A correlated and positive model for Bayesian image deconvolution

Pierre MINIER[1], Jean-François GIOVANNELLI[1], François ORIEUX[2]
[1]IMS (Univ. Bordeaux, CNRS, B-INP) & [2]L2S (Univ. Paris-Saclay, CNRS, CentraleSupélec)

Abstract

- Construction of a positive and correlated field by noise filtering.
- Application to a truncated Gaussian and a low-pass filter.
- Independent Metropolis-Hastings sampler to retrieve parameters.

Keywords

Positivity, Correlation, Filtering, Metropolis-Hastings, Hight resolution

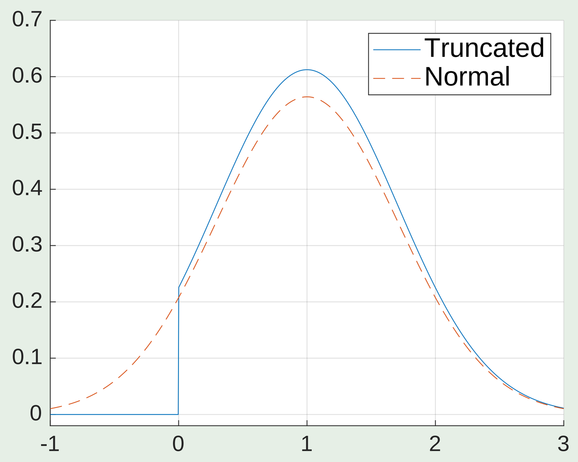
Positive field

U is a field with iid components according to a positive-support law

$$\mathbb{1}_+(\boldsymbol{u}) = \prod_{p=0}^{P-1} \mathbb{1}_{[0,+\infty]}(u_p) = \mathbb{1}_{\mathbb{R}_+^P}(\boldsymbol{u})$$

Law for the components of U

- Uniform on $[u_m, u_M]$ with $u_m \geq 0$
  - $+$ straightforward expression
  - $-$ bounded by u_M
- Truncated Gaussian on 0 $\rightsquigarrow$ chosen
- Gamma
- Levy with $\mu = 0$
  - $+$ law stability after filtering
  - $-$ undefined moments



Truncated Gaussian ($\mu = 1, \gamma = 2$)
New values ($\mu_t = 1.11, \gamma_t = 2.67$)

Filtering

A_η has positive coefficients

- preserves positivity
- adds correlation on the field U

$$\boldsymbol{x} = A_\eta \boldsymbol{u}$$

A_η controls the first and second order

- Mean : $\mathbb{E}[X] = \mu_t \sum_{n_h,n_v} a_\eta(n_h, n_v)$
- Cov : $C_X(n_h, n_v) = \gamma_t^{-1} a_\eta * a'_\eta(n_h, n_v)$
- PSD : $S_X(\nu_h, \nu_v) = \gamma_t^{-1} |\mathring{a}_\eta(\nu_h, \nu_v)|^2$

A_η represents a 2D convolution

- $a_\eta(n_h, n_v)$ the impulsion resp.
- $\mathring{a}_\eta(\nu_h, \nu_v)$ the frequency resp.

A_η has usefull properties

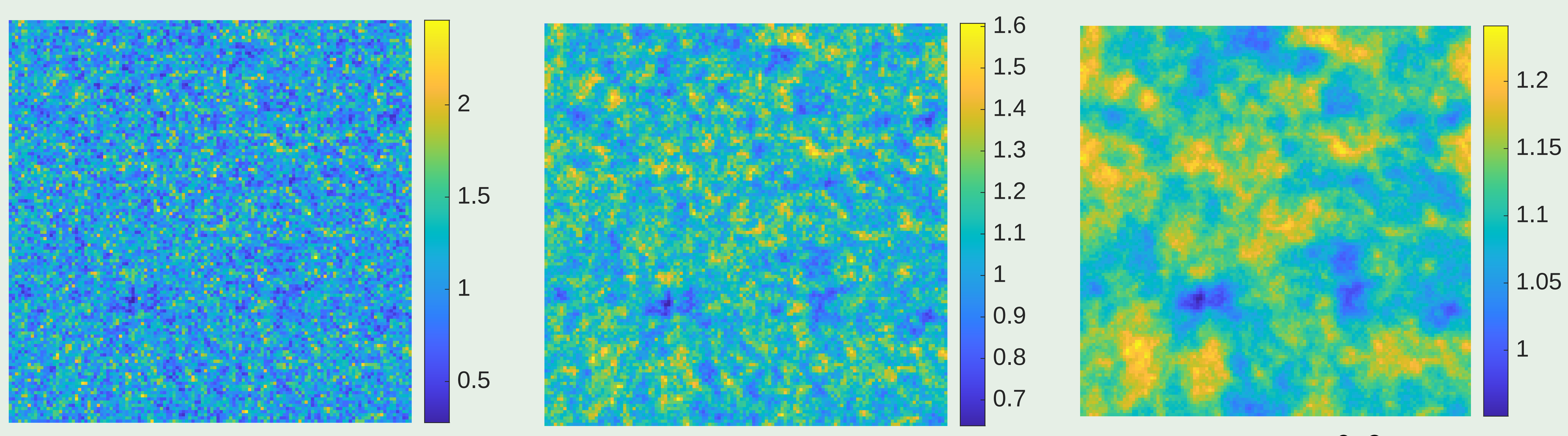
- Circulant Block-Toeplitz with Circulant Blocks (CBTCB)
- Diagonalisable in the Fourier domain
  - $\rightarrow$ F is the matrix of the discretised 2D Fourier transform
  - $\rightarrow$ Λ_η is the discretised frequency response of the filtering

$$\Lambda_\eta = F A_\eta F^\dagger = \mathrm{diag}[\lambda_\eta(p), p = 0 \cdots (P-1)]$$
$$F\boldsymbol{x} = \mathring{\boldsymbol{x}} = \Lambda_\eta \mathring{\boldsymbol{u}} = \Lambda_\eta F \boldsymbol{u}$$

with $\lambda_\eta(p) = \mathring{a}_\eta(n_h/N, n_v/N)$ the discretised frequency response.

Example of images

$$S_X(\nu_h, \nu_v) = \gamma_t^{-1} \times (1-\eta)^2 \left[1 + \eta^2 - 2\eta \cos\left(2\pi\sqrt{\nu_h^2 + \nu_v^2}\right)\right]^{-1}$$



$\eta = 0.2$ $\quad$ $\eta = 0.5$ $\quad$ $\eta = 0.8$

Change of variables

Filtering corresponds to a multivariate change of variables.

$$\begin{aligned} f_X(x) &= |\det A_\eta|^{-1} f_U(A_\eta^{-1} x)\, \mathbb{1}_+(A_\eta^{-1} x) \\ &= |\det \Lambda_\eta|^{-1} f_U(F^\dagger \Lambda_\eta^{-1} \mathring{x})\, \mathbb{1}_+(F^\dagger \Lambda_\eta^{-1} \mathring{x}) \end{aligned}$$

Filtered truncated Gaussian case

$$\begin{aligned} f_{X|\Theta}(x|\boldsymbol{\theta}) &= |\det A_\eta|^{-1} K^P \exp\left(-\frac{\gamma}{2}\|A_\eta^{-1}x - \mu\mathbb{1}\|^2\right) \mathbb{1}_+(A_\eta^{-1}x) \\ &= |\det \Lambda_\eta|^{-1} K^P \exp\left(-\frac{\gamma}{2}\sum_{p=0}^{P-1}\left|\frac{\mathring{x}_p}{\lambda_\eta(p)} - \mu\mathring{\mathbb{1}}_p\right|^2\right) \mathbb{1}_+(F^\dagger \Lambda_\eta^{-1} \mathring{\boldsymbol{x}}) \end{aligned}$$

- $\boldsymbol{\theta} = [\mu, \gamma, \eta]$ the parameter vector
- K the partition function of U

$$K = \sqrt{\frac{\gamma}{2\pi}} \times \frac{2}{1 + \mathrm{erf}\left(\sqrt{\frac{\gamma}{2}}\mu\right)}$$

- μ the mean before the truncation of U
- γ the precision (inverse variance) before the truncation of U

Estimation of parameters

Given a direct observation of X, we aim to estimate $\boldsymbol{\theta} = [\mu, \gamma, \eta]$

$$\pi_{\Theta|X}(\boldsymbol{\theta}|x) = \frac{f_{X|\Theta}(x|\boldsymbol{\theta})\ \pi_\Theta(\boldsymbol{\theta})}{\int_\Theta f_{X,\Theta}(x, \boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}}$$

- $\pi_{\Theta|X}$ the posterior describing our problem
- $f_{X|\Theta}$ the likelihood (filtered and truncated Gaussian)
- π_Θ the prior knowledge about the parameters
- $\int_\Theta f_{X,\Theta}(x, \boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}$ the normalising constant

Uniform and independent priors are considered for $\boldsymbol{\theta}$.

Independent Metropolis-Hastings

Sampling $\pi_{\Theta|X}$ using a Metropolis-Hastings does not require $\int_\Theta f_{X,\Theta}(x, \boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta}$.

$$\pi_{\Theta|X}(\boldsymbol{\theta}|x) \propto f_{X|\Theta}(x|\boldsymbol{\theta})\ \pi_\Theta(\boldsymbol{\theta})$$

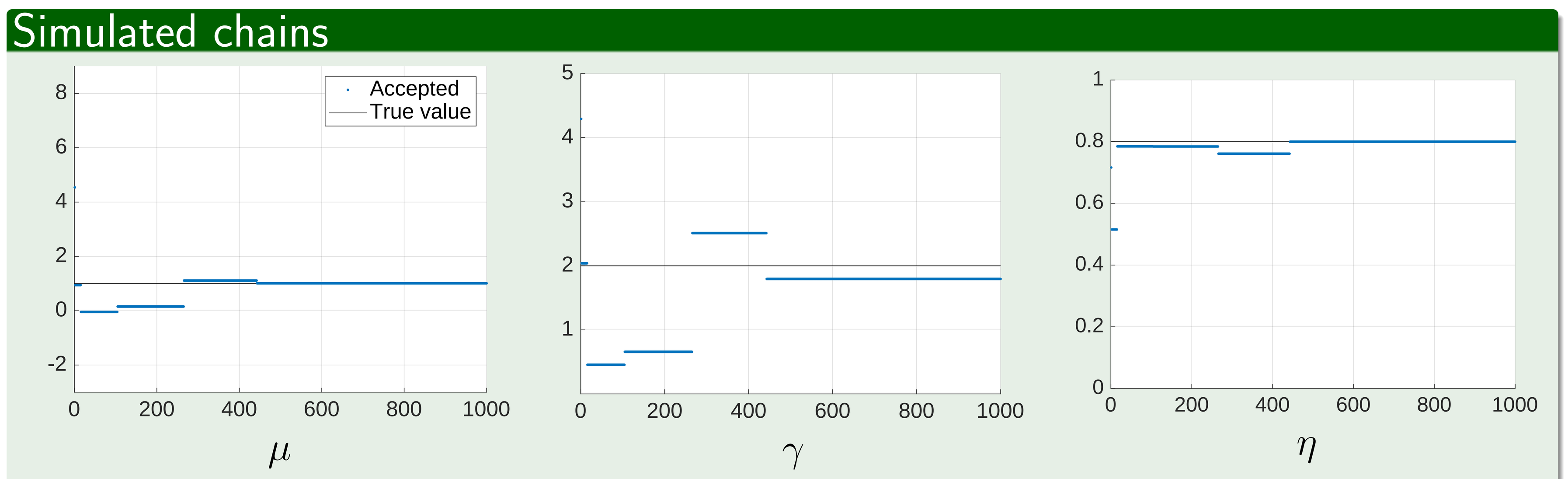
By using prior distribution as the proposed law, only the (log-)posterior is needed.

$$\begin{aligned} LP(\boldsymbol{\theta}) = &-\sum_{p=0}^{P-1} \log(|\lambda_\eta(p)|) + P\log(K) - \frac{\gamma}{2}\sum_{p=0}^{P-1}\left|\frac{\mathring{x}_p}{\lambda_\eta(p)} - \mu\mathring{\mathbb{1}}_p\right|^2 \\ &+ \log\left(\mathbb{1}_{[\mu_m,\mu_M]\times[0,\gamma_M]\times[\eta_m,\eta_M]}(\boldsymbol{\theta})\right) \\ &+ \log(\mathbb{1}_+(F^\dagger \Lambda_\eta^{-1} \mathring{\boldsymbol{x}})) \end{aligned}$$

Pseudo code

```
1: repeat
2:     Sample θ_0 from π_Θ
3: until 𝟙_+(F†Λ_{η0}^{-1} x̊) = 0
4: for k = 1 to K do
5:     Sample θ^p from π_Θ
6:     if 𝟙_+(F†Λ_{ηp}^{-1} x̊) = 0 do
7:         θ_k = θ_{k−1}
8:     else
9:         Compute acceptance ratio α = exp(min(0, LP(θ^p) − LP(θ_{k−1})))
10:        Sample u from Uniform(0, 1)
11:        if u < α do
12:            θ_k = θ^p
13:        else
14:            θ_k = θ_{k−1}
15:        end if
16:    end if
17: end for
```

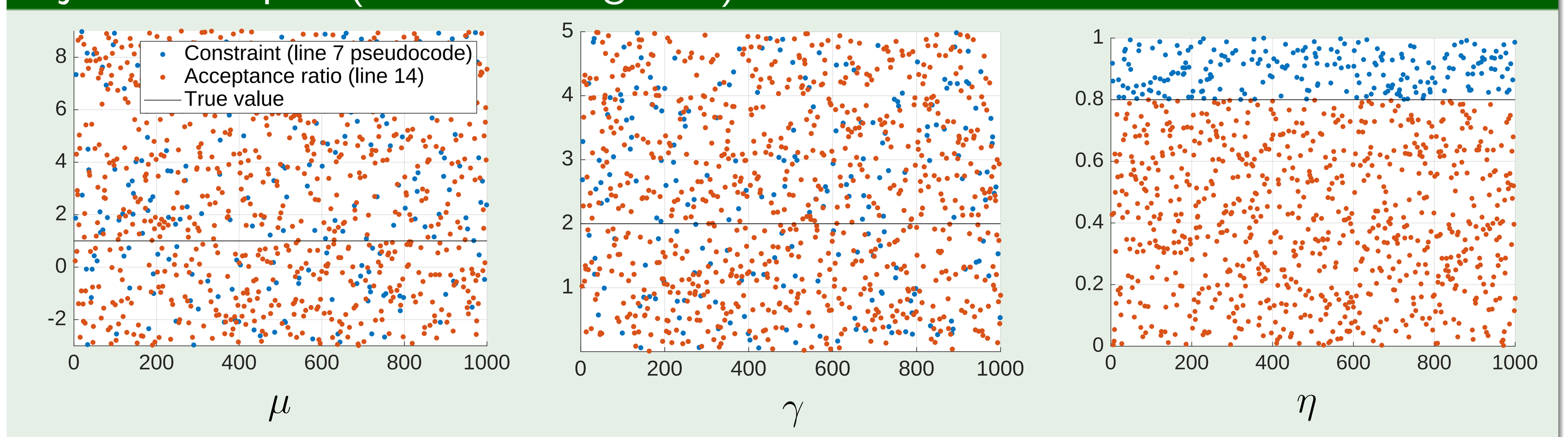
Simulated chains



Obtained samples for θ

Issue: low acceptance rate (0.6%) $\rightsquigarrow$ lack of diversity.

Rejected samples (under investigation)



η appears to play a key role.

Perspectives

- Analysing $\mathbb{1}_+(F^\dagger \Lambda_{\eta^p}^{-1} \mathring{\boldsymbol{x}}) = \mathbb{1}_+(F^\dagger \Lambda_{\eta^p}^{-1} \Lambda_{\eta^*} \mathring{\boldsymbol{u}})$ $\rightsquigarrow$ new filter $\Lambda_{\eta^p}^{-1} \Lambda_{\eta^*}$
- Random Walk Metropolis $\theta^p = \theta_{k-1} + \varepsilon$ with $\varepsilon \sim \mathcal{N}(0, \sigma^2 I)$
- Another filtering: $\boldsymbol{x} = \frac{1}{\sqrt{\alpha}} A_\eta \boldsymbol{u} + \beta\mathbb{1}$ (with $\mu = 0$ and $\gamma = 1$)
- Application to images in the astronomical, medical or industrial field.

Bibliography

- Ke Chen. *Matrix Preconditioning Techniques and Applications*. 2005.
- John Burkardt. *The Truncated Normal Distribution*. 2023.
- Siddhartha Chib. *Understanding the Metropolis-Hastings Algorithm*. 1995.