Lattice polygons

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Abstract

Polygons with n vertices, where $n \geq 3$, that lie on a lattice are called lattice n-gons. These lattice polygons turn out to be a highly restrictive class of objects, as many standard properties of polygons cannot be satisfied by lattice polygons. For instance, squares are the only regular lattice polygons in the planar integer lattice $\mathbb{Z}^2$. In a higher dimensional integer lattice $\mathbb{Z}^d$ with $d \geq 3$, the only regular lattice polygons are triangles, squares, and hexagons. Equilateral or equiangular lattice polygons are subject to similar limitations. This dissertation starts with an introduction to elementary geometry and lattices, establishing the notation and conventions used throughout. The subsequent chapter presents prior research on the existence of regular and semiregular lattice polygons, including proof strategies and an overview of the extensions of the theory on lattice polygons. Previous results are then extended to a non-Euclidean geometry, namely the taxicab geometry. We show that taxicab-equilateral lattice n-gons exist in $\mathbb{Z}^2$ for every integer n, with $n \geq 3$. The final chapter concludes with an original investigation into the minimal side length for these lattice polygons and poses open problems for further research on lattice polygons in non-Euclidean geometries.

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Chapter 1

Introduction

Lattices are an old topic of research in mathematics with authors such as Minkowski et al. [24] writing about them in the late nineteenth century and many of the results used in this dissertation coming from the first half of the twentieth century (see Schoenberg [31], Ehrhart [10], and Scherrer [30]). Furthermore, lattices have direct applications in the sciences, such as in computer graphics Ren Ding [29] and in describing crystal lattice structures in chemistry Misra [25]. Despite their apparent simplicity, they underline deep connections in various branches of mathematics, including discrete geometry, number theory and graph theory.

The objective of this dissertation is to investigate the existence of various lattice polygons. Specifically, for a fixed lattice, we explore for which values of n there exist lattice n-gons satisfying a given set of properties. Various proofs of the same results are presented to better understand the different approaches authors have taken in solving problems in lattice theory. A further objective of this dissertation is to demonstrate how specific geometric arguments may be replaced by more general number theoretical approaches. Additionally, novel geometric results are obtained by exploring a new class of objects called taxicab-equilateral lattice polygons. The intended audience consists of undergraduate mathematics students who are familiar with non-Euclidean geometry and number theory.

Chapter 1 serves as an introduction to a metric based approach to formalising geometry largely taken from Millman and Parker [23]. Alternative presentations of the same topics can be found in the works of Moise [26], Krause [17], and Divjak [9]. The reader is assumed to be familiar with elementary geometric facts and arguments. Nonetheless, this introduction will be useful for motivating the extension to the theory of lattice polygons that will be presented in Chapter 3. We define what is meant by a geometry (Definition 1.9) and use it to show that the taxicab plane is an example of a geometry (Example 1.11). Finally, lattice polygons are defined as being polygons whose vertices belong to a lattice.

In Chapter 2, the typical methods used to tackle problems in the theory of lattice polygons are explored. Most of the proofs are not new, but some have been corrected, expanded or modified to make some connections clearer. We begin by giving a detailed proof of Scherrer's theorem (Scherrer [30]) about the nonexistence of certain regular lattice polygons (Theorem 2.2). The proof is then compared to that of Chrestenson's [15] very similar statement which uses some deep results in number theory that can be found in Appendix A. Following the historical development of the theory on lattice polygons Ball [1] and Honsberger [14], we explore the existence of equilateral lattice polygons and equiangular lattice polygons in $\mathbb{Z}^2$. We present Beeson's theorem [2] that characterises angles determined by lattice points in $\mathbb{Z}^d$ for all integers d, $d \geq 2$. This result is then combined with some additional observations by Maehara and Martini [22] to complete the characterisation of equiangular lattice polygons in higher dimensional integer lattices. Finally, a brief overview is given of the directions authors have already extended the theory of lattice polygons. These include the existence of lattice polyhedra – the three-dimensional generalisation of polygons – as explored by Ehrhart [11] and Scott [32] and transformations of the lattice examined by Teuffel [34], Beeson [2], and Maehara and Martini [22].

Chapter 3 begins by defining taxicab-equilateral, taxicab-equiangular and taxicab-regular polygons. We motivate the search for analogous characterisations to those in Chapter 2, by presenting some contemporary results on taxicab-regular polygons that are not necessarily lattice polygons. We present the findings of Çolakoğlu and Kaya [6] and Yüksel and Özcan [35], who explore the correspondence between Euclidean-regular polygons and taxicab-regular polygons in $\mathbb{R}^2$ and $\mathbb{R}^3$. They find that only squares and certain octagons are both Euclidean and taxicab regular in $\mathbb{R}^2$. Similarly, Hanson [13] shows that there are no taxicab-regular pentagons. The proof strategies seen in Chapter 2 are then adapted to show Theorem 3.8 which states that there exists a taxicab-equilateral lattice n-gon in $\mathbb{Z}^2$ for all $n \geq 3$. In particular, the proof of Theorem 3.8 is based on the proof of Theorem 2.7 with additional details that apply to both, hence refining Hoffman's proof [14]. The problem of finding the smallest taxicab-equilateral lattice n-gon is then explored. This minimal side length is proven to be bounded below by $\lceil n/4 \rceil$, with equality when $n = 3$ or n is even. An upper bound is obtained when n is odd, although the examples in Figure B.2 indicate that this bound is far from optimal. This dissertation concludes by briefly discussing potential topics for further research on non-Euclidean lattice polygons.

1.1 Overview on elementary geometry

Throughout this dissertation, elements of $\mathbb{R}^d$, with $d \geq 2$, are viewed as both points and vectors. In this section we define what we mean by a geometry using the metric approach introduced by Birkhoff [3] in 1932. The basic idea is to first define an incidence geometry as a collection of two sets – a set of points and a set of lines – together with a set of relationships called axioms between those two sets. A geometry is then defined by adding notions of distance and angle measure onto an incidence geometry.[1] The set of axioms that underpin a geometry are presented incrementally allowing for examples to illustrate various familiar concepts such as lines, angles and planes.

Definition 1.1. An **incidence geometry** $\{\mathscr{P}, \mathscr{L}\}$ consists of a non-empty set $\mathscr{P}$, whose elements are called **points**, together with a collection $\mathscr{L}$ of non-empty subsets of $\mathscr{P}$, called **lines**, such that:

1. for all distinct points $A, B \in \mathscr{P}$, there exists a unique line $l \in \mathscr{L}$, such that $A, B \in l$;
2. every line contains at least two points;
3. there exist at least three points $A, B, C \in \mathscr{P}$ such that not all belong to the same line.

[1] Note that our definition of geometry corresponds to what Millman and Parker [23] refer to as a protractor geometry. For alternative presentations of these concepts, see Birkhoff [3, pp. 329–345], Moise [26], Krause [17, pp. 696–697] or Divjak [9].

The line containing A and B is denoted $\overleftrightarrow{AB}$. Three points are called **collinear** if they belong to the same line and **non-collinear** otherwise.

Example 1.2. Let $\mathscr{L}_E$ be the collection of subsets of $\mathbb{R}^2$ of the form $\{(x, y) \in \mathbb{R}^2 : x = a\}$ or $\{(x, y) \in \mathbb{R}^2 : y = mx + b\}$, where a, m, b are real numbers. The former correspond to vertical lines, while the latter correspond to non-vertical lines. It can be shown that the **Cartesian plane** $\{\mathbb{R}^2, \mathscr{L}_E\}$ is an incidence geometry [23, Proposition 2.1.4.].

It will be convenient throughout this dissertation to view the plane, and any other set of points, as a vector space. For instance, the points $A = (x_1, y_1)$ and $B = (x_2, y_2) \in \mathbb{R}^2$ are thought of as row (or column) vectors $\mathbf{A} = (x_1, y_1)$ and $\mathbf{B} = (x_2, y_2)$ so that the usual vector operations can be used, namely

1. $\mathbf{A} + \mathbf{B} = (x_1 + x_2, y_1 + y_2)$,
2. $r\mathbf{A} = (rx_1, ry_2)$ for all real numbers r and
3. $\mathbf{A} \cdot \mathbf{B} = x_1x_2 + y_1y_2$.

The line $\overleftrightarrow{AB}$ then becomes $\{\mathbf{X} \in \mathbb{R}^2 : \mathbf{X} = \mathbf{A} + t(\mathbf{B} - \mathbf{A}),\ \text{for some } t \in \mathbb{R}\}$.

We now append the first of the two measurement tools to allow us to talk about distances in the geometry.

Definition 1.3. Let X be a non-empty set. The function $d : X \times X \to \mathbb{R}$ is a **metric** on X if for all $x, y, z \in X$

1. $d(x, y) \geq 0$,
2. $d(x, y) = 0 \iff x = y$,
3. $d(x, y) = d(y, x)$ and
4. $d(x, y) \leq d(x, z) + d(z, y)$.

Any function satisfying the above conditions defines a way of measuring distances. The following is a commonly used example of a metric and unless otherwise stated, it will always be used when referring to distances.

Example 1.4. The **Euclidean metric** on $\mathbb{R}^n$ is given by

$$d_E(\mathbf{x}, \mathbf{y}) = \left(\sum_{i=1}^{n} (x_i - y_i)^2 \right)^{1/2},$$

where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

The Euclidean metric corresponds to the most "natural" method for measuring distances in the sense that the distance between two points in the plane is given "as the crow flies".[2]

Definition 1.5. A **metric geometry** $\{\mathscr{P}, \mathscr{L}, d\}$ consists of an incidence geometry $\{\mathscr{P}, \mathscr{L}\}$ and a metric d on $\mathscr{P}$ such that for every line $l \in \mathscr{L}$, there exists a bijective function $f : l \to \mathbb{R}$, called a **ruler**, such that $|f(A) - f(B)| = d(A, B)$ for all $A, B \in l$.

[2]For more information on metrics and the Euclidean metric, see Loeb [19, p. 109] or Millman and Parker [23, p. 28].

Intuitively, the ruler maps the line for which it has been defined onto the real number line $\mathbb{R}$ while preserving distances.

Example 1.6. The Cartesian plane with the Euclidean distance d_E on $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$ is a metric geometry called the **Euclidean plane** and is denoted by $\mathscr{E} = \{\mathbb{R}^2, \mathscr{L}_E, d_E\}$. The ruler for the line $\overleftrightarrow{AB}$ is defined by $f(A + t(B - A)) = t \cdot d_E(A, B)$ [23, Proposition 3.1.4.].

Note that we may use AB as a shorthand for $d(A, B)$, but, since different metrics can be defined on the same set, we will usually specify the metric used. In $\mathbb{R}^2$, we may also write $\|\mathbf{AB}\|$ for the distance $d_E(A, B)$.

Within a metric geometry, many important intuitive concepts can now be formalised. We say that B is **between** A and C if these are collinear points satisfying $d(A, C) = d(A, B) + d(B, C)$. The **segment** from A to B is

$$\overline{AB} = \{P \in \mathscr{P} : P \text{ is between } A \text{ and } B\}.$$

The **ray** (or **half-line**) from A to B is

$$\overrightarrow{AB} = \{P \in \mathscr{P} : P \text{ is between } A \text{ and } B\} \cup \{P \in \mathscr{P} : \ B \text{ is between } A \text{ and } P\}.$$

The **angle** given by the non-collinear points A, B and C is

$$\widehat{ABC} = \overrightarrow{BA} \cup \overrightarrow{BC}.$$

The **triangle** given by the non-collinear points A, B and C is

$$ABC = \overline{AB} \cup \overline{BC} \cup \overline{AC}.$$

Lastly, a subset of points $\mathscr{A} \subset \mathscr{P}$ is **convex** if for all points $P, Q \in \mathscr{A}$, $\overline{PQ} \subset \mathscr{A}$.

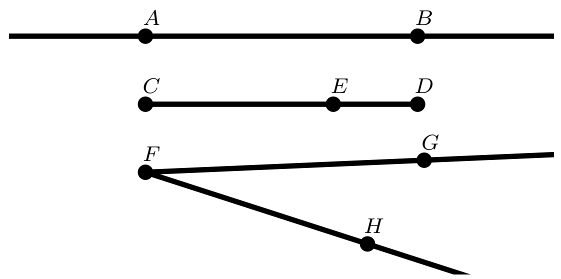


Figure 1.1: The line $\overleftrightarrow{AB}$, segment $\overline{CD}$ (with E between C and D), the ray $\overrightarrow{FG}$ and the angle $\widehat{HFG}$.

The following definition encapsulates the intuitive idea that every line must have "two sides". This idea does not follow from the axioms of a metric geometry, and must therefore be added to the definition of a geometry if we wish to use it [23, Proposition 4.3.5.].

Definition 1.7. A metric geometry $\{\mathscr{P}, \mathscr{L}, d\}$ is called a **Pasch geometry** if for every line $l \in \mathscr{L}$ there exist sets, called **half-planes**, $H_1, H_2 \subset \mathscr{P}$ such that:

1. H_1 and H_2 are convex;

2. $H_1 \cap H_2 = \emptyset$;

3. $H_1 \cup H_2 = \mathscr{P} \setminus l$;

4. if $A \in H_1$ and $B \in H_2$, then $\overleftrightarrow{AB} \cap l \neq \emptyset$.

Having defined rays in terms of collinearity and angles in terms of rays, we can now use the notion of a half-plane in a Pasch geometry to define the **interior of an angle**. The interior of $\widehat{ABC}$ is the intersection of the half-plane determined by $\overleftrightarrow{AB}$ containing C and the half-plane determined by $\overleftrightarrow{BC}$ containing A. We can now state the final axioms of a geometry.

Definition 1.8. Let r_0 be a strictly positive real number and $\mathscr{A}$ be the set of all angles in a Pasch geometry. The function $m : \mathscr{A} \to \mathbb{R}$ is called an **angle measure** based on r_0 if the following hold:

1. for all $\widehat{ABC} \in \mathscr{A}$, $0 < m(\widehat{ABC}) < r_0$;

2. for all ray $\overrightarrow{AB}$ and half-plane H defined by $\overrightarrow{AB}$, if θ is a real number with $0 < \theta < r_0$, then there exists a unique ray $\overrightarrow{AC}$ such that $C \in H$ and $m(\widehat{BAC}) = \theta$;

3. if D is in the interior of $\widehat{ABC}$, then $m(\widehat{ABD}) + m(\widehat{DBC}) = m(\widehat{ABC})$.

The second statement can be thought of as saying that angles are in one-to-one correspondence with real numbers between zero and r_0.

Definition 1.9. A **(protractor) geometry** $\{\mathscr{P}, \mathscr{L}, d, m\}$ consists of a Pasch geometry $\{\mathscr{P}, \mathscr{L}, d\}$ and an angle measure m.

Example 1.10. By the familiar formula for scalar products, $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \|\mathbf{b}\| \cos\theta$, where θ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$, we have $\theta = \cos^{-1}(\mathbf{a} \cdot \mathbf{b}/\|\mathbf{a}\|\|\mathbf{b}\|)$.

It can now be shown that,

$$m_E(\widehat{ABC}) = \cos^{-1}\left(\frac{(\mathbf{A} - \mathbf{B}) \cdot (\mathbf{C} - \mathbf{B})}{\|\mathbf{AB}\|\|\mathbf{BC}\|}\right)$$

defines an angle protractor based on π for $\{\mathbb{R}^2, \mathscr{L}_E, d_E\}$ [23, Proposition 5.1.2.]. Hence we redefine the **Euclidean plane** to be the protractor geometry $\mathscr{E} = \{\mathbb{R}^2, \mathscr{L}_E, d_E, m_E\}$.

For the sake of convenience, we may use $\widehat{ABC}$ to mean $m(\widehat{ABC})$ so the term "angle" might refer to a geometric object or a real number.

Note that in keeping with Millman and Parker [23] and Moise [26], the points defining an angle had to be non-collinear. As a result, angles were not permitted to be lines or rays. We now choose to diverge from these authors by adopting instead the convention that angles refer to **directed angles** where the two rays are not interchangeable.[3] Specifically, $\widehat{ABC}$ now refers to the positive rotation required to map $\overrightarrow{BA}$ to $\overrightarrow{BC}$. By convention we choose to take anti-clockwise to be the positive sense of rotation and take the measure of the angle modulo 2π.

From the definition we get $\widehat{ADB} + \widehat{BDC} = \widehat{ADC}$ and $\widehat{ABC} = -\widehat{CBA}$ for all points A, B, C, D Chick [5, p. 503] Moreover, a directed angle may have zero measure, in which case it corresponds to a ray, or π measure, in which case it corresponds to a line.

One of the advantages of this metric approach to formalising geometry is that the preceding work can now be used to define a wide variety of new geometric models.

[3] More formally, a directed angle is an ordered pair of rays (see Moise [26, p. 67] and Chick [5]). A formal treatment of rotation can be found in Millman and Parker [23, Section 11.5].

Example 1.11. The taxicab metric on $\mathbb{R}^2$ is defined by

$$d_T((x_1,y_1),(x_2,y_2)) = |x_1 - x_2| + |y_1 - y_2|.$$

Intuitively, the taxicab metric mimics the distance of the route a taxi would take in a densely girded city with streets going west-to-east and north-to-south.

Define the **taxicab plane** by $\mathscr{T} = \{\mathbb{R}^2, \mathscr{L}_E, d_T, m_E\}$, where $\{\mathbb{R}^2, \mathscr{L}_E\}$ is the typical Cartesian plane and m_E is the same angle measure as for the Euclidean plane from Example 1.2. We now verify that the taxicab plane is a geometry in the sense of Definition 1.9.

1. $\{\mathbb{R}^2, \mathscr{L}_E\}$ is an incidence geometry.

This follows immediately since we simply reuse the Cartesian plane from Example 1.2.

2. $\{\mathbb{R}^2, \mathscr{L}_E, d_T\}$ is a metric geometry.

It is easy to show that d_T is indeed a metric [23, Proposition 2.2.1.]. We claim that for any line $\overleftrightarrow{AB} = \{A + t(B-A) : t \in \mathbb{R}\}$, where $A = (x_1,y_1), B = (x_2,y_2) \in \mathbb{R}^2$, the function $f : \overleftrightarrow{AB} \to \mathbb{R}$ given by $f(A + t(B-A)) = t \cdot d_T(A,B)$ is a ruler for $\overleftrightarrow{AB}$.

Note that the function f is well-defined since for every $P \in \overleftrightarrow{AB}$, there exists a unique real number t such that $P = A + t(B-A)$. To see this, suppose that $P = A + r(B-A)$ and $P = A + s(B-A)$ for $r, s \in \mathbb{R}$. Then

$$(0,0) = P - P = A + r(B-A) - (A + s(B-A)) = (r-s)(B-A).$$

Since $A \neq B$, we must have $r = s$, as required.

Furthermore, f satisfies the ruler axiom from Definition 1.5. Let $P = A + r(B-A)$ and $Q = A + s(B-A)$ be points in $\overleftrightarrow{AB}$. Then, $P = (x_1 + r(x_2 - x_1), y_1 + r(y_2 - y_1))$ and $Q = (x_1 + s(x_2 - x_1), y_1 + s(y_2 - y_1))$. So

$$\begin{aligned}
d_T(P,Q) &= |x_1 + r(x_2 - x_1) - (x_1 + s(x_2 - x_1))| + |y_1 + r(y_2 - y_1) - (y_1 + s(y_2 - y_1))| \\
&= |(r-s)(x_2 - x_1)| + |(r-s)(y_2 - y_1)| \\
&= |r-s| \cdot (|x_1 - x_2| + |y_1 - y_2|) \\
&= |(r-s) \cdot d_T(A,B)| \\
&= |r \cdot d_T(A,B) - s \cdot d_T(A,B)| \\
&= |f(P) - f(Q)|.
\end{aligned}$$

The function f is surjective since for every real number r, $A + \frac{r}{d_T(A,B)}(B-A) \in \overleftrightarrow{AB}$ and

$$f\left(A + \frac{r}{d_T(A,B)}(B-A)\right) = \frac{r}{d_T(A,B)} \cdot d_T(A,B) = r.$$

Finally, to show that the function f is injective, assume that $P, Q \in \overleftrightarrow{AB}$ are such that $f(P) = f(Q)$. Then $r \cdot d_T(A,B) = s \cdot d_T(A,B)$. Since $A \neq B$, $d_T(A,B) \neq 0$, and so $r = s$.

3. $\{\mathbb{R}^2, \mathscr{L}_E, d_T\}$ is a Parsch geometry.

It can be seen that betweenness in $\{\mathbb{R}^2, \mathscr{L}_E, d_T\}$ is the same as in the Euclidean plane $\mathscr{E}$ [17, p. 699] As a result, the segments and convex sets in $\{\mathbb{R}^2, \mathscr{L}_E, d_T\}$ are the same as those in $\mathscr{E}$. The statement now follows from the fact that $\{\mathbb{R}^2, \mathscr{L}_E, d_E\}$ is a Pasch geometry.

4. $\{\mathbb{R}^2, \mathscr{L}_E, d_T, m_E\}$ is a geometry.

Finally, note that Definition 1.9 does not include any additional requirements linking the angle measure to the metric. Therefore, as m_E is an angle measure for $\{\mathbb{R}^2, \mathscr{L}_E, d_E\}$, it must also be an angle measure for $\{\mathbb{R}^2, \mathscr{L}_E, d_T\}$.

Many steps in the previous example relied on noticing the similarities between $\mathscr{E}$ and $\mathscr{T}$, but the latter differs from the Euclidean plane in one important way. To see this, we need to complete our exploration of geometries.

Definition 1.12. In a protractor geometry, the triangles ABC and DEF are **congruent**, denoted $ABC \cong DEF$, if there exists a bijection $f : \{A, B, C\} \to \{D, E, F\}$ such that, denoting $A' = f(A)$, $B' = f(B)$ and $C' = f(C)$, the following hold:

1. $AB = A'B'$, $BC = B'C'$ and $AC = A'C'$;

2. $\widehat{CAB} = \widehat{C'A'B'}$, $\widehat{ABC} = \widehat{A'B'C'}$ and $\widehat{BCA} = \widehat{B'C'A'}$.

A protractor geometry satisfies the **Side-Angle-Side Axiom (SAS)** if $AB = DE$, $BC = EF$ and $\widehat{ABC} = \widehat{DEF}$ implies that $ABC \cong DEF$.

A **neutral (or absolute) geometry** is a protractor geometry which satisfies SAS.

The Euclidean plane is a neutral geometry [23, Proposition 6.1.3.], but the following example shows that the taxicab plane does not satisfy SAS.

Example 1.13. Consider the triangles ABC and $A'B'C'$ in $\mathscr{T}$, where $A = (0, 0)$, $B = (1, 1)$, $C = (-1, 1)$, $A' = (2, 0)$, $B' = (4, 0)$ and $C = (2, -2)$. Note that $d_T(A, B) = d_T(A', B') = 2$, $d_T(A, C) = d_T(A', C') = 2$ and $\widehat{BAC} = \widehat{B'A'C'} = \pi/2$. Also $ABC \ncong A'B'C'$ since

$$d_T(B, C) = 2 \neq 4 = d_T(B', C').$$

So $\mathscr{T}$ does not satisfy SAS and hence is not a neutral geometry.

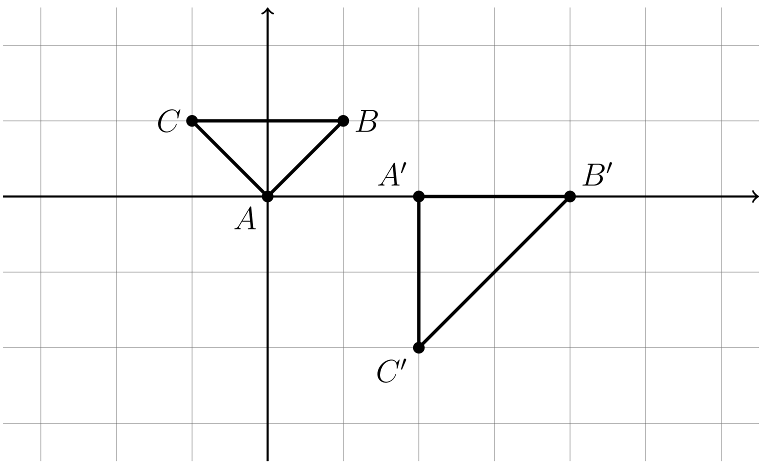


Figure 1.2: Non-congruent triangles in the taxicab plane

Since we will require higher dimensional geometry in Chapter 2, let us simply note that the ideas explored so far apply equally well to three-dimensional geometries (see Moise [26, Chapter 2]).

This section concludes by recalling a few useful trigonometric identities which can be found in [4, Sections 2.7.2.2, 2.7.2.3 and 3.2.1.2].

Proposition 1.14. *Let α, β and θ be real numbers. Then*

$$\sin(2\theta) = 2\sin(\theta)\cos(\theta)$$

$$\tan(\alpha - \beta) = \frac{\tan(\alpha) - \tan(\beta)}{1 + \tan(\alpha)\tan(\beta)}$$

Proposition 1.15 (Cosine law)**.** *If a,b and c are the side lengths of a triangle and γ is the angle opposite the edge with side length c (see Figure 1.3), then $c^2 = a^2 + b^2 - 2ab\cos(\gamma)$.*

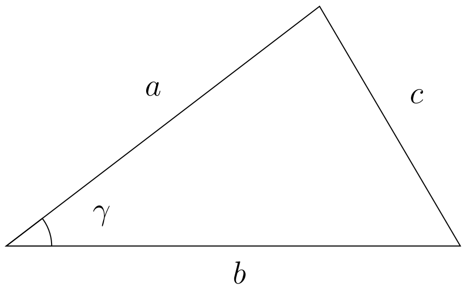


Figure 1.3: Illustration of cosine law

1.2 Definition and properties of polygons

Intuitively, a polygon should be a closed broken line lying in a plane. A more formal treatment of polygons using the same metric approach as in Section 1.1 can be found in [23, Chapter 10].

Definition 1.16. A subset P of a geometry is called a **polygon (of degree** $n \geq 3$**)** if there are n distinct coplanar points $P_1, P_2, \ldots, P_n$, called **vertices**, such that

1. $P = \overline{P_1P_2} \cup \overline{P_2P_3} \cup \cdots \cup \overline{P_{n-1}P_n} \cup \overline{P_nP_1}$,
2. these segments, called **edges**, intersect only at endpoints and
3. no two edges with a vertex in common are collinear.

A polygon with n vertices is called an **n-gon**. Two vertices are said to be **consecutive** if they belong to the same segment. Similarly, two edges are said to be **consecutive** if they have a vertex in common. We may attribute to each edge $\overline{P_1P_2}$ a vector $\mathbf{P - 2 - P_1}$ which we call its **edge vector**. Note that there are two choices for the edge vectors of a given edge.

Note that there are many definitions of polygons, and not all agree. Bronshtein et al. [4, Section 3.1.5.1] and Moise [26, p. 184] define polygons as closed plane figures, in which case polygons must contain the region of the plane they enclose. Others, such as Coxeter and Greitzer [7, p. 51] or Millman and Parker [23, p. 248], define polygons to be unions of segments. Moreover, while Coxeter and Greitzer [7] and Moise [26] allow polygons to intersect themselves (in other words, they may be **non-simple**), this is not permitted in the definition of Millman and Parker [23]. Finally, Coxeter and Greitzer [7] ensure that no three consecutive vertices are collinear (in other words, polygons are not allowed to be **degenerate**), while Millman and Parker [23] do not impose such a restriction.

The definition we use implies that polygons are made up of segments, so they do not contain their own interior, must be non-degenerate and cannot be self-intersecting.

Example 1.17. Of the following figures, only the first two are polygons. ABC is a triangle and $DEFGH$ is a pentagon. The remaining two figures are not polygons since $IJKL$ is degenerate (J, K and L are consecutive collinear vertices) and $MNOP$ is non-simple (MN and OP intersect at a non-vertex point).

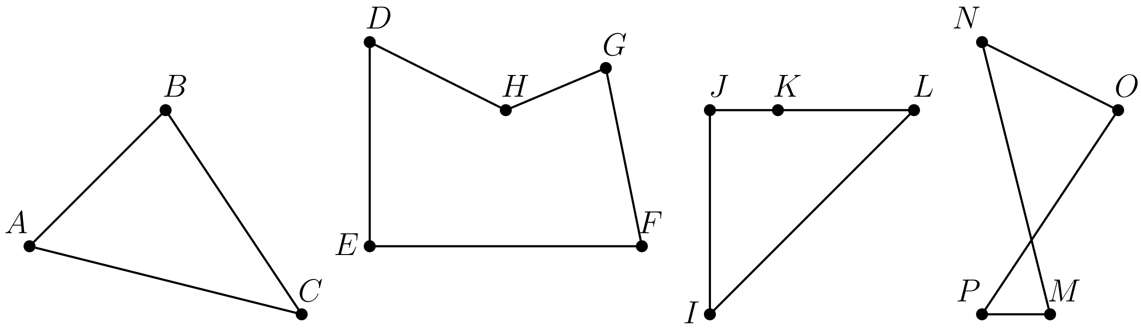


Figure 1.4: Four figures, two of which are polygons

Definition 1.18. Let P_1, P_2 and P_3 be consecutive vertices of a polygon P (see Figure1.5). The **interior angle** of P at P_2 is the angle $\widehat{P_3P_2P_1}$. The **exterior angle** of P at P_2 is the angle $\widehat{DP_2P_3}$, where D is a point such that P_2 is between P_1 and D.

Note that every vertex has two edges that could be extended to form its exterior angle, but corresponding angles are opposite so these are equal [26, p. 117].

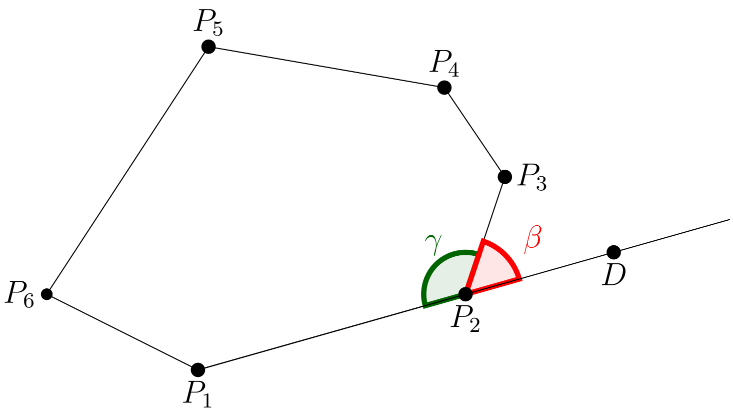


Figure 1.5: Interior angle γ, in green, and exterior angle β, in red, for the vertex P_2

Definition 1.19. A polygon is **convex** if every interior angle is less than or equal to π.

So $DEFGH$, in Example 1.17, is not convex since $\widehat{GHD} \geq \pi$. From now on, the term polygon will exclusively refer to convex polygons.[4]

We may now categorise polygons with regards to some of their properties.

Definition 1.20. A polygon is **equilateral** if all its edges have the same length, **equiangular** if all its interior angles are equal and **regular** if it is both equiangular and equilateral.

[4]Note that, by Definition 1.16, polygons may not be degenerate so interior angles must always be different to π. In other words, every polygon that is convex must have interior angles strictly less than π.

We recall some important properties of angles in a polygon [4, Section 3.1.5.1].

Proposition 1.21. *Let P be an n-gon, where $n \geq 3$.*

The sum of the interior angles in P is equal to $\pi(n-2)$.

The sum of the exterior angles in P is equal to 2π.

Moreover, if P is regular, then all interior angles of P are equal to $(n-2)\pi/n$ and all exterior angles of P are equal to $2\pi/n$.

1.3 Introduction to lattice polygons

We now define what we mean by a lattice.

Definition 1.22. Let d be an integer, with $d \geq 2$. The **d-dimensional integer lattice** is the set of points of $\mathbb{R}^d$ having integral coordinates

$$\mathbb{Z}^d = \{(x_1, x_2, \ldots, x_d) : \forall i \in \{1, \ldots, d\}, x_i \in \mathbb{Z}\}.$$

More generally, a **d-dimensional lattice** is a set of points that can be obtained by applying an invertible linear transformation to the d-dimensional integer lattice.

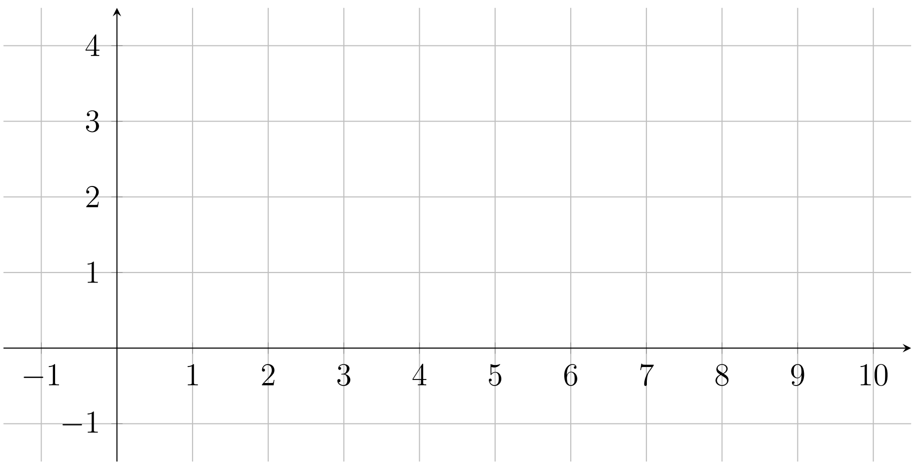


Figure 1.6: The intersection of the grid lines are the elements of the 2-dimensional integer lattice $\mathbb{Z}^2$

Since elements of $\mathbb{R}^d$ can be thought of as points as well as vectors, d-dimensional lattices can alternatively be defined as a discrete additive subgroup of $\mathbb{R}^d$. We then obtain the following definitions:[5]

Definition 1.23. The k-dimensional lattice L generated by the linear independent vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^d$ is

$$L = \{m_1\mathbf{v}_1 + m_2\mathbf{v}_2 + \cdots + m_k\mathbf{v}_k : \forall i \in \{1, \ldots, k\}, m_i \in \mathbb{Z}\}.$$

A lattice contained in another lattice L' is called a **sublattice** of L'.

A lattice is called **planar** if it is 2-dimensional.

A planar lattice L generated by $\mathbf{a}$ and $\mathbf{b}$ such that $\mathbf{a} \cdot \mathbf{b} = 0$ is called **rectangular**. Furthermore, if $|\mathbf{a}| = |\mathbf{b}|$, then L is called a **square lattice**.

[5] See Maehara and Martini [22] for an alternative definition of a lattice relying on basis vectors instead.

If instead the angle between $\mathbf{a}$ and $\mathbf{b}$ is equal to $\pi/3$, then L is called a **triangular lattice**.

A lattice L is called **integral** if for all $\mathbf{x}, \mathbf{y} \in L$, $\mathbf{x} \cdot \mathbf{y}$ is an integer.

The main object of study for this dissertation can now be defined.

Definition 1.24. Let L be a lattice and P be a polygon. We say that P is a **lattice polygon** in L if all the vertices of P are elements of L.

Alternatively, we say that P is in L or L contains P. For conciseness, if a polygon is in an integer lattice, then the integer lattice may be omitted when it is clear from the context. Previous research suggests that being a lattice polygon is a highly restrictive condition, and such objects display distinct behaviours from standard polygons [33]. In particular, the following chapters explore several properties that lattice polygons cannot satisfy.

Chapter 2

Known results

The following chapter presents an overview of past research on lattice polygons, including some of the major highlights on the theory of lattice polygons and show how the arguments used to solve problems have evolved over time. They also contain some useful strategies for working with lattice polygons which will be reused later in Chapter 3.

2.1 Regular lattice polygons

We begin by giving a few examples of lattice polygons in $\mathbb{Z}^2$ and $\mathbb{Z}^3$, then give two distinct proofs of essentially the same results (Theorem 2.2 and Theorem 2.5) about the existence of regular lattice polygons. Although Theorem 2.5 generalises Theorem 2.2, the latter – much older statement – still presents a compelling geometric proof relying on nothing more than basic trigonometry. Theorem 2.5 requires some more advanced knowledge of number theory that will be reused in Section 2.2. Moreover, the proof of Theorem 2.2 is more detailed than that of Scherrer [30] or Hadwiger, Debrunner, and Klee [12].

Example 2.1. The following figures (squares, rectangle, isosceles triangle and hexagon) are all lattice polygons in the planar integer lattice $\mathbb{Z}^2$. Note that the squares can be embedded in more than one way.

A natural question arising from Example 2.1 is whether there are any other regular polygons in $\mathbb{Z}^2$. The surprising answer to this question was given by Scherrer [30] as early as 1946, although the proof is refined, using notes from Hadwiger, Debrunner, and Klee [12].

Theorem 2.2 (Scherrer [30, p. 97])**.** *Let d be an integer with $d \geq 2$. There do not exist regular lattice n-gons in $\mathbb{Z}^d$ for $n \geq 5$, $n \neq 6$.*

Proof. Let $n \geq 5$ with $n \neq 6$ and $d \geq 2$. Assume for contradiction that there exists a regular lattice n-gon P in $\mathbb{Z}^d$, with side length $s > 0$ and vertices $P_1, P_2, \ldots, P_n \in \mathbb{Z}^d$. Since P is regular, all its interior angles are equal to $\alpha = (n-2)\pi/n$ and its exterior angles are equal to $\beta = 2\pi/n$. We aim to show that P can be "shrunk" to a strictly smaller regular lattice polygon.

Let P' be the polygon defined by the vertices $P'_1, P'_2, \ldots, P'_n$ obtained by translating $P_1, P_2, \ldots, P_{n-1}, P_n$ by the edge vectors

$$\mathbf{P}_3 - \mathbf{P}_2, \quad \mathbf{P}_4 - \mathbf{P}_3, \quad \ldots \quad , \mathbf{P}_1 - \mathbf{P}_n, \quad \mathbf{P}_1 - \mathbf{P}_2,$$

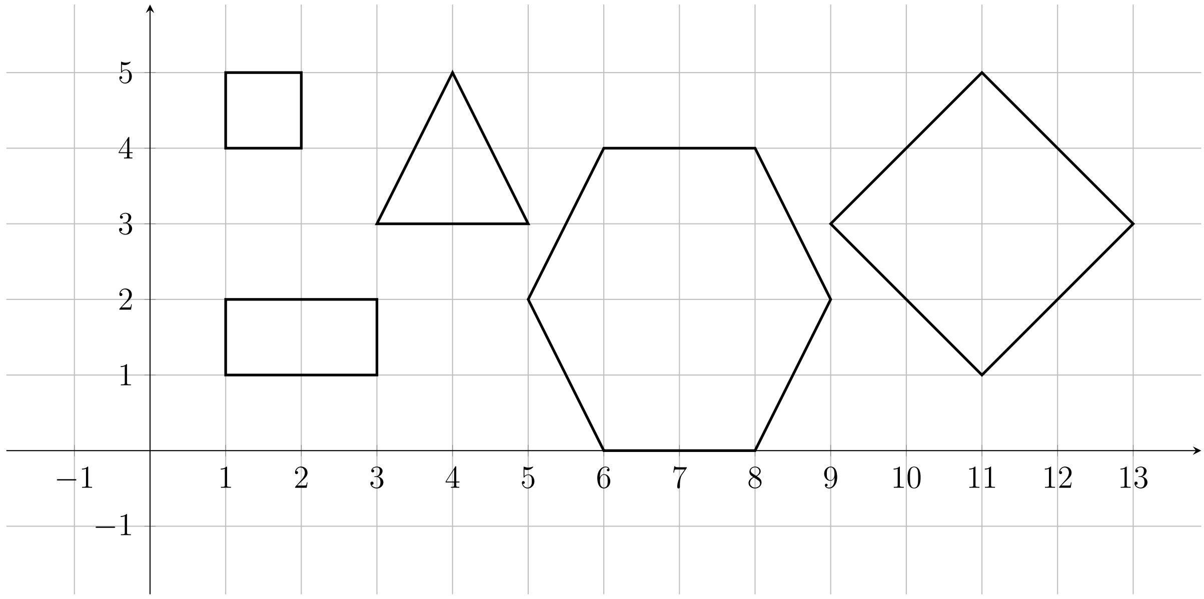


Figure 2.1: Lattice polygons in the planar integer lattice including two squares

respectively (see Figure 2.2). By construction, P' must be a regular lattice n-gon so we may denote its side length by s'. It remains to show that $|s'| < |s|$.

First consider the case when $n \geq 7$. Let us verify that P_1, P_1' and P_2 are collinear. By construction, $\overleftrightarrow{P_1P_1'}$ and $\overleftrightarrow{P_2P_3}$ are parallel lines both intersected by $\overleftrightarrow{P_1P_2}$ so $\widehat{P_2P_1P_1'}$ is equal to the exterior angle of P at P_2, which is β. A similar argument for the lines $\overleftrightarrow{P_2P_2'}$ and $\overleftrightarrow{P_3P_4}$ gives $\widehat{P_3P_2P_2'} = \beta$. Since P_2' is in the interior of $\widehat{P_3P_2P_1}$, $\widehat{P_2'P_2P_1} = \widehat{P_3P_2P_1} - \widehat{P_3P_2P_2'} = \alpha - \beta$. Furthermore, note that $P_1P_2P_2'$ is an isosceles triangle so $\widehat{P_2P_1P_2'} = \widehat{P_1P_2'P_2}$ and the sum of the interior angles of the triangle must be equal to π. Hence $\pi = 2\widehat{P_2P_1P_2'} + \alpha - \beta$, which simplifies into $\widehat{P_2P_1P_2'} = \beta$. So, the rays $\overrightarrow{P_1P_1'}$ and $\overrightarrow{P_1P_2'}$ coincide. Hence P_1, P_1' and P_2 are collinear, as claimed.

We now relate the side lengths s and s' by examining the triangle $P_1P_2P_2'$. Let B be the point of intersection of the altitude starting at P_2' and the segment $\overline{P_1P_2}$. Since BP_2P_2' has a right angle at B,

$$BP_2' = P_2P_2' \sin(\widehat{P_2'P_2P_1}) = s\sin(\alpha - \beta). \tag{2.1}$$

Similarly, examining the right-angled triangle P_1BP_2' we get

$$BP_2' = (P_1P_1' + P_1'P_2)\sin(\widehat{P_2P_1P_2'}) = (s + s')\sin(\beta). \tag{2.2}$$

Equating (2.1) and (2.2) gives

$$s' = s\left(\frac{\sin(\alpha - \beta)}{\sin(\beta)} - 1\right). \tag{2.3}$$

Some care has to be taken in the special case when $n = 5$ (see Figure 2.2).[1] A similar argument as before shows that

$$s' = s\left(1 - \frac{\sin(\alpha - \beta)}{\sin(\beta)}\right). \tag{2.4}$$

[1]The original paper incorrectly claims that $s'/s = |1 - \sin^2(\pi/n)|$.

In either case, taking absolute values of (2.3) or (2.4) gives

$$|s'| = |s| \left| \frac{\sin(\alpha - \beta)}{\sin(\beta)} - 1 \right|. \tag{2.5}$$

Repeatedly using the fact that $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$ for all real numbers θ gives

$$\begin{aligned} \frac{\sin(\alpha-\beta)}{\sin(\beta)} &= \frac{\sin((n-4)\pi/n)}{\sin(2\pi/n)} \\ &= \frac{\sin(4\pi/n)}{\sin(2\pi/n)} \\ &= 2\sin(2\pi/n). \end{aligned} \tag{2.6}$$

Using (2.6), equation (2.5) can now be simplified into

$$|s'| = |s| \, |2\sin(2\pi/n) - 1| \, . \tag{2.7}$$

Since $n \geq 5$, we have $0 < 2\pi/n \leq 2\pi/5$ and so

$$\begin{aligned} 0 = \sin(0) < \sin(2\pi/n) \leq \sin(2\pi/5) < 1&, \\ 0 < 2\sin(2\pi/n) < 2&, \\ -1 < 2\sin(2\pi/n) - 1 < 1&. \end{aligned} \tag{2.8}$$

By combining (2.7) and (2.8), $|s'| < |s|$, as required.

Finally, repeating the above argument on P', we obtain a sequence of regular lattice polygons whose side lengths are strictly decreasing, which is a contradiction since a lattice polygon cannot have an arbitrarily small side length.[2] □

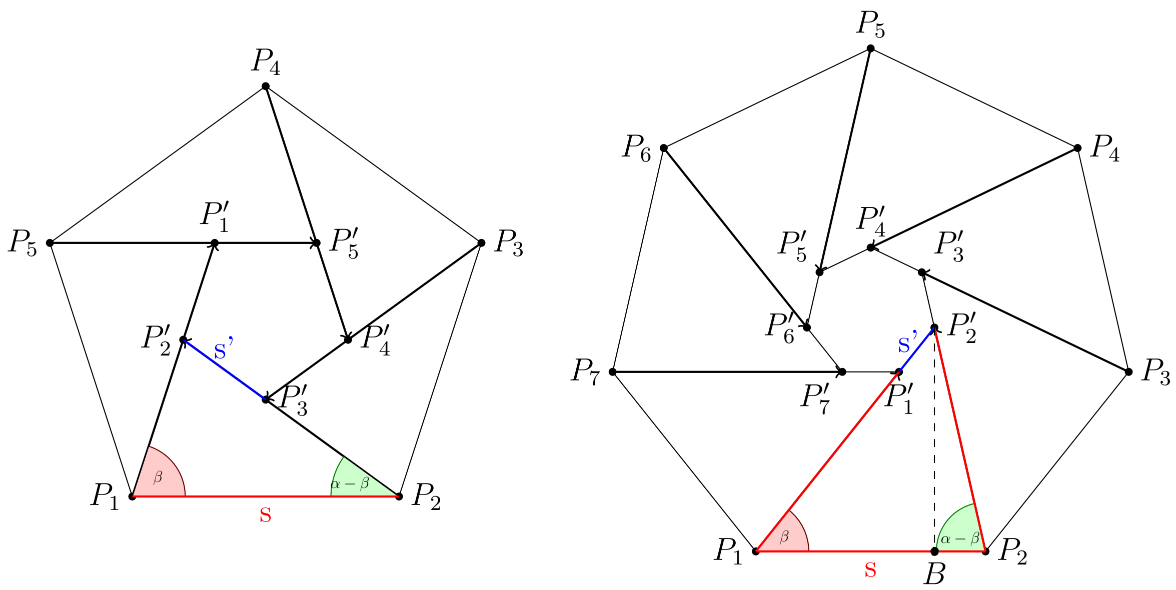


Figure 2.2: Illustration of proof for the regular pentagon and heptagon

By combining Theorem 2.2 with an observation by Hadwiger, Debrunner, and Klee [12], we obtain the following immediate corollary.

[2]Figure B.1 in Appendix B shows why this strategy does not produce smaller polygons when n is equal to $3, 4$ or 6.

Corollary 2.3 (Hadwiger, Debrunner, and Klee [12, Theorem 5]). *There exists a regular lattice n-gon in $\mathbb{Z}^2$ if and only if $n = 4$.*

In other words, squares are the only regular lattice polygons in $\mathbb{Z}^2$.

Proof. Example 2.1 gives two examples of regular lattice 4-gons. So, by Theorem 2.2, it suffices to show that $\mathbb{Z}^2$ cannot contain any regular triangles or hexagons.

Assume for contradiction that there exists a regular lattice 3-gon in $\mathbb{Z}^2$ with vertices $A = (x_1, y_1)$, $B = (x_2, y_2)$ and $C = (x_3, y_3)$. Then, using the definition of the Euclidean distance (Definition 1.4), $\|AB\| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ so $\|AB\|^2$ is an integer. Hence, on one hand, the polygon has area $\|AB\|^2\sqrt{3}/4$ which must be irrational. On the other hand, by the determinant formula [4, Equation 3.300], the area of of the triangle is also equal to

$$\frac{1}{2}\left|\left(\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right)\right| = \frac{1}{2}\left|x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\right|,$$

which is rational.

The case for $n = 6$ follows by a similar argument since a regular hexagons with side length s has area $3s^2\sqrt{3}/2$. □

Example 2.4. Although there are no regular lattice 3-gons or 6-gons in $\mathbb{Z}^2$, both can be embedded in $\mathbb{Z}^3$. The polygon defined by the vertex points $(0, 0, 0), (4, 1, 1)$ and $(1, 4, 1)$ is a regular lattice triangle, while the polygon defined by $(0, 0, 0)$, $(7, -2, 1)$, $(12, 3, 3)$, $(10, 10, 4)$, $(3, 12, 3)$ and $(-2, 7, 1)$ is a regular lattice hexagon.

The fact that regular lattice triangles can be embedded in $\mathbb{Z}^3$ but not in $\mathbb{Z}^2$ was known by Klamkin in 1962. Seemingly unaware of Scherrer's proof, he asked if any regular lattice polygon can be embedded in an integer lattice of sufficiently high dimension. Chrestenson provided the answer, summarising our previous results by using a powerful statement from number theory.

Theorem 2.5 (Klamkin and Chrestenson [15, Problem 5014]). *Let n and d be integers.*

1. *There exists a regular lattice n-gon in $\mathbb{Z}^2$ if and only if $n = 4$.*
2. *There exists a regular lattice n-gon in $\mathbb{Z}^d$ for $d \geq 3$ if and only if $n = 3, 4, 6$.*

In other words, squares are the only regular lattice polygons in $\mathbb{Z}^2$. Triangles, squares and hexagons are the only regular lattice polygons in Z^d (where $d \geq 3$), in which case $d = 3$ suffices.

Proof. Assume that there exists a regular lattice n-gon with side length $s_1 > 0$, shortest diagonal of length $s_2 > 0$ and interior angle $\alpha = (n - 2)\pi/n$. By the law of cosines,

$$\begin{aligned} s_2^2 &= 2s_1^2 - 2s_1^2\cos(\alpha) \\ &= 2s_1^2 - 2s_1^2\cos\left(\frac{(n-2)\pi}{n}\right) \\ &= 2s_1^2 - 2s_1^2\cos\left(\pi - \frac{2\pi}{n}\right) \\ &= 2s_1^2 + 2s_1^2\cos\left(-\frac{2\pi}{n}\right) \\ &= 2s_1^2 + 2s_1^2\cos\left(\frac{2\pi}{n}\right). \end{aligned}$$

Rearranging gives

$$2\cos(2\pi/n) = s_2^2/s_1^2 - 2.$$

By the same argument as in the proof of Corollary 2.3, s_1^2 and s_2^2 are integers, so $2\cos(2\pi/n))$ is rational. Now by Lehmer's lemma (Lemma A.1), $2\cos(2\pi/n))$ must be an algebraic integer of degree $\phi(n)/2$, where ϕ is Euler's totient function. Also, since rational numbers must have degree equal to one, $\phi(n) = 2$. By examining the table in Appendix A, the only possible values for n are $3, 4$ and 6. Furthermore, by the observation made in Corollary 2.3, $n = 3$ or $n = 6$ is impossible in $\mathbb{Z}^2$.

The converse implication is immediate from Example 2.1 and Example 2.4. □

2.2 Semiregular lattice polygons

The requirement for a lattice polygon to be regular proved very restrictive: only triangles, squares and hexagons may be regular lattice polygons in $\mathbb{Z}^3$. Most astonishingly, increasing the dimension of the lattice any further does not yield more varied regular polygons. Instead, let us relax this condition by considering **semiregular** lattice polygons, which are lattice polygons that are equilateral or equiangular (but not necessarily both). These have been studied in more recent works by Ball [1], Honsberger [14], Beeson [2], and Maehara [21].

2.2.1 Equilateral lattice polygons

We now explore the other type of semiregular lattice polygons.

The following lemma will be used to prove a number theoretic result by Ball [1], which corresponds to one of the implications in the subsequent theorem due to Dean Hoffman [14].

Lemma 2.6 (Ball [1, Lemma 1]). *If p and q are odd integers, then $4 \nmid (p^2 + q^2)$.*

Proof. If p and q are odd, then $p = 2n + 1$ and $q = 2m + 1$ for some integers n and m. So $p^2 + q^2 = (2n + 1)^2 + (2m + 1)^2 = 4n^2 + 4n + 4m^2 + 4m + 2$, which is not divisible by 4. □

Theorem 2.7 (Honsberger [14, Theorem 1]). *There exists an equilateral lattice n-gon in $\mathbb{Z}^2$ if and only if n is even.*

Proof. The forward implication comes from [1, p. 120]. Let n be an odd integer and assume for contradiction that there exists an equilateral lattice n-gon in $\mathbb{Z}^2$ with side length d and edge vectors $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

Since the polygon is closed, the sum of the vectors must be equal to $\mathbf{0}$ and by definition of the Euclidean distance Definition 1.16 and Definition 1.20,

$$\sum_{i=1}^{n} x_i = x_1 + x_2 + \cdots + x_n = 0 \tag{2.9}$$

$$\sum_{i=1}^{n} y_i = y_1 + y_2 + \cdots + y_n = 0 \tag{2.10}$$

$$x_1^2 + y_1^2 = x_2^2 + y_2^2 = \cdots = x_n^2 + y_n^2 = d^2. \tag{2.11}$$

Squaring then adding (2.9) and (2.10) together gives

$$\begin{aligned}
0 &= \left(\sum_{i=1}^{n} x_i\right)^2 + \left(\sum_{i=1}^{n} y_i\right)^2 \\
&= \sum_{1\le i,j\le n} x_i x_j + \sum_{1\le i,j\le n} y_i y_j \\
&= \sum_{\substack{1\le i,j\le n\\ i\ne j}} x_i x_j + \sum_{1=1}^{n} x_i^2 + \sum_{\substack{1\le i,j\le n\\ i\ne j}} y_i y_j + \sum_{1=1}^{n} y_i^2 \\
&= \sum_{\substack{1\le i,j\le n\\ i\ne j}} (x_i x_j + y_i y_j) + \sum_{1=1}^{n} (x_i^2 + y_i^2) \\
&= 2 \sum_{1\le i<j\le n} (x_i x_j + y_i y_j) + nd^2,
\end{aligned} \tag{2.12}$$

where the final equality holds by (2.11). Rearranging (2.12) gives

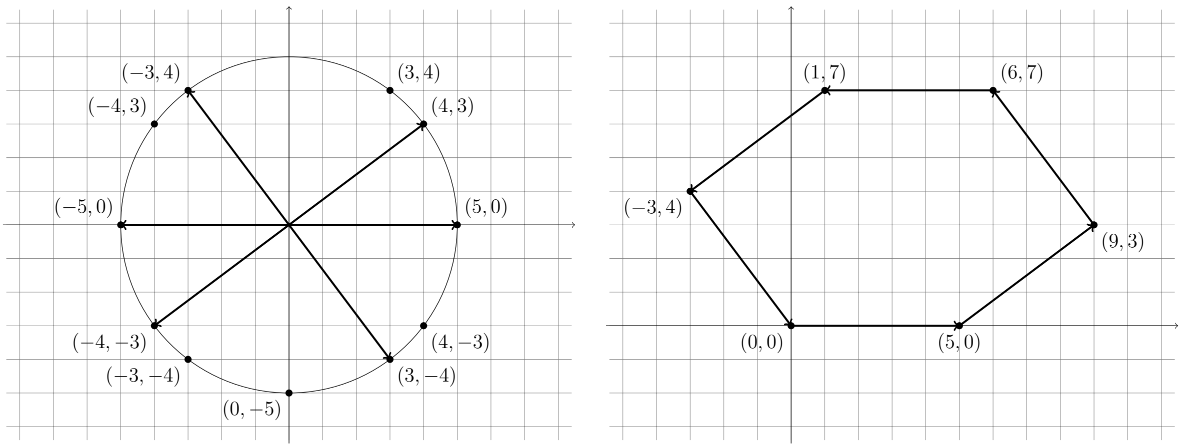
$$-2 \sum_{1\le i<j\le n} (x_i x_j + y_i y_j) = nd^2. \tag{2.13}$$

Since the left hand side of (2.13) is even and by assumption, n is odd, d^2 must be even as well. So, the left hand sides of (2.11) must be even that is for all $i \in \{1, 2, \ldots, n\}$, x_i and y_i are both odd or both even. Suppose that x_i and y_i are both odd for all i, then the sum in the left hand side of (2.13) is even. So, $4 \mid d^2$, which is a contradiction to Lemma 2.6. Instead, suppose that x_i and y_i are both even for all i, then $(x_i/2, y_i/2)$ is a lattice point for all i. The polygon defined by these points is a lattice n-gon with side length $d/2$. As in the proof of Theorem 2.2, repeating this argument leads to either the same contradiction as in the first case or a sequence of lattice polygons with strictly decreasing side lengths, which is also a contradiction.

For the converse implication, let $n = 2k$ where k is a positive integer. We now recall an important property for numbers expressible as the sum of two squares. Let m be an integer written in the form $m = 2^\alpha \prod_i p_i^{\beta_i} \prod_j q_j^{\gamma_j}$ where for all i, p_i is taken over the set of prime divisors of m of the form $4k+1$ and, for all j, q_j is taken over the set of prime divisors of m of the form $4k+3$. It can be shown that if γ_j is even for all j, then the number of integral solutions (x, y) of $x^2 + y^2 = m$ is equal to $4\prod_i(\beta_i + 1)$ [28, Theorem 3.22]. Accordingly, the equation $x^2 + y^2 = 5^{k-1}$ has $4(k-1+1) = 4k = 2n$ integral solutions, since all the divisors of 5^{k-1} (namely $5^0, 5^1, \ldots, 5^{k-1}$) are of the form $4k+1$ (see Figure 2.3a). Note that if (x, y) is such a solution, then so is $(-x, -y)$. Hence the $4k = 2n$ solutions come in $2k = n$ opposite pairs. Now identifying each lattice point as a vector from the origin to said point, a lattice polygon can be constructed by adjoining any $k = n/2$ of these pairs of vectors (see Figure 2.3b). Moreover, each vector has norm $\sqrt{5^{k-1}}$, so the polygon is equilateral.[3] $\square$

We now know that equilateral lattice polygons are possible in $\mathbb{Z}^2$ only if they have an even number of vertices, but what about higher dimensional lattices? The following two propositions are adapted from Maehara and Martini [22, Lemma 4.1. and Theorem 6.1.] and extend Theorem 2.7.

[3]Dean Hoffman fails to show that the figure is indeed a convex equilateral lattice n-gon. See Chapter 3 for a similar proof strategy where these properties are verified.



(a) The circle containing all lattice points such that $x^2 + y^2 = 5^2$

(b) The hexagon obtained by adjoining the previous vectors clockwise one by one

Figure 2.3: Illustration of Hoffman's proof for $n = 6$

Lemma 2.8 (Maehara [21, Lemma 4.1.]). *Let L be a rectangular lattice. If there exists an equilateral lattice n-gon in L, then L also contains an equilateral lattice $(n+2)$-gon.*

Proof. Let $\mathbf{a}$ and $\mathbf{b}$ be the basis vectors of L and denote their respective norms by $a = \|\mathbf{a}\|$ and $b = \|\mathbf{b}\|$. Since $\mathbf{a} \cdot \mathbf{b} = 0$, we may identify L as a subset of the complex numbers $L = \{ax + iby : x, y \in \mathbb{Z}\}$. Let $z = aN + ib \in L$, where N is an integer with $N > 0$. By elementary properties of complex numbers,

$$\widehat{\overline{z}0z} = 2\tan^{-1}\left(\frac{b}{aN}\right) \to 0$$

as $N \to \infty$. So the angle $\widehat{\overline{z}0z}$ can be made arbitrarily small.

Now let P be an equilateral lattice polygon in L with edge vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ all of which can be identified as complex numbers $v_1, v_2, \ldots, v_n \in L$. We now aim to transform the lattice L in such a way that a new equilateral polygon can be created with two more edge vectors. To this end, let z be a non-zero element of L such that $\widehat{\overline{z}0z}$ is strictly smaller than the angle between any two consecutive edge vectors of P. Such a z is possible by the previous observation. Multiplying each complex lattice point by z gives $zv_1, zv_2, \ldots, zv_n$, which must be elements of L by the formula for multiplication of complex numbers. By construction of z, the vectors corresponding to the opposite lattice points $\overline{z}v_1$ and $-\overline{z}v_1$ are not parallel to any of the vectors corresponding to $zv_1, zv_2, \ldots, zv_n$.

Define P' by appending the edge vectors corresponding to $zv_1, zv_2, \ldots, zv_n, \overline{z}v_1$ and $-\overline{z}v_1$ clockwise one by one like in the proof of Theorem 2.7. Finally, since P is equilateral,

$$|v_1| = |v_2| = \cdots = |v_n|$$

and so

$$|zv_1| = |zv_2| = \cdots = |zv_n| = |\overline{z}v_1| = |-\overline{z}v_1|.$$

Hence P' must also be equilateral. □

Theorem 2.9. *Let d be an integer with $d \geq 3$. There exists an equilateral lattice n-gon in $\mathbb{Z}^d$ for all $n \geq 3$.*

Proof. By Theorem 2.7, Z^d contains an equilateral n-gon for all even integers $n \geq 3$ and $\mathbb{Z}^d$ admits $\mathbb{Z}^3$ as a sublattice. Thus, it suffices to show that there exists an equilateral lattice n-gon in $\mathbb{Z}^3$ for all odd integers $n \geq 3$. It can be shown that the triangle with vertices $(0,0,0)$, $(1,1,0)$ and $(0,1,1)$ is an equilateral lattice 3-gon in $\mathbb{Z}^3$. This triangle is contained in a planar sublattice L of $\mathbb{Z}^3$ with basis vectors $(1,1,0)$ and $(0,1,1)$. Moreover, L has a further sublattice L' with basis vectors $(1,1,0)$ and $(-1,1,2)$ where the last vector has been obtained by calculating $-(1,1,0)+2(0,1,1)$. Note that $(1,1,0)\cdot(-1,1,2) = 0$ so L' is a rectangular lattice. Finally, L' contains the equilateral triangle with vertices $(0,0,0)$, $(2,2,0)$ and $(0,2,2)$. So by Lemma 2.8, L' must contain equilateral lattice n-gons for

$$n = 3, 5, 7, \ldots$$

In other words, L' must contain equilateral lattice n-gons for all odd integers n with $n \geq 3$. Since $L' \subset L \subset \mathbb{Z}^3$, the same is true for $\mathbb{Z}^3$, as required. □

2.2.2 Equiangular lattice polygons

Here we present a simplified proof of Honsberger's result by Scott [32, Lemma 1], which concerns the other type of semiregular lattice polygons, namely equiangular lattice polygons. The following formula for the angle between two vectors in the plane will be key to the proof [4, Equation 3.313b].

Lemma 2.10. *If θ is the angle between the vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ and $\begin{pmatrix} c \\ d \end{pmatrix}$, then $\tan(\theta) = \frac{ad-bc}{ac+bd}$.*

In particular, if the vectors correspond to lattice points, then $a, b, c, d \in \mathbb{Z}$ so $\tan(\theta)$ must be rational or infinite.

Theorem 2.11 (Honsberger [14, Theorem 2])**.** *There exists an equiangular lattice n-gon in $\mathbb{Z}^2$ if and only if $n = 4$ or $n = 8$.*

Proof. Let P be an equiangular lattice n-gon in $\mathbb{Z}^2$, s_1 and s_2 be the side lengths of two adjacent edges, d be the length of the third segment of the triangle determined by these sides and $\alpha = (n-2)\pi/n$ be the interior angle. By the law of cosines,

$$\begin{aligned} d^2 &= s_1^2 + s_2^2 - s_1 s_2 \cos(\alpha) \\ &= s_1^2 + s_2^2 - s_1 s_2 \cos((n-2)\pi/n) \\ &= s_1^2 + s_2^2 + s_1 s_2 \cos(2\pi/n). \end{aligned}$$

Rearranging gives

$$2\cos(2\pi/n) = \frac{d^2 - s_1^2 - s_2^2}{\sqrt{s_1^2 s_2^2}}$$

where, by the same argument as in the proof of Corollary 2.3, s_1^2, s_2^2, d^2 are integers. So $2\cos(2\pi/n))$ satisfies the monic second degree polynomial with rational coefficients

$$X^2 - (d^2 - s_1^2 - s_2^2)^2/(s_1^2 s_2^2).$$

Hence the degree of $2\cos(2\pi/n))$ is at most two. Furthermore, by Lehmer's lemma, $2\cos(2\pi/n)$ has degree $\phi(n)/2$ so $\phi(n) \leq 4$. By examining the table in Appendix A, the only possible values of n for which $\phi(n) \leq 4$ are $3, 4, 5, 6, 8, 10$ and 12. Moreover, by the formula in Lemma 2.10, $\tan(2\pi/n)$ must be rational or infinite. Computing all seven possible values of $\tan(2\pi/n)$, only $n = 4$ and $n = 8$ give non-irrational values (see Scott [32, Figure 4]).

The converse implication is immediate from Example 2.1 since it shows an equiangular quadrilateral and the equiangular octagon in Figure B.2. $\square$

Note that, as a result of Theorem 2.7 and Theorem 2.11, the only candidates for regular lattice n-gons in $\mathbb{Z}^2$ are squares and octagons. However, being a regular polygon requires being simultaneously equilateral and equiangular, which octagons cannot satisfy by Theorem 2.5. The following observation by Ball [1, Theorem 3] explains more specifically why an octagon cannot be a regular lattice polygon in $\mathbb{Z}^2$.

Example 2.12. There does not exist a regular lattice octagon in $\mathbb{Z}^2$. Let θ be the angle between a longest diagonal and an adjacent edge of a regular octagon. Then $\theta = 3\pi/8$ and so $\tan(\theta) = \sqrt{2} + 1$ which is irrational and hence a contradiction to Lemma 2.10.

Unlike Section 2.2.1, the results so far only relate to lattice polygons in the planar integer lattice. It is already known from Example 2.4 that there exist equiangular lattice triangles in $\mathbb{Z}^3$. So Theorem 2.11 does not generalise to higher dimensional lattices.

A generalisation is provided by Maehara and Martini [22]. For lattice points A, B, C in $\mathbb{Z}^d$, the angle $\widehat{ABC}$ is called a **lattice angle** in $\mathbb{Z}^d$. The following characterisation of lattice angles was originally obtained by Beeson [2] in 1992, although its presentation comes from Maehara and Martini [22, Theorem 4.1.].

Theorem 2.13 (Beeson [2])**.** *Let $\Theta_d = \{\widehat{ABC} : A, B, C \in \mathbb{Z}^d\}$ be the set of all lattice angles in $\mathbb{Z}^d$. Then*

1. $\theta \in \Theta_2 \iff \theta = \pi/2$ *or* $\tan\theta \in \mathbb{Q}$,
2. $\theta \in \Theta_4 \iff \theta = \pi/2$ *or* $\tan^2\theta = (a^2 + b^2 + c^2)/d^2$ *where* $a, b, c, d \in \mathbb{Z}$,
3. $\theta \in \Theta_5 \iff \cos^2\theta \in \mathbb{Q}$ *and*
4. $\Theta_2 \subset \Theta_3 = \Theta_4 \subset \Theta_5 = \Theta_6 = \Theta_7 = \dots$.

The proof is omitted, but can be found in Beeson [2]. Note that the first proposition corresponds to Lemma 2.6 and was already used to find the equiangular lattice polygons in Z^2. The theorem claims that all lattice angles in $\mathbb{Z}^4$ are also contained in $\mathbb{Z}^3$. All equiangular lattice polygons can now be fully described by using the remaining three propositions.

Theorem 2.14. *Let d be an integer with $d \geq 3$. There exists an equiangular lattice n-gon in $\mathbb{Z}^d$ if and only if $n \in \{3, 4, 6, 8, 12\}$.*

Proof. Let d be an integer with $d \geq 3$ and assume that there exists an equiangular lattice n-gon in $\mathbb{Z}^d$. As in the first part of the proof of Theorem 2.11, using the cosine law and Lehmer's lemma, we show that $n = 3, 4, 5, 6, 8, 10$ or 12. Moreover, the interior angles of this polygon are $(n-2)\pi/n$ and are lattice angles in $\mathbb{Z}^d$. Note that by Beeson's theorem, every lattice angle is contained in Θ_5 and so $\cos^2(2\pi/n) = \cos^2((n-2)\pi/n) \in \mathbb{Q}$. Finally, calculating $\cos^2(2\pi/n)$ for all remaining possible values of n gives

$$\begin{aligned}
\cos^2(2\pi/3) &= 1/4, & \cos^2(2\pi/4) &= 0, \\
\cos^2(2\pi/5) &= (\sqrt{5}-1)^2/16, & \cos^2(2\pi/6) &= 1/4, \\
\cos^2(2\pi/8) &= 1/2, & \cos^2(2\pi/10) &= (\sqrt{5}+1)^2/16, \\
\cos^2(2\pi/12) &= 3/4.
\end{aligned}$$

Hence, the only possible values for n are $3, 4, 6, 8$ and 12.

Conversely, we have already shown in Examples 2.1, 2.4, and Figure B.2 that equiangular lattice n-gons exist in $\mathbb{Z}^3$, and thus in $\mathbb{Z}^d$, for $n \in \{3, 4, 6, 8\}$. It remains to show that an equiangular lattice dodecagon exists in Z^3. By the "triple-and-truncate" method outlined in Maehara and Martini [22, Figure 3], an equiangular lattice $2n$-gon can be constructed from any regular lattice n-gon. Thus, since Z^3 contains a regular hexagon, it must also contain an equiangular dodecagon, as required. □

The results of the previous two sections are summarised in the following table.

	$\mathbb{Z}^2$	$\mathbb{Z}^d$ where $d \geq 3$
Regular	4	$3, 4, 6$
Equilateral	all even integers	all integers
Equiangular	$4, 8$	$3, 4, 6, 8, 12$

Table 2.1: Possible number of vertices n, with $n \geq 3$, for lattice n-gons in various integer lattices

2.3 Alternative problems on lattices

The current findings highlight the significant depth of the theory on lattice polygons However, problems relating to the lattice extend beyond the existence of lattice polygons. This section presents a brief summary of other related problems and explores a few directions in which this topic has already been extended.

One relatively intuitive question to explore is the existence of lattice polyhedra in $\mathbb{Z}^3$, where lattice polyhedra are defined analogously to lattice polygons.

Proposition 2.15 (Ehrhart [11, Proposition 4])**.** *There do not exist regular icosadra and regular dodecahedra in* $\mathbb{Z}^3$*.*

Proposition 2.16 (Scott [32, Theorem 1])**.** *The truncated tetrahedra, the truncated octahedra, and the cuboctahedra are the only possible semiregular lattice polyhedra in* $\mathbb{Z}^3$*.*

Figures for these shapes can be found in Scott [32, Figure 1 and Figure 2].

A completely different set of problems arises by considering transformations of the lattice. In fact, many of the results by Maehara [21] in Section 2.2 are obtained for a general planar integral lattice and not just for an integer lattice. Another transformation considered by Kołodziejczyk [16] is to replace the lattice by a hexagonal tiling. A simpler extension is given by applying a linear transformation to the integer lattice. This is exactly what is done by Teuffel [34] in 1975.

Proposition 2.17 (Teuffel [34, Aufgabe 709])**.** *No four distinct points of an equilateral triangular lattice can be the vertices of a square. Equivalently, no triangular lattice contains a square.*

First, a preliminary lemma is needed.

Lemma 2.18. *Let ABC be a right-angled triangle with $\widehat{BAC} = \pi/2$. Then ABC is contained in some triangular lattice if and only if $\|\mathbf{AB}\| = q\sqrt{3}\|\mathbf{AC}\|$ for some strictly positive rational number q.*

Proof. For simplicity, assume that ABC is contained in a triangular lattice with unit basis vectors $\mathbf{a}$ and $\mathbf{b}$. Then $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\| \times \|\mathbf{b}\| \times \cos(\frac{\pi}{3}) = \frac{1}{2}$. On one hand, since $\mathbf{a}$ and $\mathbf{b}$ span the lattice, $\mathbf{AB} = x\mathbf{a} + y\mathbf{b}$ and $\mathbf{AC} = u\mathbf{a} + v\mathbf{b}$ for some non-zero integers x, y, u, v. So calculating the dot product directly gives

$$\begin{aligned}
\mathbf{AB} \cdot \mathbf{AC} &= (x\mathbf{a} + y\mathbf{b}) \cdot (u\mathbf{a} + v\mathbf{b}) \\
&= x\mathbf{a} \cdot (u\mathbf{a} + v\mathbf{b}) + y\mathbf{b} \cdot (u\mathbf{a} + v\mathbf{b}) \\
&= xu\|\mathbf{a}\|^2 + xv(\mathbf{a} \cdot \mathbf{b}) + yu(\mathbf{b} \cdot \mathbf{a}) + yv\|\mathbf{b}\|^2 \\
&= xu + \frac{1}{2}xv + \frac{1}{2}yu + yv.
\end{aligned} \tag{2.14}$$

On the other hand, since $\mathbf{AB}$ and $\mathbf{AC}$ are orthogonal,

$$\mathbf{AB} \cdot \mathbf{AC} = 0. \tag{2.15}$$

Equating and rearranging (2.14) and (2.15) gives

$$x(2u + v) = -y(u + 2v). \tag{2.16}$$

Note that if $x = 0$, then $y \neq 0$ (otherwise $\|\mathbf{AB}\| = 0$), so $u + 2v = 0$ and thus $2u + v \neq 0$ (otherwise $\|\mathbf{AC}\| = 0$). So we may rearrange (2.16) to obtain $x = q(u + 2v)$ and $y = -q(2u + v)$, where $q = \frac{-y}{2u+v} \in \mathbb{Q}$. The case when $y = 0$ is similar. Finally, using the definition of the Euclidean distance,

$$\begin{aligned}
|\mathbf{AB}\|^2 &= x^2 + xy + y^2 \\
&= q^2(u + 2v)^2 - q^2(u + 2v)(2u + v) + q^2(2u + v)^2 \\
&= q^2(3u^2 + 3uv + 3v^2) \\
&= 3q^2(u^2 + uv + v^2) \\
&= 3q^2\|\mathbf{AC}\|^2.
\end{aligned}$$

Taking square roots gives the required result

$$\|\mathbf{AB}\| = |q|\sqrt{3}\|\mathbf{AC}\|.$$

Conversely, let ABC be a triangle with $\widehat{BAC} = \frac{\pi}{2}$ and $\|\mathbf{AB}\| = \frac{n\sqrt{3}}{m}\|\mathbf{AC}\|$ where n and m are strictly positive integers. Reordering gives

$$\frac{1}{2n}\|\mathbf{AB}\| = \frac{\sqrt{3}}{2}\frac{\|\mathbf{AC}\|}{m}.$$

By using the formula for the height of a triangle (see Bronshtein et al. [4, Table 3.2]), $\frac{1}{2n}\overline{AB}$ is the height of an equilateral triangle with vertex A and an edge parallel to $\overleftrightarrow{AC}$ with side length $\|\mathbf{AC}\|/m$. Define $\mathbf{a}$ and $\mathbf{b}$ to be the vectors corresponding to the edges of the equilateral triangle containing A. Then, by construction $\mathbf{a} \cdot \mathbf{b} = \pi/3$, $\mathbf{AB} = 2n\mathbf{a} + 2n\mathbf{b}$ and $\mathbf{AC} = m\mathbf{a} - m\mathbf{b}$. So ABC is contained in the triangular lattice defined by the vectors $\mathbf{a}$ and $\mathbf{b}$. $\square$

Main theorem. Assume for contradiction that $ABCD$ is a square inscribed in an equilateral triangular lattice. Then the triangle ABD has a right angle $\widehat{BAD}$. By the previous lemma, there exists a rational number q such that $\|\mathbf{AB}\| = q\sqrt{3}\|\mathbf{AD}\|$. However, since $ABCD$ is a square, $\|\mathbf{AB}\| = \|\mathbf{AD}\|$. So $q = 1/\sqrt{3}$, which is a contradiction. $\square$

Chapter 3

A new extension to the theory of lattice polygons

One area into which the theory of lattice polygons has not yet ventured, is non-Euclidean geometry, which will be the subject of the remainder of this dissertation. We begin by defining the taxicab geometry[1] and define novel analogous objects to those studied in Chapter 2. Previously seen proof strategies are then expanded in order to find the values of n for which there exist taxicab-equilateral lattice n-gons in $\mathscr{L}^2$.

3.1 The taxicab metric

Originally due to Minkowski [24], the taxicab metric generates one of the simplest non-Euclidean geometries in the sense that it is only "one axiom away" [17] from the standard Euclidean geometry By Example 1.13, the taxicab geometry does not satisfy SAS.

Definition 3.1. The **taxicab** (or Manhattan) metric on $\mathbb{R}^n$ is given by

$$d_T(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$$

where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

Example 3.2. The taxicab unit circle $C_T(\mathbf{0}, 1)$ is the set of points with a taxicab distance of 1 to the origin. The taxicab distance between $(0, 0)$ and $(2, 1)$ is 3.

Equipped with this new definition of a distance, Definition 1.20 may be modified.

Definition 3.3. Let P be a polygon.

1. P is **taxicab-equilateral** if all its edges have the same taxicab length.

2. P is **taxicab-equiangular** if all its interior angles are equal.

3. P is **taxicab-regular** if it is taxicab-equilateral and taxicab-equiangular.

[1] See Krause [17, p. 697] for a more detailed description of the taxicab geometry.

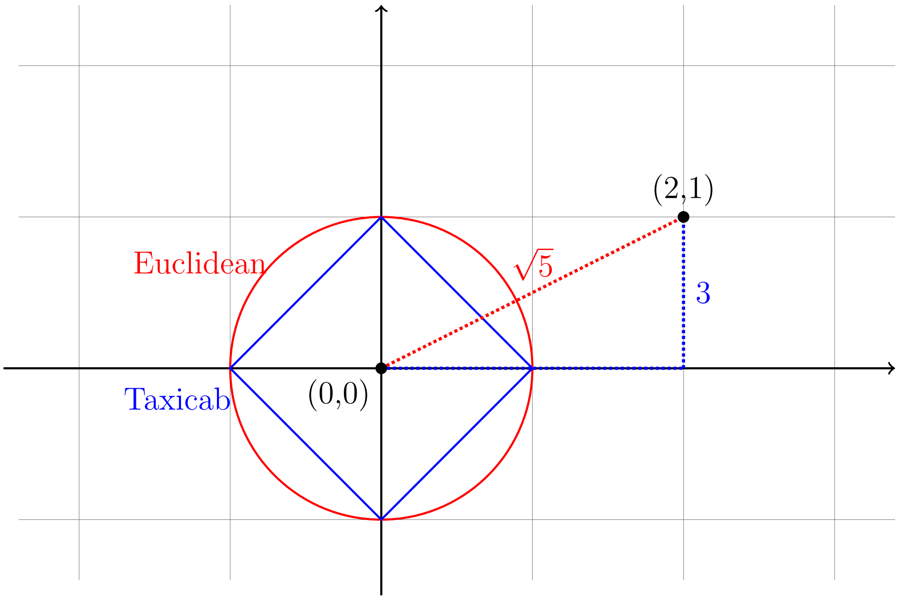


Figure 3.1: Unit circles and distance between points using the Euclidean metric (in red) and taxicab metric (in blue)

Note that the taxicab geometry has been defined using the same angle measure as that of Euclidean geometry. Therefore, taxicab-equiangular polygons coincide perfectly with equiangular ones.

It is already known that equipping the plane $\mathbb{R}^2$ with the taxicab metric changes the availability of certain polygons. Çolakoğlu and Kaya [6] explore the correspondence between regular and taxicab-regular polygons.

Proposition 3.4 (Çolakoğlu and Kaya [6, Proposition 5 and 6, Theorem 7 and 13])**.** *Let P be a polygon in the plane $\mathbb{R}^2$.*

If P is Euclidean-regular, then P is taxicab-regular if and only if P is one of

1. *a quadrilateral or*
2. *an octagon with an axis of symmetry passing through two distinct vertices parallel to one of*
   (a) $x = 0$,
   (b) $y = 0$,
   (c) $y = x$ *or*
   (d) $y = -x$.

Conversely, if P is taxicab-regular, then P is Euclidean-regular if and only if it is one of the aforementioned polygons.

Additionally, it has been demonstrated that there are no taxicab-regular n-gons (not necessarily lattice polygons) for $n = 3$ or $n = 5$ in $\mathbb{R}^2$ Hanson [13, pp. 1085, 1092], although the non-existence of taxicab-regular polygons with an odd number of vertices remains an open problem [6, Theorem 14]. Yüksel and Özcan [35] investigate similar problems in three-dimensional space. In particular, they show that there exists a taxicab-regular $2n$-gon in $\mathbb{R}^3$ for all $n \geq 2$. These results are certainly interesting in their own right, but the proof strategies used are not easy to adapt to the lattice. They do however hint at the possibility of genuinely different results in the taxicab geometry than those found in Chapter 2.

3.1.1 Finding taxicab-equilateral lattice polygons

As a reminder, the angular measures of the taxicab and Euclidean geometries agree with each other. So, by Theorem 2.11, the only candidates for taxicab-equiangular lattice polygons in $\mathbb{Z}^2$ are the 4-gons and 8-gons. For $3 \leq n \leq 12$, taxicab-equilateral lattice n-gons are known to exist since they are constructed in Figure B.2.[2] In particular, the pentagon indicates that Theorem 2.7 does not hold for the taxicab metric and hence cannot be generalised to all metrics. Similarly, the octagon is a counterexample to a possible generalisation of Theorem 2.5. In fact, we shall show that the following procedure constructs a taxicab-equilateral lattice n-gon in $\mathbb{Z}^2$ for any $n \geq 3$.

1. Select n vectors from (0,0) to distinct lattice points of $C_T(\mathbf{0}, n-1)$ such that their sum is $\mathbf{0}$.
2. Label the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that their angle with the positive x-axis is a strictly increasing sequence.
3. Adjoin the vectors one by one to obtain a figure with vertices corresponding to the endpoints of
$$\mathbf{v}_1, \quad \mathbf{v}_1 + \mathbf{v}_2, \quad \ldots, \quad \mathbf{v}_1 + \mathbf{v}_2 + \cdots + \mathbf{v}_n.$$

It now remains to prove that such a figure can be constructed and is in fact the desired polygon, but first some additional results are required.

Lemma 3.5. *The taxicab-circle $C_T(\mathbf{0}, r) = \{\mathbf{y} \in \mathbb{R}^2 : d_T(\mathbf{0}, \mathbf{y}) = r\}$ with centre $\mathbf{0} \in \mathbb{Z}^2$ and radius $r \in \mathbb{N}$, such that $r > 0$, contains $4r$ lattice points organised in $2r$ opposite pairs.*

Proof. By Definition 3.1, for any $\mathbf{y} = (y_1, y_2) \in \mathbb{Z}^2$, $d_T(\mathbf{0}, \mathbf{y}) = |y_1| + |y_2|$. So the integer solutions to $d_T(\mathbf{0}, \mathbf{y}) = r$ are
$$(0, \pm r), \quad (\pm r, 0),$$
$$(\pm 1, \pm(r-1)), \quad \ldots, \quad (\pm(r-1), \pm 1),$$
$$(\pm 1, \mp(r-1)), \quad \ldots, \quad (\pm(r-1), \mp 1).$$
Hence, the circle $C_T(\mathbf{0}, r)$ intersects the lattice at exactly $4r$ points.

Moreover, if (y_1, y_2) is such a point, then $|-y_1| + |-y_2| = |y_1| + |y_2| = r$. So $(-y_1, -y_2)$ must also be a lattice point of $C_T(\mathbf{0}, r)$. Hence the $4r$ points are split into $2r$ opposite pairs.

□

We now show that it is possible to select the required vectors in step 1.

Lemma 3.6. *There exist n vectors corresponding to lattice points in $C_T(\mathbf{0}, n-1)$ such that their sum is $\mathbf{0}$.*

Proof. By Lemma 3.5, $C_T(\mathbf{0}, n-1)$ contains $2(n-1)$ pairs of lattice points each defining a pair of opposite vectors from $\mathbf{0}$ to said lattice points.

If n is even, then consider any $n/2$ such pairs. Since the sum of the vectors in each pair is $\mathbf{0}$, the sum of all n vectors must also be equal to $\mathbf{0}$.

[2]For our purposes, the term "constructed" is synonymous with "there exists" and should not be confused with its usage in Galois theory.

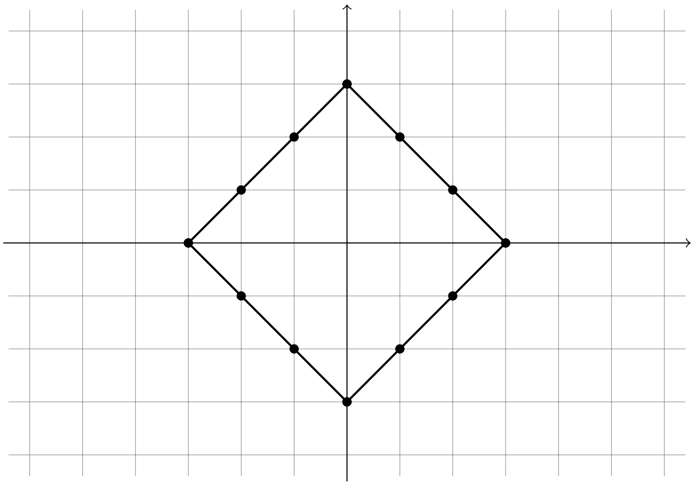


Figure 3.2: The taxicab circle with radius 3 and centre $\mathbf{0}$

Otherwise, if n is odd, then consider any three of the $4(n-1)$ vectors such that their sum is $\mathbf{0}$ say

$$\mathbf{v}_1 = (n-1, 0)\,, \quad \mathbf{v}_2 = \frac{1}{2}(-n+1, n-1) \quad \textit{and} \quad \mathbf{v}_3 = \frac{1}{2}(-n+1, -n+1)\,.$$

Observe that $\|\mathbf{v_1}\|_T = \|\mathbf{v}_2\|_T = \|\mathbf{v}_3\|_T = n-1$ and $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 = \mathbf{0}$. Now choose any $(n-3)/2$ pairs of the remaining $\frac{1}{2}[4(n-1)-6] = 2n-5$ pairs such that no pair contains $\mathbf{v}_1, \mathbf{v}_2$ or $\mathbf{v}_3$. Then, as before, the sum of all these n vectors is equal to $\mathbf{0}$. $\square$

Furthermore, these chosen vectors must satisfy an important property relating to their angles.

Lemma 3.7. *Let n be an integer, with $n \geq 3$, $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ be non-zero coplanar vectors and $\theta_1, \theta_2, \ldots, \theta_n$ be their respective angle with the positive x-axis. If $0 \leq \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ and $\sum_{i=1}^{n} \mathbf{v}_i = \mathbf{0}$, then for all $i \in \{1, 2, \ldots, n\}$, $\theta_{i+1} - \theta_i < \pi$ (where θ_{n+1} denotes θ_1).*

Proof. Assume for contradiction that there exists a $j \in \{1, 2, \ldots, n\}$ such that $\theta_{j+1} - \theta_j \geq \pi$. One one hand, rotating the first j vectors by θ_j clockwise gives

$$-\pi < \pi - \theta_{j+1} \leq -\theta_j \leq \theta_1 - \theta_j < \theta_2 - \theta_j < \cdots < \theta_j - \theta_j = 0, \tag{3.1}$$

where the lower bound is obtained by rearranging $\theta_{j+1} - \theta_j \geq \pi$ and using the fact that $\theta_1 \geq 0$ and $\theta_{j+1} < 2\pi$. Similarly, rotating the final $n-j$ vectors by θ_j clockwise gives,

$$\pi \leq \theta_{j+1} - \theta_j < \theta_{j+2} - \theta_j < \cdots < \theta_n - \theta_j < 2\pi - \theta_j < 2\pi. \tag{3.2}$$

By examining (3.1) and (3.2),

$$\sin(\theta_i - \theta_j) \leq 0, \tag{3.3}$$

for all $i \in \{1, 2, \ldots, n\}$, and with equality only possible when $i = j$ or $i = j+1$.

On the other hand, since the sum of the vectors is $\mathbf{0}$, it remains unchanged when rotated by θ_j. So, the y-component of the sum of the rotated vectors now becomes

$$\sum_{i=1}^{n} \|\mathbf{v}_i\| \sin(\theta_i - \theta_j) = 0. \tag{3.4}$$

Finally, by assumption, all vectors are non-zero so (3.4) can be simplified into

$$\sum_{i=1}^{n} \sin(\theta_i - \theta_j) = 0,$$

which is a contradiction of (3.3). $\square$

We are now finally in a position to prove the promised result.

Theorem 3.8. *For all integers n, such that $n \geq 3$, there exists a taxicab-equilateral lattice n-gon in $\mathbb{Z}^2$.*

Proof. Let $n \geq 3$ and consider the figure P obtained using the above method. By Lemma 3.6, we may choose n vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ such that $\sum_{i=1}^{n} \mathbf{v}_i = \mathbf{0}$. So P is a closed figure with n vertices.

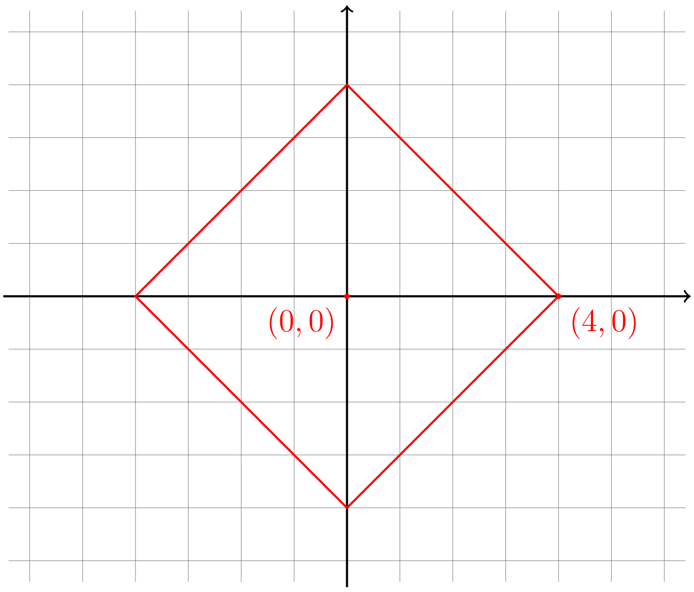
Moreover, suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ are labelled such that their respective angles with the positive x-axis $\theta_1, \theta_2, \ldots, \theta_n$ is a strictly increasing sequence. Then, by Lemma 3.7, the angle between any consecutive vectors, and thus every exterior angle of P, is strictly less than π. Also, by definition, the sum of the exterior and interior angle at every vertex is equal to π. So the interior angles of P satisfy $0 < \pi - (\theta_{i+1} - \theta_i) < \pi$ for all $i \in \{1, 2, \ldots, n\}$. Hence, by definition, P is convex.

Finally, by construction, each vector has magnitude $n - 1$ and each vertex is a lattice point. So, in particular, P is a taxicab-equilateral lattice polygon with n vertices. $\square$

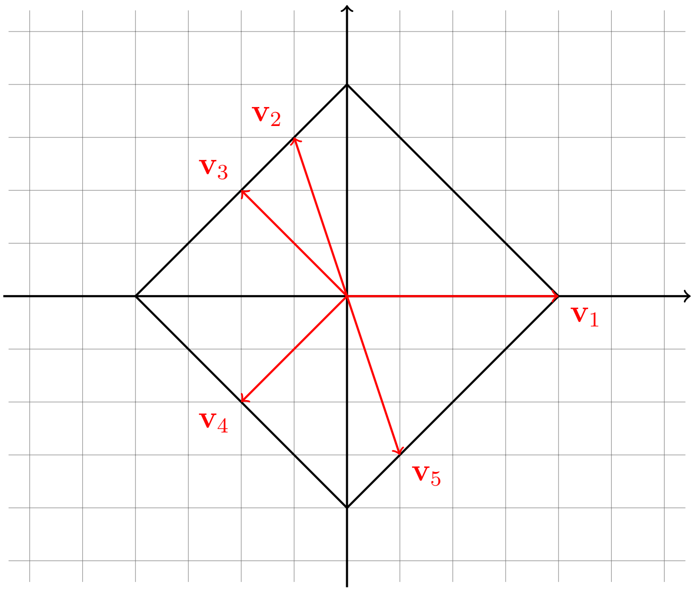
Corollary 3.9. *For all integers n and d such that $n \geq 3$ and $d \geq 2$, there exists a taxicab-equilateral lattice n-gon in $\mathbb{Z}^d$.*

Proof. This follows immediately from Theorem 3.8 since there exists a taxicab-equilateral lattice n-gon in $\mathbb{Z}^2$ for all $n \geq 3$. $\square$

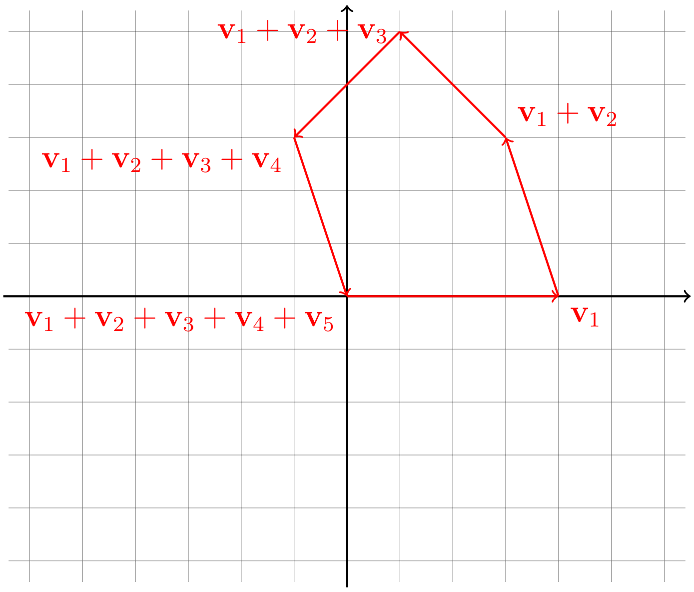
Example 3.10. We use the method to construct a taxicab-equilateral lattice pentagon (corresponding to $n = 5$). First construct the taxicab-circle $C_T(\mathbf{0}, 4)$.



Choose five vectors from the origin to lattice points of the circle such that their sum is equal to $\mathbf{0}$. We choose $\mathbf{v}_1 = (4, 0)$, $\mathbf{v}_2 = (-1, 3)$, $\mathbf{v}_3 = (-2, 2)$, $\mathbf{v}_4 = (-2, 2)$ and $\mathbf{v}_5 = (1, -3)$.



Adjoin the five vectors one by one to obtain a taxicab-equilateral lattice pentagon defined by the points $(4, 0)$, $(3, 3)$, $(1, 5)$, $(-1, 3)$ and $(0, 0)$.



3.1.2 Optimising the size of the polygons

The radius chosen for the taxicab-circle (and therefore the side length of the polygon) in Theorem 3.8 appears excessive. For instance, to construct a taxicab-equiangular 4-gon, the procedure gives a polygon with a side length of three, despite the fact that a taxicab-equilateral square can be easily constructed with a side length of one (see Figure B.2). In fact, the side lengths of the taxicab-equilateral lattice n-gons in Figure B.2 for $n = 3, 2, \ldots, 12$ seem to always be less than what would be created by our original procedure. Define $r(n)$ to be the the minimal (taxicab) side length of a taxicab-equiangular lattice n-gon in $\mathbb{Z}^2$. The aim of this section is to find upper and lower bounds for $r(n)$.

Proposition 3.11. *If a taxicab-equilateral lattice polygon has an odd number of vertices, then it must have an even side length.*

Proof. Assume for contradiction that P is a taxicab-equilateral lattice n-gon in $\mathbb{Z}^d$ with side length m for some odd integers n and m. Now consider a colouring of the lattice given by the parity of the points. Specifically, for every lattice point $\mathbf{x} = (x_1, \ldots, x_d)$ in $\mathbb{Z}^d$, colour $\mathbf{x}$ blue if $\sum_{i=1}^{d} x_i$ is even and red otherwise. By elementary properties of number theory, $\mathbf{x}$ is blue if there is an even number of odd summands and red otherwise.

We now show that the vertices of P must alternate colour. Let $\mathbf{x} = (x_1, \ldots, x_d)$ and $\mathbf{y} = (y_1, \ldots, y_d)$ be consecutive vertices of P and $\mathbf{v} = \mathbf{y} - \mathbf{x}$ the corresponding edge vector. Since all the edges of P have an odd side length, $\|\mathbf{v}\|_T = \sum_{i=1}^{d} |y_i - x_i|$ is odd so there is an odd number of odd terms in the sum. If x is blue, then there is an even number of odd terms so y must contain an odd number of odd terms and hence be coloured red. Similarly, if x is red, then y is blue.

Finally, P has an odd number of vertices that have alternating blue and red colours. In the language of graph theory, the set of vertices and edges of P form a bipartite graph, where the bipartition is given by the colouring described above. However, since P has an odd number of vertices, this bipartite graph contains an odd cycle, which is impossible (see Diestel [8, Proposition 1.6.1.]). ☐

Clearly, the number of lattice points of the taxicab-circle in the procedure of Section 3.1.1 must be at least as big as the number of edges of a given taxicab-equilateral lattice polygon. Let us formalise this remark into the following proposition.

Proposition 3.12. *For all integers n with $n \geq 3$, $r(n) \geq \lceil n/4 \rceil$.*

Proof. By Lemma 3.5, the taxicab circle centered at the origin with radius r contains $4r$ lattice points. From the previous remark, $r(n) \geq n/4$. Furthermore, the side length of the polygon must be an integer for the vertices to remain on the lattice, so $r(n) \geq \lceil n/4 \rceil$. ☐

By combining the previous results, we get the following optimized side lengths.

Theorem 3.13. *Let n be an integer with $n \geq 3$ and $r(n)$ be the minimal side length of the taxicab-equilateral lattice n-gons. Then*

1. *if $n = 3$, then $r(n) = 2$;*
2. *if n is even, then $r(n) = \lceil n/4 \rceil$;*
3. *if n is odd, then $\lceil n/4 \rceil \leq r(n) \leq (n+3)/4$, where the upper bound is rounded up to the nearest even integer.*

Proof. For $n = 3$, a taxicab-equilateral n-gon is shown in Figure B.2 so $r(3) \leq 2$. Moreover, the taxicab circle with radius one has only four vectors corresponding to lattice points: $(1, 0)$, $(0, 1)$, $(-1, 0)$ and $(0, -1)$. By exhausting all four choices of any three of these four vectors, we conclude that no selection of vectors gives a taxicab-equilateral lattice triangle. So $r(3) = 2$.

When n is even, it suffices, by Proposition 3.12, to show that a taxicab-equilateral lattice n-gon with side length $\lceil n/4 \rceil$ can be constructed. Note that in the proof of Theorem 3.8, the choice of taxicab circle only mattered when finding sufficiently many vectors whose sum is equal to $\mathbf{0}$. Let $C = C_T(\mathbf{0}, \lceil n/4 \rceil)$ and note that $\lceil n/4 \rceil = n/4$ if n is divisible by 4 and $\lceil n/4 \rceil = (n+2)/4$ otherwise. In either case, the number of lattice points in C must be greater than or equal to n. So, just as in the proof of Lemma 3.6, we may choose any $n/2$ pairs of opposite vectors corresponding to pairs of lattice points. Then the sum of all these vectors must be equal to $\mathbf{0}$.

The case when n is odd requires a bit more care. The lower bound is simply given by Proposition 3.12. To prove the upper bound, let r be equal to $(n+3)/4$ rounded up to the nearest even integer. It now suffices to show that a taxicab-equilateral lattice n-gon with side length r can be constructed. Consider the three vectors $(r, 0)$, $\frac{1}{2}(-r, r)$ and $\frac{1}{2}(-r, -r)$. Note that their sum is equal to $\mathbf{0}$ and they all belong to the taxicab circle $C(\mathbf{0}, r)$. This circle has $(4r-6)/2$ remaining pairs of vectors. Since $r \geq (n+3)/4$, $(4r-6)/2 \geq (n-3)/2$. So, as in the case when n is even, we may choose any $(n-3)/2$ pairs of opposite vectors. $\square$

Rather disappointingly, the above theorem gives an exact formula for $r(n)$ when n is even or $n = 3$, but leaves a range of possible values for when n is odd. Looking at Table B.1, we conjecture that the upper bound in Theorem 3.13 is in fact equal to $r(n)$.

3.2 Further extensions

This dissertation concludes by indicating a few potential directions for further extensions of the ideas presented in this chapter so far.

By definition of the taxicab plane, taxicab-equiangular polygons correspond to Euclidean-taxicab polygons. Moreover, since for all integers n with $n \geq 3$, a taxicab-equilateral lattice polygon exists, all the taxicab-equiangular polygons are candidates to be regular. Thus, Table 2.1 indicates all the taxicab-equiangular lattice polygons. Note that, as in Example 2.12, the existence of equilateral n-gons and equiangular n-gons for a given value of n does not imply the existence of regular n-gons. We propose the following problem: find the values of n and d for which there exist a taxicab-regular lattice n-gon in L^d.

Note that the taxicab geometry is far from being the only non-Euclidean geometry that could be used to explore problems in the lattice. For example, the **chessboard** (or Chebyshev) metric on $\mathbb{R}^n$ is given by

$$d_C(\mathbf{x}, \mathbf{y}) = \max_i(|x_i - y_i|)$$

where $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{Z}^n$. Just as for the taxicab plane 1.11, it can be shown that the chessboard plane defined by $\{\mathbb{R}^2, \mathscr{L}_E, d_C, m_E\}$, where $\{\mathbb{R}^2, \mathscr{L}_E\}$ is the Cartesian plane, d_C is the chessboard metric on $\mathbb{R}^2$ and m_E is the Euclidean angle measure is a protractor geometry Millman and Parker [23, Problem 8 in Problem Set 5.1].

The following figure shows a chessboard-regular lattice octagon in $\mathbb{Z}^2$, where the term chessboard-regular is defined analogously to taxicab-regular polygons in Definition 3.3.

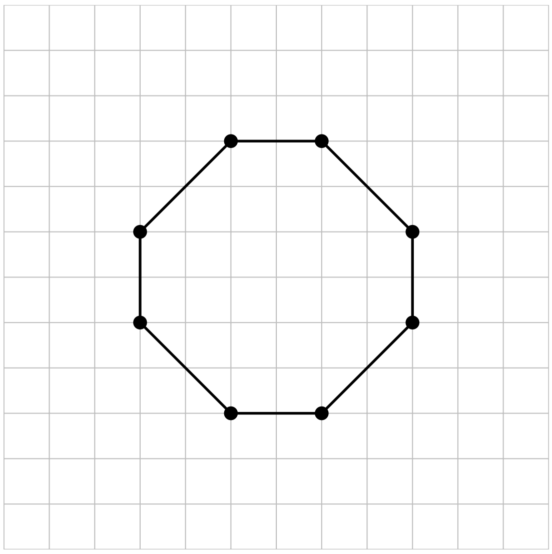


Figure 3.3: Chessboard-regular lattice octagon in $\mathbb{Z}^2$

Appendix A

Number theory

The aim of the section is to present the necessary background for the proof of Theorem 2.5 by providing a brief introduction to number theory and an important result by Lehmer [18].

An **algebraic number** is a number that satisfies a non-zero polynomial of the from $X^n + a_1 X^{n-1} + \cdots + a_n$ with rational coefficients $a_1, \ldots, a_n \in \mathbb{Q}$. Moreover, an algebraic number is called an **algebraic integer** if these coefficients are integers.

Polynomials with leading coefficient equal to one are called **monic**.

For example, $\sqrt{3}$ is a root of $X^2 - 3 = 0$ so it is an algebraic integer, while $\sqrt{5}/5$ is an algebraic number since it satisfies $X^2 - 1/5$.

Every algebraic number α is the root of a unique monic polynomial with rational coefficients of minimal degree called its **minimal polynomial**. The **degree** of α is a shorthand for the degree of its minimal polynomial.

Note that every rational number $p/q \in \mathbb{Q}$ satisfies $X - p/q$, so every rational number has degree equal to one. Conversely, if an algebraic number α has degree equal to one, then it must be a root of $X + p/q = 0$, where $p/q \in \mathbb{Q}$ so $\alpha \in \mathbb{Q}$. In other words, rational numbers coincide with algebraic numbers of degree equal to one.

If $\theta = 2\pi k/n$ with coprime integers k and n, then $\cos(\theta)$ is an algebraic integer. This may be proven by expanding de Moivre's identity $(\cos(\theta) + i\sin(\theta))^n = 1$ and by using the fact that $\sin^2(\theta) = 1 - \cos^2(\theta)$.

Of particular interest is the $\mathbf{n^{th}}$ **cyclotomic polynomial**

$$\Phi_n(X) = \prod_{\substack{k=1 \\ \gcd(n,k)=1}}^{n} (X - \zeta_k),$$

where $\zeta_k = e^{2\pi i k/n}$ is the $\mathbf{k^{th}}$ **root of unity**.

It turns out that $\Phi_n(X)$ is a monic irreducible polynomial with integer coefficients and degree $\phi(n)$, where $\phi(n)$ is Euler's totient function defined by the number of coprime integers less than or equal to n (see Niven [27]).

It can also be shown that $X^{-\phi(n)/2}\Phi_n(X)$ is a monic irreducible polynomial in $X + X^{-1}$ with integer coefficients and degree $\phi(n)/2$.

Finally, since $\zeta_k = e^{2\pi i k/n}$ is a root of $\Phi_n(X)$, $e^{2\pi i k/n} + e^{-2\pi i k/n} = 2\cos(2\pi k/n)$ is a root of $X^{-\phi(n)/2}\Phi_n(X)$.

This result can now be summarised by

Lemma A.1 (Lehmer [18, Theorem 1])**.** *If $n > 2$ and $\gcd(n,k) = 1$, then $2\cos(2\pi k/n)$ is an algebraic integer of degree $\phi(n)/2$.*

In the proof of Theorem 2.5, we use the fact that $2\cos(2\pi/n)$ must be rational and therefore have degree equal to one to conclude that $\phi(n)/2 = 1$. This can be done by inspecting the table found in Lucas [20, p. 395] (copied as Table A.1), which provides the values of n corresponding to the first few values of $\phi(n)$. From Table A.1, the only values of n such that $\phi(n)/2 = 1$ are $1, 2, 3, 4$ and 6. So the only possible values of n corresponding to a polygon are $3, 4$ and 6.

$\phi(n)$	n
1	1,2
2	3,4,6
4	5,8,10,12

Table A.1: Possible values of n for a given value for $\phi(n)$.

Appendix B

Further figures

In the proof of Theorem 2.2, a contradiction is obtained by constructing a new, strictly smaller, regular lattice n-gon, from a regular lattice n-gon of arbitrary side length, whenever $n \geq 5$, $n \neq 6$. It is not obvious why this strategy does not work for the remaining values of n, namely $3, 4$ and 6. The following figures show that applying the same method does not produce smaller polygons.

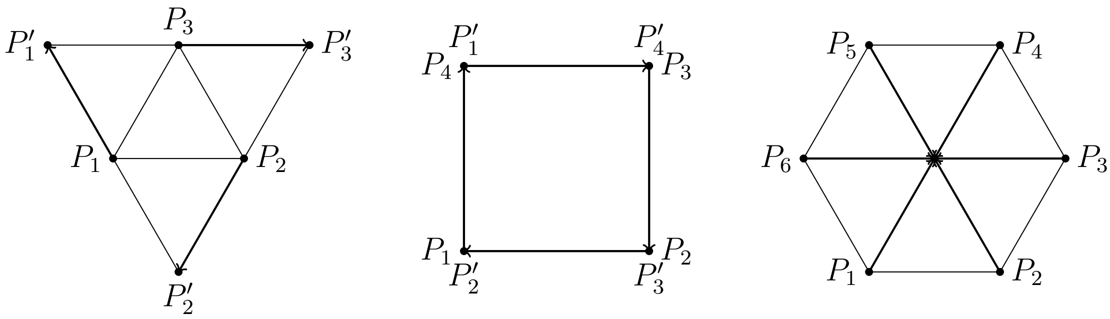


Figure B.1: Applying proof strategy to a regular triangle, square and hexagon

The new triangle has a side length that is strictly greater than that of the original. Similarly, the new square is not strictly smaller as the vertices are permuted clockwise by $\pi/2$. Finally, the translated vertices of the original hexagon all coincide with its centre, so the new figure is not a polygon.

The following figure contains examples of the taxicab-equilateral lattice n-gons in $\mathbb{Z}^2$, where $3 \leq n \leq 12$. It is worth noticing that the taxicab side length for the polygons with an odd number of vertices is always even. Also, the side lengths are always smaller than what they would have been by using the method described the proof of Theorem 3.8.

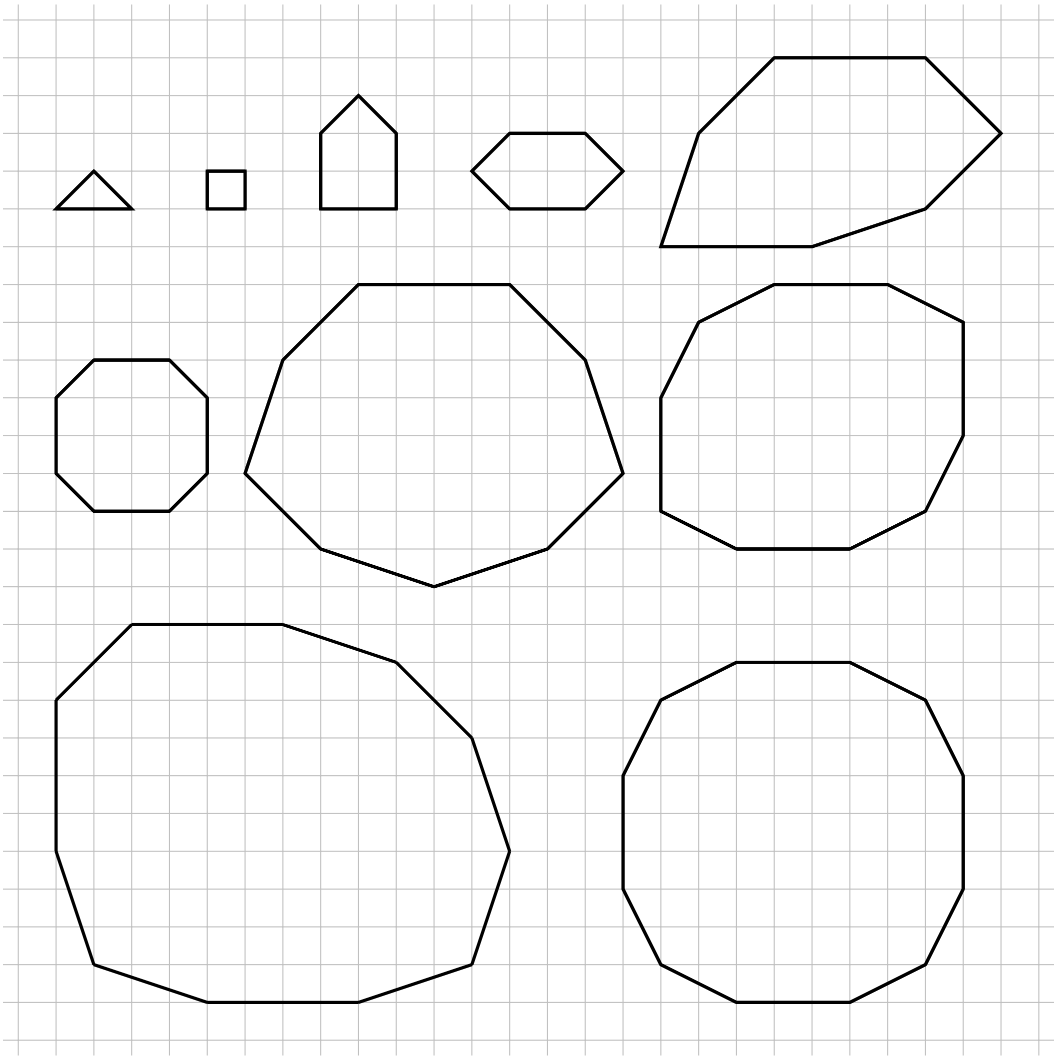


Figure B.2: Taxicab-equilateral lattice n-gons with minimal side lengths for $3 \leq n \leq 12$

n	3	4	5	6	7	8	9	10	11	12
side length	2	1	2	2	4	2	4	3	4	3

Table B.1: Side lengths of taxicab-equilateral lattice n-gons from Figure B.2

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