Statistical properties of material line elements in incompressible MHD turbulence

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Background

The deformation of material lines in turbulence is of fundamental interest and practical importance. Due to its diffusive character material lines consisting of the same set of fluid particles tend to stretch while following the fluid motion. Vortex lines and magnetic field lines in an inviscid fluid of high conductivity are examples of vector fields that are proportional to material lines. It is known analytically [1] and shown in hydrodynamic simulations [2][3] that the length of material line elements increases exponentially in time. In the present work the deformation of material lines is studied statistically by simulating infinitesimal material line elements in stationary incompressible magnetohydrodynamic (MHD) turbulence using velocity gradient time series. The velocity gradient data is obtained by tracking Lagrangian particles in a stochastically forced direct numerical simulation (DNS). In order to further understand the influence of the magnetic field on the material line deformation a method for injecting cross helicity has been devised to control the alignment of the magnetic and velocity field.

Material line elements simulation

A material line is defined as a line that always consists of the same set of particles or fluid elements. In order to study the material line dynamics statistically the lines are simplified to infinitesimal elements (Batchelor [1]) which allows for a one-point description.

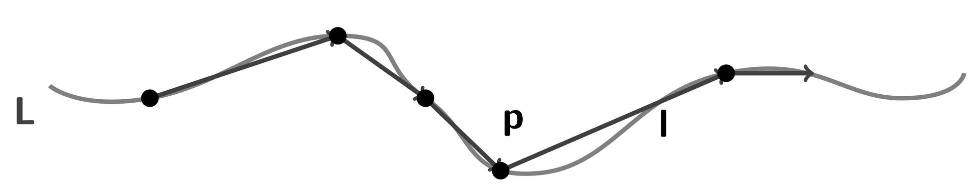

Figure 1: A material line L is approximated by line elements l which are computed for for each lagrangian particle p.

The dynamical evolution of a line element $\mathbf{l}$ is given by

$$\frac{d\mathbf{l}}{dt} = \nabla\mathbf{u}\,\mathbf{l} = \mathbf{S}\,\mathbf{l} + \mathbf{\Omega}\,\mathbf{l},$$

where velocity gradient can be split into is the symmetric part $\mathbf{S}$ (strain-rate tensor) and an antisymmetric part $\mathbf{\Omega}$(rotation-rate tensor). The line stretching rate ζ is defined as

$$\zeta \equiv \frac{d\ln(l)}{dt} = S_{ij}\hat{l}_i\hat{l}_j.$$

In the simulation lagrangian velocity gradient data $\mathbf{V}$ is first gathered for each particle and then used to evolve the corresponding line elements through the matrix $\mathbf{M}$

$$\frac{d}{dt}\mathbf{M} = \mathbf{V}\,\mathbf{M}(t), \qquad \mathbf{M}(0) = \mathbb{1},$$

$$\mathbf{l}(t) = \mathbf{M}(t)\,\mathbf{l}(0).$$

Forcing Method

The velocity gradient is obtained by tracking lagrangian particles through tricubic interpolation in a direct numerical simulation of the incompressible MHD equations,

$$\partial_t\boldsymbol{\omega} = \nabla\times[\mathbf{v}\times\boldsymbol{\omega} - \mathbf{B}\times(\nabla\times\mathbf{B})] + \nu\nabla^2\boldsymbol{\omega} + \mathbf{F}^f_\omega,$$
$$\partial_t\mathbf{B} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \lambda\nabla^2\mathbf{B} + \mathbf{F}^f_\mathbf{B},$$
$$\nabla\cdot\mathbf{v} = \nabla\cdot\mathbf{B} = 0,$$

which are solved using the pseudo spectral method. Since the MHD equations are dissipative, a stochastic forcing based on the Ornstein-Uhlenbeck Process,

$$dU(t) = -U(t)\frac{dt}{\tau_{\mathrm{corr}}} + \left(\frac{2\sigma_f^2}{\tau_{\mathrm{corr}}}\right)^{1/2} dW(t).$$

was applied on large scales to keep the system in a stationary state.

References

[1] Batchelor, G. K. *The effect of homogeneous turbulence on material lines and surfaces.* Proc. R. Soc. Lond. A, 213(1114), 349-366, 1952.

[2] Yeung, P. K., Pope, S. B. *Lagrangian statistics from direct numerical simulation of isotropic turbulence.* J. Fluid Mech., 207, 531–586, 1989.

[3] Girimaji, S. S., Pope, S. B. *Material-element deformation in isotropic turbulence.* J. Fluid Mech., 220, 427-458, 1990.

Cross helicity injection

In MHD turbulence the magnetic field $\mathbf{b}$ is coupled to the velocity field $\mathbf{v}$ through the Lorentz force $F \propto \mathbf{v}\times\mathbf{b}$ and hence affects its dynamics. The cross helicity given by

$$H_C = \int \mathbf{v}\cdot\mathbf{B}\,dV.$$

represents the orientation of two fields and therefore the coupling strength. In order to simulate this effect on the line element stretching, the large scale fields were rotated for different degrees of alignment $\sigma_C = H_C/H_C^{\max}$.

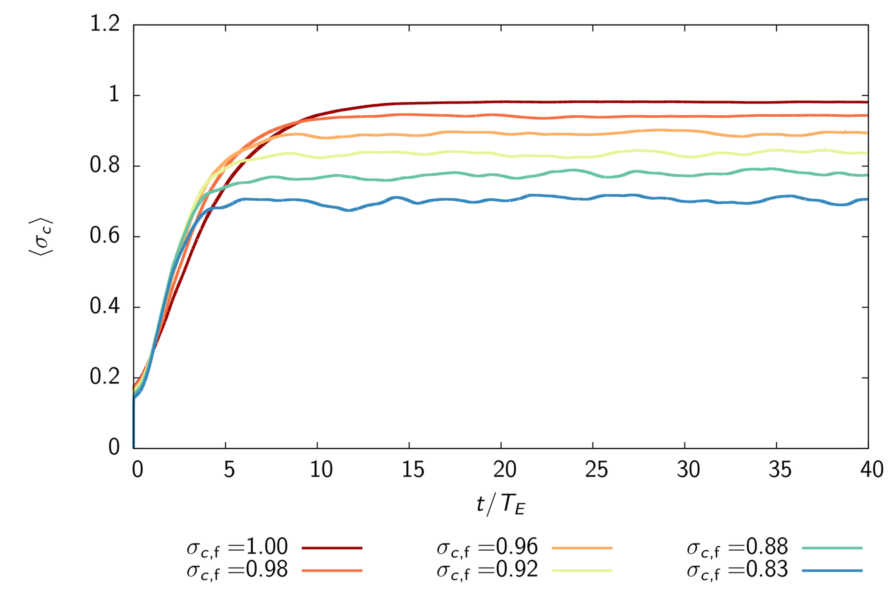

Figure 2: The average alignment over time is shown for different forcing parameters $\sigma_{c,f}$.

In case of a strong alignment the dynamical effect also be seen in the Elsässer formulation of the incompressible MHD equations,

$$\mathbf{z}^\pm = \mathbf{v}\pm\mathbf{B},$$
$$\nabla\cdot\mathbf{z}^\pm = 0$$
$$\frac{\partial}{\partial t}\mathbf{z}^\pm + (\mathbf{z}^\mp\cdot\nabla)\mathbf{z}^\pm = -\nabla P + \frac{\nu+\lambda}{2}\Delta\mathbf{z}^\pm + \frac{\nu-\lambda}{2}\Delta\mathbf{z}^\pm,$$

where the non-linear term vanishes for $\mathbf{z}^\pm \approx 0$. By reducing the non-linear interaction, the energy injected on large scales is transport less effienctly to smaller scales.

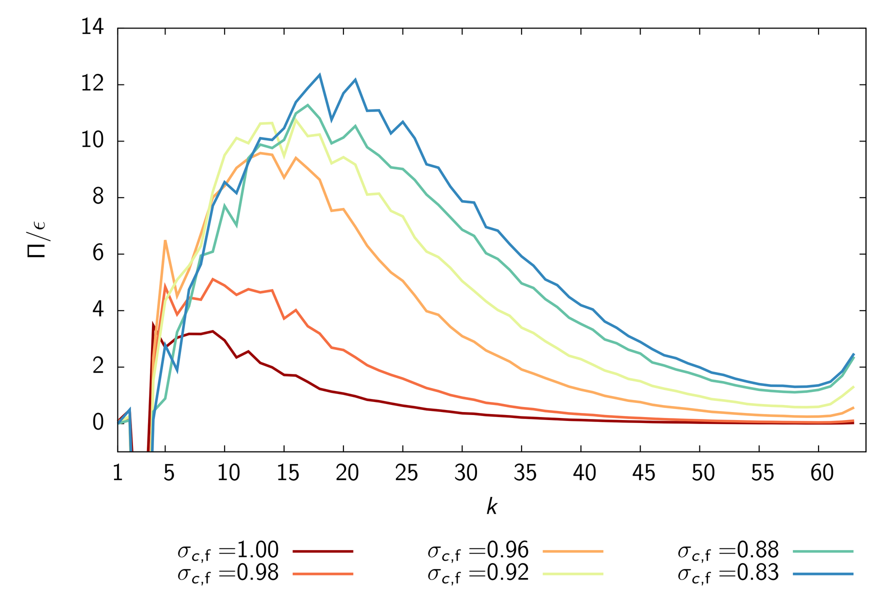

Figure 3: MHD stochastic forcing, dependency of energy dissipation evolution on cross helicity on .

Line element statistics

Strechting rates

After an initial transition phase, where the line elements orient themself along the maximum positive strain direction the strechting rates settle in a stationary state. The principal strain directions were calculated from the local strain rate tensor.

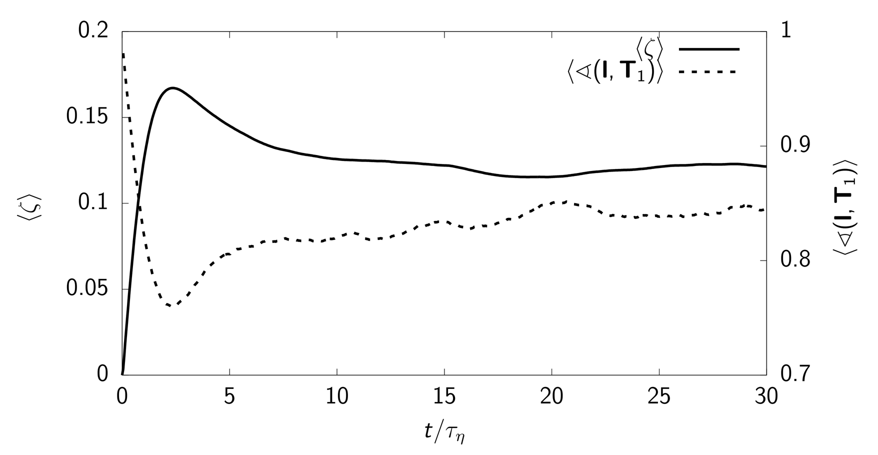

Figure 4: caption

Orientation

The probability functions in the stationary state are shown below for different angles between the line elements and the principal strain rates as well as vorticity and magnetic fields.

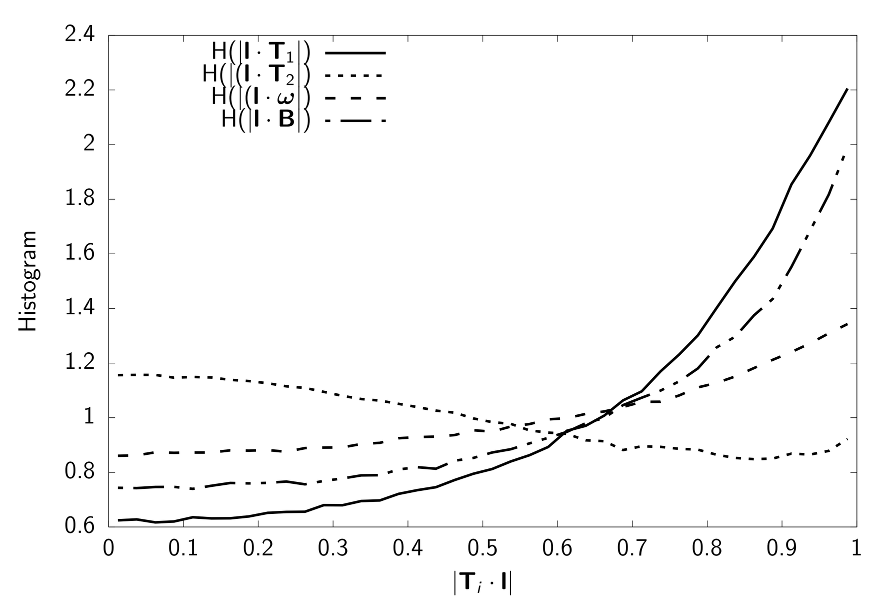

Figure 5: MHD p.d.fs for the angles between the local magnetic field and the line element orientation at steady state ($t/\tau_\eta = 20$).

Statistical distribution

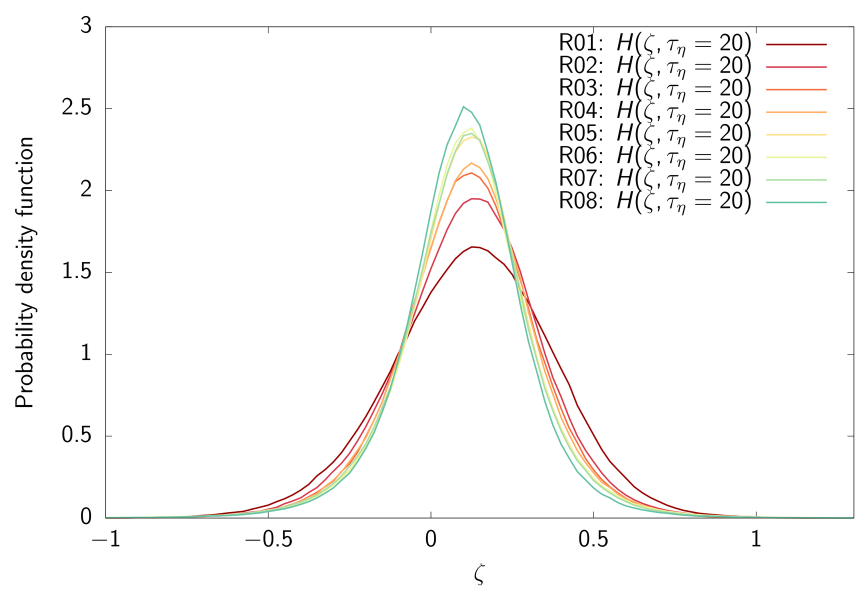

Figure 6: caption

Influence of the cross helicity

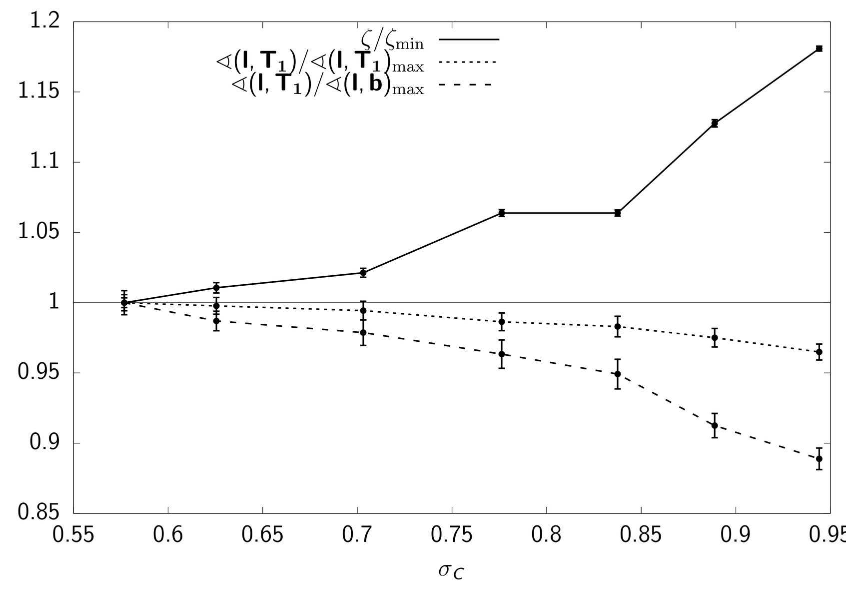

Figure 7: caption

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