v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\,d\mu}. \tag{4.3.2}$$
Any minimizer u of (4.3.2) is a weak solution to (4.3.1) for $\lambda = \lambda^\beta(\Omega)$, that is
$$\int_\Omega g(\nabla u, \nabla \varphi)\,d\mu + \beta \int_{\partial\Omega} u\varphi\,d\mathcal{H}^{n-1} = \lambda^\beta(\Omega) \int_\Omega u\varphi\,d\mu,$$
for every $\varphi \in H^1(\Omega)$. An immediate consequence of the variational characterization (4.3.2) is the fact that the function
$$\beta \in \mathbb{R} \mapsto \lambda^\beta(\Omega) \in \mathbb{R}$$

is increasing. In particular, for $\beta = 0$, the Robin boundary condition coincides with the Neumann one and $\lambda_0(\Omega) = \lambda_0^{\mathrm{N}}(\Omega) = 0$ with constant eigenfunctions. Therefore, the first Robin eigenvalue is positive for $\beta > 0$ and negative for $\beta < 0$.

Comparison theorems for the first Robin eigenvalue are widely studied in the literature. The first example of such theorems is probably the one due to Bossel in [35]: this result generalizes the Faber-Krahn inequality for the first Robin eigenvalue with $\beta > 0$ in the class of bounded open sets of the Euclidean plane $\mathbb{R}^2$. Namely, let $\Omega \subset \mathbb{R}^2$ be a bounded open set and let $B \subset \mathbb{R}^2$ be a ball having the same area, then

$$\lambda^\beta(\Omega) \geq \lambda^\beta(B). \tag{4.3.3}$$

Daners generalized the previous result in [85] for bounded open subsets of the Euclidean space $\mathbb{R}^n$.

In the context of Riemannian manifolds, one usually compares the Robin eigenvalue of a bounded domain Ω in a complete manifold M with the one of a geodesic ball in an appropriate simply connected space form. In particular, Chen, Cheng, and Li in [71] proved a Bossel-Daners inequality (4.3.3) for bounded domains of a manifold M, where either M is the hyperbolic space or it is a compact manifold whose Ricci curvature tensor satisfies a positive lower bound. As proved by Chen, li, and Wei in [72], the inequality still holds in the case in which M is a complete, non-compact, manifold whose Ricci tensor is non-negative.

In the case $\beta < 0$, Bareket in [27] famously conjectured that among all Lipschitz sets of a given area in the Euclidian plane, the ball maximizes the first Robin eigenvalue. Freitas and Krejčiřík in [96] disproved the conjecture: they proved, via an asymptotic expansion, that, for $|\beta|$ sufficiently large, the first Robin eigenvalue of an annulus is larger than the one of the ball having the same measure. At the same time, they proved that for smooth bounded subsets of the Euclidean plane, the conjecture holds true provided that β is sufficiently close to 0. However, fixing the perimeter leads to other interesting comparisons. Indeed, Antunes, Freitas, and Krejčiřík in [20] proved a comparison theorem for the first Robin eigenvalue, with $\beta < 0$, under a perimeter constraint. Namely, let Ω be a bounded open set with C^2 boundary in the Euclidean plane and let $B \subset \mathbb{R}^2$ be a ball having the same perimeter, then

$$\lambda^\beta(\Omega) \leq \lambda^\beta(B). \tag{4.3.4}$$

Bucur et al. in [47] proved that the inequality (4.3.4) holds true in any dimension provided that we restrict the class of admissible sets to the one of the convex sets, or, more in general, the inequality holds for any Lipschitz set which can be written as $\Omega \setminus K$, where Ω is open and convex and K is a closed set in Ω. Vikulova in [163] proved the result in the Euclidean space $\mathbb{R}^3$ for bounded convex sets or connected axiconvex sets whose boundary is diffeomorphic to the sphere.

In the context of Riemannian manifolds, Khalile and Lotoreichik in [119] proved the following. Let Ω be a compact, two-dimensional, simply connected Riemannian manifold with C^2 boundary and with Gauss curvature bounded from above by a non-negative constant κ_0, and let B be a geodesic disc in the simply connected space form of Gauss curvature κ_0 with the same perimeter as Ω. Then, for every $\beta < 0$, inequality (4.3.4) holds.

Finally, in Riemannian manifolds, other comparison theorems for the first Robin eigenvalue and domain monotonicity properties have been proved by Savo in [155] and by Li and Wang in [132].

The main objective of this section is to adapt the techniques of [47] to prove the following theorem. Note that we denote by $\mathcal{H}^{n-1}$ the Hausdorff measure, and we refer to Definition 4.3.14 for the definition of strong convexity.

Theorem 4.3.1. *Let $\beta < 0$, and let $\Omega \subset \mathbb{S}^n$, where the sphere is endowed with its canonical metric. Let Ω be an open set such that $\overline{\Omega}$ is strongly convex, and let D be a strongly convex geodesic ball with*

$$\mathcal{H}^{n-1}(\partial\Omega) = \mathcal{H}^{n-1}(\partial D).$$

Then

$$\lambda^\beta(\Omega) \leq \lambda^\beta(D), \tag{4.3.5}$$

and the equality holds if and only if, up to a translation, $\Omega = D$.

Notice that, thanks to [155, Theorem 5], we have that the eigenvalue is increasing with respect to the inclusion among balls, so that (4.3.5) also holds true replacing D with $\mathbb{S}^n \setminus D$.

The proof relies on the use of the method of parallel coordinates (see [148] and [80]) to construct a suitable test function on convex subsets of the sphere $\mathbb{S}^n$. Nevertheless, the main difficulty here was to recover classical results about convex sets on the sphere. In particular, the main ingredients of the proof are:

(i) convexity properties of inner and outer parallel sets;

(ii) monotonicity of perimeters with respect to the inclusion for convex sets;

(iii) Steiner's formulae;

(iv) Alexandrov-Fenchel inequality for the mean curvature.

To infer convexity properties of inner parallel and outer parallel sets (see Definition 2.1.2) we need some convexity property of the distance function provided by Bangert in his paper [24] (see Theorem 4.3.29). The monotonicity of the perimeter has been proved by Bangert in [23] (see Theorem 4.3.24). The Steiner's formulae have been extended to C^2 convex sets of the sphere by Allendoerfer in [13] and the Alexandrov-Fenchel inequality has been recently extended to C^2 convex sets of the sphere by Makowski and Scheuer in [136]. However, our result in Theorem 4.3.1 only requires the set to be convex: to avoid the constraint on the regularity of the boundary, we recover a general theory for Steiner's formulae and curvature measures introduced by Federer in [91] in $\mathbb{R}^n$ and successively generalized to simply connected space forms by Kohlmann in [123]. We are then able to generalize Alexandrov-Fenchel inequalities for general convex sets (see Corollary 4.3.49) by approximating convex sets with smooth convex sets, using a result that has been proved by Bangert in [24] (see also Theorem 4.3.32).

In addition, we are also able to adapt the techniques in [17] to prove the following stability result.

Theorem 4.3.2. *Let $\Omega \subset \mathbb{S}^n$ be an open set such that $\overline{\Omega}$ is strongly convex, and let D be a strongly convex geodesic ball such that*

$$\mathcal{H}^{n-1}(\partial\Omega) = \mathcal{H}^{n-1}(\partial D).$$

For every $\beta < 0$, let u be an eigenfunction relative to $\lambda^\beta(D)$, and let

$$u_m = \min_{p \in \overline{D}} u(p).$$

Then,

$$\frac{\lambda^\beta(D) - \lambda^\beta(\Omega)}{|\lambda^\beta(\Omega)|} \geq \frac{u_m^2}{\|u\|^2_{L^2(D)}}(|D| - |\Omega|). \tag{4.3.6}$$

The section is organized as follows. We first give introductory notions and classical tools of Riemannian manifolds and integration theory. In Section 4.3.2 we give classical results and definitions about convexity in Riemannian manifolds, with special attention to the convexity of the inner parallel sets and convex approximation. In Section 4.3.3 we give the definition of curvature measures, and we state the Steiner formula in $\mathbb{S}^n$ and the Alexandrov-Fenchel inequality. In Section 4.3.5 we prove Theorem 4.3.1 and Theorem 4.3.2. Finally, in Section 4.3.6 we discuss the limits of the proof in the hyperbolic space.

4.3.1 General notions

In the following, given a smooth, orientable Riemannian n-manifold (M, g), we will denote by d the Riemannian distance

$$d(p,q) = \min_{\substack{\gamma\in C^\infty((0,1);M)\\ \gamma(0)=p\\ \gamma(1)=q}} \int_0^1 g(\gamma'(t),\gamma'(t))\,dt$$

induced by g; by $d\mu$ its volume form which is expressed locally in coordinates as

$$d\mu = \sqrt{|\det(g_{ij})|}dx_1\dots dx_n;$$

and we will denote by $|\cdot|$ the classical Riemannian volume

$$|E| = \int_E d\mu.$$

We let TM denote the tangent bundle on M, by $\Gamma(TM)$ the sections of the bundle, namely the space of vector fields, and by T_pM the tangent space at p. We also recall that for every $(v,p)\in TM$ a *geodesic* starting from p with velocity v is the unique curve $\gamma = \gamma_{p,v}$ such that $\gamma(0) = p$, and $\gamma'(0) = v$, and such that it solves the system of equations written in local coordinates as (using the Einstein notation on repeated indices)

$$\gamma_i''(t) + \Gamma^i_{jk}(\gamma(t))\,\gamma_j'(t)\,\gamma_k'(t) = 0, \qquad\qquad i = 1,\dots,n$$

with Γ^i_{jk} representing the Christoffel symbols of the metric g. When M is complete, we can extend the geodesics $\gamma_{p,v}\in C^\infty(\mathbb{R};M)$, and we denote by

$$\exp : TM \to M \qquad \exp_p : T_pM \to M$$

the exponential map defined as

$$\exp(p,v) = \exp_p(v) = \gamma_{p,v}(1). \tag{4.3.7}$$

For every $p\in M$ we will denote the cutlocus of p in M as

$$\mathrm{Cut}(p) = \exp_p(\partial\,\mathrm{seg}(p)),$$

where

$$\mathrm{seg}(p) = \{\, v\in T_pM \mid \gamma_{p,v} \text{ minimizes the distance } d(p,\gamma_{p,v}(1)) \,\}\,.$$

We will denote by $\mathcal{H}^k$ the Hausdorff measure relative to the Riemannian distance on M. When necessary, we will denote the Hausdorff measure by $\mathcal{H}^k_g$ to highlight the dependence on the metric g.

We refer to [69, Section IV] for basic properties on this topic in the Riemannian setting. We denote by σ_n the $(n-1)$-dimensional measure of the boundary of a hemisphere in the sphere $\mathbb{S}^n$ of sectional curvature 1. Moreover, we will denote by $\langle\cdot,\cdot\rangle$ the canonical scalar product in $\mathbb{R}^n$.

We refer to [165, §2] for the following definitions.

Definition 4.3.3. Let M be a Riemannian n-manifold, and let $\Sigma \subset M$. We say that Σ is a *strongly Lipschitz submanifold* of M of dimension k if for every $p \in \Sigma$ there exist a C^1 chart (U,φ) in M around p, an open set $U' \subset \mathbb{R}^k$, and a Lipschitz function $f : U' \to \mathbb{R}^{n-k}$ such that

$$\varphi(\Sigma \cap U) = \left\{ (x, f(x)) \in \varphi(U) \;\middle|\; x \in U' \right\}.$$

Definition 4.3.4. Let M be a Riemannian n-manifold, and let $\Omega \subset M$. We say that Ω *has strongly Lipschitz boundary* if $\Omega = \overline{\mathring{\Omega}}$, and $\partial\Omega$ is a strongly Lipschitz submanifold of M of dimension $n-1$.

Definition 4.3.5. Let X, Y be two metric spaces. We say that a homeomorphism

$$f : X \to Y$$

is *locally bi-Lipschitz* if both f and f^{-1} are locally Lipschitz.

Definition 4.3.6. Let (M, g) be a Riemannian n-manifold, and let Σ be a C^2 oriented, embedded $(n-1)$-submanifold of M. We define the *second fundamental form* B of Σ in M as the 2-form such that for every $X, Y \in \Gamma(T\Sigma)$

$$B(X, Y) = g(X, \nabla_Y \nu),$$

where ∇ is the Levi-Civita connection of M, and ν is the normal to Σ.

Proposition 4.3.7. *Let M and Σ as in Definition 4.3.6. Then:*

(i) B is symmetric, namely

$$B(X, Y) = B(Y, X) \qquad \forall X, Y \in \Gamma(T\Sigma);$$

(ii) for every $\sigma \in \Sigma$ there exist $n-1$ eigenvalues $\kappa_1(\sigma) \leq \cdots \leq \kappa_{n-1}(\sigma)$ of B and we say that κ_i are the principal curvatures of Σ.

Definition 4.3.8. Let M be a Riemannian n-manifold, let Σ be a C^2 oriented, compact, embedded $(n-1)$-submanifold of M. For every $p \in \Sigma$ and for every $1 \leq j \leq n-1$, we denote by

$$H_j(p) = \sum_{1 \leq i_1 < \cdots < i_j \leq n-1} \kappa_{i_1}(p) \ldots \kappa_{i_j}(p)$$

the j-th homogeneous symmetric form of the principal curvatures, and

$$H_0(p) = 1.$$

In particular, we say that $H_1(p)$ is the *mean curvature* of Σ in p.

We now state the coarea and area formula.

Definition 4.3.9. Let V be a normed vector space of dimension n. For every $r = 1, \ldots, n$ we denote by $\bigwedge_r V$ the space of alternating r-forms on the dual V^*.

If $V = T_pM$ is a tangent space for a Riemannian n-manifold M at a point p, for every $r \leq n$ we use the notation

$$\bigwedge\nolimits_r M_p := \bigwedge\nolimits_r T_pM$$

to denote the inner product of r copies of T_pM.

Definition 4.3.10. Let (M, g) be a Riemannian n-manifold of class C^1, let (N, h) be a Riemannian k-manifold of class C^1, let

$$r = \min\{n, k\},$$

and let $f : M \to N$ be a map such that f is differentiable in $p \in M$. We define the natural extension of df_p to $\bigwedge_r M_p$ as the linear map

$$\wedge_r df_p : \bigwedge\nolimits_r M_p \to \bigwedge\nolimits_r N_{f(p)}$$

such that for every $v_1, \ldots, v_r \in T_pM$

$$\wedge_r df_p(v_1 \wedge \cdots \wedge v_r) = df_p(v_1) \wedge \ldots df_p(v_r).$$

We define the *jacobian* of f as

$$\operatorname{Jac} f(p) = \|\wedge_r df_p\|,$$

where the norm $\|\cdot\|$ denotes the operatorial norm in the space of linear applications $\mathcal{L}(\bigwedge_r M_p, \bigwedge_r N_{f(p)})$ with the respective norms $\|\cdot\|_{g,p}$ and $\|\cdot\|_{h,f(p)}$.

For the proof of the following theorem, we refer to [91, Theorem 3.1]

Theorem 4.3.11 (Coarea Formula)**.** *Let (M, g) be a Riemannian n-manifold, let (N, h) be a Riemannian k-manifold with $n \geq k$, and let $f : M \to N$ be a Lipschitz map. Then f is $\mathcal{H}^n$-a.e. differentiable and for every $\mathcal{H}^n$-integrable function $\varphi : M \to \mathbb{R}$ we have*

$$\int_M \varphi(x) \operatorname{Jac} f(x)\, d\mathcal{H}^n(x) = \int_N \int_{f^{-1}(y)} \varphi(z)\, d\mathcal{H}^{n-k}(z)\, d\mathcal{H}^k(y).$$

As a particular case we get again Theorem 2.3.1. For the following theorem we refer to [92, Theorem 3.2.5, Remark 3.2.46].

Theorem 4.3.12 (Area Formula)**.** *Let (M, g) be a Riemannian n-manifold, let (N, h) be a Riemannian k-manifold with $n \leq k$, and let $f : M \to N$ be a Lipschitz map. Then f is $\mathcal{H}^n$-a.e. differentiable and for every $\mathcal{H}^n$-measurable function $\varphi : M \to \mathbb{R}$ and we have*

$$\int_M \varphi(x) \operatorname{Jac} f(x)\, d\mathcal{H}^n(x) = \int_N \int_{f^{-1}(y)} \varphi(z)\, d\mathcal{H}^0(z)\, d\mathcal{H}^k(y).$$

As a particular case we get again Theorem 4.3.12.

4.3.2 Convexity in Riemannian manifolds

In this section, we aim to give a general overview of convexity in Riemannian manifolds, and then we will study properties of convex sets in the specific case of the sphere $\mathbb{S}^n$. In order to give some convexity definitions in the Riemannian setting, we introduce the notions of supporting cone and normal cone. (We recall the definition of the exponential map in (4.3.7).)

Definition 4.3.13. Let M be a Riemannian manifold and $C \subset M$ with non-empty interior. For every $p \in \partial C$ we define the *(local) supporting cone* of C in p as

$$\mathcal{C}_C(p) = \left\{ \xi \in T_pM \;\middle|\; \exists \varepsilon > 0 : \exp_p(t\xi) \in \mathring{C} \quad \forall t \in (0, \varepsilon) \right\},$$

and the *(internal) normal cone* as its dual cone

$$\mathcal{C}_C(p)^* = \{ \nu \in T_pM \mid \langle \nu, \xi \rangle \geq 0 \quad \forall \xi \in \mathcal{C}_C(p) \}.$$

Then, recalling that we use the notation $\overline{pq}$ to denote the minimal geodesic connecting p and q is unique in M, we give the following definitions

Definition 4.3.14. Let M be a Riemannian manifold, and let $C_1, C_2 \subset M$. We say that:

(a) C_1 is *weakly convex* if for every $p, q \in C_1$ there exists a minimal geodesic $\gamma : [a, b] \to M$ connecting p and q contained in C_1;

(b) C_1 is *strongly convex* if for every $p, q \in C_1$ there exists a unique minimal geodesic $\overline{pq}$ connecting p and q in M, and $\overline{pq} \subseteq C_1$;

(c) C_1 is *locally convex* if for every $p \in \bar{C_1}$ there exists $\varepsilon > 0$ and a metric ball $B_\varepsilon(p)$ such that $C_1 \cap B_\varepsilon(p)$ is strongly convex;

(d) C_1 is *locally strictly convex* if there exists a $\delta > 0$ such that for every point $p \in \partial C_1$ and for every $\nu \in \mathcal{C}_{C_1}(p)^*$ the following holds: there exists an hypersurface H orthogonal to ν in p such that $H \cap C_1 = \{p\}$ and its second fundamental form in p with respect to ν has eigenvalues greater than δ;

(e) C_1 is *totally convex* in C_2 if $C_1 \subseteq \mathring{C}_2$ and for every $p, q \in C_1$ and every geodesic

$$\gamma : [a, b] \to C_2$$

connecting p and q inside C_2 we have $\gamma([a, b]) \subseteq C_1$.

We refer to [70] for definitions (a)-(c), to [24] for definition (d), and to [23] for definition (e).

We now give some useful properties about convex sets in the sphere.

Remark 4.3.15. Recall that the definition of strong convexity is actually imposing some geometric constraint on the set C. For instance, on the sphere $\mathbb{S}^n$ we have that if $C \subseteq \mathbb{S}^n$ is a closed strongly convex set, then C is contained in an open hemisphere. Indeed, let $C \subseteq \mathbb{S}^n$ be a closed strongly convex set. By definition of strong convexity, we have that if $p \in C$ then necessarily the antipodal point $-p \notin C$. Therefore, we can find a plane separating C and its antipodal set $-C$: indeed,

$$\Omega^+ := \left\{ tx \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{l} t > 0, \\ x \in C \end{array} \right\}$$

and

$$\Omega^- := \left\{ tx \in \mathbb{R}^{n+1} \;\middle|\; \begin{array}{c} t > 0, \\ x \in -C \end{array} \right\}$$

are two disjoint convex cones in $\mathbb{R}^{n+1}$, and they can be separated by a plane passing through the origin. This in particular implies that C is contained in a hemisphere.

Remark 4.3.16. If $C \subset \mathbb{S}^n$ is weakly convex and it is contained in a hemisphere, then it is strongly convex, since for every couple of points $p, q \in C$ there exists a unique minimal geodesic connecting them.

Remark 4.3.17. Notice that if $C_1, C_2 \subset \mathbb{S}^n$ are two strongly convex sets such that $C_1 \subseteq C_2$, then C_1 is totally convex in C_2. Indeed, since C_2 is contained in a hemisphere, then for every couple of points $p, q \in C_1$, the unique minimal geodesic $\overline{pq}$ connecting p and q is also the unique geodesic connecting p and q contained C_2.

Notice that the definition of totally convex set becomes trivial when M is a compact manifold and we take $C_2 = M$. See for instance [23, Corollary 1] for the following

Proposition 4.3.18. *Let M be a compact connected Riemannian manifold, and let $C \subseteq M$ be a totally convex set in M. Then $C = M$.*

Remark 4.3.19. Notice that if C is strongly convex, then it is connected and locally convex.

Notice also that if C_1 is strongly convex and $C_1 \subset C_2$ is totally convex in C_2, then C_1 is strongly convex.

In $\mathbb{S}^n$, open, connected, locally convex sets contained in a hemisphere have to be strongly convex. Indeed, we can characterize weak convexity with some geometric properties of the boundary. Let us introduce the notion of supporting element (see [70, 11]).

Definition 4.3.20. Let M be a Riemannian manifold, and let $C \subseteq M$ be an open set. Let $p \in \partial C$, and for some $\nu \in T_pM$ define

$$H_p = \{ \, \xi \in T_pM \mid \langle \nu, \xi \rangle < 0 \, \} \, .$$

We say that:

(i) the half-space H_p is a *supporting element* for C in p if for every $q \in \mathring{C}$ and for every minimal geodesic

$$\gamma : [0, 1] \to M$$

such that $\gamma(0) = p$ and $\gamma(1) = q$, we have $\gamma'(0) \in H_p$;

(ii) the half-space H_p is a *locally supporting element* for C in p if there exists a neighbourhood U of p such that H_p is a supporting element for $U \cap C$ in p.

Let M be a Riemannian manifold, and for every $p \in M$, let $\mathrm{Cut}(p)$ be the cut-locus of p. We refer to [11, Proposition 2] for the following result.

Proposition 4.3.21. *Let M be a Riemannian manifold, and let $C \subset M$ be connected and open. The set C is weakly convex if and only if for every point $p \in \partial C$ there exists a locally supporting element and $C \setminus \mathrm{Cut}(p)$ is connected.*

We also have that a locally supporting element always exists for open, locally convex sets. Indeed, Cheeger and Gromoll in [70, Theorem 1.6, Lemma 1.7] proved a result summarized in Theorem 4.3.22 (see also the comments between Lemma 1.7 and Proposition 1.8); notice that Cheeger and Gromoll work with closed sets, but if C is locally convex, then also $\bar{C}$ is a locally convex set, and $\partial C = \partial \bar{C}$. On the other hand, by definition, a supporting element for $\bar{C}$ is also a supporting element for C.

Theorem 4.3.22. *Let M be a Riemannian manifold of dimension n, and let $C \subseteq M$ be a non-empty, open, locally convex set. Then ∂C is an embedded $(n-1)$-dimensional topological submanifold of M, and it has a supporting element in every point $p \in \partial C$.*

Joining Proposition 4.3.21 and Theorem 4.3.22, we get on the sphere $\mathbb{S}^n$ the following.

Proposition 4.3.23. *Let $C \subset \mathbb{S}^n$ be a closed, connected, locally convex set contained in an open hemisphere. Then C is strongly convex.*

Proof. The local convexity of C and the fact that it is connected ensure that $\mathring{C}$ is connected (see for instance [70, Lemma 1.5]). Therefore, we may apply Theorem 4.3.22 to $\mathring{C}$, so that every point $p \in \partial \mathring{C}$ admits a supporting element. Moreover, since C is contained in a hemisphere, we also have that

$$\mathring{C} \setminus \mathrm{Cut}(p) = \mathring{C},$$

which is connected. Therefore, we can apply Proposition 4.3.21, and get that $\mathring{C}$ is weakly convex, and, in particular, as in Remark 4.3.16, strongly convex. Finally, observing that closedness and local convexity ensure $C = \bar{\mathring{C}}$ (see [70, Theorem 1.6]), then C inherits the strong convexity of $\mathring{C}$.

□

The following theorem is due to Bangert in [23, Theorem 1].

Theorem 4.3.24 (Monotonicity of perimeter)**.** *Let M be a Riemannian manifold, and let $C_1, C_2 \subseteq M$ such that C_1 is totally convex in C_2, and $\mathring{C}_1 \neq \emptyset$. Assume moreover that C_2 has strongly Lipschitz boundary, and $|C_2 \setminus C_1| < +\infty$. Then*

$$\mathcal{H}^{n-1}(\partial C_1) \leq \mathcal{H}^{n-1}(\partial C_2).$$

The proof of this theorem in the Euclidean case only relies on proving that the projection onto the convex set C_1 is a 1-Lipschitz function (see for instance [39, Proposition 5.3]), while the Riemannian case requires a different proof. Even if the monotonicity theorem requires some regularity on the external set C_2, we can still prove that this is not restrictive in the case in which C_2 is locally convex. Indeed, we have the following result due to Walter in [165, Theorem 6.1].

Theorem 4.3.25. *Let M be a Riemannian manifold, and let $C \subset M$ be a closed, locally convex set. Then C has strongly Lipschitz boundary.*

Joining Theorem 4.3.24 and Theorem 4.3.25, we get:

Corollary 4.3.26. *Let $C_1, C_2 \subseteq \mathbb{S}^n$ be two closed strongly convex sets such that $\mathring{C}_1 \neq \emptyset$. If $C_1 \subseteq C_2$, then*

$$\mathcal{H}^{n-1}(\partial C_1) \leq \mathcal{H}^{n-1}(\partial C_2).$$

Proof. It is sufficient to notice that, as in Remark 4.3.17, C_1 is totally convex in C_2. Indeed, Theorem 4.3.25 ensures the strongly Lipschitz regularity of the boundary, and Theorem 4.3.24 applies. □

We now give some definitions of convexity of continuous functions on Riemannian manifolds, see for instance [107, §1] for a reference on the topic.

Definition 4.3.27. Let M be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be a continuous function. We say that:

(a) f is *convex* if for every geodesic $\gamma : [a, b] \to M$ we have $f \circ \gamma$ is convex on $[a, b]$;

(b) f is *strictly convex* if for every $p \in M$ and for every convex function $\varphi \in C^\infty(M)$ there exists an $\varepsilon > 0$ such that $f - \varepsilon\varphi$ is convex in a small neighbourhood of p.

These definitions are related to the geometry of the sublevel sets.

Proposition 4.3.28. *Let M be a Riemannian manifold, and let $f : M \to \mathbb{R}$ be a continuous function. Then:*

(i) if f is convex, then for every $t \in \mathbb{R}$ the set $\{ x \in M \mid f(x) < t \}$ is totally convex in M;

(ii) assume that f is strictly convex, and M is weakly convex; for every $t \in \mathbb{R}$, if the set $\{ x \in M \mid f(x) < t \}$ is compact, then it is locally strictly convex.

Proof. We only show *(i)*, and we refer to [24, Lemma 2.4] for the proof of *(ii)* (note that the assumption on the weak convexity of M ensures the connectedness of the sublevel set of f). Let $\gamma : [a, b] \to M$ be a geodesic, and assume that

$$f(\gamma(a)) < t \qquad\qquad f(\gamma(b)) < t.$$

Then, by the definition of convexity, for every $\alpha \in [0, 1]$,

$$f(\gamma(a + \alpha(b - a))) \leq (1 - \alpha)f(\gamma(a)) + \alpha f(\gamma(b)) < t,$$

and the assertion is proved. □

Cheeger and Gromoll, in [70, Theorem 1.10] proved that, for a given convex set C in a Riemannian manifold M with positive sectional curvatures, the distance function

$$\rho_C(x) = -d(x, \partial C)$$

is convex in $\mathring{C}$. This implies that the inner parallel sets C_t are totally convex in $\mathring{C}$. However, we will need some more refined results that can be found in [24, Theorem 2.1, Theorem 2.3], and we summarize in the following. Let $C \subset M$, and denote by $\rho = \rho_C$ the signed distance function

$$\rho_C(x) = \begin{cases} -d(x, \partial C) & \text{if } x \in \mathring{C}, \\ d(x, C) & \text{if } x \notin \mathring{C}. \end{cases}$$

Then we have

Theorem 4.3.29. *Let M be a Riemannian manifold, and let C be a connected, compact, locally convex set. Then the following hold:*

(i) *if C is locally strictly convex, then there exists $\delta > 0$ such that the function*

$$\rho_C + \frac{1}{2}\rho_C^2$$

is strictly convex on $\mathring{C}^\delta \setminus C$;

(ii) *if the sectional curvatures on C are negative, then there exists $\delta > 0$ such that the function*

$$\rho_C + \frac{1}{2}\rho_C^2$$

is strictly convex on $\mathring{C}^\delta \setminus C$;

(iii) *if the sectional curvatures on C are positive, then the function*

$$\rho_C - \log(-\rho_C)$$

is strictly convex on $\mathring{C}$.

Remark 4.3.30. Despite [24, Theorem 2.1] only proves *(i)*, result *(ii)* directly follows from the same proof using a negative upper bound on the sectional curvatures to conclude (see also the proof of [24, Corollary 2.6]).

Corollary 4.3.31. *Let $C \subset \mathbb{S}^n$ be a closed strongly convex set. Then:*

(i) *if C is strongly convex and locally strictly convex, then for small $\delta > 0$ we have that the outer parallel sets $(C)^t$ are strongly convex and locally strictly convex for every $t < \delta$;*

(ii) *the inner parallel sets $(C)_t$ are locally strictly convex and strongly convex for every $t > 0$.*

Proof. By the condition *(i)* in Theorem 4.3.29 we get that $(C)^t$, for small values of t is locally convex. Indeed, for every interior point p of $(C)^t$ it is sufficient to observe that a small strongly convex ball contained in $(C)^t$ always exists. If $p \in \partial(C)^t$, since we can find a small strongly convex ball B contained in $(\mathring{C})^\delta \setminus C$, then the convexity of the function $\rho_C + \rho_C^2/2$ ensures that $B \cap (C)^t$ is strongly convex.

Moreover, C is connected and contained in a hemisphere, as already seen in Remark 4.3.15. Therefore, $(C)^t$, for small t, is connected and contained in the same hemisphere, which implies, by Proposition 4.3.23, that $(C)^t$ is strongly convex. Finally, by Proposition 4.3.28, we also get that for small t the set $(C)^t$ is locally strictly convex.

Let us now study the inner parallels $(C)_t$. Analogously to the case of the outer parallels, condition *(iii)* in Theorem 4.3.29 yields that the inner parallel sets $(C)_t$ are locally strictly convex. Moreover, the convexity of the function $\rho_C - \log(-\rho_C)$ ensures that the sets $(C)_t$ are totally convex in $\mathring{C}$ (see *(i)* in Proposition 4.3.28). Since C is strongly convex, then the total convexity of $(C)_t$ in $\mathring{C}$ gives that the inner parallel sets $(C)_t$ are strongly convex. □

Now we state an approximation theorem proved by Bangert in [24, Theorem 2.2, Corollary 2.5, Corollary 2.6].

Theorem 4.3.32. *Let M be a Riemannian manifold, and let $C \subset M$ be a connected, compact, locally convex set such that $\mathring{C} \neq \emptyset$. Moreover, assume that either:*

(a) C is locally strictly convex;

(b) the sectional curvatures are positive on C;

(c) the sectional curvatures are negative on C;

then there exists a sequence of connected, compact, locally convex sets C_k with C^∞ boundaries such that

$$\lim_{k\to+\infty} d^{\mathrm{H}}(C_k, C) + d^{\mathrm{H}}(\partial C_k, \partial C) = 0.$$

Remark 4.3.33. Results *(b)* and *(c)* are a direct consequence of *(a)* and Theorem 4.3.29 joint with Proposition 4.3.28: in the case *(b)* one approximates the inner parallel sets, while in the case *(c)* one approximates the outer parallel sets.

Corollary 4.3.34. *Let $C \subset \mathbb{S}^n$ be a closed strongly convex set such that $\mathring{C} \neq \emptyset$. Then there exists a sequence of closed strongly convex sets C_k with C^∞ boundaries such that*

$$\lim_{k\to+\infty} d^{\mathrm{H}}(C_k, C) + d^{\mathrm{H}}(\partial C_k, \partial C) = 0.$$

Proof. Since the sectional curvatures in $\mathbb{S}^n$ are positive, Theorem 4.3.32 applies and we find an approximating sequence of connected, compact, locally convex sets C_k with C^∞ boundaries and such that $\mathring{C}_k \neq \emptyset$. Therefore, the Hausdorff convergence also allows us to assume that C_k are contained in the same hemisphere in which C is contained. By Proposition 4.3.23, we get that C_k are strongly convex. $\square$

4.3.3 Curvature measures

In this section we define curvature measures introduced in $\mathbb{R}^n$ by Federer in [91] and explicitly computed by Zähle in [168], while successively extended to simply connected space forms by Kohlmann in [123].

As a first step, we define sets of positive reach. Given a Riemannian manifold M, for every $p \in M$ and for every $r > 0$ we denote by $B_r(p)$ the metric ball centered in p of radius $r > 0$. For small enough $r > 0$ we have that $B_r(p)$ coincides with the geodesic ball $\exp_p(B_r(0))$.

Definition 4.3.35. Let M be a Riemannian manifold, and let $\Omega \subset M$ be a non-empty set. For every $p \in M$ we call *metric projection* of p onto Ω any $q \in \bar{\Omega}$ such that

$$d(p, q) = d(p, \Omega).$$

When it is unique we write $q = \pi_\Omega(p)$.

Definition 4.3.36. Let M be a Riemannian manifold, and let $\Omega \subset M$ be a non-empty set. For every $q \in \Omega$ we define the *reach* of q with respect to Ω as

$$\mathcal{R}(q) = \sup\{\, r > 0 \mid \forall p \in B_r(q) \text{ there exists a unique metric projection of } p \text{ onto } \Omega \,\};$$

we define the *reach* of Ω as

$$\mathcal{R}(\Omega) = \inf_{q\in\Omega} \mathcal{R}(q);$$

we say that Ω is *of positive reach* if $\mathcal{R}(\Omega) > 0$.

In $\mathbb{R}^n$ we have that every convex set C is a set of positive reach with

$$\mathcal{R}(C) = +\infty.$$

On simply connected space forms similar but slightly different results hold for connected, compact, locally convex sets. First, we state a result due to Walter in [164, Theorem 1].

Proposition 4.3.37. *Let M be a Riemannian manifold, and let $C \subseteq M$ be a closed locally convex set. Then C is of positive reach.*

In the specific case of simply connected space forms we have the following result that can be found in [124, Lemma 2.2].

Proposition 4.3.38. *Let M be a simply connected space form of curvature κ, and let $C \subseteq M$ be a connected, compact, locally convex set. Then:*

(i) if $\kappa < 0$, then
$$\mathcal{R}(C) = +\infty;$$

(ii) if $\kappa > 0$, then
$$\mathcal{R}(C) \geq \frac{\pi}{2}\frac{1}{\sqrt{\kappa}}.$$

Definition 4.3.39. Let (M, g) be a Riemannian manifold, let $\Omega \subseteq M$ be a set of positive reach, and let U be an open neighborhood of Ω such that the metric projection π_Ω is well defined for every $p \in U$. We denote by

$$\nu_\Omega(p) = \frac{\exp^{-1}_{\pi_\Omega(p)}(p)}{\|\exp^{-1}_{\pi_\Omega(p)}(p)\|_g} \in TM,$$

and we define the *unit normal bundle*

$$\mathcal{N}(\Omega) = \nu_\Omega(U \setminus \Omega).$$

Since we are going to integrate over $\mathcal{N}(\Omega)$, we need some regularity property of the normal bundle, that is proved in [165, Theorem 4.3]. In the following, we are equipping TM with the canonical Sasaki metric.

Theorem 4.3.40. *Let M be a Riemannian n-manifold, and let $\Omega \subseteq M$ be a set of positive reach. Then $\mathcal{N}(\Omega)$ is a strongly Lipschitz $(n-1)$-submanifold of TM. Moreover, if Ω is compact, then $\mathcal{N}(\Omega)$ is compact and there exists $\eta = \eta(\Omega) > 0$ such that for every $0 < r < \eta$, we have that*

$$\nu_\Omega|_{\partial(\Omega)^r} : \partial(\Omega)^r \to \mathcal{N}(\Omega)$$

is a locally bi-Lipschitz homeomorphism.

Definition 4.3.41. Let M be a Riemannian n-manifold, let Σ be a C^2 oriented, compact, embedded $(n-1)$-submanifold of M of positive reach, and let $v \in \mathcal{N}(\Sigma)$. Let us denote by

$$\Pi : TM \to M$$

the canonical projection such that $\Pi(p,\xi)=p$ for every $(p,\xi)\in TM$. For every $1\le j\le n-1$ we denote by $\kappa_j(v):=\kappa_j(\Pi(v))$ the principal curvatures of Σ in $\Pi(v)$, and by

$$H_j(v)=\sum_{1\le i_1<\dots<i_j\le n-1}\kappa_{i_1}(v)\dots\kappa_{i_j}(v)$$

the j-th homogeneous symmetric form of the principal curvatures. We also denote by

$$H_0(v)=1.$$

Definition 4.3.42. Let M be a complete Riemannian n-manifold of constant sectional curvature κ. We define the functions

$$\mathrm{sn}_\kappa(t)=\begin{cases}\dfrac{1}{\sqrt{-\kappa}}\sinh(\sqrt{-\kappa}t) & \text{if } \kappa<0,\\ t & \text{if } \kappa=0\\ \dfrac{1}{\sqrt{\kappa}}\sin(\sqrt{\kappa}t) & \text{if } \kappa>0,\end{cases}$$

and $\mathrm{cn}_\kappa=\mathrm{sn}'_\kappa$. We also let for $1\le j\le n$

$$L_j(t):=\int_0^t \mathrm{cn}_\kappa^{n-j}(t)\,\mathrm{sn}_\kappa^{j-1}(t)\,dt,$$

and

$$L_0(t)=1.$$

The following theorem is due to Kohlmann in [123], but we also point out that Allendoerfer in [13] proves a Steiner formula for regular convex sets in spheres with different techniques.

Theorem 4.3.43 (Steiner formula on simply connected space forms)**.** *Let M be a simply connected space form of dimension n and curvature κ, and let $\Omega\subset M$ be a set of positive reach. Let U be an open set in which the metric projection π_Ω is well defined. For every $j=0,\dots,n$ there exist Radon measures $M_j(\Omega\,;\cdot)$ on U such that the following hold: if $E\subset M$ is a bounded Borel set, and $s>0$ is such that*

$$\pi_\Omega^{-1}(E)\cap\overline{(\Omega)^s}\subset U,$$

then we have

$$\mathcal{H}^{n-1}(\pi_\Omega^{-1}(E)\cap\partial(\Omega)^s)=\sum_{r=0}^{n-1}\mathrm{cn}_\kappa^r(s)\,\mathrm{sn}_\kappa^{n-1-r}(s)M_r(\Omega;E),$$

and

$$|\pi_\Omega^{-1}(E)\cap(\Omega)^s|=\sum_{r=0}^{n}L_{n-r}(s)M_r(\Omega;E).$$

In particular,

$$M_n(\Omega\,;E)=|\Omega\cap E|$$

Moreover, if $\partial\Omega$ is a C^2 compact, embedded $(n-1)$-submanifold of M, then for every bounded Borel set $E\subset M$ and for every $r=0,\dots,n-1$ we have

$$M_r(\Omega;E)=\int_{\partial\Omega\cap E}H_{n-1-r}(p)\,d\mathcal{H}^{n-1}(p).\tag{4.3.8}$$

In the following, we denote by $M_r(\Omega)$ the total measure, namely

$$M_r(\Omega) = M_r(\Omega; M).$$

Remark 4.3.44. Notice that for sets Ω such that $\partial\Omega$ is of class C^2 the definition of the *curvature measures* M_r given in [123, Theorem 2.7] is equivalent to (4.3.8). Indeed, let Ω be such that $\partial\Omega$ is a C^2 compact, embedded $(n-1)$-submanifold of M. Using the explicit definition of the measures M_r in [123], we have

$$M_r(\Omega; E) = \int_{\mathcal{N}(\Omega)\cap\Pi^{-1}(E)} H_{n-1-r}(v) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\kappa_i(v)^2}} d\mathcal{H}^{n-1}(v),$$

where $H_j(v)$ and $\kappa_i(v)$ are defined in Definition 4.3.41. We start by noticing that the regularity on $\partial\Omega$ ensures that

$$\nu_\Omega\big|_{\partial\Omega} : \partial\Omega \to TM$$

is a C^1 map, and using Area Formula (Theorem 4.3.12) with the change of variables $\nu_\Omega(p) = v$, we have

$$M_r(\Omega; E) = \int_{\partial\Omega\cap E} H_{n-1-r}(p) \prod_{i=1}^{n-1} \frac{1}{\sqrt{1+\kappa_i(p)^2}}\, J\nu_\Omega \, d\mathcal{H}^{n-1}.$$

In particular, as Kohlmann computed in [123, Equation (2.6) for $\varepsilon = 0$] (to help the reader compare the following equation with Kohlmann's, we recall that: $j_1 = \mathrm{sn}_\kappa$ and $j_2 = \mathrm{cn}_\kappa$, while the functions f_i are defined in [123, Equation (1.20)]),

$$\operatorname{Jac} \nu_\Omega = \prod_{i=1}^{n-1} \sqrt{1+\kappa_i(p)^2},$$

and we have (4.3.8).

Remark 4.3.45. Let g denote the metric on the simply connected space form M. In the following, we explicit the dependence on the metric. Notice that if Ω is a set of positive reach with $\partial\Omega$ strongly Lipschitz, then we have that for every open set E

$$M^g_{n-1}(\Omega; E) = \mathcal{H}^{n-1}_g(\partial\Omega \cap E). \tag{4.3.9}$$

Indeed, since the Steiner formula holds, then for every open set E we have

$$M^g_{n-1}(\Omega; E) = \lim_{s\to 0^+} \frac{|(\Omega^s \setminus \Omega)\cap E|_g}{s},$$

which is the definition of (relative) Minkowski perimeter (or $(n-1)$-dimensional Minkowski content). If $M = \mathbb{R}^n$ equipped with the Euclidean metric, then the equality (4.3.9) is classical (see for instance [18, Theorem 2.106]). If M is a generic simply connected space form, then it is possible to obtain (4.3.9) from the Euclidean case using normal coordinates. Let $p_0 \in \partial\Omega$ and let $\varepsilon > 0$, there exists a $\delta = \delta(p_0, \varepsilon) > 0$ such that we can define an exponential normal chart mapping onto $\mathcal{U}_\varepsilon = \exp^{-1}_{p_0}(B_\delta(p_0))$ such that the metric g in coordinates is given by $g_{ij} = \delta_{ij} + O(\varepsilon)$. In particular, if g_e denotes the Euclidean metric on $\mathbb{R}^n$, the diffeomorphism

$$\mathrm{Id} : (\mathcal{U}_\varepsilon, g) \to (\mathcal{U}_\varepsilon, g_e)$$

is a bi-Lipschitz function with

$$\mathrm{Lip}(\mathrm{Id}) \leq 1+\varepsilon \qquad \mathrm{Lip}(\mathrm{Id}^{-1}) \leq 1+\varepsilon.$$

Therefore, if we denote by $\mathcal{H}^{n-1}_g$ the $(n-1)$-Hausdorff measure with respect to the metric g, by $|\cdot|_g$ the Riemannian volume, by $\mathcal{H}^{n-1}_e$ the $(n-1)$-Hausdorff measure with respect to the Euclidean metric g_e, and by $|\cdot|_e$ the Lebesgue measure on $\mathbb{R}^n$, then we get for every Borel set $A \subset \mathcal{U}_\varepsilon$ the following estimates (up to changing ε)

$$(1+\varepsilon)^{-1}\mathcal{H}^{n-1}_g(A) \leq \mathcal{H}^{n-1}_e(A) \leq (1+\varepsilon)\mathcal{H}^{n-1}_g(A),$$

$$(1+\varepsilon)^{-1}|A|_g \leq |A|_e \leq (1+\varepsilon)|A|_g,$$

$$\left\{\, x \in \mathcal{U}_\varepsilon \;\middle|\; d_g(x,A) < (1+\varepsilon)^{-1} \,\right\} \subset \{\, x \in \mathcal{U}_\varepsilon \mid d_e(x,A) < s \,\} \subset \{\, x \in \mathcal{U}_\varepsilon \mid d_g(x,A) < (1+\varepsilon)s \,\}\,. \tag{4.3.10}$$

Using the estimates (4.3.10) and the fact that the equality (4.3.9) holds on $\mathbb{R}^n$ for $\mathcal{H}^{n-1}_e$, then we get (up to choosing a smaller ε) for every $r < \delta$,

$$(1-\varepsilon)\mathcal{H}^{n-1}_g(\partial\Omega \cap B_r(p_0)) \leq M^g_{n-1}(\Omega; B_r(p_0)) \leq (1+\varepsilon)\mathcal{H}^{n-1}_g(\partial\Omega \cap B_r(p_0)), \tag{4.3.11}$$

where $B_r(p_0)$ denotes the Euclidean ball of radius r centered in p_0. Equation (4.3.11) in particular implies that the measure $M_{n-1}(\Omega;\cdot)$ is absolutely continuous with respect to $\mathcal{H}^{n-1}_g\big|_{\partial\Omega}$, and that there exists a $\mathcal{H}^{n-1}_g$-measurable density ρ such that

$$M_{n-1}(\Omega;E) = \int_{\partial\Omega\cap E} \rho \, d\mathcal{H}^{n-1}_g$$

with

$$1-\varepsilon \leq \rho \leq 1+\varepsilon.$$

Sending ε to 0 we get $\rho = 1$ and (4.3.9).

Remark 4.3.46. Notice that in the case $\kappa = 0$ we get the Steiner polynomial

$$|\pi_\Omega^{-1}(E) \cap (\Omega)^s| = |\Omega \cap E| + \sum_{k=1}^{n} \frac{s^k}{k} M_{n-k}(\Omega;E).$$

We now give a continuity property for the curvature measures, and we refer to [124, Theorem 2.4] for the proof.

Theorem 4.3.47. *Let M be a simply connected space form of dimension n and curvature κ. Let $\Omega_k \subset M$ be a sequence of compact sets with non-empty boundaries. Let us assume that for some $\delta > 0$ and for some compact set $\Omega \subset M$ we have*

$$\mathcal{R}(\Omega_k) \geq \delta \qquad\qquad \lim_k d^{\mathrm{H}}(\Omega_k,\Omega) = 0.$$

Then for every $r = 0,\dots,n$

$$M_r(\Omega_k;\cdot) \rightharpoonup M_r(\Omega;\cdot)$$

in the sense of Radon measures.

Finally, we state an Alexandrov-Fenchel inequality on the sphere comparing the curvature measure M_{n-2} with M_{n-1}, and we refer to [136, Theorem 1.5] for the proof of the regular case.

Theorem 4.3.48. *Let $\Omega \subset \mathbb{S}^n$ be a closed strongly convex set with C^2 boundary. Then*

$$\left(\frac{M_{n-2}(\Omega)}{(n-1)\sigma_n}\right)^2 \geq \left(\frac{M_{n-1}(\Omega)}{\sigma_n}\right)^{\frac{2(n-2)}{n-1}} - \left(\frac{M_{n-1}(\Omega)}{\sigma_n}\right)^2,$$

and the equality holds if and only if Ω is a geodesic ball.

Corollary 4.3.49. *Let $\Omega \subset \mathbb{S}^n$ be a closed strongly convex set. Then*

$$\left(\frac{M_{n-2}(\Omega)}{(n-1)\sigma_n}\right)^2 \geq \left(\frac{M_{n-1}(\Omega)}{\sigma_n}\right)^{\frac{2(n-2)}{n-1}} - \left(\frac{M_{n-1}(\Omega)}{\sigma_n}\right)^2.$$

Proof. If $\partial\Omega$ is of class C^2, then the result follows from Theorem 4.3.48.

For the general case, let Ω be a closed strongly convex set. By Corollary 4.3.34, we can find closed strongly convex sets Ω_k with smooth boundaries such that

$$\lim_k d^{\mathrm{H}}(\Omega_k, \Omega) = 0.$$

In particular, we have

$$\left(\frac{M_{n-2}(\Omega_k)}{(n-1)\sigma_n}\right)^2 \geq \left(\frac{M_{n-1}(\Omega_k)}{\sigma_n}\right)^{\frac{2(n-2)}{n-1}} - \left(\frac{M_{n-1}(\Omega_k)}{\sigma_n}\right)^2. \tag{4.3.12}$$

Since Ω_k are strongly convex, we have by Proposition 4.3.38

$$\mathcal{R}(\Omega_k) \geq \frac{\pi}{2}.$$

Therefore, we can apply Theorem 4.3.47, and passing to the limit in (4.3.12), the assertion follows. □

4.3.4 Isoperimetric inequality

Definition 4.3.50 (Minkowski Perimiter)**.** Let $\Omega \subset \mathbb{S}^n$ be a Borel set. We define the *lower Minkowski content* as

$$\mathrm{Mink}_-(\Omega) := \liminf_{s\to 0^+} \frac{|\Omega^s| - |\Omega|}{s}.$$

We now state the isoperimetric inequality on spheres in terms of Minkowski content. The original proof of this result is due to Schmidt in [156] (see also [152, Theorem 3.15, Theorem 1.52, Theorem 5.18]).

Theorem 4.3.51. *Let $\Omega \subset \mathbb{S}^n$ be a measurable set, and let $B \subset \mathbb{S}^n$ be a geodesic ball having the same measure as Ω. Then,*

$$\mathrm{Mink}_-(\partial B) \leq \mathrm{Mink}_-(\partial\Omega),$$

and the equality holds if and only if Ω is a geodesic ball.

In particular, Remark 4.3.45 ensures the following corollary for strongly convex sets of the sphere.

Corollary 4.3.52. *Let $\Omega \subset \mathbb{S}^n$ be an open set such that $\bar{\Omega}$ is strongly convex. Let $B \subset \mathbb{S}^n$ be a geodesic ball having the same measure as Ω, then*

$$\mathcal{H}^{n-1}(\partial B) \leq \mathcal{H}^{n-1}(\partial\Omega),$$

and the equality holds if and only if Ω is a geodesic ball.

4.3.5 Proof of the main theorem

In this section, for strongly convex sets $\Omega \subset \mathbb{S}^n$, we denote by $P(\Omega) = \mathcal{H}^{n-1}(\partial\Omega)$.

Let $\Omega \subset \mathbb{S}^n$ be an open set with strongly Lipschitz boundary, then the variational characterization (4.3.2) is well posed and the minimizers are weak solutions of (4.3.1).

Remark 4.3.53. Let D be a geodesic ball of center q and radius $R > 0$ in the sphere $\mathbb{S}^n$. We recall that the eigenfunctions relative to the first eigenvalue $\lambda^\beta(D)$ are all proportional. Therefore, by the rotational symmetry of D and the rotational invariance of the equation (4.3.1), we have that all the first eigenfunctions on D are radial. Precisely, $u(p) = \psi(d(p,q))$ for some function ψ solution to the one-dimensional problem

$$\begin{cases} \psi'' + (n-1)\cot(r)\psi' + \lambda^\beta(D)\psi = 0 & r \in (0,R), \\ \psi'(0) = 0, \\ \psi'(R) + \beta\psi(R) = 0. \end{cases}$$

Moreover letting $\phi(\rho) = \psi(R-\rho)$ we can write u as a function of the distance from the boundary of the ball D, indeed for every $p \in D$

$$d(p, \partial D) = R - d(p,q),$$

so that $u(p) = \phi(d(p,\partial D))$.

For every $\Omega \subset \mathbb{S}^n$, we denote by R_Ω its inradius, that is

$$R_\Omega = \max_{p\in\Omega} d(p,\partial\Omega).$$

We have the following

Lemma 4.3.54. *Let $\Omega \subset \mathbb{S}^n$ be a closed, strongly convex set, and let $\Omega_t = (\Omega)_t$. Then for almost every $t \in (0, R_\Omega)$ the function $P(\Omega_t)$ is differentiable and*

$$-\frac{d}{dt}P(\Omega_t) \geq (n-1)\left(\sigma_n^{\frac{2}{n-1}} P(\Omega_t)^{\frac{2(n-2)}{n-1}} - P(\Omega_t)^2\right)^{\frac{1}{2}}. \tag{4.3.13}$$

Proof. From the strong convexity of Ω, by Corollary 4.3.31 we have that the for every $R_\Omega > s > t > 0$ the inner parallel sets Ω_t and Ω_s are strongly convex, so that Corollary 4.3.26 ensures that

$$P(\Omega_s) \leq P(\Omega_t).$$

In particular, the function $t \mapsto P(\Omega_t)$ is monotonic decreasing and hence it is differentiable almost everywhere. Fix $t \in (0, R_\Omega)$, for every $s > 0$ sufficiently small, by Corollary 4.3.31, we have that the sets $(\Omega_t)^s$ are strongly convex. Moreover, since by definition

$$(\Omega_t)^s \subseteq \Omega_{t-s},$$

and both are strongly convex, we can apply Corollary 4.3.26 again, so that

$$P((\Omega_t)^s) \leq P(\Omega_{t-s}).$$

In particular, we get for almost every $t \in (0, R_\Omega)$

$$-\frac{d}{dt}P(\Omega_t) = \lim_{s\to 0^+} \frac{P(\Omega_{t-s}) - P(\Omega_t)}{s} \geq \lim_{s\to 0^+} \frac{P((\Omega_t)^s) - P(\Omega_t)}{s} = \frac{d}{ds}P((\Omega_t)^s)\Big|_{s=0}.$$

By the Steiner formula (Theorem 4.3.43), we have

$$\frac{d}{ds}P((\Omega_t)^s)\Big|_{s=0} = M_{n-2}(\Omega_t).$$

Hence, (4.3.13) follows from the Alexandrov-Fenchel inequality Corollary 4.3.49. □

In order to prove Theorem 4.3.1 we need a comparison result that relates $P((\Omega)_t)$ and $P((B)_t)$.

Lemma 4.3.55. *Let $f : [a, b] \to \mathbb{R}$ be a monotone decreasing function, and let $g : [a, b] \to \mathbb{R}$ be an absolutely continuous function. Assume that there exists a Lipschitz function $F : \mathbb{R} \to \mathbb{R}$ such that*

$$\begin{cases} f(a) \leq g(a), \\ f'(t) \leq F(f(t)) & \text{for a.e. } t \in (a, b), \\ g'(t) = F(g(t)) & \text{for a.e. } t \in (a, b), \end{cases}$$

then $f(t) \leq g(t)$ for every $t \in [a, b]$.

Proof. First we recall that since f is decreasing then for every $t, s \in (a, b]$ such that $t < s$, (see for instance [18, Corollary 3.29])

$$f(s) - f(t) \leq f(s^-) - f(t^+) \leq \int_t^s f'(\rho)\, d\rho \leq \int_t^s F(f(\rho))\, d\rho, \tag{4.3.14}$$

where we used the notation $f(s^\pm) = \lim_{\varepsilon\to 0^\pm} f(s+\varepsilon)$. On the other hand, for g we have the equality

$$g(s) - g(t) = \int_t^s g'(\rho)\, d\rho = \int_t^s F(g(\rho))\, d\rho. \tag{4.3.15}$$

Subtracting (4.3.15) to (4.3.14), and letting $w(t) = f(t) - g(t)$, then

$$\begin{aligned} w(s) - w(t) &\leq \int_t^s (F(f(\rho)) - F(g(\rho)))\, dt \\ &\leq L \int_t^s |w(\rho)|\, d\rho, \end{aligned} \tag{4.3.16}$$

where L is the Lipschitz constant of F. We also notice that by the monotonicity of f and the continuity of g we have

$$w(s) \leq w(s^-) \qquad \forall s \in (a, b], \tag{4.3.17}$$

$$w(s^+) \leq w(s) \qquad \forall s \in [a, b). \tag{4.3.18}$$

By contradiction, let us assume that for some $t_0 \in (a, b]$ we have $w(t_0) > 0$. Then (4.3.17) ensures that for a suitable $\delta > 0$ and for every $s \in (t_0 - \delta, t_0]$ we have $w(s) > 0$. Let

$$\tau = \sup\{\, t \in [a, t_0) \mid w(t) \leq 0 \,\},$$

so that $a \le \tau \le t_0 - \delta$, and

$$w(s) > 0 \qquad \forall s \in (\tau, t_0]. \tag{4.3.19}$$

By definition of τ, we have

$$w(\tau^+) \ge 0. \tag{4.3.20}$$

We claim that $w(\tau) = 0$. Indeed, if $\tau = a$, then the initial condition yields

$$w(a) \le 0,$$

and by (4.3.20), joint with (4.3.18), we get $0 \le w(a^+) \le w(a) \le 0$. If $\tau > a$, then, by definition of τ,

$$w(\tau^-) \le 0.$$

Using (4.3.20) joint with (4.3.17) and (4.3.18), we also have $0 \le w(\tau^+) \le w(\tau^-) \le 0$, and the claim is proved.

Therefore, since $w(\tau) = 0$, (4.3.16) reads as follows: for every $s \in (\tau, t_0)$ we have

$$w(s) \le L \int_\tau^s w(\rho)\, d\rho.$$

By the integral form of the Gronwall inequality (see for instance [114, Lemma 3.2], which is a particular case of [115, Theorem 3.1]) , we get $w \le 0$ in $[\tau, t_0]$, which is in contradiction with (4.3.19). □

We are now able to prove the main theorem.

Proof of Theorem 4.3.1. Let D be a strongly convex geodesic ball such that

$$P(\Omega) = P(D),$$

and let R be its radius. The isoperimetric inequality (Corollary 4.3.52) and the fact that both D and Ω are contained in a hemisphere, ensure that $|\Omega| \le |D|$. Since R_Ω is the radius of the biggest ball contained in Ω, we also obtain $R_\Omega \le R$, and the equality holds if and only if Ω is a ball of radius R_Ω.

For every $t \in (0, R_\Omega)$, let

$$\Omega_t = (\Omega)_t, \qquad \text{and} \qquad D_t = (D)_t$$

be the inner parallel sets of Ω and D respectively. Then from Lemma 4.3.54 we have that

$$\frac{d}{dt} P(\Omega_t) \le -(n-1)\left(\sigma_n^{\frac{2}{n-1}} P(\Omega_t)^{\frac{2(n-2)}{n-1}} - P(\Omega_t)^2\right)^{\frac{1}{2}},$$

while, by direct computation, the same estimate holds for the perimeter of D_t with the equality sign

$$\frac{d}{dt} P(D_t) = -(n-1)\left(\sigma_n^{\frac{2}{n-1}} P(D_t)^{\frac{2(n-2)}{n-1}} - P(D_t)^2\right)^{\frac{1}{2}}.$$

The comparison lemma (Lemma 4.3.55) ensures that for every $t \in (0, R_\Omega)$

$$P(\Omega_t) \le P(D_t). \tag{4.3.21}$$

Let u be an eigenfunction on D and let $\phi\colon [0,R]\to\mathbb{R}$ be as in Remark 4.3.53, then for every $p\in D$

$$u(p)=\phi(d(p,\partial D)).$$

For every $p\in\Omega$, let us define

$$v(p)=\phi(d(p,\partial\Omega)),$$

so that $v\in H^1(\Omega)$, and

$$\lambda^\beta(\Omega)\le\frac{\displaystyle\int_\Omega|\nabla v|^2\,d\mu+\beta\int_{\partial\Omega}v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\,d\mu}.$$

By direct computation, we have that

$$\int_{\partial\Omega}v^2\,d\mathcal{H}^{n-1}=\phi^2(0)P(\Omega)=\int_{\partial D}u^2\,d\mathcal{H}^{n-1}.$$

While, using coarea formula (Theorem 4.3.11) with $f(p)=d(p,\partial\Omega)$ and (4.3.21), we have

$$\int_\Omega v^2\,d\mu=\int_0^{R_\Omega}\phi^2(t)P(\Omega_t)\,dt\le\int_0^{R_\Omega}\phi^2(t)P(D_t)\,dt\le\int_D u^2\,d\mu,\tag{4.3.22}$$

and

$$\int_\Omega|\nabla v|^2\,d\mu=\int_0^{R_\Omega}(\phi'(t))^2P(\Omega_t)\,dt\le\int_0^{R_\Omega}(\phi'(t))^2P(D_t)\,dt\le\int_D|\nabla u|^2\,d\mu.$$

Then

$$\int_\Omega|\nabla v|^2\,d\mu+\beta\int_{\partial\Omega}v^2\,d\mathcal{H}^{n-1}\le\int_D|\nabla u|^2+\beta\int_{\partial D}u^2\,d\mathcal{H}^{n-1}<0.$$

Hence,

$$\lambda^\beta(\Omega)\le\frac{\displaystyle\int_\Omega|\nabla v|^2\,d\mu+\beta\int_{\partial\Omega}v^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2\,d\mu}\le\frac{\displaystyle\int_D|\nabla u|^2\,d\mu+\beta\int_{\partial D}u^2\,d\mathcal{H}^{n-1}}{\displaystyle\int_D u^2\,d\mu}=\lambda^\beta(D).$$

Finally, if the equality $\lambda^\beta(\Omega)=\lambda^\beta(D)$ holds, then the equality in (4.3.22) gives that $R_\Omega=R$, which implies that Ω is a geodesic ball of radius R_Ω. □

Following the approach of [17] we now prove Theorem 4.3.2.

Proof of Theorem 4.3.2. As in the proof of Theorem 4.3.1, let $\phi\colon[0,R]\to\mathbb{R}$ be such that $u(p)=\phi(d(p,\partial D))$ and let $v(p)=\phi(d(p,\partial\Omega))$. In order to obtain (4.3.6) we can better estimate the L^2-norm of the test v. Indeed,

$$\begin{aligned}\int_\Omega v^2\,d\mu&=\int_0^{R_\Omega}\phi^2(t)P(\Omega_t)\,dt\le\int_0^{R_\Omega}\phi^2(t)P(D_t)\,dt\\&=\int_0^{R}\phi^2(t)P(D_t)\,dt-\int_{R_\Omega}^{R}\phi^2(t)P(D_t)\,dt\le\int_D u^2\,d\mu-u_m^2\int_{R_\Omega}^{R}P(D_t)\,dt\\&\le\int_\Omega u^2\,d\mu-u_m^2(|D|-|\Omega|)=\int_\Omega u^2\,d\mu\left(1-\frac{u_m^2}{\|u\|^2_{L^2(D)}}(|D|-|\Omega|)\right),\end{aligned}$$

where we have used that, by definition of inradius, for a suitable ball B_{R_Ω} of radius R_Ω we have $B_{R_\Omega} \subseteq \Omega$. Therefore, computations analogous to the ones done in Theorem 4.3.1 lead to

$$\begin{aligned}
\lambda^\beta(\Omega) &\leq \frac{\displaystyle\int_\Omega |\nabla v|^2 \, d\mu + \beta \int_{\partial\Omega} v^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_\Omega v^2 \, d\mu} \\
&\leq \frac{\displaystyle\int_D |\nabla u|^2 \, d\mu + \beta \int_{\partial D} u^2 \, d\mathcal{H}^{n-1}}{\displaystyle\int_D u^2 \, d\mu \left(1 - \frac{u_m^2}{\|u\|^2_{L^2(D)}}(|D| - |\Omega|)\right)} \\
&= \lambda^\beta(D)\left(1 - \frac{u_m^2}{\|u\|^2_{L^2(D)}}(|D| - |\Omega|)\right)^{-1}.
\end{aligned}$$

So that, reordering the terms, (4.3.6) is proved. □

4.3.6 Further remarks

In this section, we show that the same arguments used to prove Theorem 4.3.1 cannot be used for strongly convex sets in the hyperbolic setting. Even though it is possible to generalize the Alexandrov-Fenchel inequalities to the hyperbolic space under strict convexity assumptions (see for instance [166, Theorem 1.1]), the main difficulty here is to extend Corollary 4.3.31 to the hyperbolic space $\mathbb{H}^n$. In particular, we can construct convex sets for which the inner parallel sets are not convex. To show an example, let us fix some notation. Let

$$\|x\|_e = \sqrt{\langle x, x\rangle}$$

be the Euclidean norm, and let $\mathbb{H}^n$ be represented in the Poincaré half-space model:

$$\begin{gathered}
\mathbb{H}^n = \{\, (\hat{x}, x_n) \in \mathbb{R}^n \mid x_n > 0 \,\}, \\
g_x(v, w) = \frac{\langle v, w\rangle}{x_n^2}, \\
d(x, y) = 2 \operatorname{arcsinh}\left(\frac{\|x - y\|_e}{2\sqrt{x_n y_n}}\right).
\end{gathered}$$

We also recall the shape of the geodesics in $\mathbb{H}^n$: let $p, q \in \mathbb{H}^n$, if $\hat{p} = \hat{q}$, then the geodesic γ_{pq} connecting the two is the vertical line passing through p and q; if $\hat{p} \neq \hat{q}$, then the geodesic γ_{pq} connecting p and q is the unique circular arc touching orthogonally the plane $\{x_n = 0\}$. We could describe the circular arc (non-parametrized by arc length) as

$$\begin{aligned}
\gamma_{pq} : [t_p, t_q] &\longrightarrow \mathbb{H}^n \\
t &\longmapsto x_0 + R\left(t\hat{w}, \sqrt{1 - t^2}\right),
\end{aligned} \tag{4.3.23}$$

where $x_0 = (\hat{x}_0, 0)$ is the center of the circular arc, $R = \|x_0 - p\|_e$ is the radius of the arc,

$$\hat{w} = \frac{\hat{p} - \hat{q}}{\|\hat{p} - \hat{q}\|_e},$$

and

$$[t_p, t_q] \subset (-1, 1),$$
$$\gamma_{pq}(t_p) = p \qquad\qquad \gamma_{pq}(t_q) = q.$$

We now divide the construction into three simple steps.

Step 1: Consider the cylinder

$$C = \left\{ (\hat{x}, x_n) \in \mathbb{R}^n \;\middle|\; \|\hat{x}\|_e \leq 1 \right\}.$$

C is convex:

consider $p, q \in C$, and let γ_{pq} be the geodesic connecting the points. If $\hat{p} = \hat{q}$, then we can represent the geodesic (non-parametrized by arc length) as

$$\gamma_{pq}(t) = (\hat{p}, t),$$

and obviously $\gamma_{pq}(t) \in C$ for every t. If $\hat{p} \neq \hat{q}$, then for the geodesic $\gamma_{pq} = (\hat{\gamma}_{pq}, \gamma^n_{pq})$ we have that $\hat{\gamma}_{pq}([t_p, t_q])$ is the segment joining $\hat{p}$ and $\hat{q}$ in $\mathbb{R}^{n-1}$, so that

$$\|\hat{\gamma}_{pq}(t)\|_e \leq \max\{\|\hat{p}\|_e, \|\hat{q}\|_e\} \leq 1,$$

for every $t \in [t_p, t_q]$.

Step 2. For any fixed vertical line $r(\hat{x}_0) = \{ (\hat{x}, x_n) \mid \hat{x} = \hat{x}_0 \}$, the level sets of the distance from $r(\hat{x}_0)$ are cones:

it is sufficient to notice that for every point $(\hat{x}_0, x_n)$ the geodesics orthogonal to $r(\hat{x}_0)$ in $(\hat{x}_0, x_n)$ are all contained in the hemisphere of radius x_n centered in $x_0 = (\hat{x}_0, 0)$, so that, with a direct computation,

$$(r(\hat{x}_0))^t = \{ x \in \mathbb{R}^n \mid d(x, r_0) \leq t \} = \{ (\hat{x}, x_n) \in \mathbb{R}^n \mid \|\hat{x} - \hat{x}_0\|_e \leq \sinh(t)\, x_n \}.$$

Step 3: the inner parallel sets $(C)_\delta$ are not convex for every choice of $\delta > 0$.

Indeed, notice that

$$(C)_\delta = C \cap \bigcap_{\|\hat{x}_0\|_e = 1} \overline{\mathbb{H}^n \setminus (r(\hat{x}_0))^\delta} = \left\{ (\hat{x}, x_n) \in \mathbb{R}^n \;\middle|\; \begin{array}{c} x_n > 0, \\ \|\hat{x}\|_e \leq 1 - \sinh(\delta)\, x_n. \end{array} \right\}$$

If for instance we take $q \in (C)_\delta$ such that $\|\hat{q}\|_e = 1 - \sinh(\delta) q_n$ and $p = (0, 1/\sinh(\delta)))$, then the minimal geodesic $\gamma_{pq} = (\hat{\gamma}, \gamma^n)$ connecting p and q lies outside the cone $(C)_\delta$: we can write, for $t \in [t_p, t_q]$,

$$\gamma(t) = x_0 + R\Big(t\hat{w}, \sqrt{1 - t^2}\Big)$$

as defined in (4.3.23); notice that by concavity

$$\begin{aligned} \gamma^n(t) &> \gamma^n(t_p) + \frac{t - t_p}{t_q - t_p}(\gamma^n(t_q) - \gamma^n(t_p)) \\ &= p_n + \frac{t - t_p}{t_q - t_p}(q_n - p_n) \qquad\qquad \forall t \in (t_p, t_q), \end{aligned} \tag{4.3.24}$$

and that, since $\hat{p} = 0$, then $\hat{w}$ and $\hat{x}_0$ are proportional, then $\|\hat{\gamma}(t)\|_e$ is linear in t, so that it is of the form

$$\|\hat{\gamma}_{pq}(t)\|_e = \|\hat{p}\|_e + \frac{t - t_q}{t_p - t_q}\|\hat{q}\|_e. \tag{4.3.25}$$

Using the fact that $p, q \in \partial(C)_\delta$, and in particular (4.3.24) and (4.3.25), which implies that the geodesic is not contained in $(C)_\delta$.

Chapter 5

Approximation

5.1 On the optimal shape of a thin insulating layer

The results of this section are contained in the paper [7].

Energy efficiency has emerged as one of the most pressing issues in recent years, as it is critical to achieving sustainable development, lowering greenhouse gas emissions, and mitigating climate change, all while boosting economic growth and increasing quality of life. Thermal insulation is important in the context of energy efficiency because it helps to reduce heat transfer and energy losses in buildings and industrial processes, resulting in significant energy and cost savings.

This section addresses the topic of thermal insulation of a solid, kept at constant temperature, by displacing around it an insulator. Roughly speaking, the optimal insulation is the one minimizing, at thermal equilibrium, the heat rate loss per unit time across the exterior boundary. More specifically, in our model heat exchange with the environment occurs through convection, which is by far the most common mechanism in real world applications.

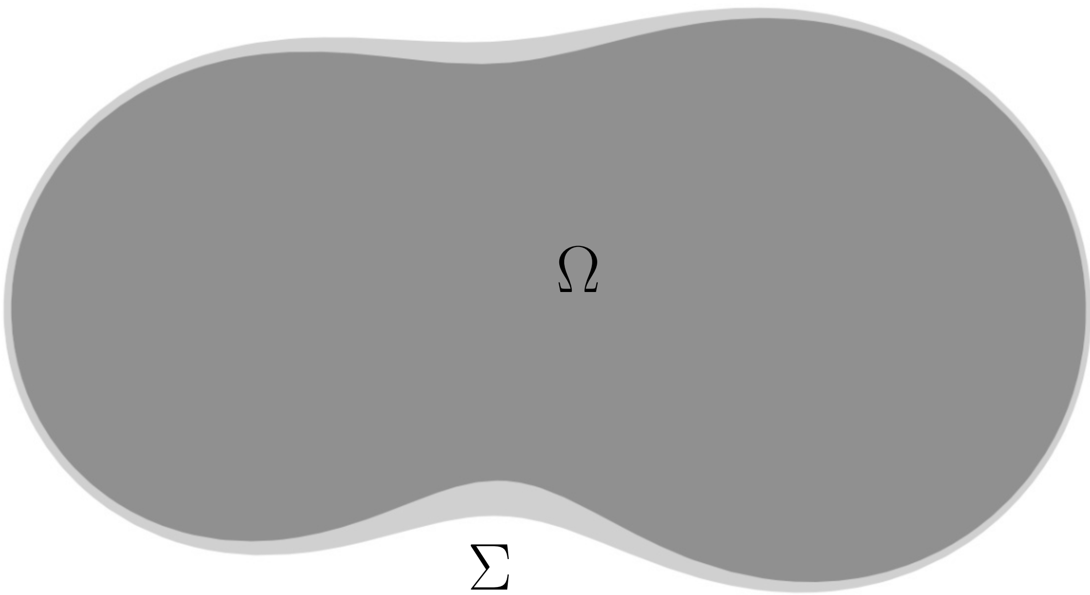
In order to contain costs of insulation, the volume of the insulator is prescribed. The resulting mathematical model gives rise to a free boundary problem which has been previously studied in [65] and for more general heat transfer mechanisms (including for instance also radiation) in [58]. In particular, the former paved the way on how to prove the existence of an optimal distribution of insulator and the regularity of its a priori unknown boundary.

However, although a solution always exists for every set to be insulated and every amount of insulator, very little is known about the optimal shape of the insulator, and the only completely solved case happens to be the radial one.

This section addresses the problem of qualitatively and quantitatively describing how to displace the insulator, and to do so, it restricts the analysis to "small insulation thickness", an approximation reasonable in many contexts, such as in the case of buildings and other big structures.

Throughout the section, Ω represents the body to insulate, in which the temperature is fixed. Since the problem is invariant under temperature scaling and translations, without loss of generality the temperature of the body will be 1, while the environment temperature will be 0. If convection is the leading mechanism of thermal exchange with the environment, then the heat rate loss per unit time and unit surface is proportional to the temperature jump across the surface element separating the insulator (or the body) with the environment. The constant of proportionality will be denoted by $\beta > 0$. If Σ denotes the insulator (see figure), then at equilibrium the temperature distribution in Σ is an harmonic function which we denote by u. On the portion of boundary that Σ shares with Ω the function u is equal to 1. While on the portion of boundary which Σ shares with the environment,

Figure 5.1: Body Ω with a thin insulating layer Σ



u satisfies a Robin condition

$$\frac{\partial u}{\partial \nu} + \beta u = 0,$$

where ν is outer normal of the boundary of Σ. In fact, according to Fourier law, which holds inside Σ, the heat flux per unit time is $-Du$, and therefore the flux across the surface element at the boundary is $-\frac{\partial u}{\partial \nu}$. On the other hand, according to convection law, the heat flux per unit time across the surface element is equal to βu (β times the jump of u). Continuity of the heat flux enforces to equalize these expressions and the Robin b.c. naturally arises.

In summary, u is a continuous function in $\Omega \cup \Sigma$ which solves

$$\begin{cases} \Delta u = 0 & \text{in } \Sigma, \\ u = 1 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} + \beta u = 0 & \text{on } \partial\Sigma \setminus \partial\Omega. \end{cases}$$

We can also characterize the function u as the minimizer of the energy functional

$$\mathcal{E}(v) = \int_\Sigma |\nabla v|^2 \, dx + \beta \int_{\partial(\Omega \cup \Sigma)} v^2 \, d\mathcal{H}^{n-1},$$

among all functions $v \in H^1(\Omega \cup \Sigma)$ such that $v \equiv 1$ in Ω.

For a given Ω, our ultimate goal would be to find the shape of Σ which minimizes

$$\int_{\partial(\Omega \cup \Sigma)} u \, d\mathcal{H}^{n-1}$$

among all Σ of prescribed measure.

So far, such a problem seems to be out of reach, so we restricted our analysis to the case in which the layer of the insulating material and the conductivity of the insulator are both very small.

More precisely, let $\Omega \subset \mathbb{R}^n$ be a smooth bounded, open set, and let $h\colon \partial\Omega \to \mathbb{R}$ be a positive function. Denoting by ν the exterior unit normal to the boundary of Ω, we define

$$\Sigma_\varepsilon := \{\, \sigma + t\nu(\sigma) \mid \sigma \in \partial\Omega,\, 0 < t < \varepsilon h(\sigma) \,\}$$

and we denote by $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$. Here, the non negative parameter ε is meant to be small and we want to investigate the limit as ε vanishes. But in order to have this limit non trivial, we also assume that the conductivity of the insulator is small, namely, the heat flux inside Σ_ε is $-\varepsilon Du$. In practice, we are assuming that we use a "small quantity" of a "very good" insulator.

All at once we can consider the minimization of the following energy functional

$$\mathcal{F}_\varepsilon(v,h) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1},$$

where $v \in H^1(\Omega_\varepsilon)$, with $v = 1$ in Ω. Here, the small parameter ε in front of the first integral is encoding the fact that the conductivity of the insulator is small. For given h, a minimum $u_{\varepsilon,h}$ of

$$\min \left\{ \mathcal{F}_\varepsilon(v,h) \;\middle|\; v \in H^1(\Omega_\varepsilon),\, v = 1 \text{ in } \Omega \right\} \tag{5.1.1}$$

solves the boundary value problem:

$$\begin{cases} \Delta u_{\varepsilon,h} = 0 & \text{in } \Sigma_\varepsilon, \\ u_{\varepsilon,h} = 1 & \text{in } \Omega, \\ \varepsilon \dfrac{\partial u_{\varepsilon,h}}{\partial \nu_\varepsilon} + \beta u_{\varepsilon,h} = 0 & \text{on } \partial\Omega_\varepsilon \setminus \partial\Omega, \end{cases}$$

where ν_ε is the exterior unit normal to the boundary of Ω_ε.

Similar problems have been studied before in the context of thermal insulation in [40], [97], [10], and more recently in [44] and [86]. The limit has been performed in several ways. In our case, we are going to use Γ-convergence. But in order to extract as much information as possible about the problem we are going to perform a first order expansion in ε [21], which, to our knowledge, has never been exploited in this context.

The volume of insulator we displace is εm, for some $m > 0$, and we define the volume constraint by defining the space

$$\mathcal{H}_m = \mathcal{H}_m(\partial\Omega) = \left\{ h \in L^1(\partial\Omega) \;\middle|\; \begin{array}{l} \displaystyle\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \le m \\ h \ge 0 \end{array} \right\}. \tag{5.1.2}$$

Our problem reduces to finding the best configuration of insulating material surrounding Ω, that is

$$\min \left\{ \mathcal{F}_\varepsilon(v,h) \;\middle|\; \begin{array}{l} v \in H^1(\Omega_\varepsilon), \\ v = 1 \text{ in } \Omega, \\ h \in \mathcal{H}_m \end{array} \right\}. \tag{5.1.3}$$

Following argument similar to those used in [86], it can be proved that, for any fixed Lipschitz function $h : \partial\Omega \to (0, +\infty)$, as $\varepsilon \to 0^+$, the functional $\mathcal{F}_\varepsilon(\cdot, h)$ Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, to the functional

$$\mathcal{F}_0(h) = \beta \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}.$$

Then, in view of the convexity of the functional with respect to h,

$$\min \{ \mathcal{F}_0(h) \mid h \in \mathcal{H}_m \}$$

is achieved by the constant $h = m/P(\Omega)$, where $P(\Omega)$ denotes the perimeter of Ω.

Displacing the insulator uniformly around the boundary is somehow the trivial solution, the one suggested by common sense, and very common when insulating buildings. However, it is mathematically not satisfactory at all. As we expect that portions of boundary with higher (mean) curvature are less convenient to insulate with respect to those with lower curvature. Such an idea is strongly suggested by the radial cases (see for example [87, Proposition 5.1]).

But such a kind of evidence is lost when performing the Γ-limit and therefore we decided to push our analysis a bit further. Let

$$K_0 = \left\{ v \in L^2(\mathbb{R}^n) \;\middle|\; v = 1 \text{ in } \Omega \right\},$$

our main result is a first-order asymptotic development by Γ-convergence (see definition 2.7.3) for the functional $\mathcal{F}_\varepsilon$. We denote by H the mean curvature of Ω (see definition 2.2.6) and we prove the following

Theorem 5.1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with C^3 boundary, and fix a C^2 positive function $h\colon \partial\Omega \to (0, +\infty)$. Then the functional*

$$\delta\mathcal{F}_\varepsilon(\cdot, h) = \frac{\mathcal{F}_\varepsilon(\cdot, h) - \mathcal{F}_0(h)}{\varepsilon}$$

Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to

$$\mathcal{F}^{(1)}(v, h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{H_\Omega h(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} & \text{if } v \in K_0, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_0. \end{cases} \tag{5.1.4}$$

The section is planned as follows. In section 5.1.1 we prove theorem 5.1.1. Thereafter, in section 5.1.2 we fix Ω and we deal with the minimum problem

$$\inf \left\{ \mathcal{F}_0(h) + \varepsilon \mathcal{F}^{(1)}(h) \;\middle|\; h \in \mathcal{H}_m \right\},$$

where $\mathcal{F}^{(1)}(h) = \mathcal{F}^{(1)}(\chi_\Omega, h)$. As we already mentioned, the problem above, is a first-order approximation of the problem eq. (5.1.3) with respect to $\varepsilon > 0$. Indeed we have that (see remark 2.7.4)

$$\mathcal{F}_\varepsilon(u_\varepsilon, h) = \mathcal{F}_0(h) + \varepsilon \mathcal{F}^{(1)}(h) + R(\Omega, h, \varepsilon),$$

where u_ε is the minimizer to eq. (5.1.1), and

$$\lim_{\varepsilon \to 0^+} \frac{R(\Omega, h, \varepsilon)}{\varepsilon} = 0.$$

In particular, we will prove that, as the intuition suggests, if ε is small enough then the optimal configuration for the insulating layer concentrates close to the points of $\partial\Omega$ where the mean curvature is relatively small. Finally, in section 5.1.2 we discuss the behaviour of the functional $\mathcal{F}_0 + \varepsilon\mathcal{F}^{(1)}$ under various geometrical constraints (volume, perimeter, quermassintegral) on the set Ω.

5.1.1 The Γ-limit

Setting of the problem

Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lispchitz function $h \colon \partial\Omega \to \mathbb{R}$. We recall that

$$\Sigma_\varepsilon = \{\, \sigma + t\nu(\sigma) \mid \sigma \in \partial\Omega,\, 0 < t < \varepsilon h(\sigma) \,\},$$

and

$$\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon.$$

We notice that if d_0 is the one defined in Proposition 2.1.6, and we extend

$$h(x) = h(\pi_\Omega(x))\chi_{\Sigma_\Omega(d_0)},$$

then $\Sigma_\varepsilon = \Sigma_\Omega(\varepsilon h)$. Our assumptions on $\partial\Omega$ ensure that there exists $\varepsilon_0 = \varepsilon_0(\Omega, h)$ such that, if $0 < \varepsilon \le \varepsilon_0$, the map

$$\phi \colon (\sigma, t) \longmapsto \sigma + t\nu(\sigma)$$

is invertible, that is, for every $x \in \Sigma_\varepsilon$ there exist unique $\pi_\Omega(x) \in \partial\Omega$ and $t(x)$, with $0 \le t(x) \le \varepsilon h(\pi_\Omega(x))$, such that

$$x = \pi_\Omega(x) + t(x)\nu(\pi_\Omega(x)).$$

Therefore, we can extend h and ν on Σ_ε as $h(x) = h(\pi_\Omega(x))$, and $\nu(x) = \nu(\pi_\Omega(x))$ respectively. Moreover, for every $x \in \mathbb{R}^n \setminus \Omega$, let

$$d(x) = d(x, \partial\Omega) = \inf_{y \in \partial\Omega} |x - y|$$

be the distance from $\partial\Omega$, then we have that $d(x) = t(x)$ for every $x \in \Sigma_\varepsilon$.

Remark 5.1.2. If $f : \Omega_\varepsilon \to \mathbb{R}$ is a positive Borel function, we have

$$\int_{\Sigma_\varepsilon} f(x)\,dx = \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} f(\sigma + t\nu)\Big(1 + tH_\Omega(\sigma) + \varepsilon^2 R_1(\sigma, t, \varepsilon)\Big)\,dt\,d\mathcal{H}^{n-1} \tag{5.1.5}$$

and

$$\int_{\partial\Omega_\varepsilon} f(\sigma)\,d\mathcal{H}^{n-1} = \int_{\partial\Omega} f(\sigma + \varepsilon h\nu)\Big(1 + \varepsilon h(\sigma)H_\Omega(\sigma) + \varepsilon^2 R_2(\sigma, \varepsilon)\Big)\,d\mathcal{H}^{n-1}, \tag{5.1.6}$$

where the remainder terms R_1 and R_2 are bounded functions, i.e. there exists $Q > 0$ such that $|R_1|, |R_2| \le Q$. Indeed, applying coarea formula (theorem 2.3.1) with the distance function d we have

$$\int_{\Sigma_\varepsilon} f(x)\,dx = \int_0^{+\infty} \int_{\{\, d=t \,\} \cap \Sigma_\varepsilon} \chi_{\Sigma_\varepsilon}(\sigma) f(\sigma)\,d\mathcal{H}^{n-1}(\sigma)\,dt.$$

Then we use the area formula on surfaces (theorem 2.2.4) with the map $\phi(x,t) = x + t\nu(x)$ which maps the set $\partial\Omega$ in the set $\{ d = t \} \setminus \Omega$, and use remark 2.2.7 to estimate the tangential Jacobian of ϕ as

$$1 + tH_\Omega(\sigma) + \varepsilon^2 R_1(\sigma, t, \varepsilon).$$

Finally, we apply Fubini's theorem to get eq. (5.1.5). Analogously we obtain eq. (5.1.6).

Let

$$K_\varepsilon = \left\{ v \in H^1(\Omega_\varepsilon) \;\middle|\; v = 1 \text{ in } \Omega \right\}, \tag{5.1.7}$$

and

$$K_0 = \left\{ v \in L^2(\mathbb{R}^n) \;\middle|\; v = 1 \text{ in } \Omega \right\}, \tag{5.1.8}$$

and consider the functional

$$\mathcal{F}_\varepsilon(v,h) = \begin{cases} \varepsilon \displaystyle\int_{\Sigma_\varepsilon} |\nabla v|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1} & \text{if } v \in K_\varepsilon, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_\varepsilon. \end{cases}$$

denoting by

$$\mathcal{F}_0(v,h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1} & \text{if } v \in K_0, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_0, \end{cases}$$

following the approach of [86], we have the following

Proposition 5.1.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a Lipchitz function $h\colon \partial\Omega \to (0,+\infty)$. Then $\mathcal{F}_\varepsilon(\cdot,h)$ Γ-converges, as $\varepsilon \to 0^+$, in the strong $L^2(\mathbb{R}^n)$ topology, to $\mathcal{F}_0(\cdot,h)$.*

Proof. We start by proving the Γ-liminf inequality: Let $v \in L^2(\mathbb{R}^n)$ and let $v_\varepsilon \in L^2(\mathbb{R}^n)$ such that v_ε converges to v in $L^2(\mathbb{R}^n)$ as $\varepsilon \to 0^+$. Up to passing to a sub-sequence, we can assume that

$$\liminf_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(v_\varepsilon, h) = \lim_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(v_\varepsilon, h),$$

moreover, we can assume that such a limit is finite and that $v_\varepsilon \in K_\varepsilon$. Therefore we have that $v \in K_0$ and, by eq. (5.1.5), eq. (5.1.6) we have that

$$\int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \, dx \geq \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu)|^2 (1 - \varepsilon Q_0) \, dt \, d\mathcal{H}^{n-1} \tag{5.1.9}$$

and

$$\int_{\partial\Omega_\varepsilon} v_\varepsilon^2 \, d\mathcal{H}^{n-1} \geq \int_{\partial\Omega} v_\varepsilon^2(\sigma + \varepsilon h(\sigma)\nu)(1 - \varepsilon Q_0) \, d\mathcal{H}^{n-1} \tag{5.1.10}$$

for some constant $Q_0 > 0$. On the other hand, we have that, for $\mathcal{H}^{n-1}$-almost every $\sigma \in \partial\Omega$,

$$\begin{aligned} \int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu)|^2 \, dt \geq& \frac{1}{\varepsilon h(\sigma)} \left(\int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu)| \, dt \right)^2 \\ \geq& \frac{(v_\varepsilon(\sigma + \varepsilon h\nu) - 1)^2}{\varepsilon h(\sigma)}, \end{aligned}$$

then by Young's inequality, we have that, for every $\lambda > 0$ and for $\mathcal{H}^{n-1}$-almost every $\sigma \in \partial\Omega$,

$$\int_0^{\varepsilon h(\sigma)} |\nabla v_\varepsilon(\sigma + t\nu)|^2 \, dt \geq \frac{(1-\lambda) v_\varepsilon(\sigma + \varepsilon h \nu)^2}{\varepsilon h(\sigma)} + \frac{1}{\varepsilon h(\sigma)} \left(1 - \frac{1}{\lambda}\right). \tag{5.1.11}$$

Putting together eq. (5.1.9), eq. (5.1.10) and eq. (5.1.11) we finally have

$$\mathcal{F}_\varepsilon(v_\varepsilon, h) \geq \int_{\partial\Omega} \left(\left(\frac{1-\lambda}{h} + \beta \right) v_\varepsilon(\sigma + \varepsilon h\nu)^2 + \frac{1}{h}\left(1 - \frac{1}{\lambda}\right) \right) d\mathcal{H}^{n-1} - \varepsilon Q_0 R_\varepsilon(\varepsilon, v_\varepsilon),$$

where, if ε is sufficiently small, using again eq. (5.1.5) and eq. (5.1.6), we have

$$R_\varepsilon(\varepsilon, v_\varepsilon) \leq 2\mathcal{F}_\varepsilon(v_\varepsilon, h).$$

Finally, letting $\lambda = \lambda(\sigma) = 1 + \beta h(\sigma)$, and passing to the limit as $\varepsilon \to 0^+$ we have that

$$\liminf_{\varepsilon \to 0^+} \mathcal{F}_\varepsilon(v_\varepsilon, h) \geq \beta \int_{\partial\Omega} \frac{1}{1 + \beta h} \, d\mathcal{H}^{n-1} = \mathcal{F}_0(v, h)$$

and the Γ-liminf inequality is proved.

Γ-limsup inequality: Let $v \in L^2(\mathbb{R}^n)$, if $v \notin K_0$ the Γ-limsup inequality is trivial, therefore let $v \in K_0$. Let

$$v_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\[2ex] 1 - \dfrac{\beta d(x)}{\varepsilon(1 + \beta h(x))} & \text{if } x \in \Sigma_\varepsilon, \\[2ex] v(x) & \text{if } x \notin \Omega_\varepsilon, \end{cases}$$

where we recall that, if $x = \sigma + t\nu(\sigma)$, then $h(x) = h(\sigma)$. Trivially v_ε converges to v in $L^2(\mathbb{R}^n)$ and $v_\varepsilon \in K_\varepsilon$. For every $x \in \Sigma_\varepsilon$,

$$\nabla v_\varepsilon(x) = -\frac{\beta \nabla d(x)}{\varepsilon(1 + \beta h(x))} + \frac{\beta^2 d(x) \nabla h(x)}{\varepsilon(1 + \beta h(x))^2}.$$

Recalling that $0 \leq d \leq \varepsilon h$, $\nabla d = \nu$ and $\nabla h \cdot \nu = 0$, we have

$$|\nabla v_\varepsilon|^2 = \frac{\beta^2}{\varepsilon^2 (1 + \beta h)^2} + \frac{\beta^4 d^2 |\nabla h|^2}{\varepsilon^2 (1 + \beta h)^4} \leq \frac{\beta^2}{\varepsilon^2 (1 + \beta h)^2} + \frac{\beta^4 h^2 |\nabla h|^2}{(1 + \beta h)^4},$$

where the second term is bounded since h is Lipschitz. Hence, substituting $\varepsilon h \tau = t$ in eq. (5.1.5), we get

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} |\nabla v_\varepsilon|^2 \, dx &\leq \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{1}{(1 + \beta h)^2} \, dx + \varepsilon C |\Sigma_\varepsilon| \\ &\leq \int_{\partial\Omega} \int_0^1 \frac{\beta^2 h}{(1 + \beta h)^2} (1 + \varepsilon Q_0) \, d\tau \, d\mathcal{H}^{n-1} + \varepsilon C |\Sigma_\varepsilon| \\ &= \int_{\partial\Omega} \frac{\beta^2 h}{(1 + \beta h)^2} \, d\mathcal{H}^{n-1} + o(\varepsilon). \end{aligned}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$v_\varepsilon(\sigma + \varepsilon h(\sigma)\nu(\sigma)) = \frac{1}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} v_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2}(1+\varepsilon Q_0)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2}\, d\mathcal{H}^{n-1} + o(\varepsilon).$$

Hence we have

$$\mathcal{F}_\varepsilon(v_\varepsilon, h) \leq \beta \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1} + o(\varepsilon),$$

so that

$$\limsup_{\varepsilon\to 0^+} \mathcal{F}_\varepsilon(v_\varepsilon, h) \leq \beta \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

In the following, for simplicity, we will denote by

$$\mathcal{F}_0(h) = \beta \int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}.$$

It can be deduced from the more general results in [86] that the minimum in the class of functions h with a given mass of such functional is achieved when h is constant. We include a direct proof of this statement in the following

Proposition 5.1.4. *Let Ω be a bounded open set with Lipschitz boundary, let $P(\Omega) = P$, and let $m > 0$. Then the problem*

$$\min\{\, \mathcal{F}_0(h) \mid h \in \mathcal{H}_m \,\} \tag{5.1.12}$$

admits

$$h_0 = \frac{m}{P}$$

as the unique solution.

Proof. Let $h \in \mathcal{H}_m$. By Holder's inequality, we have that

$$\begin{aligned} P = \int_{\partial\Omega} d\mathcal{H}^{n-1} &\leq \left(\int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}\right)^{1/2} \left(\int_{\partial\Omega} (1+\beta h)\, d\mathcal{H}^{n-1}\right)^{1/2} \\ &\leq \left(\int_{\partial\Omega} \frac{1}{1+\beta h}\, d\mathcal{H}^{n-1}\right)^{1/2} (P+\beta m)^{1/2}, \end{aligned}$$

so that

$$\mathcal{F}_0(h) \geq \frac{\beta P^2}{P+\beta m} = \mathcal{F}_0(h_0).$$

Finally, the uniqueness of the solution is given by the strict convexity of the function

$$x \longmapsto \frac{1}{1+\beta x}$$

for $x \geq 0$. □

Let $H = H_\Omega$ be the mean curvature of Ω, we aim to show that

$$\delta\mathcal{F}_\varepsilon(\cdot, h) = \frac{\mathcal{F}_\varepsilon(\cdot, h) - \mathcal{F}_0(h)}{\varepsilon}$$

Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, to

$$\mathcal{F}^{(1)}(v, h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{Hh(2+\beta h)}{2(1+\beta h)^2}\, d\mathcal{H}^{n-1} & \text{if } v \in K_0, \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n) \setminus K_0, \end{cases}$$

where K_0 is the set defined in eq. (5.1.8).

Proof of theorem 5.1.1

Let Ω be a bounded, open set with C^3 boundary, and fix a positive C^2 function $h\colon \partial\Omega \to \mathbb{R}$. In this section, we study the Γ-convergence of the family of functionals

$$\delta\mathcal{F}_\varepsilon(v) = \frac{\mathcal{F}_\varepsilon(\cdot, h) - \mathcal{F}_0(h)}{\varepsilon}, \tag{5.1.13}$$

and we prove theorem 5.1.1. In the following we consider the functions $h, H_\Omega\colon \partial\Omega \to \mathbb{R}$ extended on the set Σ_ε as $h(\sigma + t\nu) = h(\sigma)$ and $H_\Omega(\sigma + t\nu) = H_\Omega(\sigma)$.

For every $\varepsilon > 0$ let $u_\varepsilon \in K_\varepsilon$ be the minimizer to $\mathcal{F}_\varepsilon$, where K_ε is defined in eq. (5.1.7). By the assumptions on Ω and h, we have that u_ε is a $C^2(\Sigma_\varepsilon)$ function and it is a solution to

$$\begin{cases} -\Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ u_\varepsilon = 1 & \text{on } \partial\Omega, \\ \varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \beta u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5.1.14}$$

Let $\alpha \in (0, \alpha_0)$, where

$$\alpha_0 = 1 - \max_{\sigma\in\partial\Omega} \frac{\beta h(\sigma)}{1 + \beta h(\sigma)},$$

and let $\varepsilon > 0$, then on Σ_ε we define

$$w_{\varepsilon,\alpha}(x) = \begin{cases} 1 - \left(\dfrac{d(x)}{\varepsilon h(x)}\right)^{1-\alpha} \dfrac{\beta h(x)}{(1-\alpha)(1+\beta h(x))} & \text{if } H_\Omega(x) \geq 0, \\[2ex] 1 - \left(\dfrac{d(x)}{\varepsilon h(x)}\right)^{1+\alpha} \dfrac{\beta h(x)}{(1+\alpha)(1+\beta h(x))} & \text{if } H_\Omega(x) < 0. \end{cases}$$

Then we have that $w_{\varepsilon,\alpha} > 0$ and the following holds

Proposition 5.1.5. *For every $\alpha \in (0, \alpha_0)$ there exists $\varepsilon_\alpha > 0$ such that if $0 < \varepsilon < \varepsilon_\alpha$, then*

$$H_\Omega(x) u_\varepsilon(x) \geq H_\Omega(x) w_{\varepsilon,\alpha}(x) \qquad \textit{for } x \in \Sigma_\varepsilon.$$

Proof. Fix $\alpha \in (0, \alpha_0)$. For simplicity, we denote by

$$v_\gamma := 1 - \left(\frac{d(x)}{\varepsilon h(x)}\right)^\gamma \frac{\beta h(x)}{\gamma(1+\beta h(x))},$$

and we aim to show that there exists an $\varepsilon_\alpha > 0$ such that for any $0 < \varepsilon < \varepsilon_\alpha$, we have that $v_{1-\alpha}$ is a subsolution to eq. (5.1.14), while $v_{1+\alpha}$ is a supersolution to the same problem. Namely,

$$\begin{cases} -\Delta v_{1-\alpha} \leq 0 & \text{in } \Sigma_\varepsilon, \\ v_{1-\alpha} = 1 & \text{on } \partial\Omega, \\ \varepsilon \dfrac{\partial v_{1-\alpha}}{\partial \nu_\varepsilon} + \beta v_{1-\alpha} \leq 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \qquad \begin{cases} -\Delta v_{1+\alpha} \geq 0 & \text{in } \Sigma_\varepsilon, \\ v_{1+\alpha} = 1 & \text{on } \partial\Omega, \\ \varepsilon \dfrac{\partial v_{1+\alpha}}{\partial \nu_\varepsilon} + \beta v_{1+\alpha} \geq 0 & \text{on } \partial\Omega_\varepsilon. \end{cases} \tag{5.1.15}$$

In the following, we will always assume that $\varepsilon < 1$. Let us recall that

$$\Omega_\varepsilon = \left\{ x \in \mathbb{R}^n \;\middle|\; \frac{d(x)}{h(x)} \leq \varepsilon \right\}, \qquad \partial\Omega_\varepsilon = \{\, x \in \mathbb{R}^n \mid d(x) - \varepsilon h(x) = 0 \,\}.$$

By standard computations we get

$$\nabla\left(\frac{d}{h}\right) = \frac{\nabla d}{h} - \frac{d\nabla h}{h^2}, \qquad \left|\nabla\left(\frac{d}{h}\right)\right| = \frac{1}{h}\sqrt{1 + \left(\frac{d}{h}\right)^2 |\nabla h|^2}.$$

Then, recalling that $\nabla d = \nu$ and that $\nabla h \cdot \nu = 0$, the normal ν_ε to the set Ω_ε is given by

$$\nu_\varepsilon = \frac{1}{\sqrt{1 + \varepsilon^2 |\nabla h|^2}} (\nu - \varepsilon \nabla h).$$

By direct computations, for any $\gamma \in (0,2) \setminus \{1\}$ we have

$$\Delta v_\gamma = -\frac{\beta h}{\gamma \varepsilon^\gamma (1+\beta h)} \Delta\left[\left(\frac{d}{h}\right)^\gamma\right] - \frac{2}{\gamma\varepsilon^\gamma} \nabla\left[\left(\frac{d}{h}\right)^\gamma\right] \cdot \nabla\left[\frac{\beta h}{1+\beta h}\right] - \frac{1}{\gamma\varepsilon^\gamma}\left(\frac{d}{h}\right)^\gamma \Delta\left[\frac{\beta h}{1+\beta h}\right]. \tag{5.1.16}$$

We then compute

$$\nabla\left[\left(\frac{d}{h}\right)^\gamma\right] = \gamma\left(\frac{d}{h}\right)^{\gamma-1}\left(\frac{\nu}{h} - \frac{d\nabla h}{h^2}\right), \qquad \nabla\left[\frac{\beta h}{1+\beta h}\right] = \frac{\beta \nabla h}{(1+\beta h)^2}, \tag{5.1.17}$$

from which we get

$$\nabla\left[\left(\frac{d}{h}\right)^\gamma\right] \cdot \nabla\left[\frac{\beta h}{1+\beta h}\right] = -\gamma\left(\frac{d}{h}\right)^\gamma \frac{\beta|\nabla h|^2}{h(1+\beta h)^2}. \tag{5.1.18}$$

In addition, we have

$$\begin{aligned} \Delta\left[\left(\frac{d}{h}\right)^\gamma\right] &= \gamma(\gamma-1)\left(\frac{d}{h}\right)^{\gamma-2}\left|\nabla\left(\frac{d}{h}\right)\right|^2 + \gamma\left(\frac{d}{h}\right)^{\gamma-1}\left(\frac{\Delta d}{h} - \frac{d\Delta h}{h^2} + 2\frac{d|\nabla h|^2}{h^3}\right) \\ &= \gamma\left(\frac{d}{h}\right)^{\gamma-2}\left(-\frac{1-\gamma}{h^2} - \left(\frac{d}{h}\right)^2 \frac{(1-\gamma)|\nabla h|^2}{h^2} + \frac{d}{h}\frac{\Delta d}{h} - \left(\frac{d}{h}\right)^2 \frac{\Delta h}{h} + \left(\frac{d}{h}\right)^2 \frac{2|\nabla h|^2}{h^2}\right), \end{aligned} \tag{5.1.19}$$

so that, by eq. (5.1.16), eq. (5.1.19), and eq. (5.1.18), we get

$$\left(\frac{d(x)}{h(x)}\right)^{2-\gamma}\varepsilon^{\gamma}\Delta v_{\gamma}(x)=\frac{\beta(1-\gamma)}{h(x)(1+\beta h(x))}+R_1(x,\varepsilon,\gamma), \tag{5.1.20}$$

where $R_1(x,\varepsilon,\gamma)$ is a suitable remainder term. Since $d\le\varepsilon h$,

$$0<\inf_{\Sigma_\varepsilon} h\le\sup_{\Sigma_\varepsilon} h<+\infty,$$

and $|\nabla h|$, Δh, Δd are bounded, then there exist $C_\gamma,\varepsilon_0>0$ such that

$$|R_1(x,\varepsilon,\gamma)|\le C_\gamma\varepsilon \tag{5.1.21}$$

for any $\varepsilon<\varepsilon_0$. Thus, using eq. (5.1.21) in eq. (5.1.20) we have that there exists $\varepsilon_\alpha>0$ such that if $0<\varepsilon<\varepsilon_\alpha$, then

$$-\Delta v_{1-\alpha}<0,\qquad -\Delta v_{1+\alpha}>0. \tag{5.1.22}$$

On the other hand, for every $x\in\partial\Omega_\varepsilon$, since $d(x)=\varepsilon h(x)$ and eq. (5.1.17) hold true, we get

$$\nabla v_\gamma(x)=-\frac{\beta}{\varepsilon(1+\beta h(x))}(\nu(x)-\varepsilon\nabla h(x))-\frac{\beta\nabla h(x)}{\gamma(1+\beta h(x))^2},$$

which yields

$$\begin{aligned}\frac{\partial v_\gamma}{\partial\nu_\varepsilon}(x)&=-\frac{\beta\sqrt{1+\varepsilon^2|\nabla h|^2}}{\varepsilon(1+\beta h)}-\frac{\beta}{\gamma}\nu_\varepsilon\cdot\frac{\nabla h}{(1+\beta h)^2}\\&=-\frac{\beta\sqrt{1+\varepsilon^2|\nabla h|^2}}{\varepsilon(1+\beta h)}+\frac{\beta\varepsilon|\nabla h|^2}{\gamma(1+\beta h)^2\sqrt{1+\varepsilon^2|\nabla h|^2}},\end{aligned} \tag{5.1.23}$$

while

$$v_\gamma(x)=1-\frac{\beta h(x)}{\gamma(1+\beta h(x))}. \tag{5.1.24}$$

Hence, we get by eq. (5.1.23) and eq. (5.1.24)

$$\varepsilon\frac{\partial v_\gamma}{\partial\nu_\varepsilon}+\beta v_\gamma=-(1-\gamma)\frac{\beta^2 h}{\gamma(1+\beta h)}+R_2(\sigma,\varepsilon,\gamma)\quad\text{on }\partial\Omega_\varepsilon,$$

where, as before, up to choosing a smaller ε_0,

$$|R_2(\sigma,\varepsilon,\gamma)|\le C_\gamma\varepsilon.$$

Again, for small enough ε, on $\partial\Omega_\varepsilon$ we get

$$\varepsilon\frac{\partial v_{1-\alpha}}{\partial\nu_\varepsilon}+\beta v_{1-\alpha}<0,\qquad\qquad \varepsilon\frac{\partial v_{1+\alpha}}{\partial\nu_\varepsilon}+\beta v_{1+\alpha}>0. \tag{5.1.25}$$

Finally, joining eq. (5.1.22) and eq. (5.1.25), by standard comparison results for elliptic operators the proposition is proved. □

We can now prove theorem 5.1.1.

Proof of theorem 5.1.1. For simplicity we denote by $H(\sigma) = H_\Omega(\sigma)$. We start by proving the Γ-liminf inequality: without loss of generality, we can prove the inequality for the sequence of minimizers u_ε. Here we recall the definitions of $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$, omitting the dependence on h.

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1}, \tag{5.1.26}$$

$$\mathcal{F}_0 = \beta \int_{\partial\Omega} \frac{1}{1+\beta h} \, d\mathcal{H}^{n-1}. \tag{5.1.27}$$

By eq. (5.1.5) and eq. (5.1.6) we have

$$\int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2 \, dx \geq \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu)|^2 \Big(1 + tH(\sigma) - \varepsilon^2 Q\Big) \, dt \, d\mathcal{H}^{n-1} \tag{5.1.28}$$

and

$$\frac{\beta}{\varepsilon} \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 \, \mathcal{H}^{n-1} \geq \frac{\beta}{\varepsilon} \int_{\partial\Omega} u_\varepsilon^2(\sigma + \varepsilon h(\sigma)\nu(\sigma)) \Big(1 + \varepsilon h(\sigma) H(\sigma) - \varepsilon^2 Q\Big) \, d\mathcal{H}^{n-1} \tag{5.1.29}$$

for some constant $Q > 0$. For ε sufficiently small, for every $\sigma \in \partial\Omega$, and $0 < t < \varepsilon h(\sigma)$, we have that $1 + tH(\sigma) > 0$, so that, using Holder's inequality and integrating by parts,

$$\begin{aligned}
\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu)|^2 (1 + tH(\sigma)) \, dt &\geq \frac{1}{\varepsilon h} \left(\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu)| \sqrt{1 + tH} \, dt \right)^2 \\
&\geq \frac{1}{\varepsilon h} \left(\int_0^{\varepsilon h(\sigma)} \frac{d}{dt}(u_\varepsilon(\sigma + t\nu)) \sqrt{1 + tH} \, dt \right)^2 \\
&\geq \frac{1}{\varepsilon h} \left(u_\varepsilon(\sigma + \varepsilon h\nu) \sqrt{1 + \varepsilon h H} - \left(1 + \int_0^{\varepsilon h(\sigma)} \frac{H u_\varepsilon(\sigma + t\nu)}{2\sqrt{1 + tH}} \, dt \right) \right)^2.
\end{aligned}$$

Up to choosing a smaller ε, we can apply Young's inequality, having that for every $\lambda > 0$

$$\begin{aligned}
\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu)|^2 (1 + tH(\sigma)) \, dt \geq & \frac{(1-\lambda)(1 + \varepsilon h H) u_\varepsilon(\sigma + \varepsilon h\nu)^2}{\varepsilon h} \\
&+ \frac{1}{\varepsilon h} \left(1 - \frac{1}{\lambda}\right) \left(1 + \int_0^{\varepsilon h(\sigma)} \frac{H u_\varepsilon(\sigma + t\nu)}{2\sqrt{1 + tH}} \, dt \right)^2.
\end{aligned} \tag{5.1.30}$$

We then have, joining eq. (5.1.26),eq. (5.1.28), eq. (5.1.30), eq. (5.1.29), and eq. (5.1.27),

$$\begin{aligned}
\delta\mathcal{F}_\varepsilon(u_\varepsilon) = \frac{\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{F}_0}{\varepsilon} \geq & \int_{\partial\Omega} \frac{1}{\varepsilon h(\sigma)} ((1-\lambda)(1 + \varepsilon h H) + \beta h (1 + \varepsilon h H)) u_\varepsilon^2(\sigma + \varepsilon h\nu) \, d\mathcal{H}^{n-1} \\
&+ \int_{\partial\Omega} \frac{1}{\varepsilon h} \left(\left(1 - \frac{1}{\lambda}\right) \left(1 + \int_0^{\varepsilon h} \frac{H u_\varepsilon(\sigma + t\nu)}{2\sqrt{1 + tH}} \, dt \right)^2 - \frac{\beta h}{1 + \beta h} \right) d\mathcal{H}^{n-1} \\
&- Q\varepsilon R(\varepsilon, u_\varepsilon)
\end{aligned} \tag{5.1.31}$$

where, if ε is small enough,

$$R(\varepsilon, u_\varepsilon) = \varepsilon \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma + t\nu)|^2 \, d\mathcal{H}^{n-1} + \beta \int_{\partial\Omega} u_\varepsilon(\sigma + \varepsilon h(\sigma)\nu(\sigma))^2 \, d\mathcal{H}^{n-1}$$
$$\leq 2\mathcal{F}_\varepsilon(u_\varepsilon).$$

Letting $\lambda = \lambda(\sigma) = 1 + \beta h(\sigma)$ in eq. (5.1.31), and using the inequality $(1+x)^2 \geq 1 + 2x$,

$$\delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_{\partial\Omega} \frac{\beta h H}{\varepsilon(1+\beta h)} \int_0^\varepsilon \frac{u_\varepsilon(\sigma + th\nu)}{\sqrt{1 + thH}} \, dt \, d\mathcal{H}^{n-1} + O(\varepsilon).$$

Moreover, for every $t \in (0, \varepsilon)$ we have that $(1 + thH)^{-1/2} = 1 + O(\varepsilon)$, so that

$$\delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH}{(1+\beta h)} \fint_0^\varepsilon u_\varepsilon(\sigma + th\nu) \, dt \, d\mathcal{H}^{n-1} + O(\varepsilon). \tag{5.1.32}$$

Finally, let $\alpha \in (0,1)$ and let

$$\gamma = \gamma(\sigma) = \begin{cases} 1 - \alpha & \text{if } H(\sigma) \geq 0 \\ 1 + \alpha & \text{if } H(\sigma) < 0. \end{cases}$$

Let us recall that

$$w_{\varepsilon,\alpha}(\sigma + th(\sigma)\nu(\sigma)) = 1 - t^\gamma \frac{\beta h(\sigma)}{\varepsilon^\gamma \gamma(1 + \beta h(\sigma))}.$$

By proposition 5.1.5 we have that for every $0 < \varepsilon < \varepsilon_\alpha$

$$\delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH}{(1+\beta h)} \fint_0^\varepsilon w_{\varepsilon,\alpha}(\sigma + th\nu) \, dt \, d\mathcal{H}^{n-1} + O(\varepsilon)$$
$$= \int_{\partial\Omega} \left(1 - \frac{\beta h}{(1+\beta h)\gamma(\gamma+1)}\right) \frac{\beta h H}{1 + \beta h} \, d\mathcal{H}^{n-1} + O(\varepsilon),$$

so that

$$\liminf_{\varepsilon \to 0^+} \delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_{\partial\Omega} \left(1 - \frac{\beta h}{(1+\beta h)\gamma(\gamma+1)}\right) \frac{\beta h H}{1 + \beta h} \, d\mathcal{H}^{n-1}.$$

Letting α go to 0, we have that γ tends to 1, and

$$\liminf_{\varepsilon \to 0^+} \delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \beta \int_{\partial\Omega} \frac{hH(2 + \beta h)}{2(1+\beta h)^2} \, d\mathcal{H}^{n-1},$$

and the Γ-Liminf is proved.

We now prove the Γ-limsup inequality:
Let

$$\varphi_\varepsilon(x) = \begin{cases} 1 & \text{if } x \in \Omega, \\ 1 - \dfrac{\beta d(x)}{\varepsilon(1 + \beta h(x))} & \text{if } x \in \Sigma_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases}$$

where we recall that if $x = \sigma + t\nu(\sigma)$, then $h(x) = h(\sigma)$. We have that $\varphi_\varepsilon \in H^1(\Omega)$ and φ_ε converges in $L^2(\mathbb{R}^n)$, to the characteristic function of Ω. Computing the gradient of φ_ε, for any $x \in \Sigma_\varepsilon$,

$$|\nabla \varphi_\varepsilon|^2 \leq \frac{\beta^2}{\varepsilon^2(1+\beta h)^2} + C,$$

where we used again the boundedness of h, and the fact that $d \leq \varepsilon h$. Hence, substituting $\varepsilon h \tau = t$ in eq. (5.1.5), and noticing that $d(\sigma + \varepsilon\tau h(\sigma)\nu(\sigma)) = \varepsilon h \tau$, we get

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} |\nabla \varphi_\varepsilon|^2 \, dx \leq & \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{1}{(1+\beta h)^2} \, dx + \varepsilon C |\Sigma_\varepsilon| \\ \leq & \int_{\partial\Omega} \int_0^1 \frac{\beta^2 h}{(1+\beta h)^2} (1 + \varepsilon\tau h H + \varepsilon^2 Q) \, d\tau \, d\mathcal{H}^{n-1} + \varepsilon C |\Sigma_\varepsilon| \\ = & \int_{\partial\Omega} \frac{\beta^2 h (2 + \varepsilon h H)}{2(1+\beta h)^2} \, d\mathcal{H}^{n-1} + O(\varepsilon^2). \end{aligned} \tag{5.1.33}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\varphi_\varepsilon(\sigma + \varepsilon h(\sigma)\nu(\sigma)) = \frac{1}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega} \frac{\beta}{(1+\beta h)^2} (1 + \varepsilon h H + \varepsilon^2 Q) \, d\mathcal{H}^{n-1}. \tag{5.1.34}$$

Finally, joining, eq. (5.1.33), and eq. (5.1.34) we have

$$\begin{aligned} \delta\mathcal{F}_\varepsilon(\varphi_\varepsilon, h) = \frac{\mathcal{F}_\varepsilon(u_\varepsilon, h) - \mathcal{F}_0(h)}{\varepsilon} \leq & \int_{\partial\Omega} \left(\frac{\beta^2 h^2 H}{2(1+\beta h)^2} + \frac{\beta h H}{(1+\beta h)^2} \right) d\mathcal{H}^{n-1} + O(\varepsilon) \\ = & \, \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2} \, d\mathcal{H}^{n-1} + O(\varepsilon) \end{aligned}$$

so that

$$\limsup_{\varepsilon \to 0^+} \delta\mathcal{F}_\varepsilon(\varphi_\varepsilon) \leq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2} \, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

Remark 5.1.6. Despite in theorem 5.1.1 we require $h > 0$, it is still meaningful to study the approximating functional $\mathcal{F}_0(h) + \varepsilon\mathcal{F}^{(1)}(h)$ relaxing the constraint to $h \geq 0$. Indeed, even though the positivity of the function h is crucial in the proof of the theorem 5.1.1, it is still possible to prove the following: let Ω be a bounded, open set with C^3 boundary, fix a non-negative C^2 function $h \colon \partial\Omega \to [0, +\infty)$, and fix an exponent $\theta \in (1, 2)$; then, carefully retracing all the steps of the proof of theorem 5.1.1, we can still prove that the functional $\delta\mathcal{F}_\varepsilon(\cdot, h + \varepsilon^\theta)$ Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to $\mathcal{F}^{(1)}(\cdot, h)$, the same functional defined in eq. (5.1.4).

5.1.2 Properties of the first order development

Let Ω be a bounded, open set with $C^{1,1}$ boundary, and let $H = H_\Omega$. Consider the functional

$$\mathcal{G}_\varepsilon(\Omega, h) = \beta \int_{\partial\Omega} \left(\frac{1}{1+\beta h} + \varepsilon H \frac{h(2+\beta h)}{2(1+\beta h)^2} \right) d\mathcal{H}^{n-1}.$$

In the following, we drop the dependence on the set Ω and we write $\mathcal{G}_\varepsilon(h)$ in place of $\mathcal{G}_\varepsilon(\Omega, h)$.

The optimal thickness

For every $m > 0$, we will consider the problem

$$\inf \{ \mathcal{G}_\varepsilon(h) \mid h \in \mathcal{H}_m \}, \tag{5.1.35}$$

where the set $\mathcal{H}_m$ is defined in eq. (5.1.2).

Remark 5.1.7. Assume that a non-zero continuous solution $\mu \in L^1(\partial\Omega)$ to problem eq. (5.1.35) exists, so that the set $U = \{ \mu > 0 \}$ is open. Then for every $\psi \in C_c^\infty(U)$ with zero mean, and for every $\eta \in \mathbb{R}$ sufficiently small, we can consider the variation $\mu + \eta\psi$ which leads to the Euler-Lagrange equation

$$\int_{\partial\Omega} \left(-\frac{\beta}{(1+\beta\mu)^2} + \frac{\varepsilon H}{1+\beta\mu} - \varepsilon H \frac{\beta\mu(2+\beta\mu)}{(1+\beta\mu)^3} \right) \psi \, d\mathcal{H}^{n-1} = 0.$$

The previous equation yields

$$\frac{c}{\beta}(1+\beta\mu)^3 - (1+\beta\mu) + \frac{\varepsilon H}{\beta} = 0,$$

for some constant $c \in \mathbb{R}$.

Let $\varepsilon > 0$, in the following we will assume that

$$\sup_{\partial\Omega} \frac{\varepsilon H}{\beta} \leq \frac{2}{3}. \tag{5.1.36}$$

Let

$$H_0 = \inf_{\partial\Omega} H$$

and

$$k_0 = 1 - \frac{\varepsilon H_0}{\beta}.$$

For every $k \in (0, k_0)$, let

$$\Gamma_k = \left\{ \sigma \in \partial\Omega \;\middle|\; \frac{\varepsilon H(\sigma)}{\beta} < 1 - k \right\}$$

and consider

$$P_k(y, \sigma) = ky^3 - y + \frac{\varepsilon H(\sigma)}{\beta}.$$

Notice that, by the choice of k_0, the set Γ_k is always non-empty.

Proposition 5.1.8. *Let eq. (5.1.36) hold true. Then, for every $k \in (0, k_0)$, and $\sigma \in \Gamma_k$, in the interval $(1, +\infty)$ there exists a unique $y_k(\sigma)$ such that*

$$P_k(y_k(\sigma), \sigma) = 0, \tag{5.1.37}$$

and there exists $z_k > 1$ independent of σ such that

$$\max\left\{\frac{1}{\sqrt{3k}}, 1\right\} \leq y_k(\sigma) \leq z_k, \tag{5.1.38}$$

and

$$\lim_{k \to k_0^-} z_k = 1. \tag{5.1.39}$$

Moreover, for every $k_1 < k_2$ and $\sigma \in \Gamma_{k_2}$ we have that

$$y_{k_2}(\sigma) < y_{k_1}(\sigma). \tag{5.1.40}$$

Proof. For any fixed $\sigma \in \Gamma_k$ we have that $P_k(1, \sigma) < 0$, and in addition, for $k \geq 1/3$, the polynomial $P_k(y, \sigma)$ is strictly increasing in $y \geq 1$, while for $k < 1/3$ we have that

$$\frac{\partial}{\partial y} P_k(y, \sigma) < 0 \qquad \text{if } y \in \left[1, \frac{1}{\sqrt{3k}}\right),$$
$$\frac{\partial}{\partial y} P_k(y, \sigma) > 0 \qquad \text{if } y \in \left(\frac{1}{\sqrt{3k}}, +\infty\right).$$

Therefore, in the interval $(1, +\infty)$ there exists a unique zero $y_k(\sigma)$ of the polynomial $P_k(\cdot, \sigma)$, and

$$y_k(\sigma) \geq \frac{1}{\sqrt{3k}}.$$

Notice in addition that for every $y > 1$, we have that $y < y_k(\sigma)$ if and only if $P_k(y, \sigma) < 0$. Hence, if we choose z_k to be the unique real number in $(1, +\infty)$ such that

$$k z_k^3 - z_k + \frac{\varepsilon H_0}{\beta} = 0, \tag{5.1.41}$$

then

$$P_k(z_k, \sigma) = \frac{\varepsilon H(\sigma)}{\beta} - \frac{\varepsilon H_0}{\beta} \geq 0,$$

and we have that eq. (5.1.38) holds.

We now prove eq. (5.1.39). We first observe that z_k is decreasing in k: let $k_1 < k_2$, so that

$$k_1 z_{k_2}^3 - z_{k_2} + \frac{\varepsilon H_0}{\beta} < k_2 z_{k_2}^3 - z_{k_2} + \frac{\varepsilon H_0}{\beta} = 0 = k_1 z_{k_1}^3 - z_{k_1} + \frac{\varepsilon H_0}{\beta},$$

which ensures

$$z_{k_2} < z_{k_1},$$

indeed the polynomial $k_1 y^3 - y + \varepsilon H_0/\beta$ is negative on $(1, z_{k_1})$ and it is non-negative on $[z_{k_1}, +\infty)$. We now have that there exists

$$z = \lim_{k \to k_0^-} z_k,$$

and, passing to the limit in eq. (5.1.41) and recalling that by definition $\beta k_0 = \varepsilon H_0$, we get that z solves the equation

$$k_0(z^3 - 1) - z + 1 = 0.$$

From eq. (5.1.36), we have that $k_0 > 1/3$, so that $z = 1$ is the unique solution in $[1, +\infty)$ to the previous equation, proving eq. (5.1.39).

Finally, in order to prove eq. (5.1.40), let $k_1 < k_2$ and $\sigma \in \Gamma_{k_1} \cap \Gamma_{k_2} = \Gamma_{k_2}$, then eq. (5.1.37) ensures that

$$P_{k_1}(y_{k_2}, \sigma) < P_{k_2}(y_{k_2}, \sigma) = 0,$$

from which

$$y_{k_2}(\sigma) < y_{k_1}(\sigma).$$

□

Let $k \in (0, k_0)$, and let y_k be as in proposition 5.1.8, we define

$$\mu_k(\sigma) = \begin{cases} \dfrac{1}{\beta}(y_k(\sigma) - 1) & \text{if } \sigma \in \Gamma_k. \\ 0 & \text{if } \sigma \in \partial\Omega \setminus \Gamma_k, \end{cases}$$

Notice that, by eq. (5.1.40), μ_k is decreasing in k.

We have the following

Proposition 5.1.9. *Let eq. (5.1.36) hold true. Then, for every $m > 0$,there exists a unique $k = k_m \in (0, k_0)$ such that*

$$\int_{\partial\Omega} \mu_k \, d\mathcal{H}^{n-1} = m.$$

Proof. We first prove that the function

$$m(k) = \int_{\partial\Omega} \mu_k \, d\mathcal{H}^{n-1}$$

is continuous. Fix $k \in (0, k_0)$ and let $\delta > 0$, then

$$\beta(m(k) - m(k+\delta)) = \int_{\Gamma_{k+\delta}} (y_k - y_{k+\delta}) \, d\mathcal{H}^{n-1} + \int_{\left\{ 1-k-\delta \leq \frac{\varepsilon H}{\beta} < 1-k \right\}} (y_k - 1) \, d\mathcal{H}^{n-1}. \tag{5.1.42}$$

By definition, we have that for every $\sigma \in \partial\Omega$

$$\lim_{\delta \to 0^+} \chi_{\Gamma_{k+\delta}}(\sigma) = \chi_{\Gamma_k}(\sigma).$$

Let $\sigma \in \Gamma_k$, then the function $y_{k+\delta}(\sigma)$ is defined for small enough $\delta < \delta_\sigma$, and by the implicit function theorem and the regularity of $P_k(y, \sigma)$, we get

$$\lim_{\delta \to 0^+} y_{k+\delta}(\sigma) = y_k(\sigma).$$

Therefore, by eq. (5.1.38), the monotonicity of z_k, and the dominated convergence theorem we have

$$\lim_{\delta \to 0^+} \int_{\Gamma_{k+\delta}} (y_k - y_{k+\delta}) \, d\mathcal{H}^{n-1} = 0. \tag{5.1.43}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\lim_{\delta\to 0^+} \chi_{\left\{\, 1-k-\delta \le \frac{\varepsilon H}{\beta} < 1-k \,\right\}}(\sigma) = 0,$$

which entails

$$\lim_{\delta\to 0^+} \int_{\left\{\, 1-k-\delta \le \frac{\varepsilon H}{\beta} < 1-k \,\right\}} (y_k - 1)\, d\mathcal{H}^{n-1} = 0. \tag{5.1.44}$$

Joining eq. (5.1.42), eq. (5.1.43), and eq. (5.1.44), we get

$$\lim_{\delta\to 0^+} m(k) - m(k+\delta) = 0.$$

We now fix $k \in (0, k_0)$, $\delta > 0$, and we compute

$$\beta(m(k-\delta) - m(k)) = \int_{\Gamma_k} (y_{k-\delta} - y_k)\, d\mathcal{H}^{n-1} + \int_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}} (y_{k-\delta} - 1)\, d\mathcal{H}^{n-1}. \tag{5.1.45}$$

As in the previous case, by the implicit function theorem, for every $\sigma \in \Gamma_k$,

$$\lim_{\delta\to 0^+} y_{k-\delta}(\sigma) = y_k(\sigma).$$

By the dominated convergence theorem,

$$\lim_{\delta\to 0^+} \int_{\Gamma_k} (y_{k-\delta} - y_k)\, d\mathcal{H}^{n-1} = 0. \tag{5.1.46}$$

On the other hand, we have that for every $\sigma \in \partial\Omega$

$$\lim_{\delta\to 0^+} \chi_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}}(\sigma) = \chi_{\left\{\, \frac{\varepsilon H}{\beta} = 1-k \,\right\}}(\sigma),$$

and, for every σ such that $\varepsilon H(\sigma) = \beta(1-k)$, we may use the monotonicity of $y_{k-\delta}$ and then we pass to the limit in eq. (5.1.37), having that

$$\lim_{\delta\to 0^+} y_{k-\delta}(\sigma) = 1,$$

which entails

$$\lim_{\delta\to 0^+} \int_{\left\{\, 1-k \le \frac{\varepsilon H}{\beta} < 1-k+\delta \,\right\}} (y_{k-\delta} - 1)\, d\mathcal{H}^{n-1} = 0. \tag{5.1.47}$$

Joining eq. (5.1.45), eq. (5.1.46), and eq. (5.1.47), we get

$$\lim_{\delta\to 0^+} m(k-\delta) - m(k) = 0,$$

thus concluding the proof of the continuity of M.

Finally, by monotonicity, and by eq. (5.1.38), and eq. (5.1.36), we have that

$$\lim_{k\to 0^+} m(k) = +\infty, \qquad \lim_{k\to k_0^-} m(k) = 0.$$

and the proposition is proved. $\square$

Theorem 5.1.10. *Let eq. (5.1.36) hold true. Then, for every $m > 0$, the function μ_{k_m} is the unique minimizer to problem eq. (5.1.35).*

Proof. Let $h\colon \partial\Omega \to R$ with $h \geq 0$ and

$$\int_{\partial\Omega} h \, d\mathcal{H}^{n-1} \leq m.$$

For every $t \in [0,1]$ consider

$$h_t = \mu_k + t(h - \mu_k)$$

and

$$g(t) = \mathcal{G}_\varepsilon(h_t).$$

We claim that $g(t)$ is increasing in t. By explicit computation, we have that

$$g'(t) = \beta \int_{\partial\Omega} \frac{\varepsilon H - \beta(1+\beta h_t)}{(1+\beta h_t)^3}(h - \mu_k) \, d\mathcal{H}^{n-1}.$$

From eq. (5.1.36) we have that, for every $\sigma \in \partial\Omega$ the function

$$x \mapsto \frac{\varepsilon H - \beta(1+\beta x)}{(1+\beta x)^3}$$

is increasing on $[0, +\infty)$, so that

$$g'(t) \geq \beta \int_{\partial\Omega} \frac{\varepsilon H - \beta(1+\beta \mu_k)}{(1+\beta \mu_k)^3}(h - \mu_k) \, d\mathcal{H}^{n-1}.$$

Notice that for every $\sigma \in \partial\Omega$

$$\frac{\varepsilon H - \beta(1+\beta \mu_k)}{(1+\beta \mu_k)^3}(h - \mu_k) \geq -\beta k(h - \mu_k).$$

Indeed, if $\sigma \in \Gamma_k$, then by eq. (5.1.37),

$$\varepsilon H(\sigma) - \beta(1 + \beta\mu_k(\sigma)) = -\beta k(1 + \beta\mu_k(\sigma))^3$$

and equality holds. On the other hand, if $\sigma \in \partial\Omega \setminus \Gamma_k$, then $\mu_k(\sigma) = 0$ and $\varepsilon H(\sigma) \geq \beta(1-k)$, so that

$$\begin{aligned}
\frac{\varepsilon H(\sigma) - \beta(1+\beta \mu_k(\sigma))}{(1+\beta \mu_k(\sigma))^3}(h(\sigma) - \mu_k(\sigma)) &= (\varepsilon H(\sigma) - \beta)h(\sigma) \\
&\geq -\beta k h(\sigma) \\
&= -\beta k(h(\sigma) - \mu_k(\sigma)).
\end{aligned}$$

Then we have

$$g'(t) \geq \beta^2 k \int_{\partial\Omega} (\mu_k - h) \, d\mathcal{H}^{n-1} \geq 0,$$

and the claim is proven. In particular, we have that

$$\mathcal{G}_\varepsilon(\mu_k) = g(0) \leq g(1) = \mathcal{G}_\varepsilon(h)$$

that is, μ_k is a minimizer for problem eq. (5.1.35). Finally, by eq. (5.1.36), we have that, for every $\sigma \in \partial\Omega$, the function

$$x \in [0, +\infty) \mapsto \frac{1}{1+\beta x} + \varepsilon H(\sigma)\frac{x(2+\beta x)}{2(1+\beta x)^2}$$

is strictly convex, thus problem eq. (5.1.35) admits a unique minimizer. □

Remark 5.1.11. Notice that the optimal configuration μ_{k_m} concentrates where the mean curvature is smaller: for simplicity, let us write $k = k_m$, and let us take $\sigma_1, \sigma_2 \in \Gamma_k$ such that

$$H(\sigma_1) < H(\sigma_2).$$

Noticing that

$$P_k(y_k(\sigma_1), \sigma_1) = 0 = P_k(y_k(\sigma_2), \sigma_2),$$

we get

$$ky_k(\sigma_1)^3 - y_k(\sigma_1) > ky_k(\sigma_2)^3 - y_k(\sigma_2).$$

By eq. (5.1.38), we can use the monotonicity of the function $y \mapsto ky^3 - y$ on $[1/\sqrt{3k}, +\infty)$, getting

$$y_k(\sigma_1) > y_k(\sigma_2),$$

so that $\mu_k(\sigma_1) > \mu_k(\sigma_2)$.

Remark 5.1.12. Notice that, if Ω is a ball, we have that the mean curvature is constant, so that $\Gamma_k = \partial\Omega$, and μ_k is a constant function. Hence, if eq. (5.1.36) holds, problem eq. (5.1.35) admits as unique solution the constant function

$$\mu = \frac{m}{P(\Omega)}.$$

Remark 5.1.13. Notice that, if for every $\sigma \in \partial\Omega$

$$\frac{\varepsilon H(\sigma)}{\beta} \geq 2, \tag{5.1.48}$$

then the optimal configuration is given by $\mu \equiv 0$. Indeed if eq. (5.1.48) holds, for every $\sigma \in \partial\Omega$ the function

$$x \in [0, +\infty) \mapsto \frac{1}{1 + \beta x} + \frac{\varepsilon H(\sigma)}{\beta} \frac{\beta x(2 + \beta x)}{2(1 + \beta x)^2}$$

reaches its minimum for $x = 0$.

Minimization with perimeter constraint

Fix $\varepsilon > 0$. For every $P > 0$ and $m > 0$ we define,

$$\mathcal{K}_P = \left\{ \Omega \subset \mathbb{R}^n \;\middle|\; \begin{array}{l} \Omega \text{ open and bounded with } C^{1,1} \text{ boundary} \\ H_\Omega \geq 0 \\ P(\Omega) = P \end{array} \right\},$$

where H_Ω is the mean curvature of Ω. Let $\Omega \in \mathcal{K}_P$, and let $h \in \mathcal{H}_m(\partial\Omega)$, where $\mathcal{H}_m$ is the set defined in eq. (5.1.2), then we study the functional

$$\mathcal{G}_\varepsilon(\Omega, h) = \beta \int_{\partial\Omega} \left(\frac{1}{1 + \beta h} + \varepsilon H_\Omega \frac{h(2 + \beta h)}{2(1 + \beta h)^2} \right) d\mathcal{H}^{n-1}.$$

We will now consider the problem

$$\inf \left\{ \mathcal{G}_\varepsilon(\Omega, h) \mid (\Omega, h) \in \mathcal{K}_P \times \mathcal{H}_m(\partial\Omega) \right\}. \tag{5.1.49}$$

Definition 5.1.14 (Cookie Shape)**.** For any $r, R > 0$ we define the *cookie shape*

$$C_{r,R} = \{ (x', x_n) \mid -f_{r,R}(x') \le x_n \le f_{r,R}(x') \},$$

where

$$f_{r,R}(x') = \begin{cases} r & |x'| \le R, \\ \sqrt{r^2 - (|x'| - R)^2} & R < |x'| \le R + r. \end{cases}$$

Remark 5.1.15. For every $r, R > 0$, we have that $C_{r,R}$ is a convex set with $C^{1,1}$ boundary and

$$H = H_{C_{r,R}} \ge \frac{1}{r}\chi_{\{H>0\}}.$$

Moreover,

$$P(C_{r,R}) = 2\omega_{n-1}\left(R^{n-1} + (n-1)r\int_0^1 \frac{(r\rho + R)^{n-2}}{\sqrt{1-\rho^2}}\, d\rho \right). \tag{5.1.50}$$

We observe that the function $P(C_{r,R})$ is increasing in r and R.

Theorem 5.1.16. *For every $P, m > 0$ we have*

$$\inf \{ \mathcal{G}_\varepsilon(\Omega, h) \mid (\Omega, h) \in \mathcal{K}_P \times \mathcal{H}_m(\partial\Omega) \} = \frac{\beta P^2}{P + \beta m}$$

and the infimum is asymptotically achieved by a sequence of thin cookie shapes.

Proof. For every $(\Omega, h) \in \mathcal{K}_P \times \mathcal{H}_m(\partial\Omega)$,

$$\mathcal{G}_\varepsilon(\Omega, h) \ge \beta \int_{\partial\Omega} \frac{1}{1 + \beta h}\, d\mathcal{H}^{n-1} \ge \frac{\beta P^2}{P + \beta m},$$

where in the second inequality we used the fact that $\mathcal{F}_0$ is minimized by the constant function $h_0 = m/P$ (see proposition 5.1.4). Let $r_k > 0$ be a decreasing sequence with

$$\lim_k r_k = 0;$$

let R_k be such that, for every k,

$$P(C_{r_k,R_k}) = P.$$

Then R_k is increasing in k and

$$\lim_k R_k = \left(\frac{P}{2\omega_{n-1}} \right)^{\frac{1}{n-1}}.$$

Consider

$$h_k(\sigma) = \begin{cases} \dfrac{m}{2\omega_{n-1}R_k^{n-1}} & \text{if } H(\sigma) = 0, \\ 0 & \text{if } H(\sigma) > 0. \end{cases}$$

We have that

$$\mathcal{G}_\varepsilon(C_{r_k,R_k}, h_k) = \frac{2\beta\omega_{n-1}R_k^{n-1}}{1 + \frac{\beta m}{2\omega_{n-1}R_k^{n-1}}} + \beta\Big(P(C_{r_k,R_k}) - 2\omega_{n-1}R_k^{n-1} \Big).$$

Passing to the limit for k to infinity, we have

$$\lim_k \mathcal{G}_\varepsilon(C_{r_k,R_k}, h_k) = \frac{\beta P^2}{P + \beta m} = \min_{h \in \mathcal{H}_m} \mathcal{F}_0(h).$$

□

Maximization with geometric constraints

Fix $\varepsilon, m > 0$. For every bounded, open set $\Omega \subseteq \mathbb{R}^n$ with $C^{1,1}$ boundary, we let

$$\mathcal{G}_\varepsilon(\Omega) = \inf \{ \mathcal{G}_\varepsilon(\Omega, h) \mid h \in \mathcal{H}_m \},$$

where $\mathcal{H}_m$ is the set defined in eq. (5.1.2). This section will study the maximization of $\mathcal{G}_\varepsilon(\Omega)$ with fixed quermassintegral. We refer to [157, 59] for the following

Definition 5.1.17 (Quermassintegrals)**.** Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, bounded, convex set. We define the *quermassintegrals* as the unique coefficients $W_j(\Omega)$ such that

$$|\Omega + tB| = \sum_{j=0}^{n} \binom{n}{j} W_j(\Omega) t^j,$$

where B is the unit ball in $\mathbb{R}^n$, and

$$\Omega + tB = \{ x + ty \mid x \in \Omega,\ y \in B \}.$$

In particular $W_0(\Omega)$ is the measure of Ω and $W_n(\Omega) = \omega_n$, the measure of the unit ball.

Theorem 5.1.18 (Alexandrov-Fenchel Inequality)**.** *Let $0 \leq i < j \leq n-1$, and let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, bounded, convex set, then*

$$\left(\frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}} \leq \left(\frac{W_j(\Omega)}{\omega_n} \right)^{\frac{1}{n-j}}.$$

Moreover, the inequality holds as equality if and only if Ω is a ball.

Remark 5.1.19. Let Ω be a bounded, open, convex set with C^2 boundary and nonzero Gaussian curvature, then the quermassintegral are related to the principal curvatures of the boundary of Ω. Indeed we have that for every $j = 1, \dots, n$

$$W_j = \frac{1}{n} \int_{\partial\Omega} H_{j-1}(\sigma) \, d\mathcal{H}^{n-1}$$

Here H_j denotes the j-th normalized elementary symmetric function of the principal curvatures of $\partial\Omega$, that is $H_0 = 1$ and, for every $j = 1, \dots n-1$,

$$H_j(\sigma) = \binom{n-1}{j}^{-1} \sum_{1 \leq i_1 \leq \dots \leq i_j \leq n-1} \kappa_{i_1}(\sigma) \cdots \kappa_{i_j}(\sigma),$$

where $\kappa_1(\sigma), \dots, \kappa_{n-1}(\sigma)$ are the principal curvatures at a point $\sigma \in \partial\Omega$. In particular, we have that

$$W_1(\Omega) = \frac{1}{n} P(\Omega)$$

and

$$W_2(\Omega) = \frac{1}{n(n-1)} \int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1}.$$

In the planar case, we have the following

Proposition 5.1.20. *Let $n = 2$, and let Ω be a bounded, open, simply connected set Ω with C^2 boundary such that either*

$$P(\Omega) \geq 3\pi \frac{\varepsilon}{\beta}$$

or

$$P(\Omega) \leq \pi \frac{\varepsilon}{\beta}.$$

Then

$$\mathcal{G}_\varepsilon(\Omega) \leq \mathcal{G}_\varepsilon(\Omega^*), \tag{5.1.51}$$

where Ω^* is the ball having the same perimeter as Ω.

Proof. If

$$P(\Omega) \leq \pi \frac{\varepsilon}{\beta},$$

then

$$\frac{\varepsilon H_{\Omega^*}}{\beta} \geq 2,$$

and, by remark 5.1.13, we get

$$\mathcal{G}_\varepsilon(\Omega^*) = \mathcal{G}_\varepsilon(\Omega^*, 0) = \beta P(\Omega^*).$$

On the other hand,

$$\mathcal{G}_\varepsilon(\Omega) \leq \mathcal{G}_\varepsilon(\Omega, 0) = \beta P(\Omega) = \beta P(\Omega^*),$$

which gives eq. (5.1.51). If

$$P(\Omega) \geq 3\pi \frac{\varepsilon}{\beta},$$

then

$$\frac{\varepsilon H_{\Omega^*}}{\beta} \leq \frac{2}{3},$$

and, by remark 5.1.12, we get

$$\mathcal{G}_\varepsilon(\Omega^*) = \mathcal{G}_\varepsilon(\Omega^*, m/P(\Omega^*)).$$

On the other hand,

$$\begin{aligned} \mathcal{G}_\varepsilon(\Omega) &\leq \mathcal{G}_\varepsilon(\Omega, m/P(\Omega^*)) \\ &= \beta\left(\frac{P(\Omega^*)P(\Omega)}{P(\Omega^*) + \beta m} + \frac{\varepsilon m(2P(\Omega^*) + \beta m)}{2(P(\Omega^*) + \beta m)^2} \int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1}\right). \end{aligned}$$

By Gauss-Bonnet theorem we have that

$$\int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1} = 2\pi = \int_{\partial\Omega^*} H_{\Omega^*} \, d\mathcal{H}^{n-1},$$

so that

$$\begin{aligned} \mathcal{G}_\varepsilon(\Omega) &\leq \beta\left(\frac{P(\Omega^*)^2}{P(\Omega^*) + \beta m} + \frac{\varepsilon m(2P(\Omega^*) + \beta m)}{2(P(\Omega^*) + \beta m)^2} \int_{\partial\Omega^*} H_{\Omega^*} \, d\mathcal{H}^{n-1}\right) \\ &= \mathcal{G}_\varepsilon(\Omega^*, m/P(\Omega^*)), \end{aligned}$$

which is eq. (5.1.51). □

In the general case of possibly higher dimensions, we have

Proposition 5.1.21. *Let $n \geq 3$, $2 \leq k \leq n-1$, and let Ω be a bounded, open, convex set with C^2 boundary and nonzero Gaussian curvature such that either*

$$W_k(\Omega) \geq \omega_n \left(\frac{3(n-1)\varepsilon}{2\beta} \right)^{n-k} \tag{5.1.52}$$

or

$$W_k(\Omega) \leq \omega_n \left(\frac{(n-1)\varepsilon}{2\beta} \right)^{n-k}. \tag{5.1.53}$$

Then

$$\mathcal{G}_\varepsilon(\Omega) \leq \mathcal{G}_\varepsilon(\Omega^*)$$

where Ω^* is the ball such that $W_k(\Omega) = W_k(\Omega^*)$.

Proof. Since $W_k(\Omega) = W_k(\Omega^*)$, from the Alexandrov-Fenchel inequalities, we have that for every $0 \leq i \leq k$

$$\left(\frac{W_i(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-i}} = \left(\frac{W_k(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-k}} = \left(\frac{W_k(\Omega)}{\omega_n} \right)^{\frac{1}{n-k}} \geq \left(\frac{W_i(\Omega)}{\omega_n} \right)^{\frac{1}{n-i}},$$

that is

$$W_i(\Omega) \leq W_i(\Omega^*).$$

In particular, we have that

$$P(\Omega) \leq P(\Omega^*)$$

and

$$\int_{\partial\Omega} H_\Omega \, d\mathcal{H}^{n-1} \leq \int_{\partial\Omega^*} H_{\Omega^*} \, d\mathcal{H}^{n-1}.$$

Moreover, since

$$\left(\frac{W_k(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-k}} = \left(\frac{W_1(\Omega^*)}{\omega_n} \right)^{\frac{1}{n-1}} = \left(\frac{P(\Omega^*)}{n\omega_n} \right)^{\frac{1}{n-1}} = \frac{n-1}{H_{\Omega^*}},$$

the conditions eq. (5.1.52) and eq. (5.1.53) read as

$$\frac{\varepsilon H_{\Omega^*}}{\beta} \leq \frac{2}{3},$$

and

$$\frac{\varepsilon H_{\Omega^*}}{\beta} \geq 2,$$

respectively. Therefore, the result can be obtained following the proof of proposition 5.1.20. □

Final remarks

Let $\Omega, \Omega_0 \subseteq \mathbb{R}^n$ be bounded open sets with smooth boundary such that $\overline{\Omega} \subset \overline{\Omega}_0$ and let

$$\mathrm{Cap}^\beta(\Omega, \Omega_0) = \inf \left\{ \int_{\Omega_0} |\nabla u|^2 \, dx + \beta \int_{\partial\Omega_0} u^2 \, d\mathcal{H}^{n-1} \;\middle|\; \begin{array}{l} u \in H^1(\Omega_0), \\ u \geq 1 \text{ in } \overline{\Omega} \end{array} \right\}.$$

The results in proposition 5.1.20 and proposition 5.1.21 are coherent with the ones proved in [87] for the functional

$$\mathrm{Cap}^{\beta,\delta}(\Omega) = \mathrm{Cap}^{\beta}(\Omega, \Omega + \delta B),$$

where $\Omega_0 = \Omega + \delta B$ is the Minkowski sum of Ω and the unit ball $B = B_1(0) \subset \mathbb{R}^n$. Namely, in [87], the following theorems are proved

Theorem 5.1.22. *Let Ω be a connected, bounded, open set in $\mathbb{R}^2$ with piecewise C^1 boundary. Then*

$$\mathrm{Cap}^{\beta,\delta}(\Omega) \leq \mathrm{Cap}^{\beta,\delta}(\Omega^*),$$

where Ω^* is the ball having the same perimeter as Ω.

Theorem 5.1.23. *Let $n \geq 3$ and let Ω be a bounded, open, convex set in $\mathbb{R}^n$. Then*

$$\mathrm{Cap}^{\beta,\delta}(\Omega) \leq \mathrm{Cap}^{\beta,\delta}(\Omega^*),$$

where Ω^* is the ball such that $W_{n-1}(\Omega^*) = W_{n-1}(\Omega)$.

Finally, in [58], it is proved the following

Theorem 5.1.24. *The solution to the problem*

$$\min\left\{ \mathrm{Cap}^{\beta}(\Omega, \Omega_0) \;\middle|\; \begin{array}{l} |\Omega| = \omega_n, \\ |\Omega_0| \leq M \end{array} \right\}$$

consists of two concentric balls.

The previous theorem naturally leads to the following question

Open problem 1. *Prove or disprove that the problem*

$$\min\left\{ \mathcal{G}_\varepsilon(\Omega, h) \;\middle|\; \begin{array}{l} |\Omega| = \omega_n, \\ h \in \mathcal{H}_m \end{array} \right\}$$

admits the couple (B_1, h^*) as a solution, where h^* is a constant function.

5.2 Asymptotic behavior of a diffraction problem with a thin layer

The results of this section are contained in the paper [5].

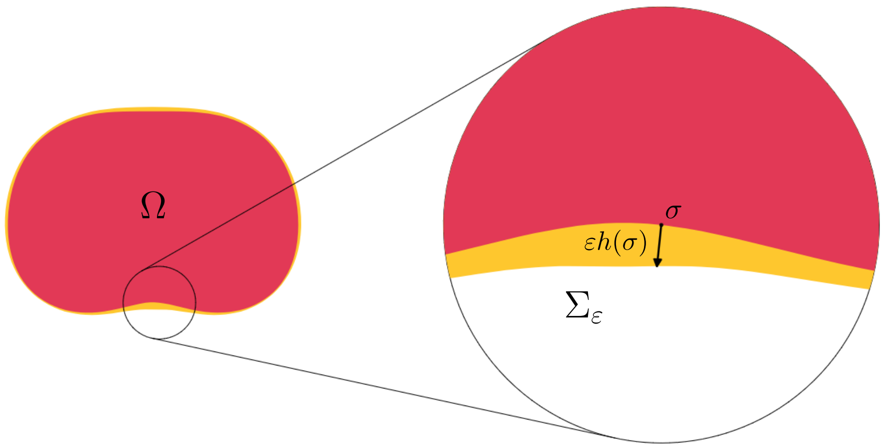
Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with $C^{1,1}$ boundary, let $\nu_0 = \nu_\Omega$ denote the outer unit normal to $\partial\Omega$, and let $h \in C^{1,1}(\partial\Omega)$. For every $\varepsilon > 0$, consider the set

$$\Sigma_\varepsilon = \left\{ \sigma + t\nu_0(\sigma) \;\middle|\; \begin{array}{c} \sigma \in \partial\Omega, \\ 0 < t < \varepsilon h(\sigma) \end{array} \right\}.$$

Let $\Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon$ and let ν_ε be the unit outer normal to $\partial\Omega_\varepsilon$. In this section, we study the asymptotic behavior, as ε goes to zero, of the solutions u_ε to the following boundary value problem:

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon^- = u_\varepsilon^+ & \text{on } \partial\Omega, \\ \dfrac{\partial u_\varepsilon^-}{\partial \nu_0} = \varepsilon \dfrac{\partial u_\varepsilon^+}{\partial \nu_0} & \text{on } \partial\Omega, \\ \Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \varepsilon \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \beta u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases} \tag{5.2.1}$$

Figure 5.2: Body Ω surrounded by a thin layer Σ_ε



where $f \in L^2(\Omega)$ and u_ε^- and u_ε^+ denote the traces of u_ε on $\partial\Omega$ in Ω and in Σ_ε respectively. As shown in [86], in Ω, the functions u_ε converges weakly in $H^1(\Omega)$ to u_0, the weak solution to the boundary value problem

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} + \dfrac{\beta}{1+\beta h} u_0 = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.2.2}$$

i.e. for every $\varphi \in H^1(\Omega)$

$$\int_\Omega \nabla u_0 \cdot \nabla \varphi \, dx + \beta \int_{\partial\Omega} \frac{u_0 \varphi}{1+\beta h} \, d\mathcal{H}^{n-1} = \int_\Omega f \varphi \, dx.$$

Let us notice that the functions u_ε are the minimizers, in $H^1(\Omega_\varepsilon)$, of the functionals

$$\mathcal{F}_\varepsilon(v,h) = \int_\Omega |\nabla v|^2 \, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} v^2 \, d\mathcal{H}^{n-1} - 2 \int_\Omega f v \, dx,$$

where $v \in H_1(\Omega_\varepsilon)$.

Our interest in this work is focused on the behavior of solutions u_ε in the set Σ_ε. This interest is mainly motivated by the following remark: it seems that the solutions to some variational problems in thin sets of the type Σ_ε (see for instance [10, 86, 7]), can be approximated with functions that are linear along the normal rays to $\partial\Omega$. We will show that, indeed, if we stretch the set Σ_ε to the set Σ_1, then the stretched solutions $\tilde{u}_\varepsilon$ converge to a function that, in Σ_1, is linear with respect to the distance from $\partial\Omega$.

In addition, we will show that this convergence property will be sufficient to study the first-order development by Γ-convergence of the functional $\mathcal{F}_\varepsilon$.

5.2.1 The main theorems

In the following it will be useful to have that for every $x \in \Sigma_\varepsilon$ the metric projection $\pi_\Omega(x)$ onto Ω is well defined, and it will be useful to work with an extension of h in the set Σ_ε which is constant along the normal directions to $\partial\Omega$. Hence, we will assume h to be a positive $C^{1,1}(\Sigma_\Omega(d_0))$ function such that $\|h\|_\infty < d_0$ and $\nabla h \cdot \nu_0 = 0$. To inspect the properties of the solution u_ε inside Σ_ε, we "stretch" the solution u_ε via a pullback on the reference set Σ_1. To be more precise, we construct a diffeomorphism

$$\Psi_\varepsilon : \Sigma_1 \to \Sigma_\varepsilon$$

by rescaling the distance from $\partial\Omega$. The approach is analogous to the dilation technique proposed in [30] for the asymptotic expansion and the derivation of the so-called effective boundary conditions. We show the following

Theorem 5.2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$, and let Ψ_ε be the stretching diffemorphism defined in Definition 5.2.3. Let u_ε be the unique weak solution to (5.2.1), and let u_0 be the unique weak solution to (5.2.2). If we let*

$$\tilde{u}_\varepsilon(z) = \begin{cases} u_\varepsilon(z) & \text{if } z \in \Omega, \\ u_\varepsilon(\Psi_\varepsilon(z)) & \text{if } z \in \Sigma_1, \end{cases}$$

then the family $\tilde{u}_\varepsilon$ is equibounded in $H^1(\Omega_1)$ and it converges weakly in $H^1(\Omega_1)$, as ε goes to 0, to the function

$$\tilde{u}_0(z) = \begin{cases} u_0(z) & \text{if } z \in \Omega, \\ u_0(\pi_\Omega(z))\left(1 - \dfrac{\beta d(z)}{1+\beta h(z)}\right) & \text{if } z \in \Sigma_1. \end{cases}$$

Notice that the limit function $\tilde{u}_0$ turns out to be linear with respect to the distance $d(z)$.

The functions u_ε are the minimizers, in $H^1(\Omega_\varepsilon)$, of the functionals

$$\mathcal{F}_\varepsilon(v,h) = \int_\Omega |\nabla v|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx,$$

related to the thermal insulation of a solid, represented by the set Ω, with a thin layer of highly insulating material, represented by the set Σ_ε. In this setting, ε is related to both the mean thickness and the conductivity of the insulating layer Σ_ε, while the function h represents the shape of the layer. Finally, the Robin boundary condition models the case in which the heat exchange with the environment occurs through convection, and u_ε is the temperature of the insulated body.

Similar problems have been studied before in the context of reinforcement problems in [40], [97], [10], and more recently in the context of thermal insulation in [44], [86] and [7].

In particular, in [86] it was proved that the family of functionals $\mathcal{F}_\varepsilon(\cdot,h)$ Γ-converges, in the strong L^2-topology, to the functional

$$\mathcal{F}_0(v,h) = \int_\Omega |\nabla v|^2\,dx + \beta\int_{\partial\Omega} \frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx.$$

Moreover, minimizing the approximated functional $\mathcal{F}_0$ with respect to both variables, the authors proved that, under an integral constraint for the function h, a minimizing couple (u_0, h_{opt}) exists and that the optimal function h_{opt}, and hence the shape of the insulating layer, should be related to the limit temperature u_0, while the dependence on the geometry of Ω is only implicit. In [7] a similar problem was studied through a first-order asymptotic development by Γ-convergence (see Definition 2.7.3), the main result is in an approximated energy functional whose minimizer h_{opt} explicitly depends on the mean curvature of $\partial\Omega$.

In the spirit of [7], we aim to study a first-order asymptotic development by Γ-convergence of $\mathcal{F}_\varepsilon$. In particular, thanks to Theorem 5.2.1, we are able to prove the following

Theorem 5.2.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Then the functional*

$$\delta\mathcal{F}_\varepsilon(\cdot, h) = \frac{\mathcal{F}_\varepsilon(\cdot, h) - \mathcal{F}_0(u_0, h)}{\varepsilon}$$

Γ-converges, in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to

$$\mathcal{F}^{(1)}(v, h) = \begin{cases} \beta \displaystyle\int_{\partial\Omega} \frac{H_\Omega h(2+\beta h)}{2(1+\beta h)^2} u_0^2 \, d\mathcal{H}^{n-1}, & \text{if } v = u_0, \\ +\infty & \text{if } v \neq u_0, \end{cases}$$

where H denotes the mean curvature of $\partial\Omega$.

Let us look at the first-order approximation in ε of the energy $\mathcal{F}_\varepsilon(u_\varepsilon, h)$, described by

$$\mathcal{G}_\varepsilon(h) = \mathcal{F}_0(u_0, h) + \varepsilon\mathcal{F}^{(1)}(u_0, h).$$

We believe that the study of $\mathcal{G}_\varepsilon$, either with a volume penalization as in [65, 56, 7] or a volume constraint as in [44, 86], will provide a deeper understanding of the optimal shape of the insulating layer. From the previous result, we conjecture that the design of the shape of the optimal insulating layer should not only be related to the limit temperature u_0, but also to the mean curvature H. However we were not able to provide an existence result for minimizers of the approximating energy.

The present section is organized as follows. In Section 2.7 we give introductory definitions and tools about hypersurfaces and Γ-convergence. In Section 5.2.2 we explicitly compute the diffeomorphism Ψ_ε and we prove Theorem 5.2.1 using some energy estimates that will be proved later on. In Section 5.2.3, using the convergence of the functions $\tilde{u}_\varepsilon$, we explicitly compute the first-order asymptotic development by Γ-convergence for the functional $\mathcal{F}_\varepsilon(\cdot, h)$. Finally, in Section 5.2.4 we prove the aforementioned energy estimates, using classical techniques in regularity theory adapted to this particular case of a diffraction problem.

5.2.2 Stretching

In this section, we construct the diffeomorphism $\Psi_\varepsilon \colon \Sigma_1 \to \Sigma_\varepsilon$, and, assuming that the family of functions $\tilde{u}_\varepsilon = u_\varepsilon \circ \Psi_\varepsilon$ converges weakly in $H^1(\Sigma_1)$ to a function $\tilde{u}_0$, we compute the limiting boundary value problem for the limit $\tilde{u}_0$.

Preliminary computations

Let Ω be a bounded open set with $C^{1,1}$ boundary. For every $x \in \Gamma_{d_0}$, we recall we have defined

$$\nu_0(x) = \nu_0(\pi_\Omega(x))$$

to be the unit outer normal to $\partial\Omega$ in $\pi_\Omega(x)$. We will denote $\nu_0 := \nu_\Omega$. Let $h \in C^{1,1}(\Sigma_\Omega(d_0))$ be a positive function such that $\nabla h \cdot \nu_0 = 0$, so that h is constant along normal radii starting from $\partial\Omega$. Up to extending $h = 0$ outside of $\Sigma_\Omega(d_0)$, we may define

$$\Sigma_\varepsilon := \Sigma_\Omega(\varepsilon h) = \{\, x \in \mathbb{R}^n \mid 0 < d(x) < \varepsilon h(x) \,\}, \qquad \Omega_\varepsilon = \overline{\Omega} \cup \Sigma_\varepsilon.$$

In particular, notice that, if $\varepsilon\|h\|_\infty < d_0$, we can write

$$\Sigma_\varepsilon = \left\{ \sigma + t\nu_0(\sigma) \;\middle|\; \begin{array}{c} \sigma \in \partial\Omega, \\ 0 < t < \varepsilon h(\sigma) \end{array} \right\}, \tag{5.2.3}$$

and the representation $x = \sigma + t\nu_0$ is unique and $d \in C^{1,1}(\Sigma_\varepsilon)$.

For simplicity's sake, we will assume that $\|h\|_\infty < d_0$ so that we can assume the distance function d to be regular on the set Σ_1.

Definition 5.2.3. We define the *stretching diffeomorphism* $\Psi_\varepsilon \in C^{0,1}(\Gamma_{d_0}; \Gamma_{\varepsilon d_0})$ as the function defined as

$$\begin{aligned} \Psi_\varepsilon(z) &= \pi_\Omega(z) + \varepsilon d(z)\nu_0(z) \\ &= z + (\varepsilon - 1)d(z)\nu_0(z). \end{aligned}$$

With this definition we have that

$$\begin{aligned} D\Psi_\varepsilon(z) &= D\pi_\Omega(z) + \varepsilon[\nu_0(z) \otimes \nu_0(z) + d(z)D\nu_0(z)] \\ &= I_n + (\varepsilon - 1)[\nu_0(z) \otimes \nu_0(z) + d(z)D\nu_0(z)], \end{aligned}$$

where I_n is the identity matrix. Moreover, Ψ_ε is invertible with

$$\Psi_\varepsilon^{-1}(x) = \pi_\Omega(x) + \frac{d(x)}{\varepsilon}\nu_0(x).$$

By direct computations, we have the following

Lemma 5.2.4. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Let $g : \mathbb{R}^n \to \mathbb{R}$ be a positive Borel function. Then*

$$\int_{\Sigma_\varepsilon} g(x)\, dx = \varepsilon \int_{\Sigma_1} g(\Psi_\varepsilon(z)) J_\varepsilon(z)\, dz, \tag{5.2.4}$$

with

$$J_\varepsilon(z) = \prod_{i=1}^{n-1} \frac{1 + \varepsilon d(z)\kappa_i(\sigma)}{1 + d(z)\kappa_i(\sigma)},$$

and $\sigma = \pi_\Omega(z)$. In addition,

$$\int_{\partial\Omega_\varepsilon} g(\xi)\, d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega_1} g(\Psi_\varepsilon(\zeta)) J_\varepsilon^\tau(\zeta)\, d\mathcal{H}^{n-1}(\zeta), \tag{5.2.5}$$

where J_ε^τ is the tangential Jacobian of Ψ_ε, and it converges uniformly, as $\varepsilon \to 0^+$, to

$$\begin{aligned} J_0^\tau(\zeta) &= \frac{1}{\sqrt{1 + |\nabla h(\zeta)|^2}} \prod_{i=1}^{n-1} \frac{1}{1 + h(\zeta)\kappa_i(\pi_\Omega(\zeta))} \\ &= \frac{1}{\sqrt{1 + |\nabla h(\zeta)|^2}} J_0(\zeta). \end{aligned}$$

Proof. Let $z \in \Gamma_{d_0}$, and let $\tau_i(z)$ be the eigenvectors of $D^2 d(z) = D\nu_0(z)$, with respective eigenvectors $\kappa_i(z)$, defined in Definition 2.1.8. We have that $\{\, \tau_1(\sigma), \dots, \tau_{n-1}(\sigma), \nu_0(\sigma)\,\}$ is a basis of eigenvectors for $D\Psi_\varepsilon(z)$, and, in particular,

$$\begin{aligned} D\Psi_\varepsilon(z)\,\tau_i(z) &= \left(1 + \frac{(\varepsilon - 1)d(z)\kappa_i(\sigma)}{1 + d(z)\kappa_i(\sigma)}\right)\tau_i(z) \\ &= \frac{1 + \varepsilon d(z)\kappa_i(\sigma)}{1 + d(z)\kappa_i(\sigma)}\tau_i(z), \end{aligned}$$

and

$$D\Psi_\varepsilon(z)\,\nu_0(z) = \varepsilon\nu_0(z).$$

Therefore, (5.2.4) follows from the area formula.

On the other hand, notice that $D\Psi_\varepsilon$ converges uniformly, as $\varepsilon \to 0^+$, to $D\pi_\Omega$, so that $\mathrm{Jac}^{\partial\Omega_1} \Psi_\varepsilon$ converges to $\mathrm{Jac}^{\partial\Omega_1} \sigma$. Therefore, we apply the area formula on surfaces (Theorem 2.2.4) to get (5.2.5), and it is left to compute the tangential Jacobian $\mathrm{Jac}^{\partial\Omega_1} \sigma$.

Let $\zeta \in \partial\Omega_1$, and let ν_1 be the unit outer normal to $\partial\Omega_1$. Recalling that $\nabla d = \nu_0$, the definition of Σ_1 ensures that

$$\nu_1(\zeta) = \frac{\nu_0(\zeta) - \nabla h(\zeta)}{\sqrt{1 + |\nabla h(\zeta)|^2}}.$$

We aim to construct an orthonormal basis for the tangent space $T_\zeta\partial\Omega_1$ with a rotation of the tangent space $T_{\pi_\Omega(\zeta)}\partial\Omega$. In the following, when possible, we will drop the dependence on ζ. Let us define the rotation operator

$$R : \mathbb{R}^n \to \mathbb{R}^n$$

as the unique linear operator having the following properties:

(i) if $w \in (\nabla h)^\perp \cap \nu_0^\perp$, then $Rw = w$;

(ii) $R\nu_0 = \nu_1$;

(iii) $\nabla h/|\nabla h|$ is mapped to a unit vector laying in the plane generated by ∇h and ν_0, and orthogonal to ν_1, namely

$$R\frac{\nabla h}{|\nabla h|} = \frac{1}{\sqrt{1 + |\nabla h|^2}}\left(\frac{\nabla h}{|\nabla h|} + |\nabla h|\nu_0\right).$$

Explicitly, we can write R as

$$\begin{aligned} R =& I_n - \nu_0 \otimes \nu_0 - \frac{\nabla h}{|\nabla h|} \otimes \frac{\nabla h}{|\nabla h|} + \nu_1 \otimes \nu_0 \\ &+ \frac{1}{\sqrt{1 + |\nabla h|^2}}\left(\frac{\nabla h}{|\nabla h|} + |\nabla h|\nu_0\right) \otimes \frac{\nabla h}{|\nabla h|}. \end{aligned}$$

This operator is a rotation in $\mathbb{R}^n$ that maps $T_{\pi_\Omega(\zeta)}\partial\Omega$ onto $T_\zeta\partial\Omega_1$, and, in particular, we define an orthonormal basis $\bar{\tau}_i$ of $T_\zeta\partial\Omega_1$ as follows: let $\tau_i = \tau_i(\pi_\Omega(\zeta))$ be an orthonormal basis of $T_{\pi_\Omega(\zeta)}\partial\Omega$, and we define

$$\bar{\tau}_s := R\tau_s = \left(I_n + \left(\frac{1}{\sqrt{1 + |\nabla h|^2}} - 1\right)\frac{\nabla h}{|\nabla h|} \otimes \frac{\nabla h}{|\nabla h|}\right)\tau_s + \frac{\nabla h \cdot \tau_s}{\sqrt{1 + |\nabla h|^2}}\,\nu_0.$$

In particular, since $D\pi_\Omega \nu_0 = 0$, then

$$(D\pi_\Omega R)\tau_s = D\pi_\Omega \left(I_n + \left(\frac{1}{\sqrt{1+|\nabla h|^2}} - 1 \right) \frac{\nabla h}{|\nabla h|} \otimes \frac{\nabla h}{|\nabla h|} \right) \tau_s.$$

Hence, recalling the evaluation of the eigenvalues of $D\pi_\Omega$ given in Remark 2.1.9, we can easily compute the determinant of the tangential gradient

$$\mathrm{Jac}^{\partial\Omega_1} \pi_\Omega(\zeta) = \left(\prod_{i=1}^{n-1} (1 + h(\zeta)\kappa_i(\pi_\Omega(\zeta))) \sqrt{1+|\nabla h(\zeta)|^2} \right)^{-1},$$

thus concluding the proof.

□

The main result

Fix $\beta > 0$. For every $0 < \varepsilon < 1$ let $u_\varepsilon \in H^1(\Sigma_\varepsilon)$ be a solution to

$$\varepsilon \int_{\Sigma_\varepsilon} \nabla u_\varepsilon \nabla \varphi \, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon \varphi \, d\mathcal{H}^{n-1} = 0, \tag{5.2.6}$$

for every $\varphi \in H^1(\Sigma_\varepsilon)$ such that $\varphi = 0$ on $\partial\Omega$. Then we have the following.

Proposition 5.2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Let u_ε be a family of weak solution to (5.2.6), and let $\tilde{u}_\varepsilon(z) = u_\varepsilon(\Psi_\varepsilon(z))$. If $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Sigma_1)$ to a function $\tilde{u}_0$, then $\tilde{u}_0$ is a solution to*

$$\int_{\Sigma_1} (\nabla \tilde{u}_0 \cdot \nu_0)(\nabla \varphi \cdot \nu_0) J_0(z) \, dz + \beta \int_{\partial\Omega_1} \tilde{u}_0(\zeta)\varphi(\zeta) \frac{J_0(\zeta)}{\sqrt{1+|\nabla h(\zeta)|^2}} \, d\mathcal{H}^{n-1}(\zeta) = 0. \tag{5.2.7}$$

for every $\varphi \in H^1(\Sigma_1)$ such that $\varphi = 0$ on $\partial\Omega$.

Proof. By definition, we have $\tilde{u}_\varepsilon \in H^1(\Sigma_1)$ and

$$\begin{aligned} \nabla u_\varepsilon|_{\Psi_\varepsilon(z)} &= D(\Psi_\varepsilon^{-1})|_{\Psi_\varepsilon(z)} \nabla \tilde{u}_\varepsilon(z) \\ &= \left(\frac{1}{\varepsilon} \nabla \tilde{u}_\varepsilon(z) \cdot \nu_0(z) \right) \nu_0(z) + \sum_{i=1}^{n-1} \left(\frac{1 + d(z)\kappa_i(\sigma)}{1 + \varepsilon d(z)\kappa_i(\sigma)} \nabla \tilde{u}_\varepsilon(z) \cdot \tau_i(z) \right) \tau_i(z). \end{aligned} \tag{5.2.8}$$

Let $\varphi \in H^1(\Sigma_1)$ with $\varphi = 0$ on $\partial\Omega$, and let $\varphi_\varepsilon(x) = \varphi(\Psi_\varepsilon^{-1}(x))$, then equation (5.2.6) yields

$$\varepsilon \int_{\Sigma_\varepsilon} \nabla u_\varepsilon \nabla \varphi_\varepsilon \, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon \varphi_\varepsilon \, d\mathcal{H}^{n-1} = 0,$$

from which, using Lemma 5.2.4 and the computation (5.2.8), we have

$$\begin{aligned} \int_{\Sigma_1} \left[(\nabla \tilde{u}_\varepsilon \cdot \nu_0)(\nabla \varphi \cdot \nu_0) + \varepsilon^2 \sum_{i=1}^{n-1} \left(\frac{1 + d\kappa_i}{1 + \varepsilon d\kappa_i} \right)^2 (\nabla \tilde{u}_\varepsilon \cdot \tau_i)(\nabla \varphi \cdot \tau_i) \right] \prod_{i=1}^{n-1} \frac{1 + \varepsilon d\kappa_i}{1 + d\kappa_i} \, dz + \\ + \beta \int_{\partial\Omega_1} \tilde{u}_\varepsilon \varphi J_\varepsilon^\tau \, d\mathcal{H}^{n-1} = 0, \end{aligned} \tag{5.2.9}$$

Passing to the limit in (5.2.9), the assertion follows. □

Remark 5.2.6 (Uniqueness)**.** For every given Dirichlet boundary condition on $\partial\Omega$, (5.2.7) admits a unique solution. Indeed, let $v_1, v_2 \in H^1(\Sigma_1)$ be two solutions to (5.2.7) such that $v_1 = v_2$ on $\partial\Omega$, and let $w = v_1 - v_2$. By linearity, we have that w is a solution to (5.2.7) with $w = 0$ on $\partial\Omega$, so that we have

$$\int_{\Sigma_1} |\nabla w \cdot \nu_0|^2 J_0 \, dz + \beta \int_{\partial\Omega_1} w^2 \frac{J_0}{\sqrt{1+|\nabla h|^2}} \, d\mathcal{H}^{n-1} = 0,$$

and since $J_0 > 0$, then $\nabla w \cdot \nu_0 = 0$ a.e. on Σ_ε. Then, for $\mathcal{L}^n$-a.e. $z \in \Sigma_1$,

$$w(z) = w(\pi_\Omega(z)) + \int_0^{d(z)} \nabla w(\pi_\Omega(z) + t\nu_0) \cdot \nu_0 \, dt = 0,$$

that is $w = 0$ and $v_1 = v_2$.

Remark 5.2.7 (Limit computation)**.** We point out that it is possible to explicitly compute the solution $\tilde{u}_0$ to (5.2.7) in terms of its values on $\partial\Omega$ as

$$\tilde{u}_0(z) = \tilde{u}_0(\pi_\Omega(z))\left(1 - \frac{\beta d(z)}{1+\beta h(z)}\right).$$

We first inspect the regular case of Ω of class C^3 to obtain the strong equation, and then we work on the general case using only the weak equation (5.2.7).

Indeed, let

$$A(z) = J_0(z)\nu_0(z) \otimes \nu_0(z),$$

and notice that when Ω is smooth, then J_0 is also smooth as it can be seen as the determinant of the smooth matrix-valued map

$$z \in \Sigma_1 \mapsto D\pi_\Omega + \nu_0 \otimes \nu_0.$$

Then, if $\tilde{u}_0$ is a regular solution to (5.2.7), it is a solution to

$$\begin{cases} \operatorname{div}(A(x)\nabla \tilde{u}_0) = 0 & \text{in } \Sigma_1, \\ \dfrac{\partial \tilde{u}_0}{\partial \nu_0} + \beta \tilde{u}_0 = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{5.2.10}$$

Moreover, we have that

$$\operatorname{div}(J_0\nu_0) = \nabla J_0 \cdot \nu_0 + J_0 \operatorname{Tr}(D\nu_0).$$

In particular, we can explicitly compute the derivative of J_0 in direction ν_0 using the local representation

$$J_0(z) = \prod_{i=1}^{n-1} \frac{1}{1 + d(z)\kappa_i(\pi_\Omega(z))},$$

and recalling that $\kappa_i \circ \pi_\Omega$ is constant along normal radii, so that, for every $\zeta \in \partial\Omega_1$,

$$\nabla J_0(\zeta) \cdot \nu_0(\zeta) = -J_0(\zeta) \operatorname{Tr}(D\nu_0(\zeta)).$$

Hence, $\operatorname{div}(J_0\nu_0) = 0$, and

$$\operatorname{div}(A\nabla u) = J_0 \nabla(\nabla u \cdot \nu_0) \cdot \nu_0.$$

Therefore, equation (5.2.10) reduces to

$$\begin{cases} \nabla(\nabla\tilde{u}_0\cdot\nu_0)\cdot\nu_0 = 0 & \text{in } \Sigma_1, \\ \dfrac{\partial\tilde{u}_0}{\partial\nu_0} + \beta\tilde{u}_0 = 0 & \text{on } \partial\Omega_1. \end{cases} \tag{5.2.11}$$

The previous computation suggests that solutions to (5.2.7) have to be linear with respect to the normal direction. Indeed, Since h is constant along the normal direction, it is sufficient to check that, for every $w \in H^1(\Sigma_1)$ the function

$$\tilde{u}(z) = w(\pi_\Omega(z))\left(1 - \frac{\beta d(z)}{1+\beta h(z)}\right)$$

is a solution to (5.2.11) in $H^1(\Sigma_1)$. Finally, we can show that the previous solution to (5.2.11) is the solution to (5.2.7) also in the case in which Ω is only $C^{1,1}$. By a change of variables and coarea formula (see Remark 5.2.12) we can rewrite the integral on Σ_1 as

$$\int_{\Sigma_1} g(z)J_0(z)\,dz = \int_{\partial\Omega}\int_0^{h(\sigma)} g(\sigma+t\nu_0)\,dt\,d\mathcal{H}^{n-1}(\sigma)$$

and the integral on $\partial\Omega_1$ as

$$\int_{\partial\Omega_1} g(\xi)\frac{J_0(\xi)}{\sqrt{1+|\nabla h|^2}}\,d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega} g(\sigma+h(\sigma)\nu_0)\,d\mathcal{H}^{n-1}(\sigma).$$

Then by direct computation, we have that

$$\nabla\tilde{u}\cdot\nu_0 = -\frac{\beta w(\pi_\Omega(z))}{1+\beta h(z)}$$

so that, for every smooth function $\varphi \in H^1(\Sigma_1)$ with $\varphi = 0$ on $\partial\Omega$,

$$\begin{aligned}
\int_{\Sigma_1}(\nabla\tilde{u}_0\cdot\nu_0)(\nabla\varphi\cdot\nu_0)J_0(z)\,dz &= -\int_{\partial\Omega}\frac{\beta w(\sigma)}{1+\beta h}\int_0^{h(\sigma)}\frac{d}{dt}(\varphi(\sigma+t\nu_0))\,dt\,d\mathcal{H}^{n-1}(\sigma) \\
&= -\int_{\partial\Omega}\frac{\beta w(\sigma)}{1+\beta h}\int_0^{h(\sigma)}\varphi(\sigma+h(\sigma)\nu_0)\,d\mathcal{H}^{n-1} \\
&= -\beta\int_{\partial\Omega}\tilde{u}_0(\sigma+h(\sigma)\nu_0)\varphi(\sigma+h(\sigma)\nu_0)\,d\mathcal{H}^{n-1} \\
&= -\beta\int_{\partial\Omega_1}\tilde{u}_0(\zeta)\varphi(\zeta)\frac{J_0(\zeta)}{\sqrt{1+|\nabla h(\zeta)|^2}}\,d\mathcal{H}^{n-1}(\zeta)
\end{aligned}$$

and $\tilde{u}$ is in fact a solution to (5.2.7).

Proof of Theorem 5.2.1

Let $f \in L^2(\Omega)$ be a non-negative function and consider the functional

$$\mathcal{F}_\varepsilon(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx & \text{if } v \in H^1(\Omega_\varepsilon), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1(\Omega_\varepsilon), \end{cases}$$

and let

$$\mathcal{F}_0(v,h) = \begin{cases} \displaystyle\int_\Omega |\nabla v|^2\,dx + \beta\int_{\partial\Omega} \frac{v^2}{1+\beta h}\,d\mathcal{H}^{n-1} - 2\int_\Omega fv\,dx & \text{if } v \in H^1(\Omega), \\ +\infty & \text{if } v \in L^2(\mathbb{R}^n)\setminus H^1(\Omega), \end{cases}$$

in [86] the authors prove the following

Theorem 5.2.8. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive Lipschitz function $h\colon \partial\Omega \to \mathbb{R}$. Then $\mathcal{F}_\varepsilon(\cdot,h)$ Γ-converges in the strong $L^2(\mathbb{R}^n)$ topology, as $\varepsilon \to 0^+$, to $\mathcal{F}_0(\cdot,h)$.*

For every $\varepsilon > 0$, let $u_{\varepsilon,h} = u_\varepsilon \in H^1(\Omega_\varepsilon)$ be the minimizer of $\mathcal{F}_\varepsilon(\cdot,h)$, then u_ε is a solution to the following boundary value problem

$$\begin{cases} -\Delta u_\varepsilon = f & \text{in } \Omega, \\ u_\varepsilon^- = u_\varepsilon^+ & \text{on } \partial\Omega, \\ \dfrac{\partial u_\varepsilon^-}{\partial \nu_0} = \varepsilon \dfrac{\partial u_\varepsilon^+}{\partial \nu_0} & \text{on } \partial\Omega, \\ \Delta u_\varepsilon = 0 & \text{in } \Sigma_\varepsilon, \\ \varepsilon\dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} + \beta u_\varepsilon = 0 & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where u_ε^- and u_ε^+ denote the trace on u_ε from the inside Ω and from the outside of Ω respectively. By the properties of Γ-convergence, we have that

$$u_\varepsilon \xrightharpoonup{H^1(\Omega)} u_0, \tag{5.2.12}$$

where $u_0 = u_{0,h}$ is the minimizer of $\mathcal{F}_0(\cdot,h)$. Namely, u_0 is the solution to

$$\begin{cases} -\Delta u_0 = f & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} + \dfrac{\beta}{1+\beta h} u_0 = 0 & \text{on } \partial\Omega. \end{cases}$$

We now prove the weak convergence of the family of functions $\tilde{u}_\varepsilon$ defined in Theorem 5.2.2. The proof revolves around some energy estimates analogous to the one proved in [40] for the solutions to a similar boundary value problem with a transmission condition.

Theorem 5.2.9. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ and $\nabla h \cdot \nu_0 = 0$. Then there exists positive constants $\varepsilon_0(\Omega)$, and $C(\Omega,h,\beta,f)$ such that if*

$$\varepsilon \|h\|_{C^{0,1}} \le \varepsilon_0 \tag{5.2.13}$$

and u_ε is the weak solution to (5.2.1), then

$$\int_\Omega |D^2 u_\varepsilon|^2\,dx + \varepsilon\int_{\Sigma_\varepsilon}|D^2 u_\varepsilon|^2\,dx + \beta\int_{\partial\Omega_\varepsilon}|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2\,d\mathcal{H}^{n-1} \le C. \tag{5.2.14}$$

Remark 5.2.10. We want to point out that the assumption

$$\nabla h(x)\cdot\nu_0(x) = 0 \qquad \forall x\in\Gamma_{d_0}$$

is not necessary to prove Theorem 5.2.9, but it makes the computations easier.

For simplicity's sake, we postpone the technicalities of the proof of Theorem 5.2.9 to Section 5.2.4, and we directly prove Theorem 5.2.1. Our aim is to have uniform H^1 estimates for $\tilde{u}_\varepsilon$, so we first show an immediate consequence of Theorem 5.2.9. For every $x\in\Gamma_{d_0}$, we denote by

$$\nabla^{\partial\Omega} u_\varepsilon(x) = \nabla u_\varepsilon(x) - (\nabla u_\varepsilon(x)\cdot\nu_0(x))\nu_0(x),$$

and we have the following.

Corollary 5.2.11. *There exists a positive constant $C = C(\Omega, h, \beta, f)$ such that*

$$\int_{\Sigma_\varepsilon}|\nabla^{\partial\Omega} u_\varepsilon|^2\,dx \le \varepsilon C.$$

Proof. We start by proving that on $\partial\Omega_\varepsilon$

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge (1-\varepsilon^2)|\nabla^{\partial\Omega} u_\varepsilon|^2. \tag{5.2.15}$$

Indeed, for every $\xi\in\partial\Omega_\varepsilon$, if $|\nabla h(\xi)| = 0$, then $\nu_0 = \nu_\varepsilon$ and the inequality is trivially true; on the other hand, if $|\nabla h(\xi)|\neq 0$ we have that

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 = |\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon\cdot\nu_\varepsilon|^2,$$

and for every $\eta>0$

$$|\nabla u_\varepsilon\cdot\nu_\varepsilon|^2 \le \frac{(1+\eta)|\nabla u_\varepsilon\cdot\nu_0|^2 + \left(1+\dfrac{1}{\eta}\right)\varepsilon^2|\nabla u_\varepsilon\cdot\nabla h|^2}{1+|\nabla h|^2}.$$

Moreover $\nabla h\cdot\nu_0 = 0$, so that

$$|\nabla u_\varepsilon\cdot\nabla h|^2 = |\nabla^{\partial\Omega} u_\varepsilon\cdot\nabla h|^2 \le (|\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon\cdot\nu_0|^2)|\nabla h|^2.$$

Therefore, for every $\eta>0$

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge \left(1 - \frac{\left(1+\dfrac{1}{\eta}\right)\varepsilon^2|\nabla h|^2}{1+|\nabla h|^2}\right)|\nabla u_\varepsilon|^2 - \frac{1+\eta-\left(1+\dfrac{1}{\eta}\right)\varepsilon^2|\nabla h|^2}{1+|\nabla h|^2}|\nabla u_\varepsilon\cdot\nu_0|^2,$$

finally, letting $\eta = |\nabla h|^2$, we have

$$|\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \ge (1-\varepsilon^2)(|\nabla u_\varepsilon|^2 - |\nabla u_\varepsilon\cdot\nu_0|^2) = (1-\varepsilon^2)|\nabla^{\partial\Omega} u_\varepsilon|^2.$$

Then, by Theorem 5.2.9 and (5.2.15), we have that

$$\varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2 \, d\mathcal{H}^{n-1} \le C. \tag{5.2.16}$$

For $x \in \Sigma_\varepsilon$ denote by

$$\xi(x) = \pi_\Omega(x) + \varepsilon h(x)\nu_0(x) \in \partial\Omega_\varepsilon,$$

so that for $\mathcal{L}^n$-a.e. $x \in \Sigma_\varepsilon$, we have that

$$\nabla^{\partial\Omega} u_\varepsilon(x) = \nabla^{\partial\Omega} u_\varepsilon(\xi(x)) - \int_{d(x)}^{\varepsilon h(x)} \frac{d}{dt}\Big(\nabla^{\partial\Omega} u_\varepsilon(\pi_\Omega(x) + t\nu_0(x))\Big)\, dt.$$

Hence,

$$|\nabla^{\partial\Omega} u_\varepsilon|(x)^2 \le C(h)\left(|\nabla^{\partial\Omega} u_\varepsilon|^2(\xi(x)) + \varepsilon \int_0^{\varepsilon h(x)} |D^2 u_\varepsilon|^2(\pi_\Omega(x) + t\nu_0(x))\, dt\right),$$

and integrating over Σ_ε, using (5.2.23) and (5.2.24), and (5.2.16), we get

$$\begin{aligned}
\int_{\Sigma_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2(x)\, dx \le{}& C \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} |\nabla^{\partial\Omega} u_\varepsilon|^2(\sigma + \varepsilon h(\sigma)\nu_0)\, ds\, d\mathcal{H}^{n-1}(\sigma) + \\
&+ C\varepsilon \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} \int_0^{\varepsilon h(\sigma)} |D^2 u_\varepsilon|^2(\pi_\Omega(x) + t\nu_0(x))\, dt\, ds\, d\mathcal{H}^{n-1}(\sigma) \\
&\le \varepsilon C\left(\int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2\, d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2\, dx\right) \\
&\le \varepsilon C.
\end{aligned}$$

□

We are now in a position to prove the H^1 convergence of the family $\tilde{u}_\varepsilon$.

Proof of Theorem 5.2.1. We recall that we are assuming without loss of generality $\|h\|_\infty < d_0$, so that $d \in C^{1,1}(\Sigma_1)$. To prove the equiboundedness in $H^1(\Omega_1)$, we decompose $\nabla\tilde{u}_\varepsilon$ into its normal part

$$(\nabla\tilde{u}_\varepsilon \cdot \nu_0)\, \nu_0,$$

and its tangential part

$$\nabla^{\partial\Omega}\tilde{u}_\varepsilon := \nabla\tilde{u}_\varepsilon - (\nabla\tilde{u}_\varepsilon \cdot \nu_0)\nu_0.$$

Using Lemma 5.2.4, since J_ε and J_ε^T are equibounded, we can find a positive constant $C = C(\Omega, h)$ such that

$$\int_{\Sigma_1} |\nabla^{\partial\Omega}\tilde{u}_\varepsilon|^2\, dz \le \frac{C}{\varepsilon} \int_{\Sigma_\varepsilon} |\nabla^{\partial\Omega} u_\varepsilon|^2\, dx, \tag{5.2.17}$$

$$\int_{\Sigma_1} |\nabla\tilde{u}_\varepsilon \cdot \nu_0|^2\, dz \le \varepsilon C \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\, dx, \tag{5.2.18}$$

and

$$\int_{\partial\Omega_1} \tilde{u}_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq C \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1}. \tag{5.2.19}$$

Moreover, by the weak convergence of u_ε in $H^1(\Omega)$, u_ε are equibounded in $L^2(\Omega)$, while by the minimality

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^2 \, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2 \, d\mathcal{H}^{n-1} &\leq \mathcal{F}_\varepsilon(0) + 2\int_\Omega f u_\varepsilon \, dx \\ &\leq \int_\Omega (f^2 + u_\varepsilon^2) \, dx \\ &\leq C\end{aligned} \tag{5.2.20}$$

Joining the inequalities (5.2.17), (5.2.18), (5.2.19), and (5.2.20) with the energy estimates in Corollary 5.2.11, we have that for some positive constant $C(\Omega, h, \beta)$

$$\int_{\Omega_1} |\nabla \tilde{u}_\varepsilon|^2 \, dz + \beta \int_{\partial\Omega_1} \tilde{u}_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq C.$$

By Poincaré's inequality with boundary term, we have that $\tilde{u}_\varepsilon$ are equibounded in $H^1(\Omega_1)$. Therefore, up to a subsequence, $\tilde{u}_\varepsilon$ converges weakly in $H^1(\Omega_1)$ to some function $\tilde{u}_0$. In particular, by the weak convergence (5.2.12) inside Ω, we have $\tilde{u}_0 = u_0$ in Ω. On the other hand, in Σ_1, using Proposition 5.2.5, we get that $\tilde{u}_0$ is a solution to (5.2.7), so that Remark 5.2.7 ensures that

$$\tilde{u}_0(z) = \tilde{u}_0(\pi_\Omega(z))\left(1 - \frac{\beta d(z)}{1 + \beta h(z)}\right).$$

Finally, since $\tilde{u}_0 \in H^1(\Omega_1)$, then it cannot jump across $\partial\Omega$, so that, since $\tilde{u}_0 = u_0$ in Ω, we necessarily have that $\tilde{u}_0(\pi_\Omega(z)) = u_0(\pi_\Omega(z))$ for a.e. $z \in \Sigma_1$, and the theorem is proved. $\square$

5.2.3 Asymptotic Development

In this section, we study the first-order asymptotic development by Γ-convergence of the functional $\mathcal{F}_\varepsilon(\cdot, h)$. Let us recall the notation introduced in Section 2.1

$$\gamma_t = \partial(\Omega \cup \Gamma_t) = \{\, x \in \mathbb{R}^n \mid d(x) = t \,\} \setminus \Omega.$$

Remark 5.2.12. Using the coarea formula (Theorem 2.3.1) with the distance function d, we have that for every $g \in L^1(\Omega_\varepsilon)$

$$\int_{\Sigma_\varepsilon} g(x) \, dx = \int_0^{+\infty} \int_{\gamma_t} g(\xi) \, \chi_{\Sigma_\varepsilon}(\xi) \, d\mathcal{H}^{n-1}(\xi) \, dt.$$

Let

$$\phi_t \colon x \in \Gamma_{d_0} \mapsto x + t\nu_0(x) \in \Gamma_{td_0},$$

then $\gamma_t = \phi_t(\partial\Omega)$, and by the area formula on surfaces (Theorem 2.2.4)

$$\begin{aligned}\int_{\Sigma_\varepsilon} g(x) \, dx &= \int_0^{+\infty} \int_{\partial\Omega} g(\sigma + t\nu_0) \chi_{\Sigma_\varepsilon}(\sigma + t\nu_0) \, \mathrm{Jac}^{\partial\Omega} \phi_t(\sigma) \, d\mathcal{H}^{n-1}(\sigma) \, dt \\ &= \int_{\partial\Omega} \int_0^{\varepsilon h(\sigma)} g(\sigma + t\nu_0) \prod_{i=1}^{n-1} (1 + t\kappa_i(\sigma)) \, dt \, d\mathcal{H}^{n-1}(\sigma).\end{aligned}$$

Similarly,

$$\int_{\partial\Omega_\varepsilon} g(\xi)\, d\mathcal{H}^{n-1}(\xi) = \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0) \prod_{i=1}^{n-1}(1+\varepsilon h(\sigma)\kappa_i(\sigma))\sqrt{1+\varepsilon^2|\nabla h|^2}\, d\mathcal{H}^{n-1}(\sigma).$$

so that we have

$$\int_{\Sigma_\varepsilon} g(x)\, dx = \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} g(\sigma+t\nu_0)\Big(1+tH(\sigma)+\varepsilon^2 R_1(\sigma,t,\varepsilon)\Big)\, dt\, d\mathcal{H}^{n-1} \tag{5.2.21}$$

and

$$\int_{\partial\Omega_\varepsilon} g(\sigma)\, d\mathcal{H}^{n-1} = \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0)\Big(1+\varepsilon h(\sigma)H(\sigma)+\varepsilon^2 R_2(\sigma,\varepsilon)\Big)\, d\mathcal{H}^{n-1}, \tag{5.2.22}$$

where the remainder terms R_1 and R_2 are bounded functions. In other words, there exists $Q = Q(\Omega,h) > 0$ such that $|R_1|, |R_2| \le Q$.

In particular, notice that there exists a positive constant $C = C(\Omega, \|h\|_{0,1})$ such that for every $0<\varepsilon<1$

$$\frac{1}{C}\int_{\Sigma_\varepsilon} g\, dx \le \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} g(\sigma+t\nu_0)\, dt\, d\mathcal{H}^{n-1} \le C\int_{\Sigma_\varepsilon} g\, dx, \tag{5.2.23}$$

and

$$\frac{1}{C}\int_{\partial\Omega_\varepsilon} g\, dx \le \int_{\partial\Omega} g(\sigma+\varepsilon h\nu_0)\, d\mathcal{H}^{n-1} \le C\int_{\partial\Omega_\varepsilon} g\, dx. \tag{5.2.24}$$

We can now follow the lead of [7] to prove Theorem 5.2.2.

Proof of Theorem 5.2.2. We start by proving the Γ-liminf inequality: without loss of generality, we can prove the inequality for the sequence of minimizers u_ε. Here we recall the definitions of $\mathcal{F}_\varepsilon$ and $\mathcal{F}_0$, omitting the dependence on h.

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \int_\Omega |\nabla u_\varepsilon|^2\, dx + \varepsilon\int_{\Sigma_\varepsilon}|\nabla u_\varepsilon|^2\, dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, d\mathcal{H}^{n-1} - 2\int_\Omega f u_\varepsilon\, dx, \tag{5.2.25}$$

$$\mathcal{F}_0(u_0) = \int_\Omega |\nabla u_0|^2\, dx + \beta\int_{\partial\Omega}\frac{u_0^2}{1+\beta h}\, d\mathcal{H}^{n-1} - 2\int_\Omega f u_0\, dx. \tag{5.2.26}$$

Moreover, notice that, by minimality of u_0,

$$\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{F}_0(u_0) \ge \varepsilon\int_{\Sigma_\varepsilon}|\nabla u_\varepsilon|^2\, dx + \beta\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, d\mathcal{H}^{n-1} - \beta\int_{\partial\Omega}\frac{u_\varepsilon^2}{1+\beta h}\, d\mathcal{H}^{n-1} \tag{5.2.27}$$

By (5.2.21) and (5.2.22) we have

$$\int_{\Sigma_\varepsilon}|\nabla u_\varepsilon|^2\, dx \ge \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma+t\nu_0)|^2\Big(1+tH(\sigma)-\varepsilon^2 Q\Big)\, dt\, d\mathcal{H}^{n-1} \tag{5.2.28}$$

and

$$\frac{\beta}{\varepsilon}\int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, \mathcal{H}^{n-1} \ge \frac{\beta}{\varepsilon}\int_{\partial\Omega} u_\varepsilon^2(\sigma+\varepsilon h(\sigma)\nu_0(\sigma))\Big(1+\varepsilon h(\sigma)H(\sigma)-\varepsilon^2 Q\Big)\, d\mathcal{H}^{n-1}. \tag{5.2.29}$$

We choose ε sufficiently small, so that for every $\sigma \in \partial\Omega$, and $0 < t < \varepsilon h(\sigma)$, we have that $1+tH(\sigma) > 0$, and, in particular, using Holder's inequality and integrating by parts,

$$\begin{aligned}\int_0^{\varepsilon h} |\nabla u_\varepsilon(\sigma + t\nu_0)|^2(1+tH)\,dt &\geq \frac{1}{\varepsilon h}\bigg(\int_0^{\varepsilon h} |\nabla u_\varepsilon(\sigma+t\nu_0)|\sqrt{1+tH}\,dt\bigg)^2 \\ &\geq \frac{1}{\varepsilon h}\bigg(\int_0^{\varepsilon h} \frac{d}{dt}(u_\varepsilon(\sigma+t\nu_0))\sqrt{1+tH}\,dt\bigg)^2 \\ &\geq \frac{1}{\varepsilon h}\bigg(u_\varepsilon(\sigma+\varepsilon h\nu_0)\sqrt{1+\varepsilon hH} - \bigg(u_\varepsilon(\sigma) + \int_0^{\varepsilon h} \frac{Hu_\varepsilon(\sigma+t\nu_0)}{2\sqrt{1+tH}}\,dt\bigg)\bigg)^2.\end{aligned}$$

We then apply Young's inequality to the absolute value of the double product, having that for every $\lambda > 0$

$$\begin{aligned}\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma+t\nu_0)|^2(1+tH(\sigma))\,dt \geq& \frac{(1-\lambda)(1+\varepsilon hH)u_\varepsilon(\sigma+\varepsilon h\nu_0)^2}{\varepsilon h} \\ &+ \frac{1}{\varepsilon h}\Big(1-\frac{1}{\lambda}\Big)\bigg(u_\varepsilon(\sigma) + \int_0^{\varepsilon h(\sigma)} \frac{Hu_\varepsilon(\sigma+t\nu_0)}{2\sqrt{1+tH}}\,dt\bigg)^2.\end{aligned} \tag{5.2.30}$$

We then have, joining (5.2.27), (5.2.28), (5.2.30), and (5.2.29),

$$\begin{aligned}\delta\mathcal{F}_\varepsilon(u_\varepsilon) =& \frac{\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{F}_0(u_0)}{\varepsilon} \\ \geq& \int_{\partial\Omega} \frac{1}{\varepsilon h(\sigma)}(1-\lambda+\beta h)(1+\varepsilon hH)u_\varepsilon^2(\sigma+\varepsilon h\nu_0)\,d\mathcal{H}^{n-1} \\ &+ \int_{\partial\Omega} \frac{1}{\varepsilon h}\Bigg(\Big(1-\frac{1}{\lambda}\Big)\bigg(u_\varepsilon(\sigma) + \int_0^{\varepsilon h} \frac{Hu_\varepsilon(\sigma+t\nu_0)}{2\sqrt{1+tH}}\,dt\bigg)^2 - \frac{\beta h\,u_\varepsilon^2(\sigma)}{1+\beta h}\Bigg)\,d\mathcal{H}^{n-1} - Q\varepsilon R(\varepsilon, u_\varepsilon)\end{aligned} \tag{5.2.31}$$

where, if ε is small enough,

$$\begin{aligned}R(\varepsilon, u_\varepsilon) &= \varepsilon\int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} |\nabla u_\varepsilon(\sigma+t\nu_0)|^2\,d\mathcal{H}^{n-1} + \beta\int_{\partial\Omega} u_\varepsilon(\sigma+\varepsilon h(\sigma)\nu_0(\sigma))^2\,d\mathcal{H}^{n-1} \\ &\leq C\int_\Omega f u_\varepsilon\,dx < C.\end{aligned}$$

Letting $\lambda = \lambda(\sigma) = 1 + \beta h(\sigma)$ in (5.2.31), and using the inequality $(a+b)^2 \geq a^2 + 2ab$, joint with the fact that $1 - \lambda^{-1} > 0$,

$$\delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \int_{\partial\Omega} \frac{\beta Hu_\varepsilon(\sigma)}{\varepsilon(1+\beta h)}\int_0^{\varepsilon h(\sigma)} \frac{u_\varepsilon(\sigma+t\nu_0)}{\sqrt{1+tH}}\,dt\,d\mathcal{H}^{n-1} + O(\varepsilon).$$

Moreover, for every $t \in (0, \varepsilon\|h\|_\infty)$ we have that $(1+tH)^{-1/2} = 1 + O(\varepsilon)$, so that

$$\delta\mathcal{F}_\varepsilon(u_\varepsilon) \geq \beta\int_{\partial\Omega} \frac{Hu_\varepsilon(\sigma)}{(1+\beta h)}\int_0^{h(\sigma)} \tilde{u}_\varepsilon(\sigma+t\nu_0)\,dt\,d\mathcal{H}^{n-1} + O(\varepsilon). \tag{5.2.32}$$

Finally, by Theorem 5.2.1 we get

$$\tilde{u}_\varepsilon \xrightarrow{L^2(\Sigma_1)} \tilde{u}_0,$$

so that

$$\int_0^{h(\sigma)} \tilde{u}_\varepsilon(\sigma + t\nu_0)\, dt \xrightarrow{L^2(\partial\Omega)} \int_0^{h(\sigma)} \tilde{u}_0(\sigma + t\nu_0)\, dt.$$

Indeed, by (5.2.23),

$$\int_{\partial\Omega} \left(\int_0^{h(\sigma)} (\tilde{u}_\varepsilon(\sigma + t\nu_0) - \tilde{u}_0(\sigma + t\nu_0))\, dt \right)^2 d\mathcal{H}^{n-1} \le C \int_{\Sigma_1} (\tilde{u}_\varepsilon - \tilde{u}_0)^2\, dz.$$

Therefore, passing to the limit in (5.2.32), and using the explicit expression of $\tilde{u}_0$, we get

$$\liminf_{\varepsilon \to 0^+} \delta\mathcal{F}_\varepsilon(u_\varepsilon) \ge \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2} u_0^2(\sigma)\, d\mathcal{H}^{n-1},$$

and the Γ-liminf is proved.

We now prove the Γ-limsup inequality.

Let

$$\varphi_\varepsilon(x) = \begin{cases} u_0(x) & \text{if } x \in \Omega, \\ u_0(\pi_\Omega(x)) \Big(1 - \dfrac{\beta d(x)}{\varepsilon(1+\beta h(x))} \Big) & \text{if } x \in \Sigma_\varepsilon, \\ 0 & \text{if } x \in \mathbb{R}^n \setminus \Omega_\varepsilon, \end{cases}$$

where we recall that $\nabla h \cdot \nu_0 = 0$. We have that $\varphi_\varepsilon \in H^1(\Omega)$ and φ_ε converges in $L^2(\mathbb{R}^n)$, to $u_0\chi_\Omega$. Since $\varphi_\varepsilon \equiv u_0$ in Ω, then by definition of the functionals (5.2.25), and (5.2.26), we can write

$$\mathcal{F}_\varepsilon(\varphi_\varepsilon) - \mathcal{F}_0(u_0) = \varepsilon \int_{\Sigma_\varepsilon} |\nabla \varphi_\varepsilon|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2\, d\mathcal{H}^{n-1} - \beta \int_{\partial\Omega} \frac{u_0^2}{1+\beta h}\, d\mathcal{H}^{n-1}. \tag{5.2.33}$$

Computing the gradient of φ_ε, for any $x \in \Sigma_\varepsilon$, letting for simplicity $\sigma = \pi_\Omega(x)$,

$$|\nabla \varphi_\varepsilon|^2(x) \le \frac{\beta^2 u_0^2(\sigma)}{\varepsilon^2(1+\beta h)^2} + C\Big(|\nabla u_0(\sigma)|^2 + u_0^2(\sigma) \Big),$$

where $C = C(h, \beta)$. Hence, by (5.2.21), and noticing that $\sigma + t\nu_0 \in \Gamma_{d_0}$ implies $d(\sigma + t\nu_0(\sigma)) = t$, we get

$$\begin{aligned} \varepsilon \int_{\Sigma_\varepsilon} |\nabla \varphi_\varepsilon|^2\, dx \le& \frac{\beta^2}{\varepsilon} \int_{\Sigma_\varepsilon} \frac{u_0^2(\sigma)}{(1+\beta h)^2}\, dx + \varepsilon C \int_{\Sigma_\varepsilon} \Big(|\nabla u_0(\sigma)|^2 + u_0^2(\sigma) \Big)\, dx \\ \le& \beta^2 \int_{\partial\Omega} \frac{u_0^2(\sigma) h}{(1+\beta h)^2} \Big(1 + \frac{\varepsilon h H}{2} \Big)\, d\mathcal{H}^{n-1} + O(\varepsilon^2). \end{aligned} \tag{5.2.34}$$

On the other hand, for every $\sigma \in \partial\Omega$,

$$\varphi_\varepsilon(\sigma + \varepsilon h(\sigma)\nu_0(\sigma)) = \frac{u_0(\sigma)}{1+\beta h(\sigma)},$$

from which we get

$$\beta \int_{\partial\Omega_\varepsilon} \varphi_\varepsilon^2 \, d\mathcal{H}^{n-1} \leq \beta \int_{\partial\Omega} \frac{u_0^2(\sigma)}{(1+\beta h)^2}(1+\varepsilon h H)\, d\mathcal{H}^{n-1} + O(\varepsilon^2). \tag{5.2.35}$$

Finally, joining (5.2.33), (5.2.34), (5.2.35), we have

$$\begin{aligned} \delta\mathcal{F}_\varepsilon(\varphi_\varepsilon) = \frac{\mathcal{F}_\varepsilon(u_\varepsilon) - \mathcal{F}_0}{\varepsilon} &\leq \beta \int_{\partial\Omega} \frac{u_0^2(\sigma) hH}{(1+\beta h)^2}\left(\frac{\beta h}{2} + 1\right) d\mathcal{H}^{n-1} + O(\varepsilon) \\ &= \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, u_0^2(\sigma)\, d\mathcal{H}^{n-1} + O(\varepsilon) \end{aligned}$$

so that

$$\limsup_{\varepsilon\to 0^+} \delta\mathcal{F}_\varepsilon(\varphi_\varepsilon) \leq \beta \int_{\partial\Omega} \frac{hH(2+\beta h)}{2(1+\beta h)^2}\, u_0^2(\sigma)\, d\mathcal{H}^{n-1}$$

and the Γ-limsup inequality is proved. □

5.2.4 Energy estimates

The aim of this section is to prove Theorem 5.2.9. The proof is mainly divided in two steps:

Step 1. We prove some H^1 uniform estimates for the minimizer u_ε (see Corollary 5.2.14).

Step 2. We prove the uniform H^2 estimates in Theorem 5.2.9 for the minimizers u_ε using a local argument similar to the approach in [40]: we focus on small neighborhoods V_{σ_0} of points $\sigma_0 \in \partial\Omega$, and we construct a diffeomorphism Φ_{σ_0} that flattens both $\partial\Omega$ and $\partial\Omega_\varepsilon$; on the flattened set $\Phi(V_{\sigma_0} \cap \Sigma_\varepsilon)$ we are able to compute energy estimates of the new functions $v_\varepsilon = u_\varepsilon \circ \Phi^{-1}$.

As in the previous sections, let u_ε be the minimizer to

$$\min\left\{ \mathcal{F}_\varepsilon(v) \,\middle|\, v \in H^1(\Omega_\varepsilon) \right\}, \tag{5.2.36}$$

where

$$\mathcal{F}_\varepsilon(v) = \int_\Omega |\nabla v|^2\, dx - 2\int_\Omega f v\, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} v^2\, d\mathcal{H}^{n-1},$$

and $f \in L^2(\Omega)$. The functions u_ε can be equivalently seen as the unique solutions to the equations

$$\int_\Omega \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \varepsilon \int_{\Sigma_\varepsilon} \nabla u_\varepsilon \cdot \nabla\varphi\, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon \varphi\, d\mathcal{H}^{n-1} = \int_\Omega f\varphi\, dx, \tag{5.2.37}$$

for every $\varphi \in H^1(\Omega_\varepsilon)$. We notice that for every $\varepsilon > 0$, by standard elliptic regularity, $u_\varepsilon \in H^2_{\mathrm{loc}}(\Omega_\varepsilon \setminus \partial\Omega)$.

Step 1: H^1 uniform estimates

We now estimate the H^1 norm of u_ε in terms of ε.

Lemma 5.2.13. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{0,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Then there exist positive constants $\varepsilon_0(\Omega)$, and $C(\Omega, \|h\|_{C^{0,1}}, \beta, f)$ such that if*

$$\varepsilon \|h\|_\infty \leq \varepsilon_0, \tag{5.2.38}$$

then for every $v \in H^1(\Omega_\varepsilon)$

$$\int_{\Omega_\varepsilon} v^2\,dx \leq C\left[\int_\Omega |\nabla v|^2\,dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\,dx + \beta \int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1}\right]. \tag{5.2.39}$$

Proof. Up to using a density argument, it is enough to prove the assertion for every smooth function in $H^1(\Omega_\varepsilon)$. Let $v \in C^1(\overline{\Omega_\varepsilon})$. For every $x \in \Sigma_\varepsilon$ we recall that we can represent

$$x = \pi_\Omega(x) + d(x)\nu_0(x),$$

and we define

$$\xi(x) := \pi_\Omega(x) + \varepsilon h(x)\,\nu_0(x).$$

This construction allows us to write for every $x \in \Sigma_\varepsilon$

$$v(x) = v(\xi(x)) - \int_{d(x)}^{\varepsilon h(x)} \frac{\partial}{\partial \nu_0} v(\pi_\Omega(x) + t\nu_0(x))\,dt.$$

Hence, by Young and Hölder inequalities,

$$v^2(x) \leq 2v^2(\xi(x)) + 2\varepsilon\|h\|_\infty \int_0^{\varepsilon h(x)} |\nabla v|^2(\pi_\Omega(x) + t\nu_0(x))\,dt. \tag{5.2.40}$$

Integrating over Σ_ε, using (5.2.23) and (5.2.24), and recalling that $\xi(x) = \xi(\pi_\Omega(x))$, we can find a positive constant $C = C(\Omega, \|h\|_{C^{0,1}})$ such that

$$\begin{aligned}\int_{\Sigma_\varepsilon} v^2\,dx \leq& C \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)} v^2(\xi(\sigma))\,ds\,d\mathcal{H}^{n-1}(\sigma) \\ &+ \varepsilon C \int_{\partial\Omega}\int_0^{\varepsilon h(\sigma)}\int_0^{\varepsilon h(\sigma)} |\nabla v|^2(\sigma + t\nu_0)\,dt\,ds\,d\mathcal{H}^{n-1}(\sigma) \\ \leq& \varepsilon C\left(\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\,dx\right).\end{aligned} \tag{5.2.41}$$

Similarly, we can integrate (5.2.40) over $\partial\Omega$ and have

$$\int_{\partial\Omega} v^2\,d\mathcal{H}^{n-1} \leq C\left(\int_{\partial\Omega_\varepsilon} v^2\,d\mathcal{H}^{n-1} + \varepsilon \int_{\Sigma_\varepsilon} |\nabla v|^2\,dx\right). \tag{5.2.42}$$

From the Poincaré inequality with trace term in Ω and the Bossel-Daners inequality, we have

$$\int_\Omega v^2\,dx \leq C_p(|\Omega|)\left(\int_\Omega |\nabla v|^2\,dx + \int_{\partial\Omega} v^2\,d\mathcal{H}^{n-1}\right), \tag{5.2.43}$$

so that joining (5.2.41) and (5.2.43), and using (5.2.42) we have the assertion.

□

Corollary 5.2.14. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, and fix a positive function $h \in C^{0,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$. Then there exists a positive constant $C = C(\Omega, h, \beta, f)$ such that if $u_\varepsilon \in H^1(\Omega_\varepsilon)$ is the minimizer to (5.2.36), then*

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, d\mathcal{H}^{n-1} \le C, \tag{5.2.44}$$

and

$$\int_{\Omega_\varepsilon} u_\varepsilon^2\, dx \le C. \tag{5.2.45}$$

Proof. For every $\eta > 0$, we can write

$$\int_\Omega |\nabla u_\varepsilon|^2\, dx + \varepsilon \int_{\Sigma_\varepsilon} |\nabla u_\varepsilon|^2\, dx + \beta \int_{\partial\Omega_\varepsilon} u_\varepsilon^2\, d\mathcal{H}^{n-1} \le \mathcal{F}_\varepsilon(0) + 2\int_\Omega f u_\varepsilon\, dx$$

$$\le \eta \int_\Omega f^2\, dx + \frac{1}{\eta}\int_\Omega u_\varepsilon^2\, dx.$$

Using (5.2.39), for a suitable choice of η, we get the result. □

Flattening the boundaries

Here we aim to construct a flattening diffeomorphism that locally transforms $\partial\Omega$ and $\partial\Omega_\varepsilon$ in subsets of parallel planes. To do so, we have to represent locally $\partial\Omega$ and $\partial\Omega_\varepsilon$.

Lemma 5.2.15 (Uniform local representation of $\partial\Omega$ and $\partial\Omega_\varepsilon$.)**.** *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h \in C^{0,1}(\Gamma_{d_0})$ such that $\nabla h \cdot \nu_0 = 0$, and let $\sigma_0 \in \partial\Omega$. There exists $\varepsilon_0 = \varepsilon_0(\Omega, \sigma_0)$ such that, if*

$$\varepsilon \|h\|_{C^{0,1}} < \varepsilon_0,$$

then there exist an open set V containing σ_0 and $\sigma_0 + \varepsilon h(\sigma_0)\nu_0(\sigma_0)$, and there exist functions $g, k_\varepsilon \colon \mathbb{R}^{n-1} \to \mathbb{R}$ such that, up to a rototranslation,

$$\Omega \cap V = \{\, (x', x_n) \mid x_n \le g(x') \,\} \cap V,$$
$$\Omega_\varepsilon \cap V = \{\, (x', x_n) \mid x_n \le g(x') + \varepsilon k_\varepsilon(x') \,\} \cap V,$$

and

$$\partial\Omega \cap V = \{\, (x', x_n) \mid x_n = g(x') \,\} \cap V,$$
$$\partial\Omega_\varepsilon \cap V = \{\, (x', x_n) \mid x_n = g(x') + \varepsilon k_\varepsilon(x') \,\} \cap V,$$

Proof. Without loss of generality we can assume that $\nu_0(\sigma_0) = \mathbf{e}_n$ and $\sigma_0 = 0$. We already know by definition that Ω can be represented locally near 0, that is, there exist a neighborhood U of σ_0 and a function $g \in C^{1,1}(\mathbb{R}^{n-1})$ with $g(0) = 0$ and $\nabla g(0) = 0$, such that

$$\Omega \cap U = \{\, (x', x_n) \in \mathbb{R}^n \mid x_n \le g(x') \,\} \cap U.$$

For every $r \in (0, 1)$ let

$$B'_r = \left\{ x' \in \mathbb{R}^{n-1} \,\middle|\, |x'| \le r \right\},$$

and $V_r = B'_r \times [-2r, 2r]$. For every $x' \in B'_r$ let

$$F_{x'} : t \in (g(x'), 2r] \longmapsto d(x', t) - \varepsilon h(x', t),$$

where we recall that $d(x)$ denotes the distance from Ω. Let us also recall that

$$\Sigma_\varepsilon = \Omega_\varepsilon \setminus \overline{\Omega} = \{\, x \in \mathbb{R}^n \mid 0 < d(x) < \varepsilon h(x) \,\}\,.$$

The definition of $F_{x'}$ gives us the possibility to characterize the property $(x', t) \in \Omega_\varepsilon$ in the equivalent way $F_{x'}(t) < 0$. As an immediate consequence, $F_{x'}(g(x')) < 0$ for every $x' \in B'_r$.

The idea of the proof can be divided in two main steps: first we show that for a right choice of ε_0 the set $\partial\Omega_\varepsilon$ cannot touch the upper base of the cylinder V_r by proving $F_{x'}(2r) > 0$ for every $x' \in B'_r$; next, we show that for a right choice of ε_0 we have that $\partial\Omega_\varepsilon$ can be represented as a graph over the whole B'_r by showing that $F_{x'}$ is strictly increasing in t for every $x' \in B'_r$

Let us choose $r = r(\sigma_0, \Omega)$ small enough so that we have $V_r \subseteq U$ and, thanks to the continuity of ν_0 in Γ_{d_0}

$$|\nu_0(x) - \mathbf{e}_n| < \frac{1}{2}, \tag{5.2.46}$$

for every $x \in V_r \cap \Gamma_{d_0}$. Note in addition that up to choosing a smaller r we can assume the vertical distance between $\partial\Omega$ and the upper base of V_r being greater than r, namely

$$2r - \max_{x' \in B'_r} g(x') > r. \tag{5.2.47}$$

Indeed, since ∇g is continuous and $\nabla g(0) = 0$, then for small enough r we also have, for every $x' \in B'_r$,

$$|\nabla g(x')| < 1,$$

which joint with the fact that $g(0) = 0$, ensures (5.2.47).

Let $x' \in B'_r$, fix $t \in (g(x'), 2r]$, and let $x = (x', t)$. We claim that under the assumption (5.2.46), we have

$$d(x) > \frac{t - g(x')}{2}. \tag{5.2.48}$$

Indeed, let $\bar{x}$ denote the point at distance $d(\bar{x}) = t - g(x')$ whose projection onto Ω is $(x', g(x'))$, namely

$$\overline{x} = (x', g(x')) + (t - g(x'))\nu_0(x', g(x')).$$

We have

$$|x - \overline{x}| = (t - g(x'))\, |\mathbf{e}_n - \nu_0(x', g(x'))| < \frac{t - g(x')}{2},$$

and, using the fact that the distance d is Lipschitzian with constant 1,

$$\begin{aligned} d(x) &\geq d(\overline{x}) - |d(x) - d(\overline{x})| \\ &\geq d(\overline{x}) - |x - \overline{x}| > \frac{t - g(x')}{2}. \end{aligned}$$

We now join (5.2.48), (5.2.47), and $\varepsilon \|h\|_\infty < \varepsilon_0$ to get that for $\varepsilon_0 < r/2$

$$F_{x'}(2r) > \frac{2r - g(x')}{2} - \varepsilon_0 > 0, \tag{5.2.49}$$

which concludes the first step.

We now prove that $F_{x'}(t)$ is monotone increasing in t. Indeed, by the assumption (5.2.46), we have that

$$\nu_0(x) \cdot \mathbf{e}_n > \frac{1}{2},$$

and

$$|\nabla h(x) \cdot \mathbf{e}_n| = |\nabla h(x) \cdot (\mathbf{e}_n - \nu_0(x))| \leq \frac{\|\nabla h\|_\infty}{2},$$

so that, since $r < 1$ and $\varepsilon_0 < r/2$, we have $\varepsilon \|\nabla h\|_\infty < 1/2$, which ensures

$$\frac{d}{dt} F_{x'}(t) = \nu_0(x', t) \cdot \mathbf{e}_n - \varepsilon \nabla h(x', t) \cdot \mathbf{e}_n > \frac{1}{4}. \tag{5.2.50}$$

Joining (5.2.49) and (5.2.50), we get that for every $x' \in B'_r$ there exists a unique $t(x') \in (g(x'), 2r)$ such that $(x', t) \in \Omega_\varepsilon \cap V_r$ if and only if $t \leq t(x')$, thus concluding the proof. □

Thanks to Lemma 5.2.15, we can now represent both $\partial\Omega$ and $\partial\Omega_\varepsilon$ as graphs, uniformly in ε, and flatten the boundaries of Ω and Ω_ε. We define the invertible map

$$\Phi_{\sigma_0} \colon (x', x_n) \in V \mapsto \left(x', \frac{x_n - g(x')}{k_\varepsilon(x')} \right) \in \tilde{V},$$

where $\tilde{V} = \Phi_{\sigma_0}(V)$. For simplicity's sake, when possible, we will drop the explicit dependence on the point $\sigma_0 \in \partial\Omega$. Notice that the map Φ indeed flattens the boundaries of Ω and Ω_ε, in the sense that

$$\Phi(\partial\Omega \cap V) = \{\, y_n = 0 \,\} \cap \tilde{V},$$

and

$$\Phi(\partial\Omega_\varepsilon \cap V) = \{\, y_n = \varepsilon \,\} \cap \tilde{V}.$$

For every $\delta > 0$ we define the cube

$$\tilde{Q}_\delta = \{\, y \in \mathbb{R}^n \mid |y_i| \leq \delta, \text{ for every } 1 \leq i \leq n \,\}.$$

Up to choosing a smaller neighborhood V of σ_0, we may assume that

$$\tilde{V} = \Phi(V) = \tilde{Q}_{\delta_0}$$

for some $\delta_0 = \delta_0(\sigma_0, \Omega) > 0$. For every $0 < \delta < \delta_0$ we define

$$Q_\delta = \Phi^{-1}(\tilde{Q}_\delta).$$

Remark 5.2.16 (Estimates for k_ε, Φ, and Φ^{-1})**.** We claim that there exists a positive constant $C = C(\Omega, \|h\|_{C^{1,1}})$ such that

$$\min k_\varepsilon \geq \frac{1}{C}, \qquad \|k_\varepsilon\|_{C^{1,1}} + \|\Phi\|_{C^{1,1}} + \|\Phi^{-1}\|_{C^{1,1}} \leq C. \tag{5.2.51}$$

Notice that, since $\Omega \subset \Omega_\varepsilon$ for every $\varepsilon > 0$, then k_ε is non negative. Moreover, for every $x = (x', x_n) \in \partial\Omega_\varepsilon \cap V$

$$\varepsilon k_\varepsilon(x') \geq d(x, \partial\Omega) \geq \varepsilon \min_{\Gamma_{d_0}} h,$$

so that k_ε is strictly bounded from below uniformly in ε. Similarly, k_ε is bounded in $C^{1,1}$ norm uniformly in ε, indeed for every $x \in \partial\Omega_\varepsilon \cap V$, we can write $x = (x', g(x') + \varepsilon k_\varepsilon(x'))$, and we have that

$$d(x, \partial\Omega_\varepsilon) = 0.$$

Differentiating with respect to x' we get

$$\nu_\varepsilon' + \nu_\varepsilon \cdot \mathbf{e}_n(\nabla g + \varepsilon \nabla k_\varepsilon) = 0, \tag{5.2.52}$$

where $\nu_\varepsilon' \in \mathbb{R}^{n-1}$ is the vector whose components are the first $n-1$ components of ν_ε. Using the fact that

$$\nu_\varepsilon = \frac{\nu_0 - \varepsilon\nabla h}{\sqrt{1 + \varepsilon^2 |\nabla h|^2}},$$

and that in V

$$\nu_0 = \left(-\frac{\nabla g}{\sqrt{1 + |\nabla g|^2}}, \frac{1}{\sqrt{1 + |\nabla g|^2}} \right),$$

we can rewrite equation (5.2.52) as

$$\nabla k_\varepsilon = \frac{(\nabla h)' + (\nabla h \cdot \mathbf{e}_n)\nabla g}{(\nu_0 - \varepsilon \nabla h) \cdot \mathbf{e}_n}. \tag{5.2.53}$$

Finally, the choice of ε_0 in Lemma 5.2.15 (see (5.2.50)) ensures us that

$$(\nu_0 - \varepsilon\nabla h) \cdot \mathbf{e}_n > \frac{1}{4}, \tag{5.2.54}$$

so that from equation (5.2.53) and (5.2.54), (5.2.51) follows.

Remark 5.2.17 (The equation in the flattened set)**.** Let u_ε be the solution to (5.2.37), fix $\sigma_0 \in \partial\Omega$, and let

$$v(y) = u_\varepsilon(\Phi^{-1}(y)),$$

then

$$v \in H^1(\{\, y_n < \varepsilon \,\} \cap \tilde V) \cap H^2_{\mathrm{loc}}\Big((\{\, y_n < \varepsilon \,\} \cap \tilde V) \setminus \{\, y_n = 0 \,\} \Big)$$

and, for all $\varphi \in H^1_0(\tilde V)$, v solves the equation

$$\int_{\{y_n<\varepsilon\}\cap\tilde V} \varepsilon(y_n) A_\varepsilon \nabla v \cdot \nabla\varphi \, dy + \beta \int_{\{y_n=\varepsilon\}\cap\tilde V} v\varphi J_\varepsilon \, d\mathcal{H}^{n-1} = \int_{\{y_n<0\}\cap\tilde V} \tilde f_\varepsilon \varphi \, dy, \tag{5.2.55}$$

where

$$\varepsilon(y_n) = \begin{cases} \varepsilon & \text{if } y_n > 0, \\ 1 & \text{if } y_n \le 0, \end{cases}$$

$$A_\varepsilon(y) = k_\varepsilon(y')(D(\Phi^{-1})(y))^{-1}(D(\Phi^{-1})(y))^{-T}, \qquad \tilde f_\varepsilon(y) = f(\Phi^{-1}(y))\, k_\varepsilon(y'),$$

and

$$J_\varepsilon(y) = \sqrt{1 + |\nabla g(y') + \varepsilon\nabla k_\varepsilon(y')|^2}.$$

Notice that A_ε is elliptic and bounded, uniformly in y and ε. Moreover, using (5.2.44) in Corollary 5.2.14, we also get that there exists a positive constant $C = C(\Omega, h, \beta, f, \sigma_0)$ such that

$$\int_{\{\, y_n<\varepsilon \,\}\cap\tilde V} \varepsilon(y_n)|\nabla v|^2 \, dy + \beta \int_{\{y_n=\varepsilon\}\cap\tilde V} v^2 \, d\mathcal{H}^{n-1} \le C. \tag{5.2.56}$$

Step 2: H^2 uniform estimates

Since we aim to prove H^2 estimates with a local approach, we define the energy quantities I_δ and $\tilde{I}_\delta$ as follows: given a function

$$\varphi \in H^1(\{ y_n < \varepsilon \} \cap \tilde{V}) \cap H^2\Big((\{ y_n < \varepsilon \} \cap \tilde{V}) \setminus \{ y_n = 0 \}\Big)$$

and $0 < \delta < \delta_0$, we denote by

$$\tilde{I}_{\delta,\sigma_0}(\varphi) = \int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_\delta} \varepsilon(y_n) |D^2 \varphi|^2 \, dy + \beta \int_{\{y_n = \varepsilon\} \cap \tilde{Q}_\delta} |\nabla_{n-1} \varphi|^2 \, d\mathcal{H}^{n-1}, \tag{5.2.57}$$

where

$$\nabla_{n-1}\varphi = \nabla\varphi - \frac{\partial \varphi}{\partial y_n} \mathbf{e}_n,$$

and, as in Remark 5.2.17,

$$\varepsilon(t) = \begin{cases} \varepsilon & \text{if } t > 0, \\ 1 & \text{if } t \leq 0. \end{cases}$$

Analogously in Ω_ε, for every

$$\varphi \in H^1(\Omega_\varepsilon \cap V) \cap H^2\Big((\Omega_\varepsilon \cap V) \setminus \partial\Omega\Big),$$

we let

$$I_{\delta,\sigma_0}(\varphi) = \int_{\Omega_\varepsilon \cap Q_\delta} \varepsilon(d(x)) |D^2 \varphi|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon \cap Q_\delta} |\nabla^{\partial\Omega_\varepsilon} \varphi|^2 \, d\mathcal{H}^{n-1}, \tag{5.2.58}$$

where we recall that

$$\nabla^{\partial\Omega_\varepsilon}\varphi = \nabla\varphi - (\nabla\varphi \cdot \nu_\varepsilon)\nu_\varepsilon.$$

When possible, we will drop the dependence on σ_0. Uniform bounds for I_δ can be read as uniform bounds for $\tilde{I}_\delta$ and viceversa. Indeed, we have the following

Lemma 5.2.18. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\varepsilon \|h\|_{C^{0,1}} \leq \varepsilon_0$ and $\nabla h \cdot \nu_0 = 0$. Then, for every $\sigma_0 \in \partial\Omega$, there exists a positive constant $C = C(\Omega, \|h\|_{C^{1,1}}, \sigma_0)$ such that if*

$$u \in H^1(Q_\delta) \cap H^2(Q_\delta \setminus \partial\Omega),$$

and

$$v(y) = u(\Phi^{-1}(y)),$$

then, for every $0 < \delta < \delta_0$,

$$I_\delta(u) \leq C\left(\tilde{I}_\delta(v) + \int_{\{y_n < \varepsilon\} \cap \tilde{Q}_\delta} \varepsilon(y_n) |\nabla v|^2 \, dy\right), \tag{5.2.59}$$

and

$$\tilde{I}_\delta(v) \leq C\left(I_\delta(u) + \int_{\Omega_\varepsilon \cap Q_\delta} \varepsilon(d(x)) |\nabla u|^2 \, dx\right). \tag{5.2.60}$$

Proof. We start by evaluating the trace term in the definition of $\tilde{I}_\delta$. By means of the change of variables $y = \Phi(x)$, we have

$$\int_{\{y_n=\varepsilon\}\cap\tilde{Q}_\delta} |\nabla_{n-1} v|^2 \, d\mathcal{H}^{n-1}(y) = \sum_{i=1}^{n-1} \int_{\partial\Omega_\varepsilon\cap V} (\nabla u \cdot w_i)^2 \, \mathrm{Jac}^{\partial\Omega_\varepsilon} \Phi \, d\mathcal{H}^{n-1}(x), \tag{5.2.61}$$

where

$$w_i = \mathbf{e}_i + (\partial_i g + \varepsilon \partial_i k_\varepsilon) \, \mathbf{e}_n.$$

The vectors w_i (that, in general, could be non-orthogonal) form a basis for the tangent plane $T_{\sigma_0}\partial\Omega_\varepsilon$. In particular, we have that there exists a positive constant $C = C(\Omega, h, \sigma_0)$ such that

$$|\nabla^{\partial\Omega_\varepsilon} u|^2 \le C \sum_{i=1}^{n-1} (\nabla u \cdot w_i)^2 \le C^2 |\nabla^{\partial\Omega_\varepsilon} u|^2. \tag{5.2.62}$$

Therefore, using the uniform bounds (5.2.51) we get that for some positive constant $C = C(\Omega, h)$

$$\frac{1}{C} \le \mathrm{Jac}^{\partial\Omega_\varepsilon} \Phi = \frac{1}{\sqrt{1 + |\nabla g + \varepsilon \nabla k_\varepsilon|^2}} \le C, \tag{5.2.63}$$

and joining (5.2.61), (5.2.62), and (5.2.63), we get

$$\int_{\partial\Omega_\varepsilon\cap Q_\delta} |\nabla^{\partial\Omega_\varepsilon} \varphi|^2 \, d\mathcal{H}^{n-1} \le C \int_{\{y_n=\varepsilon\}\cap\tilde{Q}_\delta} |\nabla_{n-1} v|^2 \, d\mathcal{H}^{n-1} \le C^2 \int_{\partial\Omega_\varepsilon\cap Q_\delta} |\nabla^{\partial\Omega_\varepsilon} \varphi|^2 \, d\mathcal{H}^{n-1}. \tag{5.2.64}$$

For what concerns the second order term, we evaluate for a.e. $y \in \{y_n < \varepsilon\} \cap \tilde{V}$,

$$\left(D^2 v(y)\right)_{ij} = \sum_{k=1}^{n} \frac{\partial^2 (\Phi^{-1})_k}{\partial y_i \, \partial y_j} \frac{\partial u}{\partial x_k} + \sum_{k,l=1}^{n} \frac{\partial (\Phi^{-1})_k}{\partial y_i} \frac{\partial^2 u}{\partial x_k \, \partial x_l} \frac{\partial (\Phi^{-1})_l}{\partial y_j},$$

so that, for some positive constant $C = C(n, \|\Phi^{-1}\|_{C^{1,1}})$,

$$|D^2 v(\Phi(x))|^2 \le C\Big(|\nabla u(x)|^2 + |D^2 u(x)|^2\Big). \tag{5.2.65}$$

In a similar way we have that, for a.e. $x \in \Omega_\varepsilon \cap V$, and for some positive constant $C = C(n, \|\Phi\|_{C^{1,1}})$,

$$|D^2 u(\Phi^{-1}(y))|^2 \le C\Big(|\nabla v(y)|^2 + |D^2 v(y)|^2\Big). \tag{5.2.66}$$

Using (5.2.65) and the uniform bounds on k_ε, we get

$$\begin{aligned} \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_\delta} \varepsilon(y_n) |D^2 v|^2 \, dy &= \int_{\Omega_\varepsilon\cap Q_\delta} \frac{\varepsilon(d(x))}{k_\varepsilon} |D^2 v(\Phi(x))|^2 \, dx \\ &\le C \int_{\Omega_\varepsilon\cap Q_\delta} \varepsilon(d(x)) (|D^2 u|^2 + |\nabla u|^2) \, d\mathcal{H}^{n-1}(x). \end{aligned} \tag{5.2.67}$$

Analogously, using (5.2.66),

$$\int_{\Omega_\varepsilon\cap Q_\delta} \varepsilon(d(x)) |D^2 u|^2 \, dx \le C \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_\delta} \varepsilon(y_n) \Big(|D^2 v|^2 + |\nabla v|^2\Big) \, d\mathcal{H}^{n-1}. \tag{5.2.68}$$

The result now follows by joining (5.2.64), (5.2.68), and (5.2.67). □

Lemma 5.2.19. *Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with $C^{1,1}$ boundary, fix a positive function $h \in C^{1,1}(\Gamma_{d_0})$ such that $\varepsilon\|h\|_{C^{0,1}} \leq \varepsilon_0$ and $\nabla h \cdot \nu_0 = 0$. If $\sigma_0 \in \partial\Omega$, and v is as in Remark 5.2.17, then*

$$v \in H^2\Big(\{\, y_n < \varepsilon \,\} \cap \tilde{Q}_{\delta_0/2} \setminus \{\, y_n = 0 \,\}\Big),$$

and there exists a positive constant $C = C(\Omega, \|h\|_{C^{1,1}}, \beta, f, \sigma_0)$ such that

$$\tilde{I}_{\delta_0/2}(v) \leq C. \tag{5.2.69}$$

Proof. Let $\xi \in C_c^\infty(\tilde{Q}_{\delta_0})$ be a non negative function with $\xi \equiv 1$ in $\tilde{Q}_{\delta_0/2}$. For $|\eta|$ small enough, we have

$$\operatorname{supp} \xi + \eta \mathbf{e}_i \subset\subset \tilde{Q}_{\delta_0}$$

for every $k = 1, \dots, n-1$. In the following, for any L^2 function ψ, we denote by

$$\Delta_k^\eta \psi(x) = \frac{\psi(x + \eta \mathbf{e}_k) - \psi(x)}{\eta}$$

the difference quotients (see for instance [88, §5.8.2]). We recall that for any couple of functions ψ_1 and ψ_2 we have

$$\Delta_k^\eta(\psi_1\psi_2)(x) = \Delta_k^\eta\psi_1(x)\psi_2(x) + \psi_1(x + \eta\mathbf{e}_k)\Delta_k^\eta\psi_2(x),$$

and if ψ_1 and ψ_2 are measurable and

$$(\operatorname{supp}\psi_1 \cap \operatorname{supp}\psi_2) \pm \eta\mathbf{e}_k \subset\subset \tilde{Q}_\delta,$$

then for every $k = 1, \dots, n-1$, it holds

$$\int_{\tilde{Q}_{\delta_0}} \psi_1(y)\Delta_k^\eta\psi_2(y)\, dy = -\int_{\tilde{Q}_{\delta_0}} \Delta_k^{-\eta}\psi_1(y)\, \psi_2(y)\, dy. \tag{5.2.70}$$

Moreover, we recall that if $\psi \in H^1(\{y_n < \varepsilon\} \cap \tilde{Q}_{\delta_0})$, then for every $U \subset\subset \tilde{Q}_{\delta_0}$ we have that

$$\int_U (\Delta_k^\eta \psi)^2\, dy \leq \int_{\tilde{Q}_{\delta_0}} |\nabla\psi|^2\, dy, \tag{5.2.71}$$

for every $|\eta| < d(U, \partial\tilde{Q}_{\delta_0})$. We aim to prove some uniform L^2 estimates for $\Delta_k^\eta(\nabla v)$, which will imply weak differentiability and uniform L^2 estimates for $\partial_k \nabla v$.

Given the equation (5.2.55), we use as a test function

$$\varphi = -\Delta_k^{-\eta}(\xi^2 \Delta_k^\eta v).$$

For small $|\eta|$, the function φ is admissible. Here we recall the weak equation

$$\underbrace{\int_{\{y_n<\varepsilon\}\cap\tilde{V}} \varepsilon(y_n) A_\varepsilon \nabla v \cdot \nabla\varphi\, dy}_{\mathcal{I}_1} + \underbrace{\beta\int_{\{y_n=\varepsilon\}\cap\tilde{V}} v\varphi J_\varepsilon\, d\mathcal{H}^{n-1}}_{\mathcal{I}_2} = \underbrace{\int_{\{y_n<0\}\cap\tilde{V}} \tilde{f}_\varepsilon \varphi\, dy}_{\mathcal{I}_3}.$$

We now estimate the integrals separately. In the following, we use the Einstein notation on repeated indices, where $i, j = 1, \dots, n$, and for simplicity we drop the dependence on ε in A_ε, whose components are denoted by a_{ij}. We recall that $a_{ij} = a_{ij}(y)$ and $J_\varepsilon = J_\varepsilon(y)$. When necessary, we will also write $a^\eta_{ij} := a_{ij}(y + \eta\mathbf{e}_k)$ and $J^\eta_\varepsilon := J_\varepsilon(y + \eta\mathbf{e_k})$.

Let us start with the higher-order term $\mathcal{I}_1$: by direct computation, we get

$$\begin{aligned} A\nabla v \cdot \nabla\varphi &= -a_{ij}\, \partial_j v\, \Delta_k^{-\eta}\, \partial_i(\xi^2 \Delta_k^\eta v) \\ &= -a_{ij}\, \partial_j v \left[\Delta_k^{-\eta}(\partial_i(\xi^2)\, \Delta_k^\eta v) + \Delta_k^{-\eta}(\xi^2 \Delta_k^\eta(\partial_i v))\right] \end{aligned}$$

Multiplying by $\varepsilon(y_n)$, integrating over $\{y_n < \varepsilon\} \cap \tilde{Q}_{\delta_0}$, and using property (5.2.70), we get (notice that $\varepsilon(y_n)$ is constant along the direction $\mathbf{e}_k$)

$$\begin{aligned} \mathcal{I}_1 &= \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\Delta_k^\eta(a_{ij}\, \partial_j v)\, \partial_i(\xi^2)\Delta_k^\eta v\, dy \\ &\quad + \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\Delta_k^\eta(a_{ij}\partial_j v)\, \xi^2\, \Delta_k^\eta(\partial_i v)\, dy \\ &= \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2\, a^\eta_{ij}\, \Delta_k^\eta(\partial_j v)\, \Delta_k^\eta(\partial_i v)\, dy + R_1 \\ &\geq C_1 \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v)\right)^2 dy - |R_1|, \end{aligned} \tag{5.2.72}$$

where C_1 is the ellipticity constant of A estimated uniformly in Remark 5.2.17, and

$$\begin{aligned} R_1 = 2&\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\Big(\Delta_k^\eta(a_{ij})\partial_j v + a^\eta_{ij}\, \Delta_k^\eta(\partial_j v)\Big)\, \xi\, \partial_i\xi\, \Delta_k^\eta v\, dy \\ &+ \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\Delta_k^\eta(a_{ij})\partial_j v\, \xi^2\, \Delta_k^\eta(\partial_j v)\, dy. \end{aligned}$$

Using Young's inequality

$$ab \leq \frac{\lambda}{2}a^2 + \frac{1}{2\lambda}b^2 \qquad \forall a, b \geq 0,$$

with a suitable choice of $\lambda > 0$, and using the uniform bounds on $\|\xi\|_{C^1}$ and $\|A_\varepsilon\|_{C^1}$ (see Remark 5.2.17 and Remark 5.2.16), we may obtain a positive constant $C = C(A, \xi)$ such that the following estimate holds

$$\begin{aligned} |R_1| \leq \frac{C_1}{2} &\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v)\right)^2 dy \\ &+ C \int_{\{y_n<\varepsilon\}\cap\operatorname{supp}\xi} \varepsilon(y_n)\left((\Delta_k^\eta v)^2 + \sum_{j=1}^n (\partial_j v)^2\right) dy. \end{aligned}$$

This estimate, joint with (5.2.72), implies

$$\mathcal{I}_1 \geq \frac{C_1}{2} \int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n \left(\Delta_k^\eta(\partial_j v)\right)^2 dy - C\int_{\{y_n<\varepsilon\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)|\nabla v|^2\, dy. \tag{5.2.73}$$

Similarly, let us work with the boundary terms $\mathcal{I}_2$, where we have

$$v\varphi J_\varepsilon = -vJ_\varepsilon \Delta_k^{-\eta}(\xi^2\Delta_k^\eta v).$$

Since $k \neq n$, we can use an analogous of property (5.2.70) also on $\{\, y_n = \varepsilon \,\} \cap \tilde{Q}_{\delta_0}$, so that

$$\begin{aligned}
\mathcal{I}_2 &= \beta \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 \Delta_k^\eta(vJ_\varepsilon)\Delta_k^\eta v \, d\mathcal{H}^{n-1} \\
&= \beta \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 (\Delta_k^\eta v)^2 J_\varepsilon^\eta \, d\mathcal{H}^{n-1} + R_2 \\
&\geq \beta C_2 \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 (\Delta_k^\eta v)^2 \, d\mathcal{H}^{n-1} - |R_2|,
\end{aligned} \tag{5.2.74}$$

where $C_2 = \inf_{Q_\delta} J_\varepsilon$ uniformly estimated in Remark 5.2.17, and

$$R_2 = \beta \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 \Delta_k^\eta(J_\varepsilon) v \Delta_k^\eta v \, d\mathcal{H}^{n-1}$$

As done for $\mathcal{I}_1$, using Young's inequality and the uniform bounds on $\|J_\varepsilon\|_{C^1}$ (see Remark 5.2.17 and Remark 5.2.16), we have that for some positive constant $C = C(J_\varepsilon, \xi)$,

$$|R_2| \leq \frac{\beta C_2}{2} \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 (\Delta_k^\eta v)^2 \, d\mathcal{H}^{n-1} + C\beta \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1}$$

This estimate, joint with (5.2.74) ensures

$$\mathcal{I}_2 \geq \frac{\beta C_2}{2} \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 (\Delta_k^\eta v)^2 \, d\mathcal{H}^{n-1} - C\beta \int_{\{\, y_n=\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1}. \tag{5.2.75}$$

Finally, we estimate the source term $\mathcal{I}_3$, recalling that

$$\tilde{f}_\varepsilon \varphi = -\tilde{f}_\varepsilon \Delta_k^{-\eta}(\xi^2\Delta_k^\eta v).$$

Using Young's inequality and property (5.2.71) for $\psi = \xi^2\Delta_k^\eta v$ (notice that $\nabla\Delta_k^\eta v = \Delta_k^\eta(\nabla v)$), then for a suitable positive constant $C = C(\tilde{f}, \xi, \delta_0)$,

$$\begin{aligned}
\mathcal{I}_3 \leq{} & \frac{C_1}{4} \int_{\{\, y_n<0 \,\}\cap\tilde{Q}_{\delta_0}} \xi^2 \sum_{j=1}^n (\Delta_k^\eta \partial_j v)^2 \, dy \\
& + C \int_{\{\, y_n<0 \,\}\cap\tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy + C \int_{\{\, y_n<0 \,\}\cap\tilde{Q}_{\delta_0}} |\nabla v|^2 \, dy \\
\leq{} & \frac{C_1}{4} \int_{\{\, y_n<\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)\xi^2 \sum_{j=1}^n (\Delta_k^\eta \partial_j v)^2 \, dy \\
& + C \int_{\{\, y_n<0 \,\}\cap\tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy + C \int_{\{\, y_n<\varepsilon \,\}\cap\tilde{Q}_{\delta_0}} \varepsilon(y_n)|\nabla v|^2 \, dy.
\end{aligned} \tag{5.2.76}$$

We can now turn back to the equation

$$\mathcal{I}_1 + \mathcal{I}_2 = \mathcal{I}_3$$

Joining (5.2.73), (5.2.75), (5.2.76), and using the fact that $\xi = 1$ in $\tilde{Q}_{\delta_0/2}$, then, for every $k = 1, \dots, n-1$,

$$\begin{aligned}\mathcal{I}_4 := \frac{C_1}{4} &\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} \varepsilon(y_n) \sum_{j=1}^{n} (\Delta_k^\eta \partial_j v)^2 \, dy \\ &+ \frac{\beta C_2}{2} \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} (\Delta_k^\eta v)^2 \, d\mathcal{H}^{n-1} \\ &\leq C \Big(\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) |\nabla v|^2 \, dy \\ &\qquad + \beta \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1} + \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy \Big).\end{aligned} \tag{5.2.77}$$

Since we have assumed the uniform H^1 estimates (5.2.44), and we have uniform estimates for $\tilde{f}_\varepsilon$, computed in Remark 5.2.17, we get that for some positive constant $C = C(\delta_0, \Omega, \xi)$

$$\mathcal{I}_4 \leq C,$$

which implies that for every $k = 1, \dots, n-1$, we have $\partial_k v \in H^1(\{y_n < \varepsilon\} \cap \tilde{Q}_{\delta_0/2})$ and

$$\begin{aligned}\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0/2}} \varepsilon(y_n) \sum_{j=1}^{n} (\partial_k \partial_j v)^2 \, dy \leq C \Big(&\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_{\delta_0}} \varepsilon(y_n) |\nabla v|^2 \, dy \\ &+ \beta \int_{\{ y_n = \varepsilon \} \cap \tilde{Q}_{\delta_0}} v^2 \, d\mathcal{H}^{n-1} + \int_{\{ y_n < 0 \} \cap \tilde{Q}_{\delta_0}} \tilde{f}_\varepsilon^2 \, dy \Big).\end{aligned}$$

It only remains to get a uniform estimate for $\partial^2_{nn} v$. We notice that since v solves (5.2.55), then almost everywhere we have

$$- \sum_{\substack{1 \leq i,j \leq n \\ (i,j) \neq (n,n)}} \varepsilon(y_n) \partial_i (a_{ij} \partial_j v) - \varepsilon(y_n) \partial_n a_{nn} \partial_n v - \varepsilon(y_n) a_{nn} \partial^2_{nn} v = \tilde{f} \chi_{\{y_n < 0\}}.$$

In particular, the fact that $a_{nn} = k_\varepsilon$ is uniformly bounded from below gives estimates for $\partial^2_{nn} v$ in terms of A_ε, $\tilde{f}$, and the other derivatives of v. Therefore, so that, for every $0 < \delta < \delta_0$, we can find a positive constant $C = C(\Omega, h, f)$ such that

$$\begin{aligned}\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_\delta} \varepsilon(y_n) |\partial^2_{nn} v|^2 \leq C \Big(&\int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_\delta} \varepsilon(y_n) \sum_{k=1}^{n-1} |\nabla \partial_k v|^2 \, dy \\ &+ \int_{\{ y_n < \varepsilon \} \cap \tilde{Q}_\delta} \varepsilon(y_n) |\nabla v|^2 \, d\mathcal{H}^{n-1} + 1 \Big).\end{aligned} \tag{5.2.78}$$

Finally, (5.2.77), joint with (5.2.78) and the bound (5.2.56), gives (5.2.69).

□

Remark 5.2.20. Let $\sigma_0 \in \partial\Omega$, by the previous lemma we have that $u_\varepsilon \in H^2(Q_{\delta_0/2} \setminus \partial\Omega)$. Moreover, putting together estimates (5.2.59) and (5.2.60) and the bounds on the first derivative ((5.2.44) and (5.2.56)), we have

$$I_\delta(u_\varepsilon) \leq C(1 + \tilde{I}_\delta(v_\varepsilon)) \leq C^2(1 + I_\delta(u_\varepsilon)). \tag{5.2.79}$$

We can finally prove Theorem 5.2.9.

Proof of Theorem 5.2.9. We recall that we have defined $\tilde{I}_{\delta,\sigma}$ and $I_{\delta,\sigma}$ in (5.2.57) and (5.2.58) respectively. For every $\sigma_0 \in \partial\Omega$, using (5.2.69) from Lemma 5.2.19, an and (5.2.79) from Remark 5.2.20, we have that there exists a constant $C = C(\Omega, h, \sigma_0, \delta_0)$ such that

$$I_{\delta_0/2,\sigma_0}(u_\varepsilon) \leq C. \tag{5.2.80}$$

From the boundedness of Ω there exist $\sigma_1, \dots, \sigma_m \in \partial\Omega$ and associated $\delta_i = \delta(\sigma_i, \Omega)$ for which estimates of the type (5.2.80) hold and such that, choosing $\varepsilon_0 = \min\{\varepsilon_0(\sigma_1), \dots, \varepsilon_0(\sigma_m)\} > 0$, we have

$$\Sigma_\varepsilon \subset \bigcup_{i=1}^m \Phi_{\sigma_i}^{-1}(\tilde{Q}_{\delta_i/2}) = V_0.$$

Let U be an open set such that $\overline{U} \subset \Omega$ and $\Omega_\varepsilon \subset U \cup V_0$ for every ε such that $\varepsilon\|h\|_{C^{0,1}} < \varepsilon_0$. Also in U we may get an estimate analogous to (5.2.80). Indeed, by standard elliptic regularity and by estimate (5.2.45), we have that

$$\int_U |D^2 u_\varepsilon|^2 \, dx \leq C\Big(\int_\Omega f^2 \, dx + \int_\Omega u_\varepsilon^2\Big) \leq C. \tag{5.2.81}$$

Let

$$I(u_\varepsilon) = \int_\Omega |D^2 u_\varepsilon|^2 \, dx + \varepsilon \int_{\Sigma_\varepsilon} |D^2 u_\varepsilon|^2 \, dx + \beta \int_{\partial\Omega_\varepsilon} |\nabla^{\partial\Omega_\varepsilon} u_\varepsilon|^2 \, d\mathcal{H}^{n-1},$$

summing from $i = 1$ to m the estimates of the type (5.2.80) and (5.2.81), we have that there exists $C = C(\Omega, h, f)$ such that

$$I(u_\varepsilon) \leq C,$$

and the assertion is proven. $\square$

Chapter 6

Stability

6.1 Sharp quantitative Talenti estimates in some special cases

The results of this section are an extract from a work in progress [8].

In this section, we will investigate some stability versions of Talenti's inequality in particular cases. Given $\Omega \subset \mathbb{R}^d$ an open set of finite volume, we will denote by u_f the unique solution to

$$\left\{ \begin{array}{rcl} -\Delta u_f & = & f \text{ in } \Omega \\ u_f & \in & H_0^1(\Omega). \end{array} \right. \tag{6.1.1}$$

The aim of Talenti's inequality is to compare u_f and $u_{f^\sharp}$ where $f^\sharp : \Omega^\sharp \to \mathbb{R}$ is the Schwarz symmetrization of f, defined on $\Omega^\sharp$ the centered ball of same volume as Ω: more precisely Talenti's inequality states that if $f \geq 0$, then

$$\forall x \in \Omega^\sharp, \;\; u_f^\sharp(x) \leq v(x),$$

where $v = u_{f^\sharp}$ solves

$$\left\{ \begin{array}{ccc} -\Delta v = f^\sharp & \text{in} & \Omega^\sharp \\ v = 0 & \text{on} & \partial\Omega^\sharp. \end{array} \right. \tag{6.1.2}$$

and u_f^* is the Schwarz symmetrization of u_f. This pointwise comparison implies in particular the following: for every $p \in [1, +\infty]$,

$$\|u_f\|_{L^p(\Omega)} \leq \|v\|_{L^p(\Omega^\sharp)}. \tag{6.1.3}$$

Moreover, as shown in [15], equality is realized in (6.1.3) for some $p \in [1, +\infty]$ only if Ω is a ball and $f = f^\sharp$ up to translations, see also [118].

In this section we are interested in quantitative versions of (6.1.3). Notice that there are two parameters that are symmetrized in these inequalities: Ω and f. Therefore, the stability inequalities we may be seeking for should take into account both the distance from f to $f^\sharp$ and from Ω to $\Omega^\sharp$: to evaluate the asymmetry of Ω for example, we denote

$$\alpha(\Omega) = \min_{x_0 \in \mathbb{R}^n} \left\{ \frac{|\Omega \Delta B_r(x_0)|}{|B_r|}, \; |B_r| = |\Omega| \right\}$$

the Fraenkel asymmetry of Ω.

As far as we know, there are only two partial results in this direction:

1. in [120], the author focuses on the case $f \equiv 1$ and shows that for every $p \in [1, +\infty]$, there exists $c = c(n, p)$ such that for every Ω bounded open set in $\mathbb{R}^n$,
$$\|v\|^p_{L^p(\Omega^\sharp)} - \|u_1\|^p_{L^p(\Omega)} \geq c\alpha(\Omega)^{2+p} \text{ if } p < +\infty, \quad \text{and} \quad \|v\|_{L^\infty(\Omega^\sharp)} - \|u_1\|_{L^\infty(\Omega)} \geq c\alpha(\Omega)^3.$$

2. in [16] the authors focus on the case $p = \infty$: it is shown that for any Ω bounded open set of $\mathbb{R}^n$ and any nonnegative $f \in L^2(\Omega)$, there exists $c = c(n, |\Omega|, f^*)$ and $\theta = \theta(n)$ such that
$$c\left(\alpha(\Omega)^3 + \inf_{x_0 \in \mathbb{R}^n} \left\|f - f^*(\cdot + x_0)\right\|^\theta_{L^1(\mathbb{R}^n)}\right) \leq \|v - u^*\|_{L^\infty(\Omega^*)}.$$

In both of these results, the obtained exponent are likely non-sharp.

The goal of this section is to provide a first quantitative result for (6.1.3) with a sharp exponent. In order to start this investigation with only one parameter (similarly to [120] where the author assumed $f = 1$), we will assume that $\Omega = B_1$ is a centered ball of radius 1, so the only parameter is f. Moreover, we will assume f to be the characteristic function of a set $E \subset B_1$: this will allow us to use a geometric approach and the framework of shape derivatives. In this framework, we obtain the following stability result:

Theorem 6.1.1. *Let $p \in [2, +\infty)$ and $r \in (0, 1)$. Then there exists $c = c(p, r) > 0$ such that for every measurable set $E \subset B_1$ such that $|E| = |B_r|$ we have*
$$\|u_{B_r}\|_p - \|u_E\|_p \geq c|E\Delta B_r|^2.$$

The assumption $p \in [2, +\infty)$ can be weakened to $p \in [1, +\infty]$, but it requires some technicalities that we avoid to tackle in the present discussion. Moreover, we show in Section 6.1.8 that the exponent 2 obtained in the previous result is sharp.

In a forthcoming work [9] we will investigate the more general case where f is not assumed to be a characteristic function.

6.1.1 Strategy of the proof

Let $B_1 \subset \mathbb{R}^n$ be the ball of radius 1 centered in 0, let $m \in (0, |B_1|)$, and let us denote by
$$\mathcal{M}_m := \left\{ V \in L^\infty(B_1) \;\middle|\; \begin{array}{l} 0 \leq V \leq 1 \\ \int_{B_1} V = m \end{array} \right\}.$$

For every $V \in L^\infty(B_1)$, we define u_V to be the minimizer to
$$\min_{\varphi \in H^1_0(B_1)} \int_{B_1} |\nabla\varphi|^2\, dx - 2\int_{B_1} V\varphi\, dx,$$
or, equivalently, u_V is the solution in $H^1_0(B_1)$ to the weak equation
$$\int_{B_1} \nabla u_V \cdot \nabla\varphi\, dx = \int_{B_1} V\varphi\, dx \qquad \forall\varphi \in H^1_0(B_1). \tag{6.1.4}$$

Our aim is to study the functional
$$\mathcal{J}(V) := \int_{B_1} j(u_V)\, dx,$$

where $j(s) = |s|^p$ with $p \geq 2$. In particular, we are interested in the problem

$$\max_{V \in \mathcal{M}} \mathcal{J}(V). \tag{6.1.5}$$

As a consequence of convexity of $\mathcal{J}$ and linearity of equation (6.1.4), we get that a maximizer exists and it is bang-bang (see Proposition 6.1.13), namely

$$\mathcal{J}(V_0) = \max_{V \in \mathcal{M}} \mathcal{J}(V),$$

and V_0 is the characteristic function of some optimal set. Moreover, by the classical Talenti comparison, a radial maximizer exists, so that if we let $B_* := B_{r_*}$ be the centered ball such that $|B_*| = m$, then we have $V_0 = \chi_{B_*}$ (uniqueness is discussed in Proposition 6.1.14), and

$$\mathcal{J}(B_*) \geq \mathcal{J}(V) \qquad \forall V \in \mathcal{M}.$$

In this section we show that the maximizer B_* is stable, in the following sense.

Theorem 6.1.2. *Let $m \in (0, |B_1|)$, let $j(s) = |s|^p$ for some $p \geq 2$. Then there exists a positive constant $C = C(j, m)$ such that for every $V \in \mathcal{M}_m$ we have*

$$\mathcal{J}(V_0) - \mathcal{J}(V) \geq C\|V - V_0\|_1^2.$$

Corollary 6.1.3. *Let $p \in (1, +\infty)$, let $m \in (0, |B_1|)$, and let C be the constant in Theorem 6.1.2 when $j(s) = s^p$. Then for every measurable set $E \subset B_1$ such that $|E| = m$ we have*

$$\|u_0\|_p - \|u_E\|_p \geq \frac{C}{p\|u_0\|_p^{p-1}}|E\Delta B_*|^2.$$

Proof of Corollary 6.1.3. We know by Theorem 6.1.2 that

$$\|u_0\|_p^p\left(1 - \left(\frac{\|u_E\|_p}{\|u_0\|_p}\right)^p\right) \geq C|E\Delta B_*|^2.$$

Then the result follows by using the inequality

$$1 - x^p \leq p(1 - x) \qquad\qquad \forall x > 0.$$

$\square$

We list here the main steps of the proof.

- **Step 1:** we prove the existence of solutions to the two problems

  $$\max_{V \in \mathcal{M}_m} \mathcal{J}(V), \qquad\qquad \max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V),$$

  where

  $$\mathcal{M}_m^\delta = \{\, V \in \mathcal{M}_m \mid \|V - V_0\|_1 = \delta \,\}.$$

  In particular, we show that for every $\delta > 0$ there exists a unique set E_δ such that $\chi_{E_\delta} \in \mathcal{M}_m^\delta$ and

  $$\mathcal{J}(V) \leq \mathcal{J}(E_\delta) \qquad \forall V \in \mathcal{M}_m$$

  (see Proposition 6.1.13). Moreover, we show that the maximizer of $\mathcal{J}$ on $\mathcal{M}_m$ is unique.

- **Step 2:** we compute the Lagrangian $\mathcal{L}_\tau$ relative to the volume constraint (see Remark 2.6.3 for the definition of *Lagrangian relative to the volume constraint*), and we show that the second order shape derivative of $\mathcal{L}_\tau$ in B_* is coercive. Namely, we prove that if $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$ and

$$\int_{\partial B_*} (\Phi \cdot \nu)\, d\mathcal{H}^{n-1} = 0,$$

then

$$\mathcal{L}_\tau''(B_*)[\Phi,\Phi] \leq -\left(1 - \frac{1}{\lambda_1(m)}\right)\|\Phi\|^2_{L^2(\partial B_*)},$$

where λ_1 is an eigenvalue related to the bi-Laplacian coming from computations inspired by [67, 68] (see Proposition 6.1.24). We show that $\lambda_1 > 1$ by choosing $\Phi \cdot \nu = Y_{1,m}$ with $1 \leq m \leq n-1$ the first non-constant eigenfunctions of the Laplacian on the sphere.

- **Step 3:** we follow the general scheme in [84], and we give an improved continuity estimate of the second order shape derivative of the Lagrangian $\mathcal{L}_\tau$ near B_* (see Proposition 6.1.31). In particular, if $L(t) = \mathcal{L}_\tau((\mathrm{Id} + t\Phi)(B_*))$, then we show that

$$|L''(t) - L''(0)| \leq \omega(\|\Phi\|_{2,p})\|\Phi\|^2_{L^2(\partial B_*)},$$

with ω a modulus of continuity (i.e. for every $\varepsilon > 0$ there exists $\eta > 0$ such that if $\|\Phi\|_{2,p} \leq \eta$ then $|\omega(\|\Phi\|_{2,p})| < \varepsilon$). The improved continuity, together with step 2, implies that B_* is stable with respect to small $W^{2,p}$ deformations (see Proposition 6.1.35).

- **Step 4:** finally, we show that local stability implies global stability Theorem 6.1.2. To do so we perform a Legendre-Fenchel transform to the functional $\mathcal{J}$, and we join it with a quantitative version of the Hardy-Littlewood inequality (see [74, 131] and the quantitative bathtub principle in [139]).

We also want to stress that the proof significantly simplifies under the assumption $j(s) = s$. Indeed, in this case, we can find a radial set A_δ that maximizes $\mathcal{J}$ under the asymmetry constraint

$$\mathcal{J}(A_\delta) = \max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V).$$

At this point, Steps 2, 3 and 4 are not needed, but it is sufficient to prove that $\mathcal{J}(B_*) - \mathcal{J}(A_\delta) \geq C\delta^2$ for some positive constant C. This computation is performed in Proposition 6.1.39.

In the general case, it is possible to prove that

$$\mathcal{J}(A_\delta) = \max_{\substack{V \in \mathcal{M}_m^\delta \\ V \text{ radial}}} \mathcal{J}(V),$$

but we were not able to prove that

$$\max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V) = \max_{\substack{V \in \mathcal{M}_m^\delta \\ V \text{ radial}}} \mathcal{J}(V),$$

which forced us to carry out Steps 2, 3, and 4.

6.1.2 Specific tools needed for the proof

We state here a classical elliptic regularity theorem that will be used often throughout the paper. We cite for instance to [102, Theorem 9.13, Theorem 8.16]) for a reference.

Theorem 6.1.4. *Let $p > n/2$, let $F \in L^p(B_1)$, and let v_F be the unique solution to the equation*

$$\int_{B_1} \nabla v_F \nabla \varphi \, dx = \int_{B_1} F\varphi \, dx.$$

There exists a positive constant $C = C(n)$ such that

$$\|u_F\|_{2,p} \leq C\|F\|_p.$$

In particular, if $F_1, F_2 \in L^p(B_1)$, then

$$\|u_{F_1} - u_{F_2}\|_{2,p} \leq C\|F_1 - F_2\|_p.$$

We refer to [39, Section 1.4] for the following notions.

Definition 6.1.5 (Legendre-Fenchel transform)**.** Let $h : \mathbb{R} \to \mathbb{R}$. We define the *Legendre-Fenchel* transform of h as the function

$$h^\star(x) = \sup_{y\in\mathbb{R}} \{ \, x \cdot y - h(y) \, \} . \tag{6.1.6}$$

When $h : \mathbb{R}^+ \to \mathbb{R}$ we consider h to be extended as $h(x) = +\infty$ for every $x < 0$. Moreover, if h is coercive, C^1 and strictly convex, then the supremum in (6.1.6) is a maximum and it is attained in the point $y = (h')^{-1}(x)$.

Theorem 6.1.6 (Fenchel-Moreau)**.** *Let $h : \mathbb{R} \to \mathbb{R}$ be convex and lower semicontinuous, then $h^{\star\star} = h$.*

Definition 6.1.7 (Spherical harmonics)**.** Let $\Delta_{\mathbb{S}^{n-1}}$ be the Laplace-Beltrami operator on the $(n-1)$-dimensional sphere of radius 1 (i.e. the spherical Laplacian). For every $k \geq 0$ we denote by Λ_k the k-th eigenvalue of the spherical Laplacian, and there exist $M(k)$ and functions $Y_{k,m}$ with $1 \leq m \leq M(k)$ such that

$$-\Delta_{\mathbb{S}^{n-1}} Y_{k,m} = \Lambda_k Y_{k,m}.$$

$M(k)$ is called the *multiplicity* of Λ_k.

The following theorem was proved directly in [139], but it can also be seen as a particular case of the quantitative Hardy-Littlewood inequality (see [74] and [131]).

Theorem 6.1.8 (Quantitative bathtub principle)**.** *Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set, let $m \in (0, |\Omega|)$, and let $u \in C^{1,\alpha}(\Omega)$ for some $\alpha \in (0,1)$. There exists a positive constant $C = C(\|u\|_{C^{1,\alpha}})$ such that the following holds: if $V \in L^1(\Omega)$ is a non-negative function such that*

$$\int_\Omega V \, dx = m,$$

and if

$$k := \min_{\{u=u^*(m)\}} |\nabla u| > 0,$$

then

$$\int_\Omega uV \leq \int_\Omega u\chi_{\{u>u^*(m)\}} - C\frac{n\omega_n r_m^{n-1}}{\mathcal{H}^{n-1}(\{u = u^*(m)\})} k\|V - \chi_{\{u>u^*(m)\}}\|_1^2,$$

where r_m is the radius of the ball of volume m.

6.1.3 Step 1: existence and uniqueness

Let us recall the definition of *fixed asymmetry* weights

$$\mathcal{M}_m^\delta := \{\, V \in \mathcal{M} \mid \|V - V_0\|_{L^1} = \delta \,\}\,.$$

In this section we prove the existence and uniqueness of a solution to the problem

$$\sup_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V). \tag{6.1.7}$$

Remark 6.1.9. If j is a convex function, then the functional $\mathcal{J}(\cdot)$ is convex.

Indeed, if we take $W_\alpha = \alpha W_1 + (1-\alpha)W_0$, then by linearity with respect to V of the equation (6.1.4), we get

$$u_{W_\alpha} = \alpha u_{W_1} + (1-\alpha)u_{W_0},$$

which, joint with the convexity of j gives

$$\mathcal{J}(W_\alpha) \leq \alpha\mathcal{J}(W_1) + (1-\alpha)\mathcal{J}(W_0).$$

Remark 6.1.10. Let h_k be a sequence of functions such that

$$h_k \xrightharpoonup{*-L^\infty(B_1)} h.$$

If there exists a set $E \subset B_1$ such that for any k

$$h_k \geq 0 \text{ in } E, \qquad\qquad h_k \leq 0 \text{ in } E^c,$$

where $E^c = B_1 \setminus E$, then

$$\lim_k \int_{B_1} |h_k|\,dx = \int_{B_1} |h|. \tag{6.1.8}$$

Indeed, it is sufficient to notice that under these assumptions,

$$h_k^+ = h_k\chi_E, \qquad\qquad h_k^- = h_k\chi_{E^c},$$

where h_k^+ and h_k^- are the positive part and the negative part of h_k respectively. In particular, by weak convergence of h_k, we get

$$h_k^+ \xrightharpoonup{*-L^\infty(B_1)} h^+ = h\chi_E,$$

$$h_k^- \xrightharpoonup{*-L^\infty(B_1)} h^- = h\chi_{E^c},$$

which implies (6.1.8)

Lemma 6.1.11. *The sets $\mathcal{M}_m, \mathcal{M}_m^\delta$ are convex and compact with respect to the weak-$*$ L^∞ topology.*

Proof. The convexity for $\mathcal{M}_m$ is immediate by definition. For the convexity of $\mathcal{M}_m^\delta$, instead, we need to show that if $W_1, W_0 \in \mathcal{M}_m^\delta$ and we define for $\alpha \in (0,1)$

$$W_\alpha = \alpha W_1 + (1-\alpha)W_0,$$

then $\|W_\alpha - V_0\|_1 = \delta$. Indeed, since $0 \le W_\alpha \le 1$, by definition of V_0 we have

$$W_\alpha \le V_0 \text{ in } B^*, \qquad\qquad W_\alpha \ge V_0 \text{ in } (B^*)^C,$$

so that explicitly computing the L^1 norms,

$$\|W_\alpha - V_0\|_1 = \alpha\|W_1 - V_0\|_1 + (1-\alpha)\|W_0 - V_0\|_1 = \delta.$$

We now prove that $\mathcal{M}_m$ is compact. If W_k weakly-$*$ converges in L^∞ to some W, then

$$m = \lim_k \int_{B_1} W_k\, dx = \int_{B_1} W\, dx.$$

For what regards $\mathcal{M}_m^\delta$, we start by noticing that for every $W \in \mathcal{M}_m$, we have

$$W \le V_0 \text{ in } B_*, \qquad\qquad W \ge V_0 \text{ in } B_*^c.$$

Therefore, if we take a sequence $V_k \in \mathcal{M}$ converging weakly-$*$ in L^∞, then the functions $V_0 - W_k$ satisfy the assumptions of Remark 6.1.10, and

$$\delta = \lim_k \|W_k - V_0\|_1 = \|W - V_0\|.$$

□

Lemma 6.1.12. *Let $j \in C^0(\mathbb{R}^+)$ be a continuous function. The functional $\mathcal{J}(\cdot)$ is continuous with respect to the weak-$*$ L^∞ convergence.*

Proof. Let W_k be a sequence converging in the weak-$*$ L^∞ sense to some function W_∞. Let $u_k := u_{W_k}$, which is the solution to

$$\int_{B_1} \nabla u_k \cdot \nabla \varphi\, dx = \int_{B_1} W_k\, \varphi\, dx \qquad\qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.9}$$

We show that u_k converges to u_{W_∞}.

Since W_k are equibounded in L^∞, Theorem 6.1.4 applies, and we have that u_k are equi-bounded in $W^{2,p}(B_1)$ for every $p > n$, namely there exists some constant $C = C(n) > 0$ such that

$$\|u_k\|_{2,p} \le C.$$

Therefore, there exists a subsequence (not relabelled) such that

$$u_k \xrightarrow{W^{1,p}(B_1)} u$$

for some $u \in W^{2,p}(B_1)$. Using also the weak-$*$ convergence of W_k, we get, passing to the limit in the Euler-Lagrange equation (6.1.9)

$$\int_{B_1} \nabla u \cdot \nabla \varphi\, dx = \int_{B_1} W_\infty\, \varphi\, dx \qquad\qquad \forall \varphi \in H_0^1(B_1),$$

which implies $u = u_{W_\infty}$. Finally, since j is continuous and u_k converges strongly in L^∞ to u, then

$$\lim_k \mathcal{J}(W_k) = \lim_k \int_{B_1} j(u_k)\, dx = \mathcal{J}(W_\infty).$$

Since the argument is valid for every choice of the subsequence, this proves the continuity. □

Proposition 6.1.13. *Let $j \in C^0(\mathbb{R}^+)$ be a convex function. Then problems (6.1.5) and (6.1.7) admit a bang-bang solution. Namely, there exists a set E such that*

$$\mathcal{J}(E) = \max_{V \in \mathcal{M}_m} \mathcal{J}(V),$$

and for every $\delta > 0$ there exists a set E_δ such that

$$\mathcal{J}(E_\delta) = \max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V).$$

Proof. By Lemma 6.1.11 and Lemma 6.1.12 we get the existence of a solution.

First we claim that there exists a maximizer $W_\delta \in \mathcal{M}_m^\delta$ which is an extremal point for $\mathcal{M}_m^\delta$. Let W be a maximizer for $\mathcal{J}$ on $\mathcal{M}_m^\delta$. If W were not an extremal point for $\mathcal{M}_m^\delta$, then we could find two extremal points $W_1, W_0 \in \mathcal{M}_m^\delta$ and $\alpha \in (0,1)$ such that

$$W = \alpha W_1 + (1-\alpha) W_0.$$

In particular, by convexity

$$\mathcal{J}(W) \le \max\{\mathcal{J}(W_1), \mathcal{J}(W_0)\},$$

which proves the claim.

It remains to prove that the extremal points of $\mathcal{M}_m^\delta$ are characteristic functions of measurable sets. Let $W \in \mathcal{M}_m^\delta$, and assume that

$$|\{\, 0 < W < 1 \,\}| > 0.$$

Since $\|W - V_0\|_1 = \delta$, there exist four pairwise disjoint subsets S_1, S_2, S_3, S_4 of B_1 such that

$$|S_1| = |S_2| > 0, \qquad\qquad |S_3| = |S_4| > 0,$$

and

$$S_1 \cup S_2 \subseteq \{\, \varepsilon < W < 1-\varepsilon \,\} \cap B_* \qquad\qquad S_3 \cup S_4 \subseteq \{\, \varepsilon < W < 1-\varepsilon \,\} \setminus B_*.$$

for some $\varepsilon > 0$. Under these assumptions we can write

$$W_0 = \frac{1}{2}(W - \varepsilon\chi_{S_1\cup S_3} + \varepsilon\chi_{S_2\cup S_4}) + \frac{1}{2}(W + \varepsilon\chi_{S_1\cup S_3} - \varepsilon\chi_{S_2\cup S_4}),$$

with $W \mp \varepsilon\chi_{S_1\cup S_3} \pm \varepsilon\chi_{S_2\cup S_4} \in \mathcal{M}_m^\delta$. This proves that a non-bang-bang function is not an extreme point for $\mathcal{M}_m^\delta$. □

Proposition 6.1.14. *Let $j \in C^1(\mathbb{R}^+)$ be a strictly convex function. Then χ_{B_*} is the unique maximizer to the problem*

$$\mathcal{J}(B_*) = \max_{V \in \mathcal{M}_m} \mathcal{J}(V).$$

Proof. Let W be a maximizer for $\mathcal{J}$ on $\mathcal{M}_m$. Since j is strictly convex, we also have that $\mathcal{J}$ is strictly convex. Under this assumption, any maximizer of $\mathcal{J}$ on $\mathcal{M}_m$ has to be an extreme point fo $\mathcal{M}_m$, and there exists E such that $W = \chi_E$. Since W is a maximizer, we have

$$0 \le \mathcal{J}(E) - \mathcal{J}(E^\sharp) = \int_0^{+\infty} \Big(j'(\mu_{u_E}(s)) - j'(\mu_{u_{E^\sharp}}(s)) \Big)\, ds \le 0,$$

where we used the monotonicity of j' and the Talenti inequality Theorem 2.4.7. Since j' is increasing, we also get

$$\mu_{u_E}(t) = \mu_{u_{E^\sharp}}(t) \qquad \text{for a.e. } t \in \mathbb{R}^+.$$

By the rigidity of Talenti inequality (Theorem 2.4.8) we obtain $E = E^\sharp$ up to negligible sets, and the proof is complete. □

6.1.4 Computation of shape derivatives

In this section we compute the shape derivative of the shape functional

$$E \longmapsto \mathcal{J}(E).$$

To that aim, we need to define the following function (see also [113, §5.8]).

Definition 6.1.15 (Adjoint State)**.** Let $j \in C^1(\mathbb{R}^+)$. For every $V \in \mathcal{M}$, we define the *adjoint state* w_V of u_V as the unique function solving in the weak sense the *adjoint problem*

$$\begin{cases} -\Delta w_V = j'(u_V) & \text{in } B_1, \\ w = 0 & \text{on } \partial B_1, \end{cases}$$

namely,

$$\int_{B_1} \nabla w_V \cdot \nabla \varphi \, dx = \int_{B_1} j'(u_V)\varphi \qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.10}$$

When $V = \chi_E$ we will write $w_E := w_{\chi_E}$.

We will follow the classical approach (see for instance [113, Theorem 5.3.2]) to compute the shape derivative of u_E, w_E, and finally $\mathcal{J}(E)$.

Let $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$ with $p > n$. Let $s = 1 - n/p$, so that for every E of class $C^{1,s}$ we have that

$$E^\Phi := (\mathrm{Id} + \Phi)(E)$$

is of class $C^{1,s}$. We then define the functions

$$u_\Phi := u_{E^\Phi} \qquad\qquad w_\Phi := w_{E^\Phi}$$
$$\hat{u}_\Phi := u_{E^\Phi} \circ (\mathrm{Id} + \Phi) \qquad\qquad \hat{w}_\Phi := w_{E^\Phi} \circ (\mathrm{Id} + \Phi),$$

where w_E is the adjoint state defined in Definition 6.1.15. If $\|\Phi\|_{2,p} < 1$, we can invert the matrix $I_n + D\Phi$, so that, defining

$$J_\Phi := \det(I_n + D\Phi) \qquad\qquad A_\Phi := J_\Phi\,(I_n + D\Phi)^{-1}(I_n + D\Phi)^{-T},$$

we have that, in the distributional sense,

$$-\operatorname{div}(A_\Phi \nabla \hat{u}_\Phi) = \chi_E\, J_\Phi, \qquad\qquad -\operatorname{div}(A_\Phi \nabla \hat{w}_\Phi) = j'(\hat{u}_\Phi)\, J_\Phi.$$

To prove the shape differentiability of $\hat{u}_\Phi$ and $\hat{w}_\Phi$ in $H_0^1(B_1)$, we prove that we are in the assumptions of the implicit function theorem in Banach spaces proving the following lemma.

Lemma 6.1.16. *Let $p > n$, let $j \in C^2(\mathbb{R})$, let $X = H_0^1(B_1) \cap W^{2,p}(B_1)$, and let*

$$\begin{aligned} \mathcal{F} : W^{2,p}(B_1;\mathbb{R}^n) \times X \times X &\longrightarrow \quad L^p(B_1) \times L^p(B_1) \\ (\Phi, \hat{u}, \hat{w}) &\longmapsto \quad (\mathcal{F}_1(\Phi, \hat{u}, \hat{w}), \mathcal{F}_2(\Phi, \hat{u}, \hat{w})), \end{aligned}$$

where

$$\mathcal{F}_1(\Phi, \hat{u}, \hat{w}) = -\operatorname{div}(A_\Phi \nabla \hat{u}) - \chi_E\, J_\Phi,$$

and

$$\mathcal{F}_2(\Phi, \hat{u}, \hat{w}) = -\operatorname{div}(A_\Phi \nabla \hat{w}) - j'(\hat{u})\, J_\Phi.$$

Then $\mathcal{F}_1$ is of class C^∞ in a neighbourhood of 0, and $\mathcal{F}_2$ is of class C^1 in a neighbourhood of 0.

Proof. First we notice that the application

$$(A, \hat{u}) \in \mathbb{R}^{n\times n} \times X \longmapsto -\operatorname{div}(A\nabla\hat{u}) \in L^p(B_1)$$

is of class C^∞ because it is linear and continuous in both the variables. Analogously, multilinearity and conitnuity in the origin imply that the applications

$$\Phi \in W^{2,p}(B_1) \longmapsto (I_n + D\Phi)^{-1} = \sum_{k=1}^{+\infty} D\Phi^k \in W^{1,p}(B_1)$$

$$\Phi \in W^{2,p}(B_1) \longmapsto J_\Phi = \det(I_n + D\Phi) \in W^{1,p}(B_1)$$

are of class C^∞ in a neighbourhood of 0. Therefore, the application

$$\Phi \in W^{2,p}(B_1) \longmapsto A_\Phi \in W^{1,p}(B_1)$$

is C^∞ in a neighbourhood of 0. Joining these regularities, we obtain that $\mathcal{F}_1$ is of class C^∞.

The same regularities shown above, joint with the fact that the application

$$\hat{u} \in X \longmapsto j'(\hat{u}) \in L^p(B_1)$$

is of class C^1, ensure that $\mathcal{F}_2$ is of class C^1. □

As a consequence, we get

Proposition 6.1.17 (Shape differentiability of $\hat{u}_\Phi$ and $\hat{w}_\Phi$)**.** *Let $p > n$, let $j \in C^2(\mathbb{R})$, let $X = H^1_0(B_1) \cap W^{2,p}(B_1)$ endowed with the $W^{2,p}$ norm, and let $\Psi \in W^{2,p}(B_1;\mathbb{R}^n)$. There exists a $\eta > 0$ such that if $\|\Psi\|_{2,p} \le \eta$, then the application*

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto \hat{u}_\Phi \in H^1_0(B_1) \cap W^{2,p}(B_1)$$

is of class C^∞ in Ψ, and the application

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto \hat{w}_\Phi \in H^1_0(B_1) \cap W^{2,p}(B_1)$$

is of class C^1 in Ψ.

Proof. If $\|\Psi\|_{2,p}$ is small enough, Lemma 6.1.16 applies to Ψ. In the case $\Psi = 0$, we notice that the functions $\hat{u}_0 = u_E$ and $\hat{w}_0 = w_E$ solve the equations

$$\mathcal{F}(0, u_E, w_E) = 0.$$

Hence, thanks to Lemma 6.1.16 and the implicit function theorem, it is only left to prove that $D_{\hat{u}}\mathcal{F}_1(0, u_E, w_E)[\cdot]$ and $D_{\hat{w}}\mathcal{F}_2(0, u_E, w_E)[\cdot]$ are isomorphisms of X onto $L^p(B_1)$.

For every $\xi \in X$ we have

$$D_{\hat{u}}\mathcal{F}_1(0, u_E, w_E)[\xi] = -\Delta\xi - \chi_E,$$

which is a diffeomorphism of X onto $L^p(B_1)$ thanks to the classical elliptic regularity (see for instance [102, Theorem 9.14]). Analogously, for every $\xi \in X$,

$$D_{\hat{w}}\mathcal{F}_2(0, u_E, w_E)[\xi] = -\Delta\xi - j'(u_E)$$

is a diffeomorphism of X onto $L^p(B_1)$. Since A_Ψ is a positive definite matrix for $\|\Psi\|_{2,p}$ small, then elliptic estimates still apply to the case $\Psi \neq 0$, and the proof is analogous. □

Proposition 6.1.18 (Shape differentiability of u_Φ)**.** *Let $p > n$, let $j \in C^2(\mathbb{R})$, and let $\Psi \in W^{2,p}(B_1;\mathbb{R}^n)$. There exists a $\eta > 0$ such that if $\|\Psi\|_{2,p} \le \eta$, then the application*

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto u_\Phi \in W_0^{1,p}(B_1)$$

is of class C^1 in Ψ. In particular, if we denote by $u_t := u_{t\Phi}$, then we have that $u_t' \in W_0^{1,p}(B_1)$ is the unique solution in $H_0^1(B_1)$ to

$$-\Delta u_t' = -\nabla \chi_{E^{t\Phi}}[\tilde{\Phi}], \tag{6.1.11}$$

where $\tilde{\Phi} = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$, and the equality has to be intended in the distributional sense.

Proof. Let $E_t = E^{t\Phi}$, and let $X = H_0^1(B_1) \cap W^{2,p}(B_1)$. By Proposition 6.1.17 we have that the application

$$\Phi \in W^{2,p}(B_1) \longmapsto \hat{u}_\Phi \in X$$

is of class C^∞. Since in particular $\hat{u}_\Psi \in W^{2,p}(B_1)$, by noticing that $u_\Phi = \hat{u}_\Phi \circ (\mathrm{Id} + t\Phi)^{-1}$, then we know that the application

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto u_\Phi \in W_0^{1,p}(B_1)$$

is C^1 in a neighbourhood of 0 (see for instance [113, Lemma 5.3.3]).

In particular, we have that there exists $u_t' \in W_0^{1,p}(B_1)$ such that

$$u_t' = \lim_{t \to 0} \frac{u_t - u_0}{t}$$

where the limit has to be intended in $W_0^{1,p}(B_1)$.

Let us recall that u_t solves the weak equation (6.1.4), namely

$$\int_{B_1} \nabla u_t \cdot \nabla \varphi \, dx = \int_{E_t} \varphi \, dx \qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.12}$$

Using Remark B.0.8 to differentiate (6.1.12), we get that u_t' solves

$$\int_{B_1} \nabla u_t' \cdot \nabla \varphi \, dx = \int_{\partial E_t} \varphi \, (\tilde{\Phi} \cdot \nu_t) \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H_0^1(B_1),$$

which is (6.1.11). □

Proposition 6.1.19 (Shape differentiability of w_Φ)**.** *Let $p > n$, let $j \in C^2(\mathbb{R})$, and let $\Psi \in W^{2,p}(B_1;\mathbb{R}^n)$. There exists a $\eta > 0$ such that if $\|\Psi\|_{2,p} \le \eta$, then the application*

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto w_\Phi \in W_0^{1,p}(B_1)$$

is of class C^1 in Ψ. In particualr, if we denote by $w_t := w_{t\Phi}$, then we have that $w_t' \in W_0^{1,p}(B_1)$ is the unique solution in $H_0^1(B_1)$ to

$$-\Delta w_t' = j''(u_t) u_t'. \tag{6.1.13}$$

Proof. As in Lemma 6.1.16, let $X = H_0^1(B_1) \cap W^{2,p}(B_1)$. By Proposition 6.1.17 we have that the application

$$\Phi \in W^{2,p}(B_1) \longmapsto \hat{w}_\Psi \in X$$

is of class C^1. Analogously to Proposition 6.1.18, we get that the application

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto u_\Phi \in W_0^{1,p}(B_1)$$

is differentiable in Ψ.

In particular, we have

$$w_t' = \lim_{t\to 0} \frac{w_t - w_0}{t},$$

where the limit has to be intended in $W_0^{1,p}(B_1)$.

The function w_t solves the weak equation (6.1.10), namely

$$\int_{B_1} \nabla w_t \cdot \nabla\varphi \, dx = \int_{B_1} f'(u_t)\varphi \, dx \qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.14}$$

Using Proposition 6.1.18, we can differentiate (6.1.14) to obtain that w_t' solves

$$\int_{B_1} \nabla w_t' \cdot \nabla\varphi \, dx = \int_{E^\Psi} j''(u_t) u_t' \, \varphi \, dx \qquad \forall \varphi \in H_0^1(B_1),$$

which proves (6.1.13). □

Proposition 6.1.20 (Shape derivative of $\mathcal{J}$)**.** *Let $p > n$, let $j \in C^2(\mathbb{R})$, let E be a set of class $W^{2,p}$, and let $\Psi \in W^{2,p}(B_1;\mathbb{R}^n)$. There exists a $\eta > 0$ such that if $\|\Psi\|_{2,p} \le \eta$, then the application*

$$\Phi \in W^{2,p}(B_1;\mathbb{R}^n) \longmapsto \mathcal{J}(E^\Phi) \in \mathbb{R}$$

is of class C^2 in Ψ. In particular, if $J(t) = \mathcal{J}(E^{t\Phi})$, then

$$J'(t) = \int_{\partial E^{t\Phi}} w_t \, (\tilde\Phi \cdot \nu_t) \, d\mathcal{H}^{n-1}, \tag{6.1.15}$$

where $w_t = w_{E^{t\Phi}}$ is the adjoint state defined in Definition 6.1.15, $\nu_t = \nu_{E^{t\Phi}}$ is the outer unit normal to $E^{t\Phi}$, and $\tilde\Phi = \Phi \circ (\mathrm{Id} + t\Phi)^{-1}$. Moreover, letting $g_t = \tilde\Phi \cdot \nu_t$,

$$J''(t) = \int_{\partial E^{t\Phi}} \Big(w_t' g_t + g_t \nabla w_t \cdot \tilde\Phi \Big) \, d\mathcal{H}^{n-1} + a_t(\Phi,\Phi), \tag{6.1.16}$$

where

$$a_t(\Phi,\Phi) = \int_{\partial E^{t\Phi}} w_t \Big(g_t \operatorname{div}(\tilde\Phi) - D\tilde\Phi\tilde\Phi \cdot \nu_t \Big) d\mathcal{H}^{n-1}.$$

Proof. Since for every $\Phi \in W^{2,p}$

$$\mathcal{J}(E^\Phi) = \int_{B_1} j(u_\Phi) \, dx,$$

then Proposition 6.1.18, joint with $j \in C^2(\mathbb{R}^+)$ is sufficient to prove the differentiability of the functional $\mathcal{J}$.

Let us denote by $u_t := u_{t\Phi}$, and by $u_0' := u_E'[\Phi]$. We begin by noticing that

$$J'(t) = \int_{B_1} j'(u_t) u_t' \, dx$$

We recall that u_t' is the solution to the weak equation (6.1.11), namely

$$\int_{B_1} \nabla u_t' \cdot \nabla \varphi \, dx = \int_{\partial E^{t\Phi}} \varphi \, g_t \, d\mathcal{H}^{n-1}, \qquad \forall \varphi \in H_0^1(B_1).$$

In particular, if we put w_t as a test function into the equation, and we use the equation (6.1.10) solved by $w_t = w_{E^{t\Phi}}$, then

$$\int_{\partial E^{t\Phi}} w_t \, g_t \, d\mathcal{H}^{n-1} = \int_{B_1} \nabla u_t' \cdot \nabla w_t \, dx = \int_{B_1} j'(u_t) u_t',$$

which proves (6.1.15), that can also be read as $J'(t) = \langle \partial_t \chi_{E_t} | w_t \rangle$.

By Leibniz rule we get

$$J''(t) = \langle \partial^2_{tt} \chi_{E_t} | w_t \rangle + \langle \partial_t \chi_{E_t} | w_t' \rangle \, ,$$

and by Remark B.0.11 we get formula (6.1.16). □

Remark 6.1.21. When we evaluate (6.1.15) in $t = 0$, we get

$$J'(0) = \int_{\partial E} w_E (\Phi \cdot \nu_E) \, d\mathcal{H}^{n-1}.$$

When Φ is orthogonal to ∂E (i.e. $\Phi = g_0 \nu_0$ on ∂E), then (6.1.16) in $t = 0$ reads

$$J''(0) = \int_{\partial E} \left(w_0' \, g_0 + w_0 H_E g_0^2 + \frac{\partial w_0}{\partial \nu_0} g_0^2 \right) d\mathcal{H}^{n-1},$$

with H_E the mean curvature of ∂E. Indeed, it is sufficient to notice that in this case

$$g_0 \operatorname{div}(\Phi) - D\Phi \Phi \nu_0 = g_0^2 \operatorname{div}^{\partial E}(\nu_0) = g_0^2 H_E.$$

The computation of the first derivative allows us to compute the optimality conditions for the maximizing problem

$$\max_{|E|=m} \mathcal{J}(E). \tag{6.1.17}$$

Remark 6.1.22 (Computation of the Lagrange multiplier)**.** We have that the Lagrange multiplier for problem (6.1.5) is given by

$$\tau = -w_0|_{\partial B_*}, \tag{6.1.18}$$

where we recall that $w_0 = w_{B_*}$ is the adjoint state defined in Definition 6.1.15.

Indeed, let us consider the Lagrangian

$$\mathcal{L}_\tau(E) := \mathcal{J}(E) + \tau |E|.$$

Since by Lemma B.0.7 we have

$$\frac{d}{dt} |B_*^{t\Phi}| \bigg|_{t=0} = \int_{\partial B_*} (\Phi \cdot \nu) \, d\mathcal{H}^{n-1},$$

then the equality

$$\mathcal{L}_\tau'(B_*)[\Phi] = \mathcal{J}'(B_*)[\Phi] + \tau \int_{\partial B_*} (\Phi \cdot \nu) \, d\mathcal{H}^{n-1} = 0, \qquad \Phi \in W^{2,p}(B_1; \mathbb{R}^n),$$

joint with the expression of the first derivative (6.1.15), implies (6.1.18).

Remark 6.1.23 (Second order shape derivative of the Lagrangian)**.** Let $L(t) = \mathcal{L}_\tau(B_*^{t\Phi})$. It will be useful in subsection 6.1.6 to have a computation of $L''(t)$ in the particular case where $\tilde{\Phi} = \Phi$ for every t. This happens when we choose Φ to be extended constantly along the normal radii starting from ∂E. When E is a set with $C^{2,s}$ boundary (in our case $E = B_*$), we can consider Φ defined as

$$\Phi(x) = g(\pi_{\partial E}(x))\nu_0(\pi_{\partial E}(x)),$$

where g is some regular function defined on ∂E, and ν is the outer unit normal to ∂E. We identify $g = g \circ \pi_{\partial E}$ and $\nu_0 = \nu_0 \circ \pi_{\partial E}$, so that

$$D\nu_0\nu_0 = 0, \qquad\qquad \nabla g \cdot \nu_0 = 0,$$

and, in particular, for every $t > 0$,

$$\tilde{\Phi} = \Phi \circ (\mathrm{Id} + t\Phi) = \Phi.$$

Under these assumptions, we have

$$J''(t) - a_t(\Phi, \Phi) = \int_{\partial E_t} \left(w_t' g + g^2 \frac{\partial w_t}{\partial \nu_t} \right) d\mathcal{H}^{n-1},$$

and

$$a_t(\Phi, \Phi) = \int_{\partial E_t} w_t(\nu_t \cdot \nu_0) g^2 \operatorname{div}(\nu_0)\, d\mathcal{H}^{n-1}.$$

We point out that $\operatorname{div} \nu_0(x)$ is not the mean curvature of ∂E in $\pi_{\partial E}(x)$, but it is the mean curvature of the boundary of the outer parallel set $\partial(\partial E)^{d(x)}$, where $d(x)$ is the distance from ∂E. With the same computation, using Remark B.0.11 (with $\varphi = 1$) to compute the derivative of the volume, we obtain

$$L''(t) = J''(t) - a_t(\Phi, \Phi) + \int_{\partial E_t} (w_t - w_0(r_*))(\nu_t \cdot \nu_0) g^2 \operatorname{div}(\nu_0)\, d\mathcal{H}^{n-1},$$

where $w_0(r_*) = w_0(x)$ for any $x \in \partial B_*$.

6.1.5 Step 2: coercivity of the second order shape derivative in B_*

Let $\Phi \in W^{2,p}(B_1; \mathbb{R}^n)$ be a deformation which is normal to ∂B_*, i.e.

$$\Phi(x) = (\Phi(x) \cdot \nu_0(x))\nu_0(x) \qquad\qquad \forall x \in \partial B_*.$$

This section is devoted to proving the following.

Proposition 6.1.24. *Let $m \in (0, |B_1|)$, let $j(s) = |s|^q$ for some $q \geq 2$, and let $L(t) := L_\tau(B_*^{t\Phi})$ be the Lagrangian of $\mathcal{J}$ associated to the volume constraint, defined in Remark 6.1.22. There exists positive constants $C = C(j, m), \eta = \eta(j, m)$ such that if $\|\Phi\|_\infty < \eta$, then*

$$L''(0) \leq -C\|\Phi \cdot \nu\|_2^2.$$

To do so, we compute $L''(0)$ in terms of $\Phi \cdot \nu_0$, defining a suitable eigenvalue problem that has been introduced in [67] (see in particular [67, Theorem III] and [68, Proposition 34]).

We recall the notations introduced in Proposition 6.1.18 and Proposition 6.1.19: $u_0' := u_{B_*}'[\Phi]$ and $w_0' := w_{B_*}'[\Phi]$. Let us notice that the assumption that Φ is normal to ∂B_* ensures that, using Remark 6.1.23,

$$L''(0) = \mathcal{L}_\tau''(B_*)[\Phi,\Phi] = -\int_{\partial B^*} g_\Phi \left|\frac{\partial w_0}{\partial \nu}\right| \left(-\left|\frac{\partial w_0}{\partial \nu}\right|^{-1} w_0' + g_\Phi\right) d\mathcal{H}^{n-1},$$

where $g_\Phi = \Phi \cdot \nu_0$. Since w_0', u_0' are the unique solution in $H_0^1(B_1)$ to the equations

$$-\Delta w_0' = j''(u_0)u_0' \qquad\qquad -\Delta u_0' = g_\Phi \, d\mathcal{H}^{n-1} \llcorner_{\partial B_*},$$

we may observe that $L''(0)$ only depends on g_Φ. For this reason, we define an analogous of $\mathcal{L}''(B_*)[\Phi,\Phi]$ in dependence of functions $g \in L^2(\partial B_*)$. Let $\rho = |\partial_\nu w_0|^{-1}|_{\partial B^*}$, and for every $g \in L^2(\partial B_*)$, let us consider $(U_g, W_g) \in H_0^1(B_1;\mathbb{R}^2)$ the unique solution to the coupled boundary value problems

$$\begin{cases} -\Delta U_g = 0 & \text{in } B_1 \setminus \partial B^* \\ -\left[\dfrac{\partial U_g}{\partial \nu}\right] = \rho g, & \text{on } \partial B^* \\ U_g = 0 & \text{on } \partial B_1. \end{cases} \qquad \begin{cases} -\Delta W_g = j''(u_0)U_g & \text{in } B_1 \setminus \partial B^* \\ W_g = 0 & \text{on } \partial B_1, \end{cases} \tag{6.1.19}$$

where

$$\left[\frac{\partial U_g}{\partial \nu}\right](x) = \lim_{\substack{y\to x \\ y\in B_*^c}} \frac{\partial U_g}{\partial \nu}(y) - \lim_{\substack{y\to x \\ y\in B_*}} \frac{\partial U_g}{\partial \nu}(y). \tag{6.1.20}$$

Note that, when $g = \Phi \cdot \nu$, we have $U_g = \rho u_0'$ and $W_g = \rho w_0'$. We define for $h, g \in L^2(\partial B_*)$

$$l_2(h,g) := \int_{\partial B_*} \frac{1}{\rho} h(W_g - g) \, d\mathcal{H}^{n-1}.$$

Such a definition allows us to write

$$L''(0) = l_2(g_\Phi, g_\Phi).$$

To estimate $L''(0)$, we gather the idea from [67] (see also [68, Lemma 35]) to diagonalize W_g with respect to g. In the following $\mathrm{tr}_{\partial B_*}$ denotes the trace operator

$$\mathrm{tr}_{\partial B_*} : W^{1,2}(B_1) \to L^2(\partial B_*).$$

If we define the operator

$$T_\rho : g \in L^2(\partial B_*) \longmapsto \mathrm{tr}_{\partial B_*}(W_g) \in L^2(\partial B_*),$$

then we get for every $h, g \in L^2(\partial B_*)$,

$$l_2(h,g) = \int_{\partial B_*} \frac{1}{\rho} h(T_\rho g - g) \, d\mathcal{H}^{n-1}.$$

The operator T_ρ is symmetric and positive. Under boundedness assumptions on j'', we can also prove that T_ρ is compact. However, we aim to get a stronger result, without requiring j'' to be bounded, and we will explicitly diagonalize T_ρ.

Remark 6.1.25. Let us assume that T_ρ is diagonalizable, i.e. there exists an orthonormal basis g_j of $L^2(B_*)$ such that

$$T_\rho g_j = \frac{1}{\lambda_j} g_j,$$

then

$$l_2(g_j, g_k) = \frac{1}{\rho}\Big(\frac{1}{\lambda_k} - 1\Big)\delta_{jk}.$$

On the other hand, these eigenvalues λ_j are actually eigenvalues for a differential operator on $H^2(B_1)\cap H^1_0(B_1)$.

Indeed, if we denote by $U_k := U_{g_k}$ and $W_k := W_{g_k}$, then the boundary value problems (6.1.19) for g_k read

$$\begin{cases} -\Delta U_k = 0 & \text{in } B_1\setminus\partial B^*,\\ -\Big[\dfrac{\partial U_k}{\partial\nu}\Big] = \rho\lambda_k W_k, & \text{on } \partial B^*,\\ U_k = 0 & \text{on } \partial B_1, \end{cases} \qquad \begin{cases} -\Delta W_k = j''(u_0)U_k & \text{in } B_1\setminus\partial B^*,\\ W_k = 0 & \text{on } \partial B_1, \end{cases} \tag{6.1.21}$$

where we recall the notation $[v](x)$ to denote the jump of the function v along ∂B_* (see (6.1.20)). In particular, under the assumption $j''(s)>0$ for $s\neq 0$ we can write U_k in terms of ΔW_k, so that the coupled systems (6.1.21) can be rewritten as a single fourth order system

$$\begin{cases} \Delta(j''(u_0)^{-1}\Delta W_k) = 0 & \text{in } B_1\setminus\partial B^*,\\ \Big[\dfrac{\partial}{\partial\nu}(j''(u_0)^{-1}\Delta W_k)\Big] = \rho\lambda_k W_k, & \text{on } \partial B^*,\\ -\Delta W_k = 0 & \text{on } \partial B_1,\\ W_k = 0 & \text{on } \partial B_1. \end{cases} \tag{6.1.22}$$

We also note that from this equation one could write a variational characterization of λ_k (see for instance formula (B.12) in [68]).

We will show that T_ρ is diagonalizable and that the spherical harmonics $Y_{k,m}$ are its eigenfunctions. Let us denote by Δ_r the differential operator defined as

$$\Delta_r\varphi = \partial_r(r^{1-n}\partial_r(r^{n-1}\varphi)),$$

and by $\mathcal{D}^k_r$ the operator defined as

$$\mathcal{D}^k_r\varphi = \Delta_r\varphi - \frac{\Lambda_k}{r^2}\varphi,$$

where we recall that Λ_k denotes the k-th eigenvalue of the spherical Laplacian $\Delta_{\mathbb{S}^{n-1}}$. We will also write for every $\varphi\in C^0((0,1)\setminus\{r_*\})$

$$[\varphi](r_*) = \varphi(r_*^+) - \varphi(r_*^-) = \lim_{r\to r_*^+}\varphi(r) - \lim_{r\to r_*^-}\varphi(r).$$

Before computing explicitly the eigenvalues, we need some computation on ρ and u_0, as follows.

Remark 6.1.26. Let us notice that we can write ρ using the following identity, that we think to be a resonable form to take into account if one wants to compare ρ and λ_1:

$$\frac{1}{\rho} = \frac{1}{r_*^{n-1}P(B_1)} \int_{B_1} j''(u_0)|\nabla u_0|^2 \, dx.$$

Indeed

$$\begin{aligned}
\frac{1}{\rho} &= \left|\frac{\partial w_0}{\partial \nu}\right|\Big|_{\partial B_*} \\
&= \frac{1}{P(B_*)} \int_{\partial B_*} \left(-\frac{\partial w_0}{\partial \nu}\right) d\mathcal{H}^{n-1} \\
&= \frac{1}{r_*^{n-1}P(B_1)} \int_{B_*} (-\Delta w_0) \, dx \\
&= \frac{1}{r_*^{n-1}P(B_1)} \int_{B_1} (-\Delta u_0) j'(u_0) \, dx \\
&= \frac{1}{r_*^{n-1}P(B_1)} \int_{B_1} j''(u_0)|\nabla u_0|^2 \, dx.
\end{aligned}$$

Remark 6.1.27. Identifying $u_0(x) = u_0(|x|)$, we can rewrite ρ in the following way

$$\frac{1}{\rho} = \frac{1}{n^2 r_*^{n-1}} \int_0^1 j''(u_0(r))(r \wedge r_*)^{2n} \, r^{1-n} \, dr. \tag{6.1.23}$$

Indeed, we just need to notice that $\nabla u_0(x) = \partial_r u_0(|x|)$ (and that the function u_0 solves

$$\begin{cases}
-\partial_r(r^{n-1}\partial_r u_0(r)) = r^{n-1}\chi_{(0,r_*)}(r) & r \in (0, R), \\
\partial_r u_0(0) = 0, \\
u_0(R) = 0.
\end{cases}$$

Hence, integrating from 0 to r, and dividing by r^{n-1}, we have

$$\partial_r u_0(r) = -r^{1-n} \int_0^{r\wedge r_*} s^{n-1} \, ds = -\frac{1}{n}(r \wedge r_*)^n \, r^{1-n},$$

which implies (6.1.23)

Proposition 6.1.28. *Let $j(s) = |s|^q$ for some $q \geq 2$. Then for every $k \in \mathbb{N}$ there exists $\varphi_k \in C^2(0, 1)$ such that*

$$\frac{1}{j''(u_0)} \mathcal{D}_r^k \varphi_k \in C^1((0, r_*) \cup (r_*, 1)) \cap C^0(0, 1),$$

and such that φ_k solves the ODE

$$\begin{cases}
\mathcal{D}_r^k \left(\dfrac{1}{j''(u_0(r))} \mathcal{D}_r^k \varphi_k\right)(r) = 0 & r \in (0, r_*) \cup (r_*, 1), \\
\varphi_k(0) = \varphi_k(1) = 0, \\
\mathcal{D}_r^k \varphi_k(0) = \lim\limits_{r \to 1^-} \dfrac{1}{j''(u_0(r))} \mathcal{D}_r^k \varphi_k(r) = 0.
\end{cases} \tag{6.1.24}$$

Moreover,

$$\lambda_1(j,m) := \frac{1}{\rho\varphi_1(r_*)}\left[\partial_r\left(\frac{1}{j''(u_0)}\mathcal{D}_r^k\varphi_1\right)\right](r_*) > 1.$$

Proof. We recall that $\mathcal{D}_r^k$ denotes the differential operator

$$\mathcal{D}_r^k = \Delta_r - \frac{\Lambda_k}{r^2}.$$

We also recall the value of the spherical Laplacian eigenvalue $\Lambda_k = k(n+k-2)$, and that we are looking for solutions φ_k to the ODE

$$\mathcal{D}_r^k\left(\frac{1}{j''(u_0)}\mathcal{D}_r^k\varphi_k\right)(r) = 0 \qquad\qquad r\in(0,r_*)\cup(r_*,1).$$

For this ODE we will find 4 independent solutions $\psi_{k,\beta}$ with $\beta\in\mathcal{I}_k$, where

$$\mathcal{I}_k = \{k,\ k+2,\ 2-n-k,\ 4-n-k\}.$$

First we notice that for every twice differentiable φ

$$\mathcal{D}_r^k\varphi = r^{1-n-k}\left(r^{n-1+2k}\left(r^{-k}\varphi\right)'\right)'.$$

Hence, for $k\neq 0$, we define the four independent functions

$$\psi_{k,\beta}(r) = r^\beta, \qquad\qquad \beta = k, 2-n-k,$$

$$\psi_{k,\beta}(r) = r^k\int_{r_*}^r s^{-(n-1+2k)}\int_0^s j''(u_0(t))\,t^{n+k+\beta-3}\,dt\,ds, \qquad\qquad \beta = k+2, 4-n-k.$$

which respectively solve the equations

$$\begin{aligned} \mathcal{D}_r^k\psi_{k,\beta} &= 0, &\qquad \beta &= k, 2-n-k,\\ \mathcal{D}_r^k\psi_{k,\beta} &= j''(u_0)\psi_{k,\beta-2}, &\qquad \beta &= k+2, 4-n-k. \end{aligned} \tag{6.1.25}$$

For $k=0$, instead, to avoid $\Psi_{0,0}=\Psi_{0,2-n}$ we need to define

$$\psi_{0,0}(r) = 1, \qquad\qquad \psi_{0,2-n}(r) = \begin{cases} 1-\log(r) & n=2,\\ r^{2-n} & n>2, \end{cases}$$

$$\psi_{0,\beta}(r) = \int_{r_*}^r s^{-(n-1)}\int_0^s j''(u_0(t))\,\psi_{0,\beta-2}(t)t^{n-1}\,dt\,ds, \qquad\qquad \beta = 2, 4-n,$$

so that (6.1.25) also hold true for $k=0$.

We now define φ_k in the following way

$$\varphi_k(r) = \begin{cases} \displaystyle\sum_{\beta\in\mathcal{I}_k} c^-_{k,\beta}\psi_{k,\beta}(r) & r\le r_*,\\ \displaystyle\sum_{\beta\in\mathcal{I}_k} c^+_{k,\beta}\psi_{k,\beta}(r) & r> r_*. \end{cases}$$

Since φ_k is determined up to scaling factors, we are allowed to assume $c^-_{k,k} = 1$ (if $c^-_{k,k} = 0$ then it can be proved that $c^{\pm}_{k,\beta} = 0$ for every k and $\beta \in \mathcal{I}_k$). We now impose the following conditions on φ_k that will uniquely determine the constants $c^{\pm}_{k,\beta}$.

$$\begin{cases} \mathcal{D}^k_r \varphi_k(0) = 0, & \text{(6.1.26a)} \\ \varphi_k(0) = 0, & \text{(6.1.26b)} \\ \lim\limits_{r \to 1^-} \dfrac{1}{j''(u_0(r))} \mathcal{D}^k_r \varphi_k(r) = 0, & \text{(6.1.26c)} \\ \dfrac{1}{j''(u_0(r))} \mathcal{D}^k_r \varphi_k(r) \in C^0(0,1), & \text{(6.1.26d)} \\ \Big(r^{-k} \varphi_k(r) \Big)' \in C^0(0,1), & \text{(6.1.26e)} \\ \varphi_k(r) \in C^0(0,1), & \text{(6.1.26f)} \\ \varphi_k(1) = 0. & \text{(6.1.26g)} \end{cases}$$

For simplicity, we only prove that the system (6.1.26) admits a unique solution for $k = 1$, but the same proof also works for $k \neq 1$ (slightly adjusting the case $k = 0$)). Therefore, we drop the dependence on k, and we write

$$\mathcal{D}_r := \mathcal{D}^1_r, \qquad \psi_\beta := \psi_{1,\beta}, \qquad \mathcal{I} := \mathcal{I}_1 = \{1, 3, 1-n, 3-n\},$$

and

$$\varphi_1(r) = \begin{cases} \displaystyle\sum_{\beta \in \mathcal{I}} c^-_\beta \psi_\beta(r) & r \le r_*, \\ \displaystyle\sum_{\beta \in \mathcal{I}} c^+_\beta \psi_\beta(r) & r > r_*. \end{cases}$$

Noticing that

$$\mathcal{D}_r \psi_\beta(0) = 0 \qquad\qquad \beta = 1, 3, 1-n,$$

we have that (6.1.26a) reads

$$c^-_{3-n} = 0. \tag{6.1.27}$$

Since $j \in C^2(\mathbb{R} \setminus \{0\})$, we have that $j''(u_0) \in L^\infty(0, r_*)$, and $\psi_3(0) = 0$. Therefore, (6.1.26b) reads

$$c^-_{1-n} = 0. \tag{6.1.28}$$

Using the equations (6.1.25) solved by the functions ψ_β, we have that (6.1.26c) reads

$$c^+_{3-n} = -c^+_3. \tag{6.1.29}$$

Moreover, using also (6.1.27) and (6.1.28) for the values of c^-_{1-n} and c^-_{3-n}, we get that (6.1.26d) reads

$$c^-_3 r_* = c^+_3 r_* + c^+_{3-n} r_*^{1-n},$$

which can be rewritten as

$$c^-_3 = (1 - r_*^{-n}) c^+_3. \tag{6.1.30}$$

We now write (6.1.26e) using the explicit expressions of the functions ψ_β, and the fact that $c_{1-n}^- = c_{3-n}^- = 0$, to obtain

$$c_3^- r_*^{-n-1}\gamma_3 = -nc_{1-n}^+ r_*^{-n-1} + c_3^+ r_*^{-n-1}\gamma_3 + c_{3-n}^+ r_*^{-n-1}\gamma_{3-n},$$

where

$$\gamma_\beta = r_*^{n+1}(r^{-1}\psi_\beta(r))'(r_*) = \int_0^{r_*} j''(u_0(t))t^{n+\beta-2}\,dt. \tag{6.1.31}$$

Using (6.1.30) and (6.1.29), we rewrite

$$c_{1-n}^+ = \frac{1}{n}c_3^+(\gamma_3 r_*^{-n} - \gamma_{3-n}). \tag{6.1.32}$$

Using again $c_{1-n}^- = c_{3-n}^- = 0$, then (6.1.26f) reads

$$r_* = c_1^+ r_* + c_{1-n}^+ r_*^{1-n},$$

which joint with (6.1.32) ensures

$$c_1^+ = 1 - \frac{c_3^+}{n}(\gamma_3 r_*^{-n} - ga_{3-n}). \tag{6.1.33}$$

Finally, we may evaluate c_3^+ thanks to equation (6.1.26g), which reads

$$c_1^+ + c_3^+\psi_3(1) + c_{1-n}^+ + c_{3-n}^+\psi_{3-n}(1) = 0.$$

Substituting the values of c_1^+, c_{1-n}^+, and c_{3-n}^+ computed respectively in (6.1.33), (6.1.32), and (6.1.29), we get

$$c_3^+ = \frac{n}{(\gamma_3 r_*^{-n} + \gamma_{3-n})(r_*^{-n} - 1) + \psi_{3-n}(1) - \psi_3(1)}. \tag{6.1.34}$$

Notice that $\psi_3(1)$ and $\psi_{3-n}(1)$ are well defined and finite. Equation (6.1.34) concludes the proof of the existence and the explicit computation of φ_1.

We can now compute the eigenvalue λ_1. Using the equations (6.1.25) solved by the functions ψ_β, we have that $\mathcal{D}_r\psi_\beta \equiv 0$ for $\beta = 1, 1-n$. In addition, since $c_{3-n}^- = 0$ (as computed in (6.1.27)), then

$$\begin{aligned}\rho\lambda_1 &= \frac{\left[\partial_r\Big((j''(u_0(r)))^{-1}\mathcal{D}_r\varphi_1\Big)\right](r_*)}{\varphi_1(r_*)}\\ &= \frac{c_3^+ - c_3^- + (1-n)c_{3-n}^+ r_*^{-n}}{r_*}.\end{aligned}$$

We can evaluate c_3^- with (6.1.30), and $c_{3-n}^+ = -c_3^+$ (by (6.1.27)), so that, substituting the value of c_3^+, computed in (6.1.34),

$$\begin{aligned}r_*\rho\lambda_1 &= nr_*^{-n}c_3^+\\ &= \frac{n^2 r_*^{-n}}{(\gamma_3 r_*^{-n} - \gamma_{3-n})(r_*^{-n} - 1) + n(\psi_{3-n}(1) - \psi_3(1))}.\end{aligned} \tag{6.1.35}$$

To conclude the proof, we need to explicitly compute the values of γ_β and $\psi_\beta(1)$ for $\beta = 3, 3-n$. First we note that by Fubini theorem

$$\begin{aligned}
\psi_\beta(1) &= \int_{r_*}^1 s^{-n-1} \int_0^s j''(u_0(t)) t^{n+\beta-2}\, dt\, ds \\
&= \int_0^1 \int_{r\vee r_*}^1 j''(u_0(t)) t^{n+\beta-2} s^{-n-1}\, ds\, dt \\
&= \frac{1}{n} \int_0^1 j''(u_0(t)) t^{n+\beta-2} ((r\vee r_*)^{-n} - 1)\, dt \\
&= \frac{r_*^{-n} - 1}{n} \gamma_\beta + \frac{1}{n} \eta_\beta,
\end{aligned} \tag{6.1.36}$$

where

$$\eta_\beta = \int_{r_*}^1 j''(u_0(t)) t^{n+\beta-2} (t^{-n} - 1)\, dt.$$

With these notations, thanks to (6.1.36), (6.1.35) reads

$$\frac{n^2}{r_* \rho \lambda_1} = \gamma_3 r_*^{-n} (1 - r_*^n)^2 + (\eta_{3-n} - \eta_3) r_*^n. \tag{6.1.37}$$

In particular, we can compute

$$\begin{aligned}
\eta_3 - \eta_{3-n} &= \int_{r_*}^1 j''(u_0(t)) (t^{n+1} - t)(1 - t^{-n})\, dt \\
&= \int_{r_*}^1 j''(u_0(t))\, t^{1-n} (1 - t^n)^2\, dt.
\end{aligned}$$

Therefore, recalling the value of γ_β in (6.1.31), and recalling the value of ρ computed in (6.1.23), we can rewrite (6.1.37) as

$$\begin{aligned}
\frac{n^2}{r_* \rho \lambda_1} &= r_*^{-n} \int_0^1 j''(u_0(t)) (t\wedge r_*)^{2n} t^{1-n} (1 - (t\vee r_*)^n)^2\, dt \\
&= \frac{n^2}{r_* \rho} \left(1 - \fint_0^1 \left(2(t\vee r_*)^n - (t\vee r_*)^{2n} \right) d\mu(t) \right),
\end{aligned} \tag{6.1.38}$$

with

$$d\mu(t) = j''(u_0(t)) (t\wedge r_*)^{2n} t^{1-n}\, dt.$$

Hence, (6.1.38) is equivalent to

$$1 - \frac{1}{\lambda_1} = \fint_0^1 \left(2(t\vee r_*)^n - (t\vee r_*)^{2n} \right) d\mu(t) > 0,$$

which concludes the proof.

$\square$

Proposition 6.1.29. *Let $j(s) = |s|^q$ for some $q \geq 2$. Let $Y_{k,m}$ be the spherical harmonics and Λ_k the respective eigenvalues (see Definition 6.1.7). The operator T_ρ can be diagonalized, and $Y_{k,m}$ form an orthornormal basis of eigenfunctions in $L^2(\partial B_*)$. In particular, for every $m = 1, \dots, M(k)$,*

$$T_\rho Y_{k,m} = \frac{1}{\lambda_k} Y_{k,m},$$

where

$$\lambda_k = \lambda_k(j, m) = \frac{1}{\rho \varphi_k(r_*)} \left[\partial_r \left(\frac{1}{j''(u_0)} \mathcal{D}_r^k \varphi_k \right) \right](r_*). \tag{6.1.39}$$

Moreover, λ_k is non-decreasing in k, and, in particular, λ_0 is the eigenvalue relative to the constant eigenfunction $Y_{0,1}$.

Proof. Let us define $|x| = r$ and $|x|^{-1}x = \theta$. To show that $Y_{k,m}(\theta)$ are eigenfunctions for T_ρ, let us define the function

$$W_{k,m}(x) = \varphi_k(r) Y_{k,m}(\theta),$$

where φ_k are given in Proposition 6.1.28. Let us notice that for every $\varphi \in C^2(0,1)$, writing the Laplacian in radial coordinates, we get

$$\Delta(\varphi(r) Y_{k,m}(\theta)) = \left(\Delta_r - \frac{1}{r^2} \Delta_{\mathbb{S}^{n-1}} \right) (\varphi(r) Y_{k,m}(\theta)) = Y_{k,m}(\theta) \left(\Delta_r - \frac{\Lambda_k}{r^2} \right) \varphi(r).$$

Therefore, we can compute

$$\Delta \left(\frac{1}{j''(u_0)} \Delta W_{k,m} \right)(x) = Y_{k,m}(\theta)\, \mathcal{D}_r^k \left(\frac{1}{j''(u_0)} \mathcal{D}_r^k \varphi_k \right)(r) = 0.$$

Analogously, we have that $W_{k,m}$ solves the boundary conditions

$$W_{k,m} = \Delta W_{k,m} = 0 \qquad \text{on } \partial B_1.$$

Finally, recalling the notation

$$[v](x) = \lim_{\substack{y \to x \\ y \in B_*^c}} v(y) - \lim_{\substack{y \to x \\ y \in B_*}} v(y), \qquad x \in \partial B_*$$

we compute on ∂B_*

$$\left[\partial_\nu \left(\frac{1}{j''(u_0)} \Delta W_{k,m} \right) \right] = \left[\partial_r \left(\frac{1}{j''(u_0)} \mathcal{D}_r^k \varphi_k \right) \right] Y_{k,m}(\theta). \tag{6.1.40}$$

This in particular implies that

$$\left[\partial_\nu \frac{1}{j''(u_0)} \Delta W_{k,m} \right] = \rho \lambda_k W_{k,m} \qquad \text{on } \partial B_*,$$

thus $Y_{k,m}$ is an eigenfunction for T_ρ relative to the eigenvalue $1/\lambda_k$. Since $\{Y_{k,m}\}_{k,m}$ is a complete system of $L^2(\partial B_*)$, this concludes the proof of the diagonalization of T_ρ.

We now prove the monotonicity property. Let us fix the right rescaling of φ_k in such a way that

$$\left[\partial_r \left(\frac{1}{j''(u_0)} \mathcal{D}_r^k \varphi_k \right) \right](r_*) = 1, \tag{6.1.41}$$

so that

$$\lambda_k = \frac{1}{\rho\, \varphi_k(r_*)}.$$

Therefore, let us divide the proof in two steps: first we show that $\varphi_k \geq 0$ for every k, and then we prove the monotonicity of φ_k studying the difference $\psi_k := \varphi_{k+1} - \varphi_k$.

Step 1: let

$$\Phi_k = \frac{1}{j''(u_0)}\left(\Delta_r - \frac{\Lambda_k}{r^2}\right)\varphi_k,$$

so that

$$\begin{cases} \Delta_r \Phi_k = \dfrac{\Lambda_k}{r^2}\Phi_k & r \in (0, r_*) \cup (r_*, 1), \\ \Phi_k(0) = \Phi_k(1) = 0 \end{cases} \tag{6.1.42}$$

We claim that $\Phi_k \leq 0$. Indeed, let us assume that $\bar{r}$ is a maximum point for Φ_k. If $\bar{r} = 1$ or $\bar{r} = 0$, the claim is trivial. On the other hand, the constraint (6.1.41) implies that

$$\Phi_k'(r_*^+) - \Phi_k'(r_*^-) = 1 > 0,$$

which ensures that r_* cannot be a maximum point, namely $\bar{r} \neq r_*$. Therefore, we may evaluate the ODE (6.1.42) in the maximum point $\bar{r}$ obtaining, after the expansion of Δ_r,

$$0 \geq \Phi_k''(\bar{r}) = \Delta_r \Phi_k(\bar{r}) = \frac{\Lambda_k}{\bar{r}^2}\Phi_k(\bar{r}),$$

which proves the claim.

Since $\Phi_k \leq 0$, then

$$\begin{cases} \Delta_r \varphi_k \leq \dfrac{\Lambda_k}{r^2}\varphi_k & r \in (0,1), \\ \varphi_k(0) = \varphi_k(1) = 0. \end{cases}$$

The same argument used for Φ_k, simplified by the fact that φ_k is of class C^2 in the whole interval $(0,1)$, brings to the conclusion $\varphi_k \geq 0$.

Step 2: let $\psi_k := \varphi_{k+1} - \varphi_k$, and let $\Psi_k := \Phi_{k+1} - \Phi_k$. We claim that $\Psi_k \geq 0$. First we notice that, since $\Phi_k \leq 0$,

$$\left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\Phi_{k+1} = 0 = \left(\Delta_r - \frac{\Lambda_k}{r^2}\right)\Phi_k \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\Phi_k,$$

so that

$$\left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)(\Phi_{k+1} - \Phi_k) \leq 0. \tag{6.1.43}$$

Since (6.1.41) holds for both k and $k+1$, then $\Psi_k \in C^1(0,1)$, but the ODE solved by Ψ_k implies $\Psi_k \in C^2(0,1)$. As done for φ_k in Step 1, (6.1.43) ensures that $\Psi_k \geq 0$.

Finally, $\Psi_k \geq 0$ reads

$$0 \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)\varphi_{k+1} - \left(\Delta_r - \frac{\Lambda_k}{r^2}\right)\varphi_k \leq \left(\Delta_r - \frac{\Lambda_{k+1}}{r^2}\right)(\varphi_{k+1} - \varphi_k),$$

where we have used in the last inequality $\Lambda_k \varphi_k \leq \Lambda_{k+1}\varphi_k$. As before, this implies $\varphi_{k+1} \leq \varphi_k$. $\square$

Proof of Proposition 6.1.24. For simplicity, we let $\{g_j\}_j \in \mathbb{N}$ be an orthonormal basis of eigenfunctions for T_ρ in such a way that

$$T_\rho g_j = \frac{1}{\tilde{\lambda}_j} g_j,$$

and $\tilde{\lambda}_j$ are non-decreasing (to be precise, $\tilde{\lambda}_j = \lambda_k$ for some $k(j)$ non-decreasing in j). We recall the notation $g_\Phi = \Phi \cdot \nu$, and that by construction

$$L''(0) = l_2(g_\Phi, g_\Phi).$$

The functional l_2 is bilinear, and as observed in Remark 6.1.25

$$l_2(g_j, g_k) = \frac{1}{\rho}\left(\frac{1}{\tilde{\lambda}_k} - 1\right)\delta_{jk},$$

where δ_{jk} is the Kronecker delta. We can decompose $g_\Phi = \sum_k \alpha_k g_k$, and since the first eigenfunction $g_0 = Y_{0,1}$ is constant (as proved in Proposition 6.1.29), then α_0 is the mean of g_Φ. Hence, using the fact that $\tilde{\lambda}_k$ is non-decreasing,

$$\begin{aligned}
L''(0) &= -\sum_{k,j=0}^{+\infty} \alpha_k \alpha_j l_2(g_k, g_j)\\
&= -\frac{1}{\rho}\alpha_0^2\left(1 - \frac{1}{\lambda_0}\right) - \frac{1}{\rho}\sum_{k=1}^{+\infty}\alpha_k^2\left(1 - \frac{1}{\tilde{\lambda}_k}\right)\\
&\leq -\frac{1}{\rho}\alpha_0^2\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right) - \frac{1}{\rho}\left(1 - \frac{1}{\tilde{\lambda}_1}\right)\|g_\Phi\|_2^2.
\end{aligned}$$

Since $\lambda_1 > 1$ from Proposition 6.1.28, to conclude the proof it is sufficient to notice that if we choose $\eta > 0$ small enough then

$$\alpha_0^2\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_0}\right) \geq -\frac{1}{2}\left(1 - \frac{1}{\lambda_1}\right)\|g_\Phi\|_2.$$

Indeed, from the volume constraint,

$$0 = \frac{1}{n}\int_{\partial B_*}((1+g_\Phi)^n - 1)\,d\mathcal{H}^{n-1} = \alpha_0 + \frac{1}{n}\sum_{k=2}^{n}\binom{n}{k}\int_{\partial B_*} g_\Phi^k\,d\mathcal{H}^{n-1} \geq |\alpha_0| - C\eta\|g_\Phi\|_2^2,$$

thus concluding the proof. □

6.1.6 Step 3: improved continuity of the second order shape derivative

We introduce here some useful notations.

Definition 6.1.30. Let X, Y be two normed vector spaces, and let

$$F : X \to Y.$$

We will write

$$F(x) = \omega_Y^X(x)$$

to indicate that

$$\lim_{\|x\|_X \to 0} \|F(x)\|_Y = 0.$$

In particular, when $Y = \mathbb{R}$ we only write $\omega_Y^X = \omega^X$. Moreover, when $Y = W^{k,p}(E;\mathbb{R}^n)$ and $X = W^{j,q}(F;\mathbb{R}^n)$ then we write

$$\omega_Y^X = \omega_{k,p,E}^{j,q,F}.$$

When $Y = L^p(E)$ we write $\omega_Y^X = \omega_{p,E}^X$, and when possible we drop the dependence on the sets E, F.

Let $L(t) = L_\tau(B_*^{t\Phi})$. The main aim of this section is to prove the following.

Proposition 6.1.31. *Let $m \in (0, |B_1|)$, let $j(s) = |s|^q$ for some $q \geq 2$, and let $p > n$. For every $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$ such that Φ is orthogonal to ∂B_* it holds*

$$L''(t) = L''(0) + \omega^{2,p}(\Phi)\, \|\Phi\|_{2,\partial B_*}^2,$$

i.e. there exists a positive constant $C = C(p,j,m)$ such that for every $\varepsilon > 0$ there exists $\eta(\varepsilon,p,j,m) > 0$ such that if

$$\|\Phi\|_{2,p} \leq \eta,$$

then

$$|L''(t) - L''(0)| \leq \varepsilon C \|\Phi\|_{L^2(\partial B_*)}^2$$

for every $t \in [0,1]$.

To simplify the computation of $L''(t)$, we want to replace Φ by a deformation which is constant along the direction normal to ∂B_*, as in Remark 6.1.23. If we let $E_t = B_*^{t\Phi}$, then $L(t) = \mathcal{L}_\tau(E_t)$ only depends on the values of Φ on $\partial\Omega$. In particular, if we have $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$ for some $p > n$, and we define $\Psi(x) = \Phi(\pi_{\partial B_*}(x))$, then, up to extending Ψ by multiplying it by a cutoff function, we have that $\Psi \in W^{2,p}(B_1;\mathbb{R}^n)$ and $\Psi = \Phi$ on ∂B_*. This implies that $B_*^{t\Phi} = B_*^{t\Psi}$ for every t, and in particular $L(t) = \mathcal{L}_\tau(B_*^{t\Psi})$. Therefore, it is not restrictive to assume that we are in the assumptions of Remark 6.1.23, and (after a change of variables)

$$L''(t) = b_t(\Phi,\Phi) + \int_{\partial B_*} J_t^\tau \left(\widehat{w_t'} g + g^2 \widehat{\nabla w_t} \cdot \hat{\nu}_t \right) d\mathcal{H}^{n-1}, \tag{6.1.44}$$

where for a generic function h we have denoted by $\hat{h} = h \circ (\mathrm{Id} + t\Phi)^{-1}$,

$$b_t(\Phi,\Phi) = \int_{\partial B_*} J_t^\tau (\hat{w}_t - w_0)(\hat{\nu}_t \cdot \nu_0) g^2\, \widehat{\mathrm{div}(\nu_0)}\, d\mathcal{H}^{n-1}, \tag{6.1.45}$$

and $J_t^\tau = \mathrm{Jac}^{\partial B_*}(\mathrm{Id} + t\Phi)$. Notice also that we used $\hat{g} = g$ and $\hat{\nu_0} = \nu_0$. In particular, when $t = 0$,

$$L''(0) = \int_{\partial B_*} \left(w_0' \, g + \frac{\partial w_0}{\partial \nu_0} g^2 \right) d\mathcal{H}^{n-1}. \tag{6.1.46}$$

To prove Proposition 6.1.31 we need some geometric estimates on $\hat{\nu}_t$ and $I_n + tD\Phi$, and estimates on $\hat{w}_t$ and $\widehat{w_t'}$ that will be resumed in the following lemmas.

Lemma 6.1.32 ([84, Lemma 4.3, Lemma 4.7],[151, Lemma 3.7])**.** *Let $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$. Then*

$$(I_n + tD\Phi)^{-1} = I_n + \omega_\infty^{2,p}(\Phi) \qquad \det(I_n + tD\Phi) = 1 + \omega_\infty^{2,p}(\Phi)$$

$$\hat{\nu}_t = \nu_0 + \omega_\infty^{2,p}(\Phi), \qquad J_t^\tau = 1 + \omega_\infty^{2,p}(\Phi),$$

Lemma 6.1.33. *Let $m \in (0, |B_1|)$, $j(s) = |s|^q$ for some $q \geq 2$, let $\Phi \in W^{2,p}(B_1; \mathbb{R}^n)$ with $p > n$, let $u_t := u_{B_*^{t\Phi}}$, and let $\widehat{u_t'} = u_t' \circ (\mathrm{Id} + t\Phi)$. Then there exist constants $C = C(r_*)$ and $\delta = \delta(r_*)$ such that if $\|\Phi\|_{2,p} \leq \delta$ then*

$$\|\widehat{u_t'}\|_{1,2} \leq C\|\Phi\|_{2,\partial B_*}. \tag{6.1.47}$$

Moreover, we have

$$\widehat{u_t'} = u_0' + \omega_{1,2}^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}. \tag{6.1.48}$$

Proof. Let us recall that, by Proposition 6.1.18, u_t' solves the equation

$$\int_{B_1} \nabla u_t' \cdot \nabla \varphi \, dx = \int_{\partial B_*^{t\Phi}} \varphi \, \tilde{\Phi} \cdot \nu_t \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H_0^1(B_1).$$

In particular, with the change of variables $x = (\mathrm{Id} + t\Phi)(y)$, we get

$$\int_{B_1} A_t \nabla \widehat{u_t'} \cdot \nabla \varphi \, dy = \int_{\partial B_*} J_t^\tau \varphi \, \hat{g}_t \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H_0^1(B_1), \tag{6.1.49}$$

where

$$J_t^\tau := \mathrm{Jac}^{\partial B_*}(\mathrm{Id} + t\Phi), \qquad A_t := \det(I_n + tD\Phi)(I_n + tD\Phi)^{-1}(I_n + tD\Phi)^{-T}.$$

Choosing $\varphi = \widehat{u_t'}$, and noticing that by Lemma 6.1.32 we have that A_t is uniformly elliptic for small $\|\Phi\|_{2,p}$, then there exists a positive constant C such that

$$\|\nabla \widehat{u_t'}\|_2^2 \leq C \int_{\partial B_*} |\hat{g}_t \widehat{u_t'}| \, d\mathcal{H}^{n-1}.$$

Using Poincaré inequality, Young inequality, and the embedding $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_*)$, we get for every $\eta > 0$

$$\|\widehat{u_t'}\|_{1,2}^2 \leq C\Big(\eta \|\hat{g}_t\|_2^2 + \frac{1}{\eta}\|\widehat{u_t'}\|_{1,2}^2\Big).$$

Formula (6.1.47) follows by choosing a suitable η, and by recalling $|\hat{g}_t| = |\Phi \cdot \hat{\nu}_t| \leq |\Phi|$.

Analogously, subtracting the weak equations (6.1.55) solved by $\widehat{u_t'}$ and u_0', we have

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0') \cdot \nabla \varphi \, dy = \int_{\partial B_*} (J_t^\tau \hat{g}_t - g_0)\varphi \, d\mathcal{H}^{n-1} \qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.50}$$

We evaluate (6.1.50) with the test function $\varphi = \widehat{u_t'} - u_0'$. We estimate the left-hand side of (6.1.50) by adding and subtracting $A_t \nabla u_0' \cdot \nabla \varphi$, and using the uniform ellipticity of A_t joint with Lemma 6.1.32, so that for some constant $C > 0$

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0') \cdot \nabla \varphi \, dy \geq C\|\nabla(\widehat{u_t'} - u_0')\|_2^2 - C\int_{B_1} \omega_\infty^{2,p}(\Phi) \Big| \nabla u_0' \cdot \nabla(\widehat{u_t'} - u_0') \Big| \, dy.$$

Thanks to (6.1.47), we get

$$\int_{B_1} (A_t \nabla \widehat{u_t'} - \nabla u_0') \cdot \nabla \varphi \, dy \geq C\|\nabla(\widehat{u_t'} - u_0')\|_2^2 - C\omega^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}^2. \tag{6.1.51}$$

On the other hand, we may estimate the right-hand side of (6.1.50) using Young's inequality, the geometric estimates in Lemma 6.1.32, and the embedding $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_*)$ as follows

$$
\begin{aligned}
\int_{\partial B_*} (J_t^\tau \hat{g}_t - g_0)\varphi \, d\mathcal{H}^{n-1} &= \int_{\partial B_*} (1 + \omega_\infty^{2,p}(\Phi))\Phi \cdot \omega_\infty^{2,p}(\Phi)(\widehat{u_t'} - u_0') \, d\mathcal{H}^{n-1} \\
&\leq C\omega^{2,p}(\Phi)\Big(\eta \|\Phi\|_{2,\partial B_*}^2 + \frac{1}{\eta}\|\widehat{u_t'} - u_0'\|_{1,2}\Big)
\end{aligned} \tag{6.1.52}
$$

for every $\eta > 0$ and a suitable constant $C > 0$. Joining (6.1.50), (6.1.51), (6.1.52), and the Poincaré inequality, with the right choice of η we get

$$
\|\widehat{u_t'} - u_0'\|_{1,2}^2 \leq \omega^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}^2,
$$

which implies (6.1.48). □

A similar argument brings to the following estimates for $\widehat{w_t'}$.

Lemma 6.1.34. *Let $m \in (0, |B_1|)$, $j(s) = |s|^q$ for some $q \geq 2$, let $\Phi \in W^{2,p}(B_1; \mathbb{R}^n)$ with $p > n$, let $w_t := w_{B_*^{t\Phi}}$ be the adjoint state defined in Definition 6.1.15, and let $\widehat{w_t'} = w_t' \circ (\mathrm{Id} + t\Phi)$. Then there exist constants $C = C(j, m)$ and $\delta = \delta(j, m)$ such that if $\|\Phi\|_{2,p} \leq \delta$ then*

$$
\|\widehat{w_t'}\|_{1,2} \leq C\|\Phi\|_{2,\partial B_*}. \tag{6.1.53}
$$

Moreover, we have

$$
\widehat{w_t'} = w_0' + \omega_{1,2}^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}. \tag{6.1.54}
$$

Proof. Let us recall that, by Proposition 6.1.19, w_t' solves the equation

$$
\int_{B_1} \nabla w_t' \cdot \nabla \varphi \, dx = \int_{B_1} j'(u_t) u_t' \varphi \, dx \qquad \forall \varphi \in H_0^1(B_1).
$$

In particular, with the change of variables $x = (\mathrm{Id} + t\Phi)(y)$, we get

$$
\int_{B_1} A_t \nabla \widehat{w_t'} \cdot \nabla \varphi \, dy = \int_{B_1} J_t \, j'(\hat{u}_t) \widehat{u_t'} \varphi \, dy \qquad \forall \varphi \in H_0^1(B_1), \tag{6.1.55}
$$

where

$$
J_t := \det(I_n + tD\Phi), \qquad A_t := J_t (I_n + tD\Phi)^{-1}(I + tD\Phi)^{-T}.
$$

By standard elliptic estimates joint with the continuity of j', the equi-boundedness of u_t, and the geometric estimates Lemma 6.1.32, we get

$$
\|\widehat{w_t'}\|_{1,2} \leq C\|\widehat{u_t'}\|_2.
$$

Therefore, (6.1.47) in Lemma 6.1.33 implies (6.1.53).

Analogously, we now estimate the norm of $\widehat{w_t'} - w_0'$ rewriting the equation as

$$
\underbrace{\int_{B_1} (A_t \nabla \widehat{w_t'} - \nabla w_0') \cdot \nabla \varphi \, dy}_{I_1} = \underbrace{\int_{B_1} (J_t \, j'(\hat{u}_t) \widehat{u_t'} - j'(u_0) u_0') \varphi \, dy}_{I_2} \qquad \forall \varphi \in H_0^1(B_1). \tag{6.1.56}
$$

We evaluate (6.1.56) with the test function $\varphi = \widehat{w_t'} - w_0'$. As done in Lemma 6.1.33 for (6.1.51),

$$I_1 \geq C\|\nabla(\widehat{w_t'} - w_0')\|_2^2 - C\omega^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}^2. \tag{6.1.57}$$

Analogously, we now estimate the right-hand side of (6.1.56). We first notice that by the shape differentiability of $\hat{u}_t$ in Proposition 6.1.17 we have

$$\hat{u}_t = u_0 + \omega_\infty^{2,p}(\Phi).$$

In particular, the Lipschitz continuity of j' ensures that

$$j'(\hat{u}_t) = j'(u_0) + \omega_\infty^{2,p}(\Phi).$$

Therefore, as done in Lemma 6.1.33 to prove (6.1.52), we use Young's inequality, the geometric estimates in Lemma 6.1.32, the embedding $W^{1,2}(B_1) \hookrightarrow L^2(\partial B_*)$, and the estimates (6.1.53), (6.1.48), so that

$$\begin{aligned} I_2 &\leq C\int_{B_1}\left|(j'(u_0) + \omega_\infty^{2,p}(\Phi))\widehat{u_t'} - j'(u_0)u_0'\right|\left|\widehat{w_t'} - w_0'\right| dy \\ &\leq C\int_{B_1}|j'(u_0)|\left|\widehat{u_t'} - u_0'\right|\left|\widehat{w_t'} - w_0'\right| dy + \omega^{2,p}(\Phi)\int_{B_1}\left|\widehat{u_t'}\right|\left|\widehat{w_t'} - w_0'\right| dy \\ &\leq C\omega^{2,p}(\Phi)\left(\eta\|\Phi\|_{2,\partial B_*}^2 + \frac{1}{\eta}\|\widehat{w_t'} - w_0'\|_{1,2}\right) \end{aligned} \tag{6.1.58}$$

for every $\eta > 0$ and a suitable constant $C > 0$. Joining (6.1.56), (6.1.57), (6.1.58), and the Poincaré inequality, with the right choice of η we get

$$\|\widehat{w_t'} - w_0'\|_{1,2}^2 \leq \omega^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}^2,$$

which implies (6.1.54). □

Proof of Proposition 6.1.31. We divide the proof in two steps. We first provide the improved continuity for b_t and then we study $L''(t) - b_t - L''(0)$.

Step 1: we prove

$$b_t(\Phi, \Phi) = \omega^{2,p}(\Phi)\|\Phi\|_{2,\partial B_*}^2. \tag{6.1.59}$$

Since $\nu_0(x) = |x|^{-1}x$, then we can rewrite (6.1.45) as

$$b_t(\Phi, \Phi) = \int_{\partial B_*} J_t^\tau(\hat{w}_t - w_0)(\hat{\nu}_t \cdot \nu_0)g^2\,\frac{n-1}{r_* + tg}\,d\mathcal{H}^{n-1}.$$

By the geometric estimates Lemma 6.1.32 we have that there exists a constant $C > 0$ such that if $\|\Phi\|_{2,p}$ is small enough, then

$$|J_t^\tau| + |\hat{\nu}_t \cdot \nu_0| + \frac{1}{r_* + tg} < C.$$

Moreover, by shape differentiability of $\hat{w}_t$, proved in Proposition 6.1.20, we have

$$\hat{w}_t = w_0 + \omega_{1,\infty}^{2,p}(\Phi). \tag{6.1.60}$$

These estimates yield

$$b_t(\Phi,\Phi) = \omega^{2,p}(\Phi)\|\Phi\|^2_{2,\partial B_*}. \tag{6.1.61}$$

Step 2: we prove that

$$L''(t) - b_t(\Phi,\Phi) = L''(0) + \omega^{2,p}(\Phi)\|\Phi\|^2_{2,\partial B_*}.$$

Here we recall the evaluation of $L''(t) - b_t(\Phi,\Phi)$ already performed in (6.1.44)

$$L''(t) - b_t(\Phi,\Phi) = \int_{\partial B_*} J_t^\tau \left(\widehat{w_t'} g + g^2 \widehat{\nabla w_t}\cdot \hat{\nu}_t\right) d\mathcal{H}^{n-1}.$$

We may rewrite

$$L''(t) - b_t(\Phi,\Phi) - L''(0) = I_1 + I_2,$$

where

$$I_1 = \int_{\partial B_*} (J_t^\tau \widehat{w_t'} - w_0')g\, d\mathcal{H}^{n-1}, \qquad I_2 = \int_{\partial B_*} g^2 \Big(J_t^\tau(\widehat{\nabla w_t}\cdot \hat{\nu}_t) - (\nabla w_0 \cdot \nu_0)\Big)\, d\mathcal{H}^{n-1}.$$

To estimate I_1, we use Lemma 6.1.32, and Lemma 6.1.34 to get, dropping the dependence on Φ inside the infinitesimal notation (i.e. $\omega_X^Y = \omega_X^Y(\Phi)$),

$$\begin{aligned} I_1 &= \int_{\partial B_*} g\Big(\Big(1+\omega_\infty^{2,p}\Big)\Big(w_0' + \omega^{2,p}_{2,\partial B_*}\|\Phi\|_{2,\partial B_*}\Big) - w_0'\Big)\, d\mathcal{H}^{n-1} \\ &= \omega^{2,p}\|\Phi\|^2_{2,B_*}, \end{aligned}$$

where we used Hölder inequality, $\|g\|_2 = \|\Phi\|_{2,\partial B_*}$, and (6.1.53) in Lemma 6.1.34.

Finally, to estimate I_2 we use again the shape differentiability (6.1.60) and we notice that

$$\widehat{\nabla w_t} = (I_n + tD\Phi)^{-1}\nabla \hat{w}_t. \tag{6.1.62}$$

Hence, using (6.1.62), (6.1.60), and the geometric estimates Lemma 6.1.32, we get, dropping again the dependence on Φ inside the infinitesimal notation

$$\begin{aligned} I_2 &= \int_{\partial B_*} g^2\Big((I_n + \omega_\infty^{2,p})(\nabla w_0 + \omega_\infty^{2,p})\cdot(\nu_0 + \omega_\infty^{2,p}) - (\nabla w_0\cdot \nu_0)\Big)\, d\mathcal{H}^{n-1}. \\ &= \omega^{2,p}\|\Phi\|^2_{2,\partial B_*}. \end{aligned}$$

□

Joining Proposition 6.1.24 with Proposition 6.1.31 we get

Proposition 6.1.35. *Let $m \in (0,|B_1|)$, let $j \in C^2(\mathbb{R}^+)$ be such that $j''(s) > 0$ for every $s > 0$, and let $\Phi \in W^{2,p}(B_1;\mathbb{R}^n)$. There exist positive constants $C = C(j,m)$, $\eta = \eta(j,m)$ such that if Φ is orthogonal to ∂B_* and*

$$\|\Phi\|_{2,p} \le \eta, \qquad |B_*^\Phi| = |B_*| = m,$$

then

$$\mathcal{J}(B_*) - \mathcal{J}(B_*^\Phi) \ge C|B_*\Delta B_*^\Phi|^2.$$

Proof. Let

$$L(t) = L_\tau(B_*^{t\Phi}).$$

By Proposition 6.1.24 we know that for some positive constant C_1 we have

$$L''(0) \leq -C_1 \|\Phi \cdot \nu\|^2_{L^2(\partial B_*)} = -C_1 \|\Phi\|^2_{L^2(\partial B_*)}. \tag{6.1.63}$$

By Proposition 6.1.31, instead, we get that for a suitable choice of η,

$$L''(t) \leq L''(0) + \frac{C_1}{2}\|\Phi\|^2_{L^2(\partial B_*)}. \tag{6.1.64}$$

Joining (6.1.63), (6.1.64), and the optimality condition $L'(0) = 0$, we obtain for some $t_0 \in (0,1)$

$$L_\tau(B_*) - L_\tau(B_*^\Phi) = L(0) - L(1) = -L''(t_0) \geq \frac{C_1}{2}\|\Phi\|^2_{L^2(B_*)}.$$

The result now follows by noticing that, since Φ is orthogonal to ∂B_*, and $\|\Phi\|_\infty$ is arbitrarily small

$$|B_* \Delta B_*^\Phi| = \frac{1}{n}\int_{\partial B_*} |(1+\Phi)^n - 1|\, d\mathcal{H}^{n-1} \leq C(n)\|\Phi\|_{L^1(\partial B_*)} \leq C(n,m)\|\Phi\|_{L^2(\partial B_*)}.$$

□

6.1.7 Step 4: local stability implies global stability

Proposition 6.1.36. *Let j be a continuous function on $\mathbb{R}$. If there exists $\delta_0 > 0$ such that*

$$\inf_{0 \leq \delta \leq \delta_0} \inf_{\substack{V \in \mathcal{M}_m \\ \|V - V_0\|_1 \geq \delta_0}} \frac{\mathcal{J}(V_0) - \mathcal{J}(V)}{\delta^2} > 0, \tag{6.1.65}$$

then Theorem 6.1.2 holds true.

Proof. Under the assumption (6.1.65), to prove Theorem 6.1.2 it is sufficient to show that there exists a positive constant $C = C(j, m)$ such that

$$\|V - V_0\|_1 \geq \delta_0.$$

implies

$$G(V) := \frac{\mathcal{J}(V_0) - \mathcal{J}(V)}{\|V - V_0\|_1^2} > C.$$

Let V_k be a minimizing sequence for the problem

$$\inf_{\substack{V \in \mathcal{M}_m \\ \|V - V_0\|_1 \geq \delta_0}} G(V).$$

By equi-boundedness, we may assume that V_k weakly-$*$ L^∞ converges to some function $V_\infty \in \mathcal{M}_m$. If we apply Remark 6.1.10 to the sequence $V_0 - V_k$ (as done in the proof of Lemma 6.1.11), we get

$$\|V_\infty - V_0\|_1 = \lim_k \|V_k - V_0\|_1 \geq \delta_0.$$

Therefore, V_∞ is still admissible, and using the uniqueness of the maximizer V_0 (Proposition 6.1.14), we obtain $\mathcal{J}(V_\infty) < \mathcal{J}(V_0)$, while from the continuity of $\mathcal{J}(\cdot)$ (Lemma 6.1.12) we get

$$\min_{\|V - V_0\|_1 \geq \delta_0} G(V) = G(V_\infty) > 0.$$

□

We now prove that we may rewrite the functional $\mathcal{J}$ in a variational formulation, which will be the key tool to prove the stability result.

Proposition 6.1.37 (Legendre-Fenchel transform)**.** *Let $j \in C^1(\mathbb{R})$ be strictly convex and coercive. Then for every $V \in L^\infty(B_1)$*

$$\mathcal{J}(V) = \max_{\varphi \in H^2(B_1)\cap H_0^1(B_1)} \left\{ \int_{B_1} V\varphi \, dx - \int_{B_1} j^\star(-\Delta\varphi)\, dx \right\}.$$

Proof. We first prove that

$$\mathcal{J}(V) \geq \sup_{\varphi \in H^2(B_1)\cap H_0^1(B_1)} \left\{ \int_{B_1} V\varphi \, dx - \int_{B_1} j^\star(-\Delta\varphi)\, dx \right\}. \tag{6.1.66}$$

Since j is convex in $\mathbb{R}^n$, then by Fenchel-Moreau theorem (Theorem 6.1.6) we have that $j = j^{\star\star}$. This in particular implies, by the definition of Legendre-Fenchel transform,

$$j(u_V) = j^{\star\star}(u_V) \geq u_V(-\Delta\varphi) - j^\star(-\Delta\varphi) \qquad\qquad \forall \varphi \in H^2(B_1) \cap H_0^1(B_1).$$

We integrate over B_1, and we integrate by parts, so that the weak equation for u_V yields (6.1.66).

For the converse inequality, we take $\varphi = w_V$ the adjoint state defined in Definition 6.1.15, and we prove that

$$\mathcal{J}(V) = \int_{B_1} V w_V - \int_{B_1} j^\star(-\Delta w_V)\, dx. \tag{6.1.67}$$

Indeed, since j is strictly convex, then j' is invertible, and the equation solved by w_V (6.1.10) can be read as

$$u_V = (j')^{-1}(-\Delta w_V). \tag{6.1.68}$$

Moreover, by the definition of Legendre-Fenchel transform,

$$j^\star(y) = (j')^{-1}(y)y - j\Big((j')^{-1}(y)\Big),$$

so that

$$j\Big((j')^{-1}(y)\Big) = (j')^{-1}(y)y - j^\star(y). \tag{6.1.69}$$

Joining (6.1.68) and (6.1.69) for $y = -\Delta w_V$, we get

$$\mathcal{J}(V) = \int_{B_1} j(u_V)\, dx = \int_{B_1} u_V(-\Delta w_V)\, dx - \int_{B_1} j^\star(-\Delta w_V)\, dx.$$

Finally, (6.1.67) follows by noticing that integration by parts and the weak equations for u_V (see (6.1.4)) give

$$\int_{B_1} u_V(-\Delta w_V)\, dx = \int_{B_1} \nabla u_V \cdot \nabla w_V \, dx = \int_{B_1} V w_V \, dx.$$

$\square$

Proof of Theorem 6.1.2. By Proposition 6.1.13 we have that for every $\delta > 0$ there exist a set E_δ solution to the problem

$$\mathcal{J}(E_\delta) = \max_{V \in \mathcal{M}_m^\delta} \mathcal{J}(V).$$

Thanks to Proposition 6.1.36 it is sufficient to prove that

$$\liminf_{\delta \to 0} \frac{\mathcal{J}(B_*) - \mathcal{J}(E_\delta)}{\delta^2} > 0.$$

Let us assume by contradiction, up to extracting a subsequence, that

$$\lim_{\delta \to 0} \frac{\mathcal{J}(B_*) - \mathcal{J}(E_\delta)}{\delta^2} = 0.$$

For every δ we define $u_\delta := u_{E_\delta}$, and $w_\delta := w_{E_\delta}$ the adjoint states defined in Definition 6.1.15, and we let

$$\tilde{E}_\delta := \{w_\delta > t_\delta\}$$

be the level set such that $|\tilde{E}_\delta| = |B_*| = m$; such a t_δ always exists becasue w_δ is C^1 and superharmonic, and it cannot have plateaus, see also Remark 2.4.5. We now show that $\tilde{E}_\delta$ is a $W^{2,p}$ deformation of B_* with $p > n$ big. By classical elliptic estimates (Theorem 6.1.4) we have that u_δ converges to u_0 in $W^{2,p}(B_1)$, and by the regularity of j we also get that w_δ converges to w_0 in $W^{2,p}(B_1)$. Moreover, since $0 < m < |B_1|$, we also have that there exists a positive constant c such that $|\nabla w_0| \geq c$ on ∂B_*. Therefore, we are in the assumptions to apply Lemma A.0.1 to w_δ and Proposition A.0.6 to $w_\delta - t_\delta$, so that the following holds: for every fixed $\varepsilon > 0$ and for $\delta < \delta_0(\varepsilon)$ we can find deformations $\Phi_\delta \in W^{2,p}(B_1; \mathbb{R}^n)$ such that $\|\Phi_\delta\|_{2,p} \leq \varepsilon$, Φ_δ are orthogonal to ∂B_*, and

$$\tilde{E}_\delta = B_*^{\Phi_\delta}.$$

Moreover, we find constants $c, C > 0$ such that

$$\begin{aligned} |\nabla w_\delta|(x) \geq c \qquad \forall x \in \partial B_* \\ \mathcal{H}^{n-1}(\{w_\delta = t_\delta\}) = \int_{\partial B_*} \operatorname{Jac}^{\partial B_*}(\operatorname{Id} + \Phi_\delta)\, d\mathcal{H}^{n-1} \leq C. \end{aligned} \tag{6.1.70}$$

Thanks to (6.1.70), the quantitative bathtub principle stated in Theorem 6.1.8 and the variational formulation computed in Proposition 6.1.37, we get

$$\begin{aligned} \mathcal{J}(E_\delta) &= \int_{E_\delta} w_\delta\, dx - \int_{B_1} j^\star(-\Delta w_\delta)\, dx \\ &\leq \int_{\tilde{E}_\delta} w_\delta\, dx - \int_{B_1} j^\star(-\Delta w_\delta)\, dx - C|E_\delta \Delta \tilde{E}_\delta|^2 \\ &\leq \mathcal{J}(\tilde{E}_\delta) - C|E_\delta \Delta \tilde{E}_\delta|^2 \end{aligned} \tag{6.1.71}$$

for some positive constant C. On the other hand, since $\|\Phi\|_{2,p} < \varepsilon$, we may choose ε small enough to apply the local stability (Proposition 6.1.35) to get

$$\mathcal{J}(B_*) - \mathcal{J}(\tilde{E}_\delta) \geq C|\tilde{E}_\delta \Delta B_*|^2. \tag{6.1.72}$$

Finally, noticing that

$$\delta = |E_\delta \Delta B_*| \leq |E_\delta \Delta \tilde{E}_\delta| + |\tilde{E}_\delta \Delta B_*|,$$

we join (6.1.71) and (6.1.72), and we get for some constant C

$$\begin{aligned} C\delta^2 &\leq 2C\Big(|E_\delta \Delta \tilde{E}_\delta|^2 + |\tilde{E}_\delta \Delta B_*|^2\Big) \\ &\leq \mathcal{J}(B_*) - \mathcal{J}(E_\delta), \end{aligned}$$

which is a contradiction. □

6.1.8 Sharpness of the exponent

We show that the exponent 2 is sharp in Theorem 6.1.2, in the sense of the following Proposition 6.1.39. We first define the following optimal asymmetric radial sets

Definition 6.1.38. Let $m \in (0, |B_1|)$. For every $\delta > 0$ we define the *optimal δ-asymmetric radial set* A_δ as

$$A_\delta = B(0; r_1(\delta)) \cup \{\, x \in \mathbb{R}^n \mid r_* \leq |x| \leq r_2(\delta)\,\},$$

where r_* is the radius of the ball of volume m, and r_1 and r_2 are chosen in such a way that

$$|A_\delta \triangle B^*| = \delta.$$

Namely, we have

$$r_1(\delta) = \frac{1}{|B_1|^{1/n}}\left(m - \frac{\delta}{2}\right)^{\frac{1}{n}},$$

and

$$r_2(\delta) = \frac{1}{|B_1|^{1/n}}\left(\frac{\delta}{2} + m\right)^{\frac{1}{n}}$$

Proposition 6.1.39. *Let $m \in (0, |B_1|)$, and let $j(s) = |s|^q$ for some $q \geq 2$. Then there exists $\bar{\delta} > 0$ and a positive constant $C = C(j, m)$ such that if $0 < \delta < \bar{\delta}$, then*

$$\frac{1}{C}\delta^2 \leq \mathcal{J}(B_*) - \mathcal{J}(A_\delta) \leq C\delta^2.$$

Proof. Let us define $V_\delta = \chi_{A_\delta}$, $V_0 = \chi_{B_*}$, and $h = V_0 - V_\delta$. For every $t \in (0,1)$, let

$$V_{\delta,t} := V_\delta + th,$$

and by definition of A_δ, we have $V_t \in \mathcal{M}_m$. In the following, we will denote $u_{\delta,t} := u_{V_{\delta,t}}$ and $w_{\delta,t} := w_{V_{\delta,t}}$ the adjoint state. Moreover, since these functions are all radial, we will identify, with a slight abuse of notation

$$u_{\delta,t}(x) = u_{\delta,t}(|x|), \qquad w_{\delta,t}(x) = w_{\delta,t}(|x|).$$

The proof is divided in two main steps: first we show that $J(t) := \mathcal{J}(V_t)$ is differentiable, so that

$$\mathcal{J}(B_*) - \mathcal{J}(A_\delta) = J(1) - J(0) = J'(\xi)$$

for some $\xi \in (0,1)$; then we show that if δ is small enough then

$$\frac{1}{C}\delta^2 \leq J'(t) \leq C\delta^2 \qquad \forall t \in (0,1). \tag{6.1.73}$$

Step 1: First we notice that, by linearity of the equation solved by $u_{\delta,t}$, and by standard elliptic uniform estimates (see Theorem 6.1.4), we can find some positive constant $C = C(n)$ such that for every $p > n/2$ we have

$$\|u_{\delta,t_2} - u_{\delta,t_1}\|_{2,p} \leq C|t_2 - t_1|\|h\|_p \leq C|t_2 - t_1|\delta^{\frac{1}{p}} \qquad \forall t_1, t_2 \in [0,1]. \tag{6.1.74}$$

Analogously, since j' is Lipschitz, using again standard elliptic regularity, we also get that for some positive constant $C = C(j)$,

$$\|w_{\delta,t_2} - w_{\delta,t_1}\|_{2,p} \leq C\|u_{\delta,t_2} - u_{t_1}\|_\infty \leq C^2|t_2 - t_1|\delta^{\frac{1}{p}}. \tag{6.1.75}$$

Inequality (6.1.74), in particular, proves that $u_{\delta,t}$ is differentiable in $H^1_0(B_1)$, with respect to t: the quotient ratios $q_s = s^{-1}(u_{\delta,t+s} - u_{\delta,t})$ are equi-bounded in $W^{2,p}$, so that, up to a subsequence, q_s converges in $H^1(B_1)$ to some function $u'_{\delta,t}$ for s that goes to 0. In addition, differentiating the weak equation solved by $u_{\delta,t}$, we have that $u'_{\delta,t}$ solves in $H^1_0(B_1)$ the equation

$$\int_{B_1} \nabla u'_{\delta,t} \cdot \nabla\varphi \, dx = \int_{B_1} h\varphi \, dx \qquad \forall \varphi \in H^1_0(B_1). \tag{6.1.76}$$

Since the solution to (6.1.76) is unique, the definition of $u'_{\delta,t}$ does not depend on the choice of the subsequence, and this proves the differentiability. This in particular implies that $J(t)$ is differentiable, and that, using the weak equation solved by the adjoint state (see Definition 6.1.15)

$$J'(t) = \int_{B_1} j'(u_{\delta,t})u'_{\delta,t} \, dx = \int_{B_1} w_{\delta,t} \, h \, dx.$$

Step 2: We now aim to show that for some constant C and for δ small enough we have (6.1.73) for every t. Since $h \neq 0$ only in the set $\{ r_1(\delta) \leq |x| \leq r_2(\delta) \}$, we have that for δ that goes to zero we need to focus our interest on the values of $w_{\delta,t}$ concentrated near $\partial B^* = \{ |x| = r_* \}$. Therefore, we prove a uniform Taylor expansion near ∂B^* of the form

$$w_{\delta,t}(r) = w_{\delta,t}(r_*) + \partial_r w_{\delta,t}(r_*)(r - r_*) + R_1(x, \delta, t),$$

with a suitably small remainder R_1. To do so, we use (6.1.75) with $t_1 = t$ and $t_2 = 1$ which implies

$$w_{\delta,t} \xrightarrow[\delta \to 0^+]{C^1(B_1)} w_0 = w_{V_0} \tag{6.1.77}$$

uniformly in t. We also have that there exists a positive constant $C = C(j)$ such that

$$\|w_{\delta,t}\|_{2,p} \leq C. \tag{6.1.78}$$

Note in addition that the C^1 convergence (6.1.77) implies that for $\delta < \bar{\delta}$ small enough we get that

$$0 < -\frac{1}{2}\partial_r w_0(r_*) \leq -\partial_r w_{\delta,t}(r_*) \leq -\frac{3}{2}\partial_r w_0(r_*), \tag{6.1.79}$$

since $\partial_r w_{V_0}(r) = 0$ if and only if $r = 0$. Using the Taylor expansion for $w_{\delta,t}$,

$$w_{\delta,t}(r) = w_{\delta,t}(r_*) + \partial_r w_{\delta,t}(r_*)(r - r_*) + R_1(x, \delta, t), \tag{6.1.80}$$

where

$$R_1(x, \delta, t) = \int_{r_*}^{r} \left(\partial_r w_{\delta,t}(s) - \partial_r w_{\delta,t}(r_*)\right) ds,$$

we claim that (6.1.78) ensures that R_1 is small enough with respect to δ. Indeed, by (6.1.78), if we choose $p > n$, then we get uniform Hölder continuity of $\partial_r w_{\delta,t}$, i.e., there exist $\alpha \in (0, 1)$, and $C = C(j, p)$ such that

$$|\partial_r w_{\delta,t}(s) - \partial_r w_{\delta,t}(r_*)| \leq C|s - r_*|^{\alpha}. \tag{6.1.81}$$

The estimate (6.1.81) ensures that for some uniform positive constant $C = C(f, p)$

$$|R_1| \leq C|r - r_*|^{1+\alpha},$$

and, in particular, since $r_2^n - r_1^n = O(r_2 - r_1) = O(\delta)$, and $|h| = \chi_{\{r_1 \leq |x| \leq r_2\}}$,then

$$\lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{B_1} |h\, R_1(|x|, \delta, t)|\, dx = \lim_{\delta \to 0^+} \frac{1}{\delta^2} \int_{r_1(\delta)}^{r_2(\delta)} n\omega_n\, r^{n-1} |R_1(r, \delta, t)|\, dr = 0, \tag{6.1.82}$$

uniformly in t. Let us now use the Taylor expansion (6.1.80) to compute $J'(t)$. Using again that $(r - r_*)h(r) = -|r - r_*|$, and using (6.1.79), we have that

$$\begin{aligned} J'(t) &= \int_{B_1} w_{\delta,t} h\, dx \\ &= w_{\delta,t}(r_*) \underbrace{\int_{B_1} h\, dx}_{=0} - \partial_r w_{\delta,t}(r_*) \int_{r_1}^{r_2} n\omega_n\, r^{n-1} |r - r_*|\, dr + \int_{B_1} h\, R_1\, dx \\ &= CR_2(r)\Big((r_2 - r_*)^2 + (r_* - r_1)^2\Big) + \int_{B_1} h\, R_1\, dx, \end{aligned}$$

with $C = C(j) > 0$

$$\frac{1}{C} < R_2(r) < C.$$

Since $r_2 - r_* = O(\delta)$ and $r_* - r_1 = O(\delta)$, then, thanks to (6.1.82) we get, up to choosing a bigger C,

$$\frac{1}{C}\delta^2 \leq J'(t) \leq C\delta^2,$$

thus concluding the proof. □

6.1.9 The case $j(s) = s$

Proposition 6.1.40. *Let $j(s) = s$ for every $s \in \mathbb{R}_0^+$, i.e. for every measurable set E*

$$\mathcal{J}(E) = \int_{B_1} u_E\, dx,$$

where $u_E \in W_0^{1,2}(B_1)$ is the solution to (6.1.4). If $|E| = |B_*|$ and $|E \Delta B_*| = \delta$, then

$$\mathcal{J}(E) \leq \mathcal{J}(A_\delta).$$

Proof. By linearity we may write

$$u_E = u_{E\cap B_*} + u_{E\setminus B_*} + u_{B_*} - u_{B_*}.$$

Using the Talenti comparison (see Theorem 2.4.7), we know that

$$u^\sharp_{E\cap B_*} \le u_{A_\delta\cap B_*} \qquad\qquad (u_{E\setminus B_*} + u_{B_*})^\sharp \le u_{A_\delta\setminus B_*} + u_{B_*}.$$

Integrating over B_1 the two inequalities, and using the equi-measurability of the Schwarz rearrangement,

$$\begin{aligned}
\mathcal{J}(E) &= \int_{B_1} u_E\,dx \\
&= \int_{B_1} u^\sharp_{E\cap B_*}\,dx + \int_{B_1} (u_{E\setminus B_*} + u_{B_*})^\sharp\,dx - \int_{B_1} u_{B_*}\,dx \\
&\le \int_{B_1} u_{A_\delta\cap B_*}\,dx + \int_{B_1} u_{A_\delta\setminus B_*} + u_{B_*}\,dx - \int_{B_1} u_{B_*}\,dx \\
&= \int_{B_1} u_{A_\delta}\,dx \\
&= \mathcal{J}(A_\delta).
\end{aligned}$$

□

Proposition 6.1.41. *There exists a positive constant $C = C(m)$ such that*

$$\mathcal{J}(B_*) - \mathcal{J}(E) \ge C|E\Delta B_*|^2$$

for every measurable set $E \subseteq B_1$ such that $|E| = m$.

Proof. By Proposition 6.1.36, it is sufficient to assume that $\delta = |E\Delta B_*|$ is small. Therefore, the result follows by joining Proposition 6.1.40 and Proposition 6.1.39. □

Appendix A

About the convergence of level sets

In this section, we take $p > n$, and we prove that if u_k is a sequence of $W^{2,p}$ functions converging to a $W^{3,p}$ function u, then, choosing suitably t, the level sets $\{u_k > t\}$ are converge to $\{u > t\}$ in $C^{1,s}$ for some s.

In what follows, we let $I_n \in \mathbb{R}^{n\times n}$ denote the identity matrix on $\mathbb{R}^n$, and we let $\mathrm{Id} : \mathbb{R}^n \to \mathbb{R}^n$ denote the identity map on $\mathbb{R}^n$, so that for every $x \in \mathbb{R}^n$ we have $D\,\mathrm{Id}(x) = I_n$. Let $E \subseteq \mathbb{R}^n$, and let $\Phi \in W^{2,p}(B_1; \mathbb{R}^n)$, then we use the notation

$$E^\Phi := (\mathrm{Id} + \Phi)E = \{\, x + \Phi(x) \mid x \in E \,\}\,.$$

Lemma A.0.1. *Let $u_k \in L^\infty(\Omega)$ be a sequence of functions, let $u \in W^{1,\infty}(\Omega)$, and let $t_k, t_* \in \mathbb{R}$ such that*

$$|\{\, u_k > t_k \,\}| = |\{\, u > t_* \,\}| \in (0, |\Omega|).$$

If

$$u_k \xrightarrow{L^\infty(\Omega)} u,$$

then $t_* = \lim_k t_k$.

Proof. Since u_k converge uniformly, they are equi-bounded. In particular this ensures that t_k are equi-bounded in $\mathbb{R}$, otherwise we would get for big k that $t_k > \sup_j \|u_j\|_\infty$, and $|\{\, u_k > t_k \,\}| = 0$. Up to passing to a subsequence, let $\bar t = \lim_k t_k$, and let $\varepsilon > 0$. By L^∞ convergence we get that for big k

$$\{\, u > \bar t + 2\varepsilon \,\} \subseteq \{\, u_k > \bar t + \varepsilon \,\} \subseteq \{\, u_k > t_k \,\}\,.$$

Letting $\mu(t) = |\{\, u > t \,\}|$, we discover

$$\mu(\bar t + 2\varepsilon) \le m.$$

Analogously

$$\{\, u > \bar t - 2\varepsilon \,\} \supseteq \{\, u_n > \bar t - \varepsilon \,\} \supseteq \{\, u_n > t_n \,\}\,,$$

and $\mu(\bar t - 2\varepsilon) \ge m$. Since $u \in W^{1,\infty}$, then the function μ is strictly decreasing in $(0, |\Omega|)$, so that

$$\mu(\bar t + 2\varepsilon) \le m \le \mu(\bar t - 2\varepsilon)$$

ensures that

$$\bar t - 2\varepsilon \le t_* \le \bar t + 2\varepsilon.$$

Since ε is arbitrary, $\bar t = t_*$. The argument does not depend on the choice of the subsequence, and the conclusion follows. $\square$

Definition A.0.2. Let $K \subset \mathbb{R}^n$. For every $t \geq 0$, we define the *outer parallel* set

$$(K)^t = \{ p \in \mathbb{R}^n \mid d(p, K) \leq t \}.$$

Lemma A.0.3. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, let $u \in C^0(\Omega)$. Then for every $\varepsilon > 0$ there exists t_0 such that if $|t| \leq t_0$ then*

$$u^{-1}(t) \subseteq (u^{-1}(0))^{\varepsilon}.$$

Proof. Let us assume by contradiction that there exist $\varepsilon > 0$ and points $y_k \in u^{-1}(t_k)$ with $\lim_k t_k = 0$ such that

$$d(y_k, u^{-1}(0)) \geq \varepsilon.$$

Up to a subsequence, we may assume that y_k converge to some point $\bar{y}$. Under these assumptions the continuity of u gives $u(\bar{y}) = 0$, so that $\bar{y} \in u^{-1}(0)$. On the other hand

$$d(\bar{y}, u^{-1}(0)) \geq \varepsilon,$$

which is a contradiction. □

Lemma A.0.4. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded set, let $u \in C^1(\bar{\Omega})$, and let*

$$\nu(x) = -\frac{\nabla u}{|\nabla u|}(x).$$

If $\Omega_1 \subset\subset \Omega$ and $k > 0$ are such that

$$|\nabla u|(x) \geq k, \qquad \forall x \in \Omega_1,$$

then there exists a positive constant $\alpha_0(d(\Omega_1, \partial\Omega), k, u)$ such that for every positive $\alpha \leq \alpha_0$

$$u(x + \alpha\nu(x)) \leq u(x) - \alpha\frac{k}{4}, \qquad u(x - \alpha\nu(x)) \geq u(x) + \alpha\frac{k}{4}, \qquad \forall x \in \Omega_1$$

Proof. First we notice that $u \in C^1(\Omega)$ ensures that ∇u is uniformly continuous, namely there exists δ_0 such that if $|z - x| \leq \delta_0$ then

$$||\nabla u|(z) - |\nabla u|(x)| \leq \frac{k}{4}.$$

Therefore, for $r < \alpha_0 := \min\{ \delta_0, r_0 \}$ we have

$$|\nabla u|(x) \geq \frac{k}{2}, \qquad \forall x \in (\Omega_1)^r.$$

Let $x \in \Omega_1$. By Lagrange theorem applied on the function $u(x + \alpha\nu(x))$ with respect to the variable $\alpha > 0$ we get for some z_α on the segment between x and $x + \alpha\nu(x)$,

$$\begin{aligned} u(x + \alpha\nu(x)) &= u(x) - \alpha|\nabla u|(x) - \alpha(|\nabla u|(z_\alpha) - |\nabla u|(x)), \\ &\leq u(x) - \frac{\alpha k}{4}. \end{aligned}$$

An analogous argument brings to the conclusion for $u(x - \alpha\nu(x))$. □

Remark A.0.5. When $\Omega_1 = u^{-1}(0)$, Lemma A.0.4 reads as follows: let $\varepsilon > 0$ small enough, and let $|t| \le \varepsilon k/4$; then for every $x \in u^{-1}(0)$ there exists $y \in \Omega$ such that $|x-y| < \varepsilon$ and $u(y) = t$. This in particular implies that

$$u^{-1}(0) \subseteq (u^{-1}(t))^{\varepsilon}, \qquad \forall |t| \le \frac{\varepsilon k}{4}.$$

This result joint with Lemma A.0.3 implies the continuity in 0 of the function

$$u^{-1} : t \in \left[-\frac{\alpha_0 k}{4}, \frac{\alpha_0 k}{4}\right] \longmapsto u^{-1}(t) \in \mathcal{P}(\mathbb{R}^n)$$

with respect to the Hausdorff distance.

Proposition A.0.6. *Let $p > n$, let $\Omega \subset \mathbb{R}^n$ be an open bounded set, and let $s = 1 - n/p$. Let $u \in W^{3,p}(B_1)$ be a function such that for some positive constant k*

$$|\nabla u|(x) \ge k, \qquad \forall x \in u^{-1}(0).$$

Let $u_j \in W^{2,p}(\Omega)$ be a sequence of functions such that

$$u_j \xrightarrow{W^{2,p}(\Omega)} u.$$

Then for every j big enough $u_j^{-1}(0)$ is a $C^{1,s}$ hypersurface and there exist deformations $\Phi_j \in W^{2,p}(\Omega;\mathbb{R}^n)$ such that

(i) Φ_j *are orthogonal to* $u^{-1}(0)$*;*

(ii) $u_j^{-1}(0) = (\mathrm{Id} + \Phi_j)(u^{-1}(0))$;

(iii) $\lim_j \|\Phi_j\|_{2,p} = 0$

Proof. First we want to extend the non-degeneracy property of the gradient of u to the functions u_j on their level sets $u_j^{-1}(0)$. We notice that by uniform convergence and by Lemma A.0.3, we have that for every $\varepsilon > 0$ there exists $t_0 = t_0(u,\varepsilon)$ and $j_0 = j_0(u,\varepsilon)$ such that

$$u_j^{-1}(0) \subset u^{-1}(-t_0, t_0) \subset (u^{-1}(0))^{\varepsilon}. \tag{A.0.1}$$

On the other hand, since ∇u is uniformly continuous in Ω and since ∇u_j uniformly converge to ∇u, then we may assume that for $j \ge j_0$ big enough and for $\varepsilon_0 > 0$ small enough

$$|\nabla u_j|(x) \ge \frac{k}{2} \qquad \forall x \in (u^{-1}(0))^{\varepsilon_0}.$$

Therefore, $|\nabla u_j|(x) > 0$ for every $x \in u_j^{-1}(0)$, which implies that the sets $u_j^{-1}(0)$ are $W^{2,p}$ hypersurfaces.

Now we construct the diffeomorphisms Φ_j. For every $x \in (u^{-1}(0))^{\varepsilon}$, let us denote by

$$\nu(x) = -\frac{\nabla u}{|\nabla u|}(x),$$

and let us define

$$F_j(x,t) = u_j(x + t\nu(x)) - u(x).$$

We show that F_j is strictly monotone in t and that it always admits a zero. First we notice that

$$\partial_t F_j(x,t) = \nabla u_j(x + t\nu(x)) \cdot \nu(x).$$

Using the uniform convergence of ∇u_j joint with the uniform continuity of ∇u, there exists a $\delta_0 > 0$ such that if $|t| \leq \delta_0$ and $j \geq j_0$ then

$$\partial_t F_j(x,t) \leq -k/4$$

uniformly in x, t and j. On the other hand, by Lemma A.0.4 with $\Omega_1 = (u^{-1}(0))^{\varepsilon_0}$,

$$u(x + \alpha_0\nu(x)) \leq u(x) - \alpha_0\frac{k}{4}, \qquad u(x - \alpha_0\nu(x)) \geq u(x) + \alpha_0\frac{k}{4},$$

By uniform convergence, for j big enough we have

$$u_j(x + \alpha_0\nu(x)) \leq u(x) - \alpha_0\frac{k}{8}, \qquad u_j(x - \alpha_0\nu(x)) \geq u(x) + \alpha_0\frac{k}{8},$$

so that $F_j(x,\alpha_0) < 0 < F_j(x,-\alpha_0)$. Hence, we discover that for every $x \in (u^{-1}(0))^{\varepsilon_0}$ there exists a unique $t_j(x) \in [-\alpha_0, \alpha_0]$ such that $F_j(x, t_j(x)) = 0$, or, equivalently

$$u_j(x + t_j(x)\nu(x)) = u(x). \tag{A.0.2}$$

We claim that $\Phi_j(x) := t_j(x)\nu(x)$ is the desired deformation (up to multiplying it by a cutoff function).

Property *(i)* follows by noticing that by construction ν is orthogonal to $u^{-1}(0)$.

Let us now prove *(ii)*. First we notice that (A.0.2) for $x \in u^{-1}(0)$ reads

$$(\mathrm{Id} + \Phi_j)(u^{-1}(0)) \subset u_j^{-1}(0).$$

It remains to prove that points y of $u_j^{-1}(0)$ are close enough to $u^{-1}(0)$ to enjoy the unique representation $y = x + t_n(x)\nu(x)$ for some $x \in u^{-1}(0)$. Indeed, by Lemma A.0.3, up to taking a bigger j_0 we also have, as done in (A.0.1),

$$u_j^{-1}(0) \subset (u^{-1}(0))^{\alpha_0},$$

so that every $y \in u_j^{-1}(0)$ can be written as

$$y = \bar{y} + d(y, u^{-1}(0))\bar{\nu}(y)$$

with $\bar{y} \in u^{-1}(0)$ and either $\bar{\nu} = \nu$ or $\bar{\nu} = -\nu$. Since $d(y, u^{-1}(0)) < \alpha_0$, then we have $d(y, u^{-1}(0)) = |t_j(\bar{y})|$ and we have proved that

$$(\mathrm{Id} + \Phi_j)(u^{-1}(0)) = u_j^{-1}(0).$$

Finally, *(iii)* follows by noticing that (A.0.1) implies that t_j converges uniformly to 0 and by computing

$$\begin{aligned}
\nabla t_j(x) &= -\frac{\partial_x F_j(x, t_j(x))}{\partial_t F_j(x, t_j(x)} \\
&= |\nabla u|(x)\frac{\nabla u_j(x + \Phi_j(x)) - t_j(x)D\nu(x)\nabla u_j(x + \Phi_j(x)) - \nabla u(x)}{\nabla u_j(x + \Phi_j(x)) \cdot \nabla u(x)},
\end{aligned}$$

followed by a bootstrap argument, which is possible thanks to the assumption $p > n$. $\square$

Appendix B

Derivatives of characteristic functions

Let Ω a bounded open set with $C^{1,1}$ boundary. We know that there exists $\varepsilon > 0$ such that it is well defined the *projection onto* $\partial\Omega$, namely the map $\pi_{\partial\Omega} : (\partial\Omega)^\varepsilon \to \partial\Omega$ such that $d(x, \partial\Omega) = |x - \pi_{\partial\Omega}(x)|$, and it is Lipschitz.

Definition B.0.1 (Tangential gradient)**.** Let Ω be a bounded open set with $C^{1,1}$ boundary, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U; \mathbb{R}^n)$. We define for $\mathcal{H}^{n-1}$-a.e. $\sigma \in \partial\Omega$ the *tangential gradient of* ϕ at σ as the the linear map $D^{\partial\Omega}\phi$ such that

$$D^{\partial\Omega}\phi(x)v = D\phi(x)v - (v \cdot \nu)D\Phi\nu, \qquad v \in \mathbb{R}^n$$

where $\nu(x) = \nu(\pi_{\partial\Omega}(x))$ is an extension of the outer unit normal to Ω.

We let $T_x\partial\Omega = \nu(x)^\perp$ denote the tangential space to $\partial\Omega$ in $x \in \partial\Omega$.

Definition B.0.2 (Tangential Jacobian)**.** Let Ω be an open bounded set with C^1 boundary, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$, and let $\phi \in C^{0,1}(U; \mathbb{R}^n)$. We define *the tangential Jacobian of* ϕ as

$$\operatorname{Jac}^{\partial\Omega}\phi = \sqrt{\det\left((D^{\partial\Omega}\phi)^T(D^{\partial\Omega}\phi)\right)},$$

where the determinant has to be intended in the space $T_\sigma\partial\Omega \otimes T_\sigma\partial\Omega$.

Remark B.0.3. We notice that it is possible to prove that

$$\operatorname{Jac}^{\partial\Omega}\phi = |D\phi^{-T}\nu_\Omega| \operatorname{Jac}\phi.$$

Definition B.0.4 (Tangential divergence)**.** Let Ω be a bounded open set with $C^{1,1}$ boundary, let $U \subseteq \mathbb{R}^n$ be an open set containing $\partial\Omega$ and let $\phi \in C^{0,1}(U; \mathbb{R}^n)$. We define the *tangential divergence of* ϕ as

$$\operatorname{div}^{\partial\Omega}\phi = \operatorname{div}(\phi) - D\phi\nu \cdot \nu.$$

In the following, we denote by $\mathcal{D}'(\mathbb{R}^n)$ the space of distributions on $\mathbb{R}^n$. When $\varphi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1(\mathbb{R}^n)$, we denote by

$$\langle f|g\rangle := \int_{\mathbb{R}^n} fg\,dx.$$

We then define the distributional gradient and hessian as follows: let $T \in \mathcal{D}'(\mathbb{R}^n)$, let $\Phi, \Psi \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$ for some $s \in (0,1)$, and let $\varphi \in C_c^\infty(\mathbb{R}^n, \mathbb{R})$; we denote by

$$\langle \nabla T[\Phi]|\varphi\rangle := -\langle T|\operatorname{div}(\varphi\Phi)\rangle,$$

$$\langle \nabla^2 T[\Phi, \Psi]|\varphi\rangle := \langle \nabla(\nabla T[\Phi])[\Psi]|\varphi\rangle - \langle \nabla T[D\Phi\Psi]|\varphi\rangle.$$

Remark B.0.5. When E is a set with $C^{1,1}$ boundary, letting $\nu = \nu_E$ denote the outer unit normal to ∂E, the divergence theorem implies

$$\nabla \chi_E[\Phi] = -(\Phi \cdot \nu)\, d\mathcal{H}^{n-1} \llcorner_{\partial E}. \tag{B.0.1}$$

In this case we extend the definition of $\nabla \chi_E[\Phi]$ to functions $\Phi \in W^{1,2}(B_1;\mathbb{R}^n)$ in such a way that (B.0.1) holds. In addition,

$$\langle \nabla^2 \chi_E[\Phi, \Psi] | \varphi \rangle = \int_{\partial E} ((\Phi \cdot \nu)\, \mathrm{div}(\varphi \Psi) + \varphi D\Phi \Psi \cdot \nu)\, d\mathcal{H}^{n-1} \tag{B.0.2}$$

we extend the definition of $\nabla^2 \chi_E[\Phi, \Psi]$ to functions $\Phi, \Psi \in W^{2,2}(B_1;\mathbb{R}^n)$ in such a way that (B.0.2) holds.

If we extend $\nu(x) = \nu(\pi_{\partial E}(x))$ and we take $\Phi = \Psi$ decomposed in tangential and normal part $\Phi = \Phi^\tau + (\Phi \cdot \nu)\nu$, then we may rewrite (B.0.2) in another form. Using the tangential divergence theorem on ∂E (see for instance [135, Formula (17.24)]), straightforward computations yield

$$\begin{aligned}\langle \nabla^2 \chi_E[\Phi, \Psi] | \varphi \rangle = &\int_{\partial E} \left((\Phi \cdot \nu)^2 \left(H\varphi + \frac{\partial \varphi}{\partial \nu} \right) - \varphi D\nu \Phi^\tau \cdot \Phi^\tau \right) d\mathcal{H}^{n-1} \\ &+ 2 \int_{\partial E} \varphi (\Phi \cdot \nu) D\Phi \nu \cdot \nu \, d\mathcal{H}^{n-1},\end{aligned} \tag{B.0.3}$$

where for every $x \in \partial E$, the symbol $H(x) = \mathrm{div}\, \nu(x)$ denotes the mean curvature of E, and we have used that $D\nu\nu = 0$,

$$\mathrm{div}(\varphi \Phi) = \mathrm{div}^{\partial E}(\varphi \Phi) + (\Phi \cdot \nu) \frac{\partial \varphi}{\partial \nu} + \varphi D\Phi \nu \nu,$$

and

$$\nabla^{\partial \Omega}(\Phi \cdot \nu) \cdot \Phi = D\nu \Phi^\tau \Phi^\tau + D\Phi \Phi^\tau \nu. \tag{B.0.4}$$

For every diffeomorphism $F \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ we define the *distributional composition* as

$$\langle T \circ F | \varphi \rangle := \langle T |\, \det(D(F^{-1}))\varphi \circ F^{-1} \rangle\,.$$

Remark B.0.6. When E is a set with Lipschitz boundary and F is a C^1 diffeomorphism, then

$$\chi_{F(E)} = \chi_E \circ F^{-1},$$

both in the pointwise and distributional sense.

Moreover, analogously to the classical chain rule for smooth functions,

$$\nabla \chi_{F(E)}[\Phi] = \nabla \chi_E[(DF)^{-1}(\Phi \circ F)] \circ F^{-1}, \tag{B.0.5}$$

Indeed, we notice that the tangential jacobian of the change of variables from ∂E to $\partial F(E)$ is given by $\mathrm{Jac}^{\partial E} F = |DF^{-T}\nu_E| \det(DF)$, and that

$$\nu_{F(E)}(F(x)) = \frac{DF(x)^{-T}\nu_E(x)}{|DF(x)^{-T}\nu_E(x)|}.$$

Therefore,

$$\begin{aligned}&\int_{\partial F(E)} \varphi(y)\, (\Phi(y) \cdot \nu_{F(E)}(y))\, d\mathcal{H}^{n-1}(y) \\ &\qquad = \int_{\partial E} \varphi(F(x))\, (\Phi(F(x)) \cdot DF(x)^{-T}\nu_E(x)) \det(DF(x))\, d\mathcal{H}^{n-1}(x),\end{aligned}$$

which is (B.0.5).

Finally, for $t \in \mathbb{R}$, let T_t be a family of distributions. If it exists, we define the partial derivative of T_t with respect to t as the limit in $\mathcal{D}'(\mathbb{R}^n)$

$$\lim_{s\to 0} \frac{1}{s}(T_{t+s} - T_t).$$

By definition of gradient, if T_t is differentiable with respect to t, then also ∇T_t is differentiable and

$$\partial_t(\nabla T_t[\Phi]) = \nabla(\partial_t T_t)[\Phi].$$

Moreover, if $\Phi_t \in C^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is C^1 with respect to t, then

$$\partial_t(\nabla T_t[\Phi_t]) = \nabla(\partial_t T_t)[\Phi_t] + \nabla T_t[\partial_t \Phi_t].$$

We refer to [113, Corollary 5.2.8] for the following classical result.

Lemma B.0.7 (Hadamard formula)**.** *Let $E \subset B_1$ be a set with Lipschitz boundary, let $F \in C^1([0,T); W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n))$, and let $F_t(x) = F(t,x)$. Then*

$$\partial_t \chi_{F_t(E)} = -\nabla \chi_{F_t(E)}[\partial_t F_t \circ F_t^{-1}]. \tag{B.0.6}$$

Remark B.0.8. In particular, when $F_t = \mathrm{Id} + t\Phi$ with $\Phi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$, we have

$$\partial_t \chi_{F_t(E)} = -\nabla \chi_{F_t(E)}[\tilde{\Phi}],$$

where $\tilde{\Phi} = \Phi \circ F_t^{-1}$.

Remark B.0.9. Let us notice that as in Remark B.0.6 we may rewrite the gradient of $\chi_{F_t(E)} = \chi_E \circ F_t^{-1}$ as

$$\nabla \chi_{F_t(E)}[\partial_t F_t \circ F_t^{-1}] = \nabla \chi_E[(DF_t)^{-1}\partial_t F_t] \circ F_t^{-1},$$

so that the Hadamard formula can be seen as a chain rule for characteristic functions. Indeed, notice also that when $f \in C^1(\mathbb{R}^n)$ then

$$\partial_t(f \circ F_t^{-1}) = \Big(\nabla f \cdot (DF_t)^{-1}\, \partial_t F_t\Big) \circ F_t^{-1}.$$

As a consequence of Hadamard formula, we may differentiate two times $\chi_{F_t(E)}$ obtaining the following.

Corollary B.0.10. *Let $E \subset B_1$ be a set with Lipschitz boundary, let $\Phi \in W^{1,\infty}(\mathbb{R}^n;\mathbb{R}^n)$, and let $F_t(x) = x + t\Phi(x)$. Then*

$$\partial^2_{tt} \chi_{F_t(E)} = \nabla^2 \chi_{F_t(E)}[\tilde{\Phi}, \tilde{\Phi}] + 2\nabla \chi_{F_t(E)}[D\tilde{\Phi}\tilde{\Phi}],$$

where $\tilde{\Phi} = \Phi \circ F_t^{-1}$.

Proof. By Hadamard formula we have

$$\partial_t \chi_{F_t(E)} = \nabla \chi_{F_t(E)}[-\tilde{\Phi}]. \tag{B.0.7}$$

Differentiating again we get

$$\partial^2_{tt} \chi_{F_t(E)} = \nabla(\partial_t \chi_{F_t(E)})[-\tilde{\Phi}] + \nabla \chi_{F_t(E)}[-\partial_t \tilde{\Phi}].$$

We now compute, using again (B.0.7) and the definition of Hessian,

$$\nabla(\partial_t \chi_{F_t(E)})[-\tilde{\Phi}] = \nabla^2 \chi_{F_t(E)}[\tilde{\Phi}, \tilde{\Phi}] + \nabla \chi_{F_t(E)}[D\tilde{\Phi}\tilde{\Phi}].$$

The result now follows by noticing that

$$\partial_t \tilde{\Phi} = -D\tilde{\Phi}\tilde{\Phi}.$$

□

Remark B.0.11. When E is a set with $C^{1,1}$ boundary, then we can make the expression of $\partial^2_{tt}\chi_{F_t(E)}$ explicit. Indeed, joining the corollary with the explicit computations of gradient and Hessian in Remark B.0.5, we get

$$\langle \partial^2_{tt}\chi_{F_t(E)} | \varphi \rangle = \int_{\partial F_t(E)} \Big((\tilde{\Phi} \cdot \nu_t) \operatorname{div}(\varphi \tilde{\Phi}) - \varphi D\tilde{\Phi}\tilde{\Phi} \cdot \nu_t \Big) \, d\mathcal{H}^{n-1}.$$

where ν_t denotes the outer unit normal to $\partial F_t(E)$.

Since the map $E \mapsto \langle \chi_E | \varphi \rangle$ is a shape functional, we could rewrite the second order derivative in a form similar to the one in [84, Remark 2.6] (see also [84, Formula (12)] for the case $\varphi \equiv 1$). In particular, we notice that by (B.0.3) we can rewrite

$$\begin{aligned}\langle \partial^2_{tt}\chi_{F_t(E)} | \varphi \rangle = &\int_{\partial F_t(E)} \left((\tilde{\Phi} \cdot \nu)^2 \left(H_t \varphi + \frac{\partial \varphi}{\partial \nu_t} \right) - \varphi D\nu_t \tilde{\Phi}^\tau \cdot \tilde{\Phi}^\tau \right) d\mathcal{H}^{n-1} \\ &- 2 \int_{\partial F_t(E)} \varphi D\tilde{\Phi}\tilde{\Phi}^\tau \cdot \nu_t \, d\mathcal{H}^{n-1},\end{aligned}$$

where H_t denotes the mean curvature of $\partial F_t(E)$, and for every vector $v \in \mathbb{R}^n$ we used the notation

$$v^\tau = v - (v \cdot \nu_t)\nu_t.$$

This formula coincides with [84, remark 2.6] noticing that (B.0.4) holds.

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