Statistical properties of material line elements in incompressible MHD turbulence

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Introduction

The deformation of material lines that are advected by turbulence is of fundamental interest and practical importance. Due to its diffusive character material lines consisting of the same set of fluid particles tend to stretch while following the fluid motion. Vortex lines and magnetic field lines in an inviscid fluid of high conductivity are examples of vector fields that are proportional to material lines. As known from analytical studies [1] and shown in hydrodynamic simulations [2][3] the length of material line elements increases exponentially in time. In the present work the deformation of material lines is studied statistically by simulating infinitesimal material line elements in stationary incompressible magnetohydrodynamic (MHD) turbulence using velocity gradient time series. The velocity gradient data is obtained by tracking Lagrangian particles in a stochastically forced direct numerical simulation (DNS). In order to further understand the influence of the magnetic field on the material line deformation a method for injecting cross helicity has been devised to control the alignment of the magnetic and velocity field.

Material line elements

A material line is defined as a line that always consists of the same set of particles or fluid elements. In order to study the material line dynamics statistically the lines are simplified to infinitesimal elements [1] which allows for an one-point description.

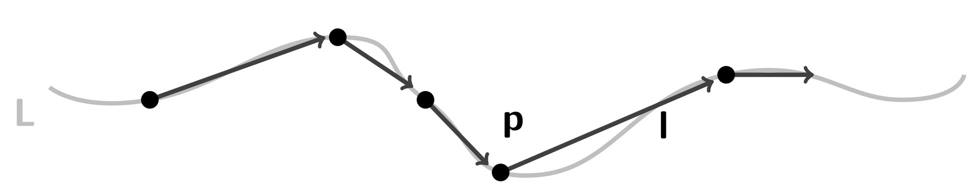

Figure 1: A material line L is approximated by line elements l which are computed for for each Lagrangian particle p.

The dynamical evolution of a line element $\mathbf{l}$ is given by

$$\frac{d\mathbf{l}}{dt} = \nabla\mathbf{v}\,\mathbf{l} = \mathbf{S}\,\mathbf{l} + \mathbf{\Omega}\,\mathbf{l}, \tag{1}$$

where the velocity gradient can be split into a symmetric part $\mathbf{S}$ (strain-rate tensor) and an antisymmetric part $\mathbf{\Omega}$ (rotation-rate tensor). The line stretching rate is defined as

$$\zeta \equiv \frac{d\ln(l)}{dt} = S_{ij}\hat{l}_i\hat{l}_j. \tag{2}$$

In the simulation, Lagrangian velocity gradient data $\mathbf{V}$ is first gathered for each particle and then used to evolve the corresponding line elements through the matrix $\mathbf{M}$

$$\frac{d}{dt}\mathbf{M} = \mathbf{V}\,\mathbf{M}(t), \qquad \mathbf{M}(0) = \mathbb{1}, \tag{3}$$

$$\mathbf{l}(t) = \mathbf{M}(t)\,\mathbf{l}(0). \tag{4}$$

MHD simulation

The velocity gradient is obtained by tracking 10^6 Lagrangian particles with tricubic interpolation in a direct numerical simulation of the incompressible MHD equations,

$$\begin{aligned} \partial_t\boldsymbol{\omega} &= \nabla\times[\mathbf{v}\times\boldsymbol{\omega} - \mathbf{b}\times(\nabla\times\mathbf{b})] + \nu\nabla^2\boldsymbol{\omega} + \mathbf{F}^f_\omega, \\ \partial_t\mathbf{b} &= \nabla\times(\mathbf{v}\times\mathbf{b}) + \lambda\nabla^2\mathbf{b} + \mathbf{F}^f_\mathbf{b}, \\ \nabla\cdot\mathbf{v} &= \nabla\cdot\mathbf{b} = 0, \end{aligned} \tag{5}$$

which are solved using the pseudo spectral method on a 128^3 grid. Since the MHD equations are dissipative, a stochastic forcing based on the Ornstein-Uhlenbeck Process,

$$dU(t) = -U(t)\frac{dt}{\tau_{\mathrm{corr}}} + \left(\frac{2\sigma_f^2}{\tau_{\mathrm{corr}}}\right)^{1/2} dW(t). \tag{6}$$

is applied on large scales to keep the system in a quasi-stationary state.

References

[1] Batchelor, G. K. *The effect of homogeneous turbulence on material lines and surfaces.* Proc. R. Soc. Lond. A, 213(1114), 349-366, 1952.

[2] Yeung, P. K., Pope, S. b. *Lagrangian statistics from direct numerical simulation of isotropic turbulence.* J. Fluid Mech., 207, 531–586, 1989.

[3] Girimaji, S. S., Pope, S. b. *Material-element deformation in isotropic turbulence.* J. Fluid Mech., 220, 427-458, 1990.

Cross helicity injection

As can be seen from the incompressible MHD equations (5), the magnetic and the velocity field couple through the $\mathbf{v}\times\mathbf{b}$ term. Therefore the coupling strength depends on the alignment of the two fields. A measure of the orientation is defined by the cross helicity given by

$$H_C = \frac{1}{2V}\int \mathbf{v}\cdot\mathbf{b}\,dV. \tag{7}$$

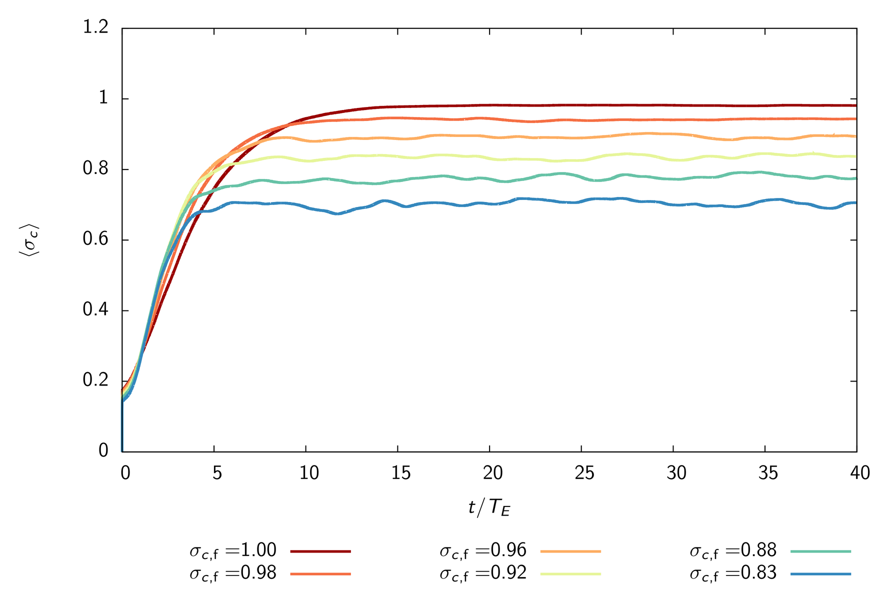

Figure 2: The average alignment of $\mathbf{v}$ and $\mathbf{b}$ over time is shown for different forcing parameters $\sigma_{c,f}$.

In this work cross helicity was injected by rotating the large scale forcing fields for different degrees of alignment $\sigma_C = H_C/H_C^{\max}$. One of the main characteristics of turbulence is the energy transport from large to small length scales, which is also reduced by an increasing alignment.

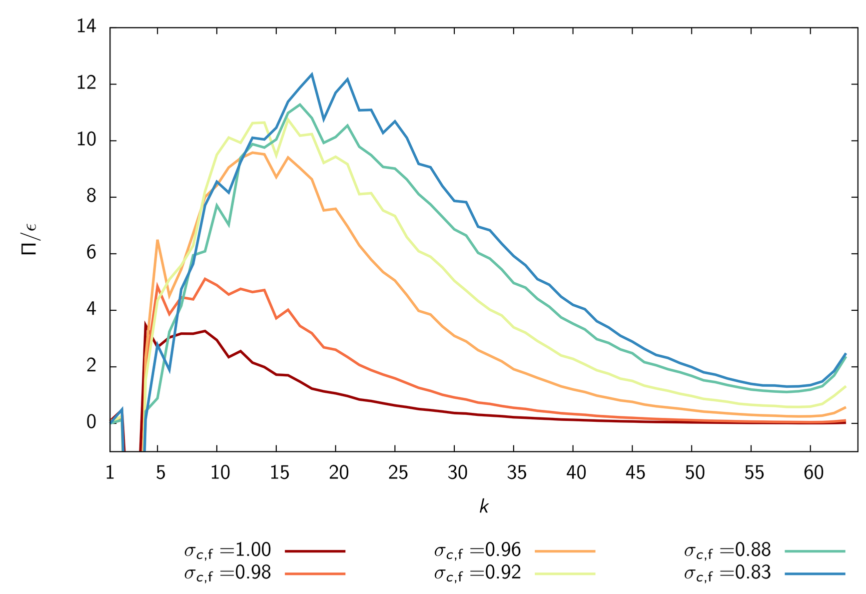

Figure 3: The spectral kinetic energy flux over the energy dissipation rate is shown for different alignment forcings.

Line element statistics

Stretching rates

The line element stretching rate was computed for each Lagrangian particle using (2) and averaged over the ensemble. Further, the alignment of the line elements with the principal strain rates was studied by finding the eigenvectors of the strain rate tensor.

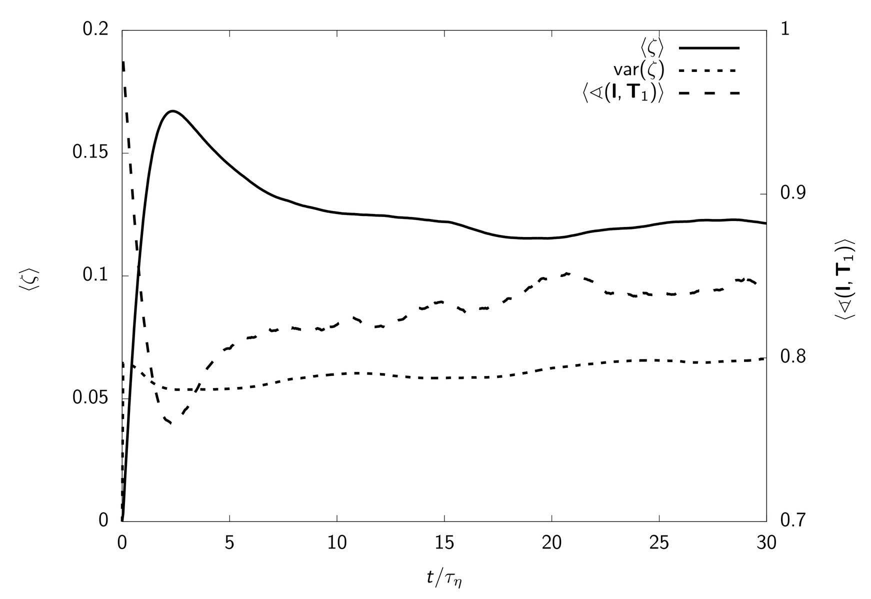

Figure 4: Temporal evolution of the average line stretching rate and angle with the maximum strain rate direction $\mathbf{T}_1$.

- After an initial transition phase the stretching rates settle into a quasi-stationary state.
- In this transition phase the line elements $\mathbf{l}$ show a strong alignment with the maximum positive strain direction $\mathbf{T}_1$.

Orientation

The histograms for different angles between the line elements and the principal strain rates as well as the vorticity and magnetic field in the stationary state are shown below.

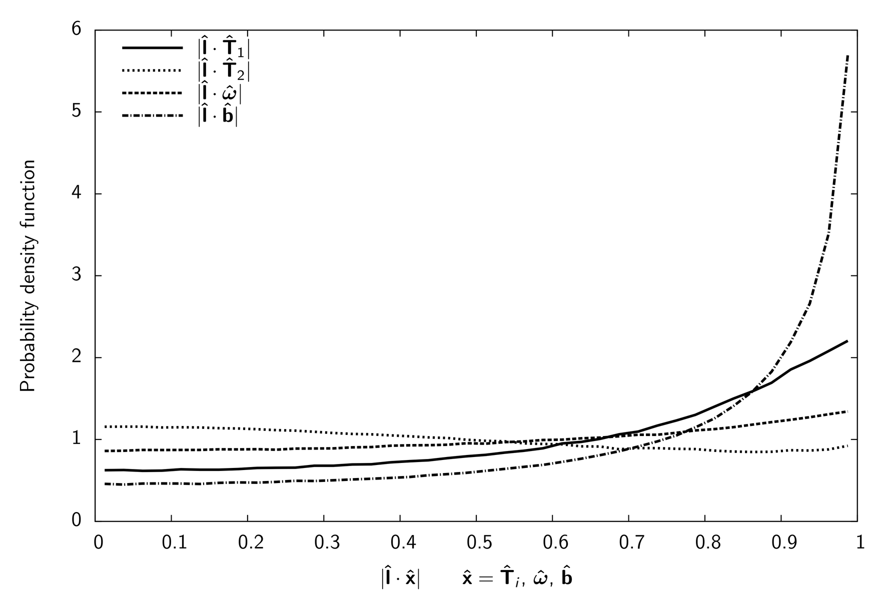

Figure 5: MHD p.d.fs for the angles between the local magnetic and voriticity field direction $\hat{\mathbf{b}}$ and the line element orientation $\hat{\mathbf{l}}$ at steady state ($t/\tau_\eta = 20$).

- In MHD turbulence the line elements show the strongest alignment with the magnetic field, followed by the maximum positive strain rate direction and the vorticity.

Influence of the cross helicity

After aligning $\mathbf{v}$ and $\mathbf{b}$ through cross helicitiy injection, its effect on material line deformation was studied by averaging the ensemble line stretching rate and angle over time for different alignments $\langle\sigma_c\rangle$.

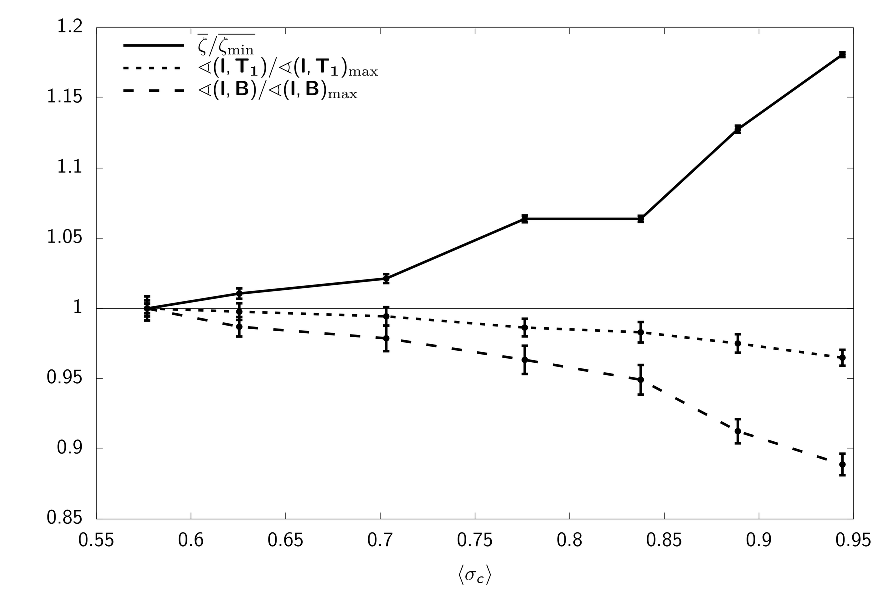

Figure 6: Time averaged stretching rates and angles are shown for different alignments $\langle\sigma_c\rangle$.

- The stretching rate $\bar{\zeta}$ increases with increasing alignment $\langle\sigma_c\rangle$.
- At the same time the angle between $\mathbf{l}$ and $\mathbf{T}_1$ or $\mathbf{b}$ decreases with increasing $\langle\sigma_c\rangle$.

$\sigma_{c,f}$	$\bar{\zeta}$	$\varepsilon(\bar{\zeta})$	$\sphericalangle(\mathbf{l},\mathbf{T}_1)$	$\varepsilon(\sphericalangle(\mathbf{l},\mathbf{T}_1))$	$\sphericalangle(\mathbf{l},\mathbf{b})$	$\varepsilon(\sphericalangle(\mathbf{l},\mathbf{b}))$
1	0.123	3.992E-3	0.840	6.152E-3	0.702	8.434E-3
0.98	0.111	1.732E-3	0.852	5.674E-3	0.752	7.673E-3
0.96	0.106	2.547E-3	0.861	6.557E-3	0.772	8.621E-3
0.92	0.100	2.196E-3	0.868	7.309E-3	0.803	1.057E-2
0.88	0.100	2.439E-3	0.871	6.272E-3	0.815	1.008E-2
0.83	0.096	3.169E-3	0.878	6.657E-3	0.828	9.117E-3

Table 1: Time averaged stretching rates and the respective errors are shown for different alignments.

Statistical distribution

The statistical distribution of the line element stretching rate was calculated in the stationary state.

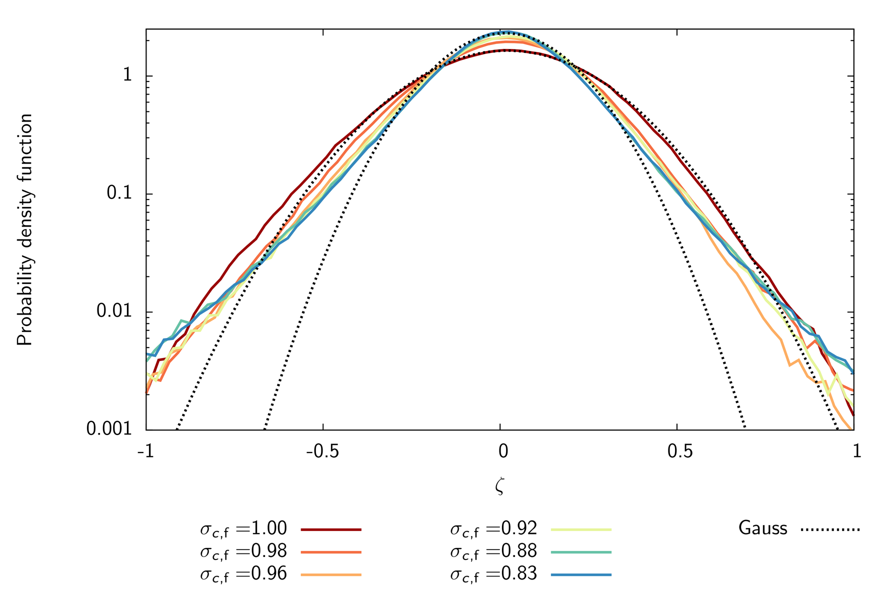

Figure 7: Probability density functions of ζ are shown for forcing parameters $\sigma_{c,f}$.

- The p.d.fs of ζ show a Gaussian shaped distribution for the maximal alignment and are stationary.
- The kurtosis of the p.d.f decreases with an increasing cross helicity fraction.