Lyon 13 May 2006

Dear Serge,

I want to detail the last construction of my previous notes:

The following hypothesis are not the most general we can hope but they are convenient. So, let us consider a Lorentzian manifold (M, g) geodesically complet. Let us denote by $\mathcal{G}$ the space of parametrized geodesics, in our case $\mathcal{G} \sim TM$, which is equiped with the inherited symplectic form ω. Let $\mathcal{T}$ the space of geodesical trajectories, as I said previously:

$$\begin{cases} \mathcal{T} = \mathcal{G} / \mathrm{Aff}^{+}(\mathbb{R}) \\ \gamma' \sim \gamma \text{ iff } \gamma'(t) = \gamma(at+b) \quad a>0 \quad b \in \mathbb{R} \end{cases}$$

Let us consider: $\omega_\gamma : T_\gamma \mathcal{G} \longrightarrow T^*_\gamma \mathcal{G}$ the symplectic form is viewed as a linear map from the tangent to the cotangent.

Then,

$$\omega^{-1}_\gamma : T^*_\gamma \mathcal{G} \longrightarrow T_\gamma \mathcal{G}$$

So ω^{-1} is a section of some bundle over $\mathcal{G}$ with fiber $L^*(T^*_\gamma \mathcal{G}; T_\gamma \mathcal{G})$, L^* means non-degenerate. Let us denote $\mathcal{L}^*$ this bundle over $\mathcal{G}$.

Now, let us consider the projectivization of $\mathcal{L}^*$, that is

$$\begin{tikzcd} \mathcal{L}^* \arrow[r] \arrow[dr] & \mathbb{P}\mathcal{L}^* = \mathcal{L}^*/]0,\infty[\arrow[d] \\ & \mathcal{G} \end{tikzcd}$$

the section $\omega^{-1} : \mathcal{G} \to \mathcal{L}^*$ give a section $[\omega^{-1}] : \mathcal{G} \to \mathbb{P}\mathcal{L}^*$ with $\omega^{-1} \sim a\,\omega^{-1}$ $a \in]0,\infty[$

the section $[\omega^{-1}]$ factorizes (since $\omega^{-1} \sim (a,b)^*\omega^{-1}$ $(a,b) \in \mathrm{Aff}^+(\mathbb{R})$)

$$\begin{tikzcd} \mathbb{P}\mathcal{L}^*(\mathcal{G}) \arrow[r] \arrow[d] & \mathbb{P}\mathcal{L}^*(\mathcal{T}) \arrow[d] \\ \mathcal{G} \arrow[r, "\pi"'] \arrow[u, bend left, "{[\omega^{-1}]}"] & \mathcal{T} \arrow[u, bend right, "\beta"'] \end{tikzcd}$$

The section β is what we remember of the symplectic structure ω on $\mathcal{G}$.

Now we can consider β as an operator on the sub spaces of the cotangent space of $\mathcal{T}$:

let $V \subset T^*_\tau\mathcal{T}$ $\qquad \left.\begin{array}{l} \beta_\tau(V) = \pi\left(\omega^{-1}_\gamma(\tilde{V})\right) \\ \text{with } \tilde{V} = \pi^* V,\ \pi(\gamma) = \tau \end{array}\right\}$

this is well defined.

Now we can consider the distribution of subspaces:

$$\tau \longmapsto \beta(T^*_\tau\mathcal{T}) = F_\tau$$

The fact is that :

a) if τ is not a light geodesic $\beta(T^*_\tau \mathcal{G}) = T_\tau \mathcal{G}$

b) if τ is a light geodesic $\beta(T^*_\tau \mathcal{G}) \subset T_\tau$ light Rays and $=$ Orth $(T_\gamma \, Aff^+(\mathbb{R})(\gamma)) / Aff^+(\mathbb{R})$ is the contact distribution on the space of light rays of M.

What I want to emphasize now is that physicists do the following :

a) they choose an hypersurface $\Sigma_c = \{ (x,v) \in TM \mid v.v = c \}$

b) they make the symplectic reduction $\Sigma_c / \ker(\omega | \Sigma_c)$

c) they get a symplectic manifold of dimension $2n-2$ even for $c = 0$

d) with the above construction the space of light rays has dimension $2n-3$ and is contact.

e) what I guess is that the space of light rays obtained by physicists is the symplectization of the contact manifold

f) the supplementary parameter obtained by physicists is the "color" of the "photon"

That's all for now Serge, All the best Yours Patrick

Patrick Iglesias-Zemmour
piz@math.huji.ac.il

Dear Serge,

Excuse me for the time I took to reply to your request. I have had a lot of stuff to fix before leaving for the US. And I'm leaving next week.

So let's come back to the question of geodesics in a pseudo riemannian manifold. My motivation was to understand how to glue together all the geodesics in only one structure or description. Because, for example, if we take the space $\mathbb{R}^4$ with the pseudo metric $dx^2 + dy^2 + dz^2 - dt^2$ (the Minkowsky space), we know that the geodesics are just lines and the spaces of lines of $\mathbb{R}^4$ is a manifold diffeomorphic to TS^3. And this manifold is splitted in three parts: the time-like geodesics, the space-like geodesics which are 2 open domains of TS^3 and the light geodesics which is a codimension 1 submanifold. We know that on the time/space geodesics we can put a symplectic structure, but what about the light geodesics? And if there exist a structure on the light geodesics how does it glue with the symplectic structures?

This was my first question.

Now I'll try to describe my answer:

1) <u>first of all</u> I had to understand the difference between "parametrized geodesics" and "unparametrized geodesics":

* "parametrized geodesics" are curves in M (M is the pseudo-riemannian manifold), that is maps $\gamma: t \mapsto x$ such that

$$\frac{\hat{d}\, v}{dt} = 0 \quad \text{with} \quad v = \frac{dx}{dt}$$

and $\hat{d}$ is the covariant derivative with respect to the Levi-Civita connexion.

So parametrized geodesics live in the space of functions from (a priori) $\mathbb{R}$ to M. They are solutions of a 2nd order ordinary differential equation, then the set of parametrized geodesics is a $2n$-dimensional space. Let us assume that the metric is complete and then this space, I denote by $\mathcal{G}_{par}$ is diffeomorphic to TM.

In the example of $M = \mathbb{R}^4$ $(+++-)$:

- $\mathcal{G}_{par} = \{ \gamma_{(x,v)} = [t \mapsto x + tv] \mid x \in \mathbb{R}^4, v \in \mathbb{R}^4 - \{0\} \}$

* "Unparametrized geodesic".

Let me open a parenthesis now: Of course in a pseudo-riemannian manifold we do not want to consider parametrized geodesics since the parametrization by the time is included in the image of the curve $im(\gamma_{(x,v)})$. It is why we have to clean up the parametrization. So we define a new set:

$$\begin{cases} \mathcal{G}_{unpar} = \mathcal{G}_{par} / \sim \\ \text{with:} \\ \gamma \sim \gamma' \text{ iff } \quad im(\gamma) = im(\gamma') \end{cases}$$

If we are lucky the space $\mathcal{G}_{unpar}$ of unparametrized geodesics is a nice manifold. This is the case for $(\mathbb{R}^4, +++-)$ for example. And in this example $\mathcal{G}_{unpar}$ is just the space of lines as we told before.

So for $(\mathbb{R}^4, +++-)$: $\begin{cases} \mathcal{G}_{par} \simeq \mathbb{R}^4 \times \mathbb{R}^4 - \{0\} & (\dim 8) \\ \mathcal{G}_{unpar} \simeq TS^3 & (\dim 6) \end{cases}$

But we can get the quotient $\mathcal{G}_{unpar}$ by the quotient of the action of the affine group of $\mathbb{R}$:

$$Aff^+(\mathbb{R}) = \{ (a, b) \in]0, \infty[\times \mathbb{R} \}$$

[Note: I consider oriented geodesics, parametrized are of course oriented by the "speed".]

Now, two geodesics γ and γ' have the same "trajectory" iff:

$$\gamma'(t) = \gamma(at + b)$$

So I have an action of $Aff^+(\mathbb{R})$ on $\mathcal{G}_{par}$:

$$(a,b)(\gamma) = [t \mapsto \gamma(at+b)]$$

And

$$\mathcal{G}^+_{unpar} = \mathcal{G}_{par} / Aff(\mathbb{R})$$

this is completely true if the geodesic flow is complete, it's more complicate if it is not complete.

So now, I make a difference between the space of "parametrized geodesics" and "geodesic trajectories" moreover oriented. I denote also $\mathcal{G}^+_{traj} = \mathcal{G}^+_{unpar}$.

2) <u>What about the structure?</u>

a) We know that there exist a symplectic structure on $\mathcal{G}_{par}$ since $\mathcal{G}_{par}$ is the set of solutions of a variational problem :

$$\delta \int_{t_0}^{t_1} \frac{1}{2} v \cdot v \, dt \quad \text{with} \quad v = \frac{dx}{dt} \text{ and } \cdot \text{ is}$$

the pseudo scalar product.

Thus we have the Cartan form

$$\begin{cases} \varpi = v \cdot dx - \frac{1}{2} v \cdot v \, dt \quad \text{on } TM \times \mathbb{R} = Y \\ y \in Y \quad y = (x, v, t) \quad \dim Y = 2n+1 \end{cases}$$

and $\mathcal{G}_{par}$ = Characteristics of $d\varpi$

To be complete means just that a solution of the distribution $y \mapsto \ker d\varpi$ projects surjectively on $\mathbb{R}$ by $y \mapsto t$ and then $\mathcal{G}_{par} = Y/\ker d\varpi = TM \times \mathbb{R} / \ker d\varpi$

and then $\mathcal{G}_{par} \simeq TM \times \{0\}$ for example. The symplectic form ω on $\mathcal{G}_{par}$ defined by $\pi^* \omega = d\varpi$ where $\pi : TM \times \mathbb{R} \to \mathcal{G}_{par}$ is the projection. Hence

$$(\mathcal{G}_{par}, \omega) \simeq (TM, d\varpi | TM \times \{0\})$$

b) now, what about the symplectic form and the action of the group $Aff(\mathbb{R})$!

let us look at the $(\mathbb{R}^4, +++-)$ example :

$$\left.\begin{array}{l} \gamma' = [t \mapsto \gamma(at+b)] = [t \mapsto x + (at+b)v] \\ \qquad\qquad\qquad\qquad\qquad \text{with } \gamma(t) = x + tv \end{array}\right\}$$

$$\gamma' = [t \mapsto x + bv + t(av)$$

Hence
$$(a,b)(\gamma_{(x,v)}) = \gamma_{(x+bv,\, av)}$$

Therefore on TM the action of $Aff(\mathbb{R})$ is described by

$$(a,b)(x,v) = (x + bv, av)$$

[On a more general manifold $x + bv$ represents the action of the geodesic flow.]

Hence
$$(a,b)^* \varpi_{(x,v)} = av \cdot (dx + b\,dv)$$
$$= a\, v \cdot dx + \frac{1}{2} a b\, d(v \cdot v)$$

$$\Rightarrow \quad \boxed{(a,b)^* \omega = a\,\omega}$$

R 1. The symplectic form is multiplied and not preserved by the group $\mathrm{Aff}^+(\mathbb{R})$.

What about the orbits of $\mathrm{Aff}^+(\mathbb{R})$? Let $O = \{(a,b)(x,v) \mid x,v \in TM\}$ an orbit of the group. We get:

$$\omega \big|_{O} = (v \cdot v) \times \omega_{aff} \quad \text{with} \quad \omega_{aff} = da \wedge db$$

R2. The Orbits of $\mathrm{Aff}^+(\mathbb{R})$ are:

- symplectic if $v \cdot v \neq 0$
- isotropic if $v \cdot v = 0$

R3. Note that the action of $\mathrm{Aff}^+(\mathbb{R})$ is always free since $v \neq 0$.

R4. We get a distinguished distribution of $2n-2$ spaces:

$$\gamma \longmapsto \mathrm{Orth}\left(T_\gamma [\mathrm{Aff}^+(\mathbb{R})(\gamma)] \right) \; (= F_\gamma)$$

F_γ is the orthogonal with respect to the symplectic form ω to the orbit of γ under the action of $Aff^+(\mathbb{R})$

We have: a) if $v \cdot v \neq 0$ F_γ is symplectic and

$$T_\gamma \mathcal{G}_{par} = T_\gamma \mathcal{O}_\gamma \oplus F_\gamma$$

$\mathcal{O}_\gamma = Aff^+(\mathbb{R})(\gamma)$

b) if $v \cdot v = 0$ F_γ is ω-isotropic and

$$T_\gamma \mathcal{O}_\gamma \subset F_\gamma$$

c) in each case $\dim F_\gamma = 2n - 2$

R 5. let $\pi : \mathcal{G}_{par} \to \mathcal{G}_{traj}$

a) if γ is space/time geodesic: $D\pi_\gamma(F_\gamma) = T_{\pi(\gamma)} \mathcal{G}_{traj}$

b) if γ is light: $D\pi_\gamma(F_\gamma) \subsetneq T_{\pi(\gamma)} \mathcal{G}_{traj}$ and

$\dim D\pi_\gamma(F_\gamma) = \dim F_\gamma - 2$ (since the action of $Aff^+(\mathbb{R})$ is free) $= 2n-4$.

c) let $\mathcal{G}_*^{ligh}$ denotes the space of light rays we have:

$$\mathrm{Orth}(T_\gamma \mathcal{G}_{par}^{light}) \subset T_\gamma \mathcal{O}_\gamma \quad , \gamma \in \mathcal{G}_{par}^{ligh}$$

Hence $F_\gamma \subset T_\gamma \mathcal{G}_{par}^{light}$

$$\Rightarrow \underbrace{D\pi_\gamma(F_\gamma)}_{\dim = 2n-4} \subset \underbrace{D\pi_\gamma(T_\gamma \mathcal{G}_{par}^{light})}_{\dim = 2n-3} \quad , \gamma \in \mathcal{G}_{par}^{ligh}$$

d) the distribution $\tau \mapsto \mathcal{F}_\tau$ $\begin{cases} \mathcal{F}_\tau = D\pi_\gamma F_\gamma \\ \tau = \pi_\gamma \end{cases}$

is :

- if τ is space/time : the tangent space
- if τ is light ; a codimension 1 subspace of the tangent space to $\mathcal{G}^{light}_{traj}$ And it is a contact distribution.

R6. On the space/time geodesic subspace the distribution $\gamma \mapsto F_\gamma$ is a connexion distribution with respect to the action of $Aff^+(\mathbb{R})$

R7. In all the case, since the fibration $\mathcal{G}_{par} \to \mathcal{G}_{traj}$ is a principal fibration of group $Aff^+(\mathbb{R})$ and $Aff^+(\mathbb{R})$ is contractible this fibration is trivial, that is $\mathcal{G}_{par} \simeq \mathcal{G}_{traj} \times Aff^+(\mathbb{R})$

3) The souvenir of the symplectic structure :

Since not everywhere $T_\gamma \mathcal{G}_{par} = T_\gamma O_\gamma \oplus F_\gamma$ we have no chance to find a symplectic structure on $\mathcal{G}_{traj}$ but we can try to remember the most of the symplectic structure on $\mathcal{G}_{par}$?

let consider ω^{-1} the 2 contravariant form defined by

$$\omega^{-1}(\underbrace{\alpha , \beta}_{\in T^*_\gamma \mathcal{G}_{par}}) = \beta(\underbrace{\omega^{-1}(\alpha)}_{\in T_\gamma \mathcal{G}_{par}}) \quad \text{with} \quad \omega : T_\gamma \mathcal{G}_{par} \to T^*_\gamma \mathcal{G}_{par}$$

this structure is not invariant by $Aff^+(\mathbb{R})$ but it's conformal class is :

$$[\omega^{-1}] = \text{class with respect to} \quad \underbrace{\Lambda}_{\in T^{\wedge 2}_\gamma(\mathcal{G}_{par})} \sim \underbrace{c}_{c \neq 0} \Lambda$$

Now $[\omega^{-1}]$ can by pushed forward to $\mathcal{G}_{tray}$ in a 2 contravariant class of some tensor, let :

$$\Lambda = \pi_* [\omega^{-1}]$$

Λ is a section of some fiber bundle. The kernel of Λ is well defined ;

- on the subspace of time/space geodesics Λ has no kernel
- on the light rays the kernel of Λ is exactly the contact distribution above.

Note that on the subspace of time/space rays the section Λ can be realized as a cosymplectic class which gives back the symplectic structure.

Last remark: There are a lot of questions still unresolved associated to these construction. From time to time I'am looking at them...