Statistical properties of material line elements in incompressible MHD turbulence

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Background

The deformation of material lines in turbulence is of fundamental interest and practical importance. Due to its diffusive character material lines consisting of the same set of fluid particles tend to stretch while following the fluid motion. Vortex lines and magnetic field lines in an inviscid fluid of high conductivity are examples of vector fields that are proportional to material lines. It is known analytically [1] and shown in hydrodynamic simulations [2][3] that the length of material line elements increases exponentially in time. In the present work the deformation of material lines is studied statistically by simulating infinitesimal material line elements in stationary incompressible magnetohydrodynamic (MHD) turbulence using velocity gradient time series. The velocity gradient data is obtained by tracking Lagrangian particles in a stochastically forced direct numerical simulation (DNS). In order to further understand the influence of the magnetic field on the material line deformation a method for injecting cross helicity has been devised to control the alignment of the magnetic and velocity field.

Material line elements simulation

A material line is defined as a line that always consists of the same set of particles or fluid elements. In order to study the material line dynamics statistically the lines are simplified to infinitessimal elements [1] which allows for a one-point description.

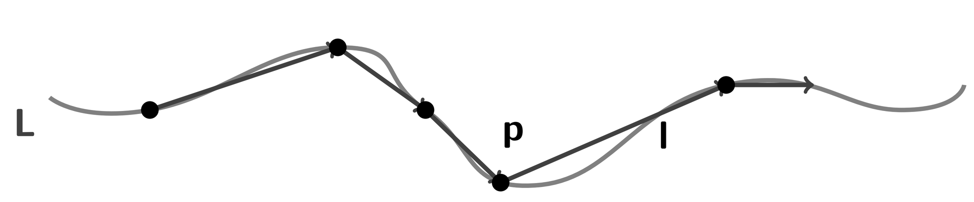

Figure 1: A material line L is approximated by line elements l which are computed for for each lagrangian particle p.

The dynamical evolution of a line element $\mathbf{l}$ is given by

$$\frac{d\mathbf{l}}{dt} = \nabla \mathbf{v}\,\mathbf{l} = \mathbf{S}\,\mathbf{l} + \boldsymbol{\Omega}\,\mathbf{l}, \tag{1}$$

where the velocity gradient can be split into is the symmetric part $\mathbf{S}$ (strain-rate tensor) and an antisymmetric part $\boldsymbol{\Omega}$(rotation-rate tensor). The line stretching rate ζ is defined as

$$\zeta \equiv \frac{d\ln(l)}{dt} = S_{ij}\hat{l}_i\hat{l}_j. \tag{2}$$

In the simulation lagrangian velocity gradient data $\mathbf{V}$ is first gathered for each particle and then used to evolve the corresponding line elements through the matrix $\mathbf{M}$

$$\frac{d}{dt}\mathbf{M} = \mathbf{V}\,\mathbf{M}(t), \qquad \mathbf{M}(0) = \mathbb{1}, \tag{3}$$

$$\mathbf{l}(t) = \mathbf{M}(t)\,\mathbf{l}(0). \tag{4}$$

Forcing Method

The velocity gradient is obtained by tracking lagrangian particles through tricubic interpolation in a direct numerical simulation of the incompressible MHD equations,

$$\begin{aligned} \partial_t \boldsymbol{\omega} &= \nabla \times [\mathbf{v} \times \boldsymbol{\omega} - \mathbf{B} \times (\nabla \times \mathbf{B})] + \nu \nabla^2 \boldsymbol{\omega} + \mathbf{F}^f_{\boldsymbol{\omega}}, \\ \partial_t \mathbf{B} &= \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B} + \mathbf{F}^f_{\mathbf{B}}, \\ \nabla \cdot \mathbf{v} &= \nabla \cdot \mathbf{B} = 0, \end{aligned} \tag{5}$$

which are solved using the pseudo spectral method. Since the MHD equations are dissipative, a stochastic forcing based on the Ornstein-Uhlenbeck Process,

$$dU(t) = -U(t)\frac{dt}{\tau_{\text{corr}}} + \left(\frac{2\sigma_f^2}{\tau_{\text{corr}}}\right)^{1/2} dW(t). \tag{6}$$

was applied on large scales to keep the system in a stationary state.

References

[1] Batchelor, G. K. *The effect of homogeneous turbulence on material lines and surfaces.* Proc. R. Soc. Lond. A, 213(1114), 349-366, 1952.

[2] Yeung, P. K., Pope, S. B. *Lagrangian statistics from direct numerical simulation of isotropic turbulence.* J. Fluid Mech., 207, 531–586, 1989.

[3] Girimaji, S. S., Pope, S. B. *Material-element deformation in isotropic turbulence.* J. Fluid Mech., 220, 427-458, 1990.

Cross helicity injection

In MHD turbulence the magnetic field $\mathbf{b}$ is coupled to the velocity field $\mathbf{v}$ through the Lorentz force $F \propto \mathbf{v} \times \mathbf{b}$ and hence affects its dynamics. The cross helicity given by

$$H_C = \int \mathbf{v} \cdot \mathbf{B}\, dV. \tag{7}$$

represents the orientation of two fields and therefore the coupling strength. In order to better understand the role of the alignment on the line element stretching, the large scale fields were rotated for different degrees of alignment $\sigma_C = H_C/H_C^{\max}$.

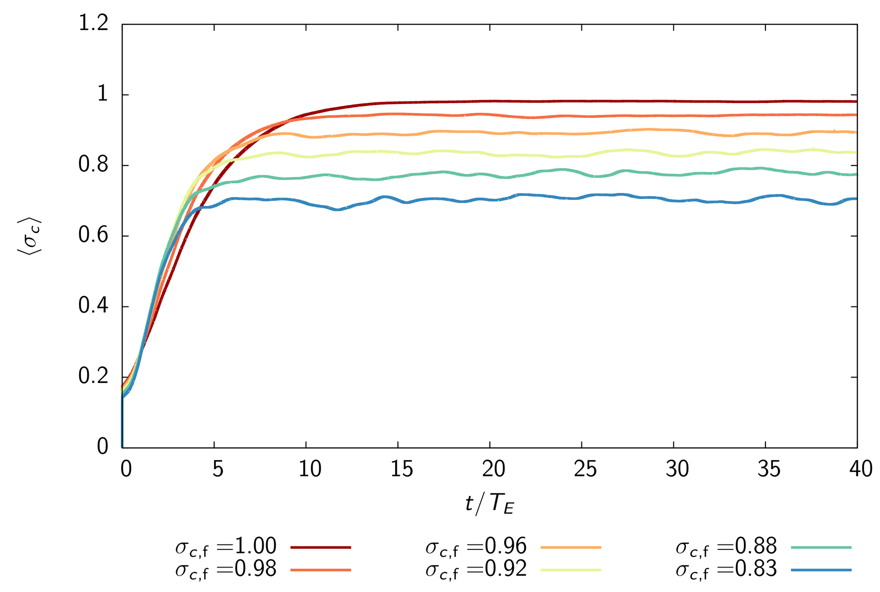

Figure 2: The average alignment over time is shown for different forcing parameters $\sigma_{c,f}$.

In case of a strong alignment the dynamical effect also be seen in the Elsässer formulation of the incompressible MHD equations,

$$\begin{aligned} \mathbf{z}^{\pm} &= \mathbf{v} \pm \mathbf{B}, \\ \nabla \cdot \mathbf{z}^{\pm} &= 0 \\ \frac{\partial}{\partial t}\mathbf{z}^{\pm} + (\mathbf{z}^{\mp} \cdot \nabla)\mathbf{z}^{\pm} &= -\nabla P + \frac{\nu + \lambda}{2}\Delta \mathbf{z}^{\pm} \\ &\quad + \frac{\nu - \lambda}{2}\Delta \mathbf{z}^{\pm}, \end{aligned} \tag{8}$$

where the non-linear term vanishes for $\mathbf{z}^{\pm} \approx 0$. By reducing the non-linear interaction, the energy injected on large scales is transport less effienctly to smaller scales.

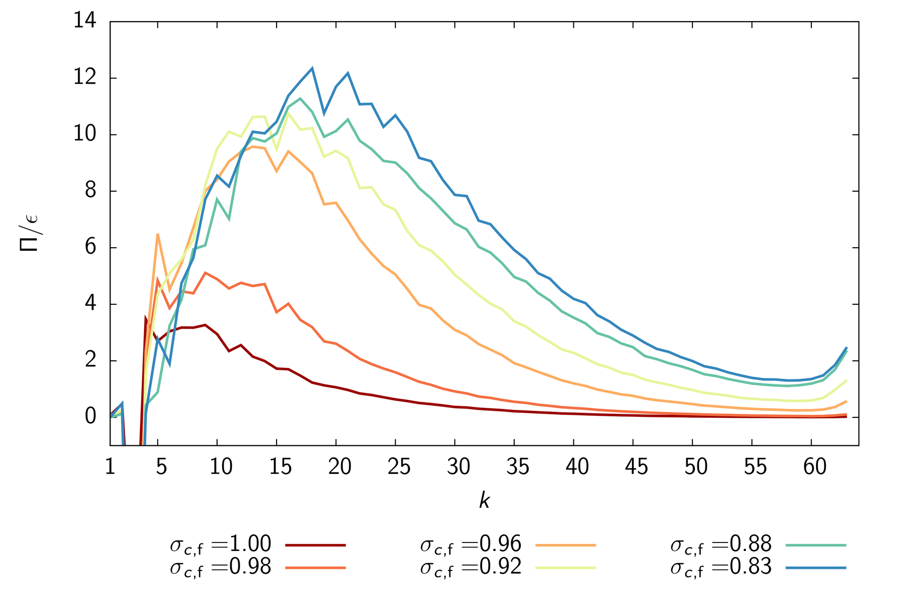

Figure 3: MHD stochastic forcing, dependency of energy dissipation evolution on cross helicity on .

Line element statistics

Strechting rates

The line element stretching rate was computed for each Lagrangian particle using Equation (2) and averaged over the ensemble. Further, the alignement of the line elements with the principal strain rates was studied by finiding the eigenvectors of the strain rate tensor.

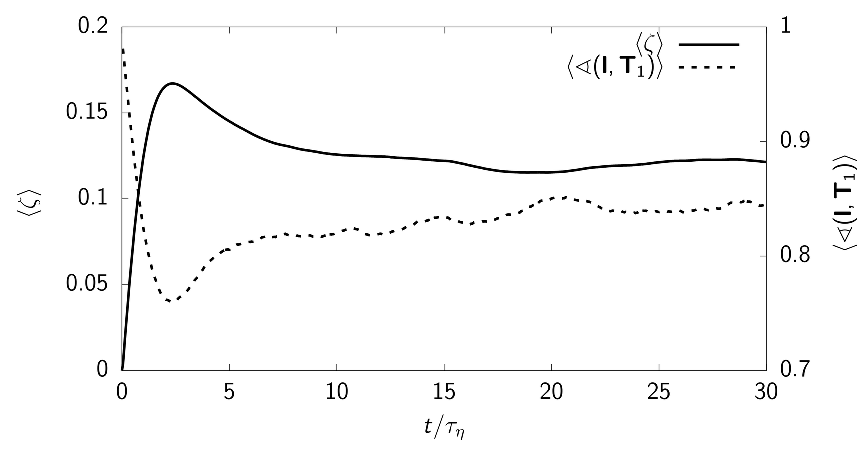

Figure 4: Temporal evolution of the average line strechting rate and orienation.

- After an initial transition phase the strechting rates settle into a stationary state.
- In this transition phase the line elements $\mathbf{l}$ show a strong alignment with the maximum postive strain direction $\mathbf{T}_1$.

Orientation

The histograms for different angles between the line elements and the principal strain rates as well as the vorticity and magnetic field in the stationary state are shown below .

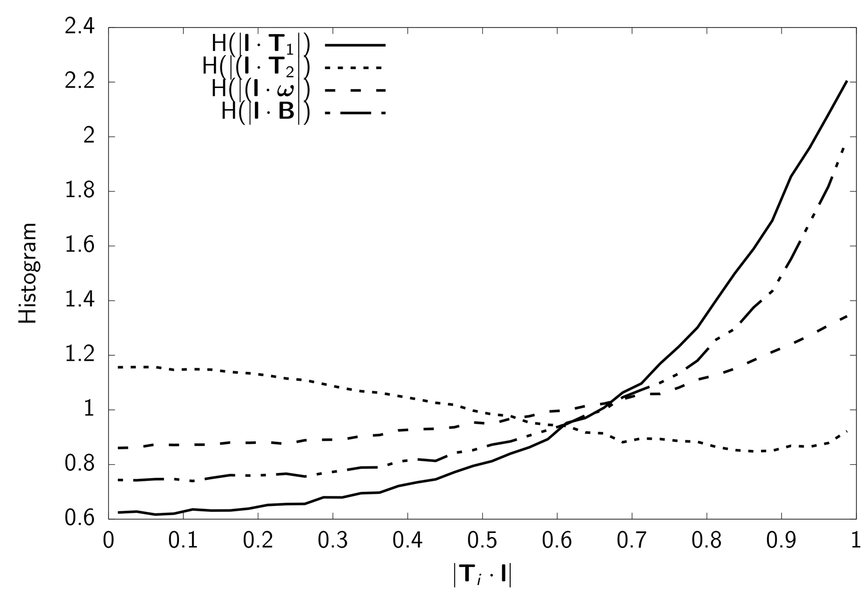

Figure 5: MHD p.d.fs for the angles between the local magnetic field and the line element orientation at steady state ($t/\tau_\eta = 20$).

- In MHD turbulence the line elements show the strongest alignement with the magnetic field, followed by the vorticity and the max. positive strain rate.

Statistical distribution

The statistical distribution of the line element strechting rate ζ was calculated in the stationary state.

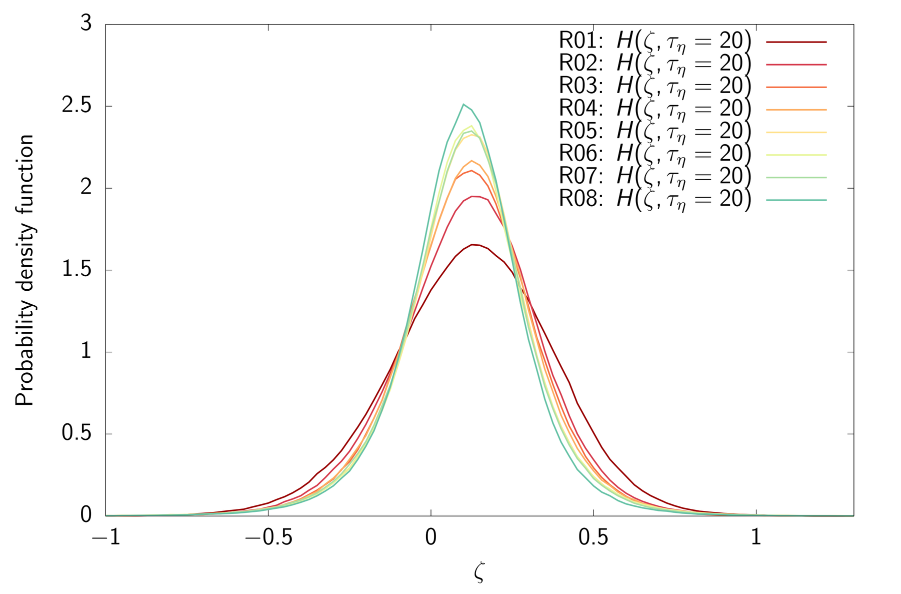

Figure 6: Probability density functions of ζ are shown for different alignments $\langle\sigma\rangle$.

- The p.d.f of ζ show a Gaussian shaped distribution and are stationary.
- The kurtosis of the p.d.f increases with an increasing cross helicity fraction.

Influence of the cross helicity

After aligning $\mathbf{v}$ and $\mathbf{B}$ through cross helictiy injection, its effect on material line stretching was investigated by averaging the ensemble line stretching rate and orientation over time for different alignments $\langle\sigma_c\rangle$.

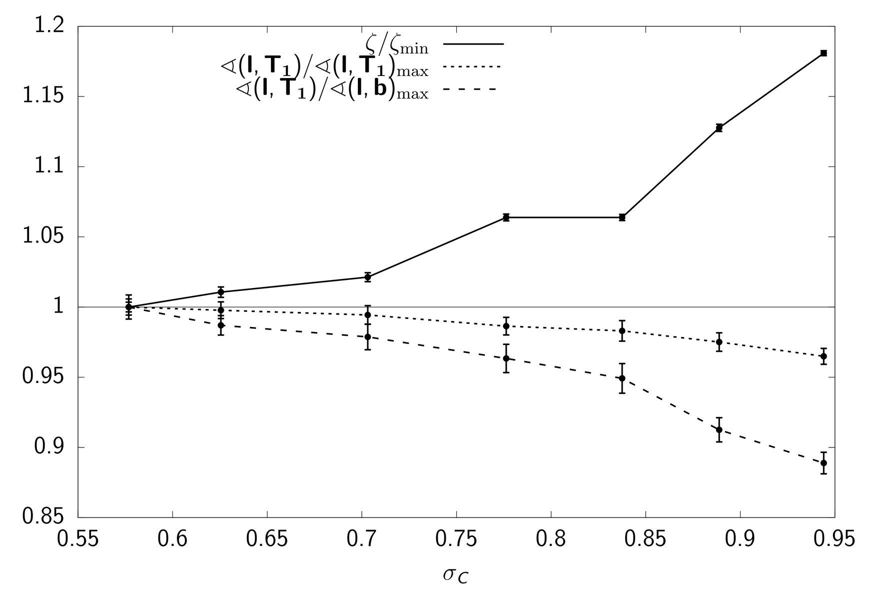

Figure 7: Time averaged stretching rates and angles are shown for different alignment $\langle\sigma_c\rangle$.

- The stretching rate $\bar{\zeta}$ increases with increasing alignment $\langle\sigma_c\rangle$.
- At the same time the angle between $\mathbf{l}$ and $\mathbf{T}_1$ or $\mathbf{B}$ decreases with increasing $\langle\sigma_c\rangle$.