How to build a gradiometer

August 31, 2024

This document is intended to act as a guide/reference for the project I carried out over the summer of 2024. As this project looks likely to be continued into becoming a masters project, I have carried out a brief write-up, so that whatever progress I have made towards the main goal, can be accessed easily, and things I have done don't have to be re-done.

1 Introduction

The gravitational field is a vector field, whose makeup is determined by mass density. Hence, measuring of local gravitation acceleration can be used to detect gravitational anomalies, typically areas of higher or lower mass density. There exist two broad classes of gravimeter: Absolute Gravimeters, these measure the value of acceleration due to gravity at one point, these are large, usually static devices, which can determine the value of gravity to a very high accuracy. The second class is what I would call gravity gradiometers, these measure relative changes in gravity from one point to another. These are usually more compact, and less accurate, or can be susceptible to drift of other forms of error, which would make them unsuitable for absolute measurements. Since the former are less relevant I will not go into detail into their designs, but some reading material can be found in the appendix.

The applications of these measurements are many, from finding subsurface mineral deposits, to detecting buried structure for archaeologists. There exist many different gravity gradiometer designs, many application specific, a few interesting ones are listed below:

- MEMs sensors: Standing for Micro-Electro-Mechanical, these devices use tiny springs and masses to act as a gravimeter. They have a sensitivity around $10^{-5} ms^{-2}$.
- Ribbon sensor gradiometry uses a light conductive ribbon, which will deform or flex due to the influences of large external masses.
- Many such devices use superconduction to measure small signals generated fluctuations caused by changes in g in other physical systems, some examples can be found in the references.
- Finally, some success has been had in the production of Strapdown-Aerial-Gravimetry, which uses a comparison between the internal and externally measured acceleration of an. [1]

2 The Big Idea

Our design for the gravimeter is based on a simple idea: The closer an object is to a test mass, the greater gravitational acceleration it will feel. Something that is capable of measuring small fluctuations in the position of a mass relative to its base is a seismometer. These seismometers use coils and magnets to measure the relative velocity between their inner, suspended mass, and the outer casing. By modelling the two seismometers as masses on springs, we can work out how they will behave in the presence of a large mass. Mathematically:

$$F_1 = kz_1 = m_1 g(z_1, t)$$
$$F_2 = kz_1 = m_2 g(z_2, t)$$

Where m_1 and m_2 are the displacements from equilibrium of the masses in seismometers 1 and 2. Then, assuming we are above the natural resonance of the mass and spring system, ω_0 we can treat the masses as being effectively free moving, and thus behave linearly:

$$m_i a_i = F_i = m_i g(z_i, t) \quad i = 1, 2$$

Hence, any small differences between the two masses cancel and we arrive at:

$$a = a_1 - a_2 = g(z_1, t) - g(z_2, t) = \delta g$$

Where δg is the relative acceleration due to gravity between the two masses in the seismometers. Of course we have had to assume damping is not present, but it would complicate things significantly if this was the case. By assuming the two seismometers are some distance d apart, and a distance z_0 away from the moving mass, we can write δg as $g(z_0 + d/2, t) - g(z_0 - d/2, t)$. Assuming $z_0 >> d$ we can Taylor expand g:

$$g(z_0, t) + \frac{\partial g(z_0, t)}{\partial z}\frac{d}{2} - g(z_0, t) - \frac{\partial g(z_0, t)}{\partial z}(-\frac{d}{2}) = \frac{\partial g(z_0, t)}{\partial z} d = -\frac{2GMd}{z_0^3}$$

Hence, we find that the acceleration measured by the device is:

$$a = \frac{2GMd}{z_0^3} \tag{1}$$

So, provided a is greater than the noise floor we obtain for our device, we can design a testing device based off these parameters.

3 Calculating Device Parameters

Our device currently consists of 2 two L4c Geophones, placed on a frame, located approximately 10cm apart vertically. These Geophones have a resonant frequency that is approximately equal to one, and a Q factor approximately equal to 0.2 (meaning they lose roughly 1/5 of their energy per cycle to damping.) In order to actually measure small changes in g we need to perform an operation known as a "coherent subtraction". This requires us to account for any small deviations in device parameters, which any subtraction would otherwise include. This allows isolation of gravitational signals, and elimination of ground noise. We construct the frame to be as solid as possible, using one inch posts attached to an optical breadboard.

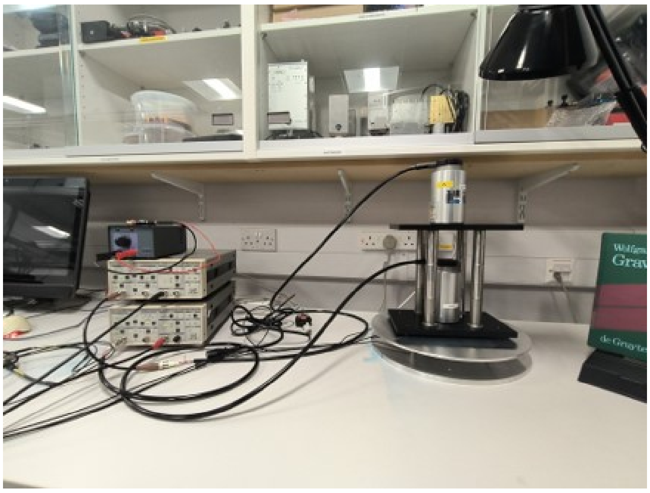


Figure 1: Image of set up used to obtain transfer functions.

The outputs of each seismometer are run through an SR560 low noise pre-amplifier to improve the SNR (Signal-Noise Ratio)[1].

3.1 Measuring Transfer Functions of the L4C's

There are several methods that can be used to obtain parameters for the L4C's. The method used here is to measure the transfer function of the L4c's using a shaker table, and fit a model using Vectfit. [2] We can model an L4C as a mass and spring system, with an output proportional to the relative velocity between the inner suspended mass and frame:

$$\ddot{x} - \beta\dot{x} - \omega_0^2 x = F(\omega t) \tag{2}$$

[1] The SR560s are set to 10x amplification, and have a bandpass filter set between 0.03 and 1000Hz

[2] A transfer function is the ratio of a components output to a unitary sinusoidal input at frequency ω. This is a complex number, as it involves a phase difference and a ratio of two magnitudes.

becomes:

$$G(i\omega) = \frac{\omega^3}{\omega_0^2 - 2i\beta\omega - \omega^2} = \frac{\omega^3}{\omega_0^2 - i\frac{\omega_0}{Q}\omega - \omega^2} \tag{3}$$

With ω^3 on top representing differentiation of a second order resonance (in units of displacement the factor on top would be ω^2, as response is expected to be flat after primary resonance). ω_0 and Q are the resonant frequency and Q factor respectively. We measure the transfer function of the seismometer by placing it on a PZT shaker table, seen below the frame in the diagram. We can then use CDS to actuate the platform vertically to measure the differences in response of each seismometer. It should be noted that we are modelling the frame as effectively rigid. Doing this over a period of 12 hours gives us the following picture:

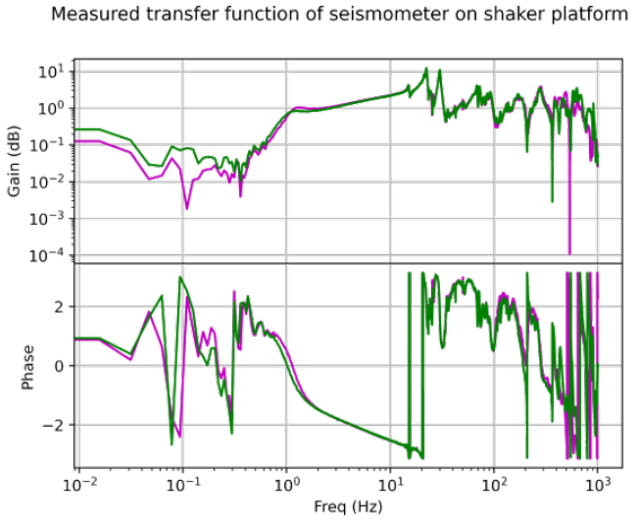


Figure 2: Transfer function measured over 12 hours, main seismometer resonance can be seen between 0.8 and 1.2 Hz

To fit these transfer functions we use a tool called VectFit. The fitting parameters used are four conjugate pole - pairs. We then use inbuilt functions within to turn these poles into a series of zeros and poles:

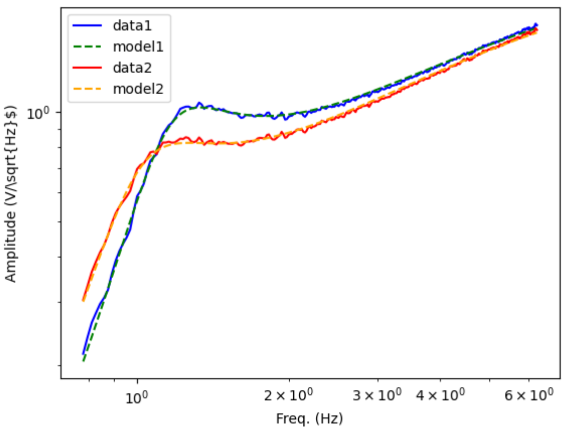


Figure 3: Fitted transfer functions to data.

The use of two conjugate pairs as opposed to one is to account for any higher frequency resonances, which would cause the model to diverge from expected behaviour. These are the resultant zeros and poles:
We can factorise 3, recognising the second order pole and third order zero in the expression, and express the zero in its factorised form. We find the values of ω_0 and Q to be:

3.2 Why did we just do that?

So why did we go to the effort of fitting the transfer functions so precisely: Mathematically it's quite simple. If want to perform a true coherent subtraction it is necessary that both seismometers appear identical in behaviour- this allows suppression of any ground motion and allows us to isolate the small change in gravitation acceleration. By applying the inverse of the transfer function of a seismometer to its own output then applying the transfer function of the other seismometer we can trick the subtraction into thinking it is subtracting two identical seismometers [3].

The reason for performing the method this way and not applying the inverse of each seismometers transfer function to it's respective output, and then subtracting, is a little more subtle. Note the third-order dependence in ω in 3, and recalling that dividing by ω in Fourier Space. This results an infinite gain at DC (i.e. no time dependence), meaning that any transients will cause the two seismometer outputs to diverge at low frequencies. For this reason the method laid out in the paragraph above is the most practical.

4 Calculating noise Levels

The main source of noise, while using CDS and the 10x amplification of an SR560 comes from mechanical noise within the mechanism of the L4c itself[2]. We are performing a subtraction between two seismometers, so we need to account for minor differences between them to estimate our thermal noise floor. The thermal noise in an L4c comes from the fluid damping inside the L4c, given by the following expression[3]:

$$S_{ff} = \frac{1}{m}\sqrt{2k_B TQ} \tag{4}$$

To get this in the correct units, acceleration in our case, we need to divide by the transfer function of our seismometer, eq. 3:

$$s_{xx} = s_{ff}\frac{\omega_0^2 - 2i\beta\omega - \omega^2}{\omega^3} \tag{5}$$

Now as we want our final plot to be in the correct units, we need to multiply by the callibration constant for the L4C, which is $277Vm^{-1}s$. Since we are considering the subraction of two L4c, and do not expect noise between them to be correlated, we need to find the Root-Mean-Square of the two seismometers.

We can obtain an estimate for the electronic noise floor by unplugging a both seismometers from our amplifiers and measuring the output on the subtraction channel. Electronic noise is hard to estimate analytically in the manner above, since it derives from several sources, like the ADC inside CDS within the amplifier, and in any cables.

5 Callibrating and Reaching the noise floor

Once we have found the transfer functions of both seismometers we will use CDS to apply the inverse to the seismometer output. We do this by using filter constructed in foton. To construct these filters we use the zpk function, which is inbuilt to foton.

We can tune the relative gain between our seismometers by using our shaker table again. We set a noise signal that goes between 0.1 and 20Hz, with a fairly large amplitude and adjust the gain in the relevant medm screen. Once we have reached the limit of four decimal places, we can write this gain into foton, and adjust the gain in there, until we see a complete overlap between each seismometer.

We can fine tune the zpk function by exciting the region around the resonances specifically. To do this we set the excitation region to be much smaller (0.6-1.4Hz) and increase the amplitude by a substantial amount. We can then attempt to adjust the positions of our zeros and poles until the coherence between the incoming noise signal and the output subtraction is as close to zero as possible. This is often quite a trick process, and requires a lot of time and patience to adjust to get just right.

Once this has been done we can compare the results of our callibration to the results of our noise estimation, for which we can get the following plot:

[3]This process is implemented digitally, using a computer program called CDS. The process of implementing filters expressed as zeros and poles in known as tustins method, and is very common in control theory. CDS is a very complex and robust program, with a lot of features, as it allows implementation of complex control loops, it is one of the controllers used in LIGO- see appendix for more.

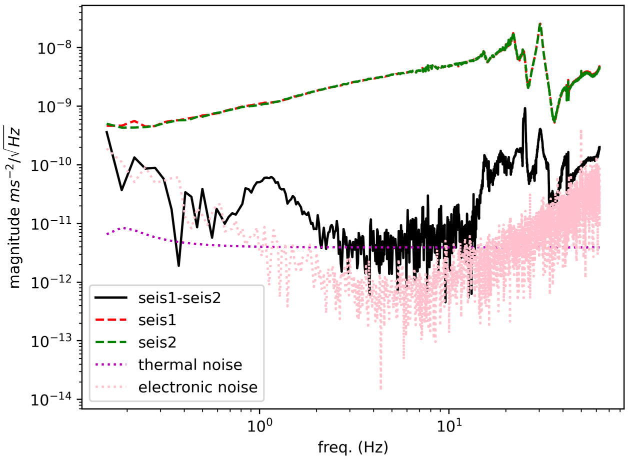


Figure 4: Plot of expected noise and measured transfer functions of gradiometer.

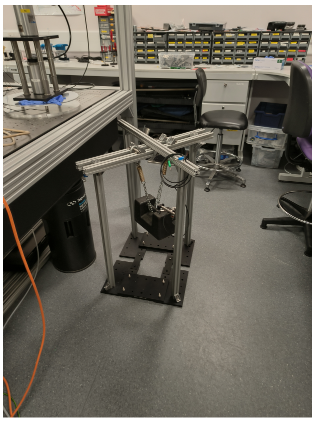
To callibrate this figure in units of acceleration I have divided both seismometers and the subtraction by the transfer function of "seis1". We can see that the transfer function of gradiometer is at the noise floor between 2 and 10Hz, with a small bump near the resonance point. This does not come as a surprise, as resonances tend to contain weird non-linear behaviour, which is hard to account for. Finally we see that the the gradiometer output tracks well with the measured noise below resonance.

6 Testing the Gradiometer

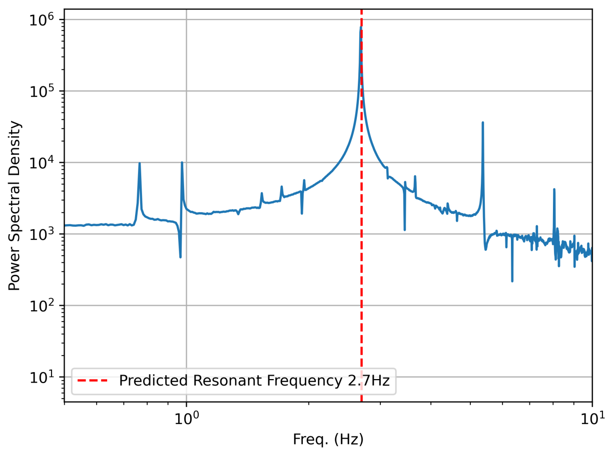
In order to test the gradiometer we need to find some way of moving a mass up and down in proximity to the device. Several methods appear to turn up results, including press-ups, weightlifting and swinging a weight side to side. Obviously none of these are particularly repeatable, so it would be ideal to find a way to construct a mass that can oscillate at a set frequency for a prolonged period of time. My method for achieving this was simple: By suspending a 20kg dumbell using two springs I was able to use a coil driver to actuate the mass up and down. By driving near its resonance point, I could minimise the amount of power needed by the coils to drive the system, and get a large motion from response from the mass. We can attempt calculate the resonant frequency of the spring by using the spring constant. The springs are at an approximately $20\deg$ angle from the vertical. They are under a load of approximately 200N. Some A-level physics will give us the component of force along the length of the spring. Under this stress the springs extend by 1.6cm from their previous length. Hence we find the spring constant to be 6.5kN/m. Using:

$$F_0 = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

We find that the resonant frequency should be around 2.7Hz (You can check for yourself!)



(a) The mass actuator used to test the gradiometer, coil driver is positioned directly underneath the COM of the dumbell in final iteration.



(b) Fourier Transform of impulse response of mass and spring system, measured using the driving coil.

Using the spectrum above we find the resonant frequency is actually closer to 2.69, and more importantly, the resonance peak is very sharp. This tells us that this resonance has a very high Q-value, meaning that the mass will keep moving for a long time after excitation is removed. So to get a good response from our system we should drive as close to resonance as we can.

Using our newly measured resonant frequency we can use CDS to drive our mass up and down. At the same time we allow our gradiometer to measure the acceleration it experiences - from which we obtain this new spectrum:

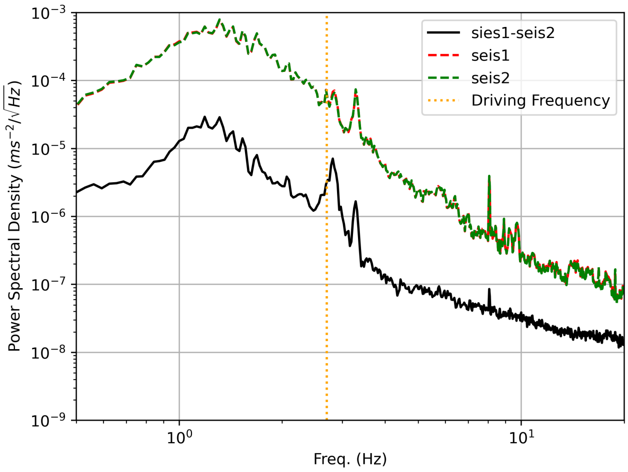


Figure 6: Power Spectral density of gradiometer readout. Note peak at the driving frequency of our mass.

Where this is a clear peak visible near 2.7Hz! We should verify that this peak does correspond to coupling through gravity as opposed to seismic noise, while we have placed the gradiometer on the table we can still expect come motion to be transmitted through the legs. Recall in eq. 1 we can estimate the magnitude of a we would expect. Since our system is to first order approximation, linear (which is verified by the lack of harmonics visible on our gradiometer PSD) we can assume that the RMS motion of the mass will be directly proportional to gradiometer output. Our amplitude of motion of the mass is around 3cm up and down. This gives an RMS value for $d = 0.012$cm. Using a distance of around 0.5m from the gradiometer to the mass we find that $a \approx 1.2e-8\text{ms}^{-2}$. We can also compute the RMS value of the peak seen in the graph above, using a bandwidth of 0.01Hz. From this we obtain a measured value of acceleration $a = 7.7e-7\text{ms}^{-2}$. Which is not as close as we would like it to be? lets investigate a bit further. We can approximate the expected RMS value by dividing a by the bandwidth squared, our bandwidth is 0.01, so we should expect to see a value around $a = 1.2e-4\text{ms}^{-2}$. Which is much closer to what we actually read on the graph. This is an interesting discrepancy, and requires further investigation.

7 Conclusion

So what have we learnt? The aim of this project was to make as much progress as possible towards constructing a drone based gradiometer. The result of the project was to show that the basic concept of such a gradiometer is a sound basis on which to base this project. We were able to subtract the motion of two seismometers in a coherent manner, that would allow elimination of seismic noise (Common Mode Rejection). We have shown, when done well, this can reach the thermal noise level expected of our design.

Appendix 1: Progressing Development- what needs to be done...

To optimise the gradiometer to be small enough to place on a drone- the end goal of this project, the following needs to be done:

- Miniturization of data acquisition hardware to be light enough to place on a drone platform. This could be done using a small Linux computer running CDS, and a compact DAQ and amplifier set up. Audio amplifiers could work quite well, but a characterisation of their noise spectrum will be necessary to ensure they perform well at this application. Intel NUC's are a potentially viable platform, as with four cores, they can quite easily be capable of running CDS, with one core for any data processing, and another core for any seismic isolation you might want to do.

- A power delivery system, capable of supplying a lightweight computer and amplifier, for a duration around 20 minutes. A large rechargeable battery could be a good option, but weight will have to be considered.
- Characterisation of how much the drones acceleration couples into your gradiometer design. Characterising this would require you to use a series of accelerometers to measure the motion of the drone through space, and see if the noise it generates will have an effect on your design. This will require you to develop data acquisition and power supply solutions first before you carry this step out. Commercial solutions do exist, see [4].
- Design of a lighter weight equivalent to an L4c, with stiffer springs to allow for a higher resonant frequency, in the region of 10Hz or so.
- is some sort of active seismic isolation necessary when the gradiometer is placed on the drone? This is dependent on how stable of a platform the drone is. It may be off interest to look into compact 6d isolation, maybe using one of these PZT-Flexure hexapod platforms [5].

Appendix 2: Simulating moving the Gradiometer over changes in Mass density.

Since we are constructing a gradiometer using a non-DC readout, we cannot detect the absolute amount of “stuff” under the ground at a paticular point, we can however detect changes or fluctuations in mass density as we move over the top. So how can we simulate, say, moving over the top of a junction between two different densities, or a large void? For this we need to be able to produce a simulation that can transform a mass density function, which associates a density at a particular point in space, with graviational potential at another point. From this we then want to move our device through this graviational potential, and calculate the output of our gradiometer. To simulate the affect of moving our device over a mass gradient we can then model moving our seismometer through this potential. The gravitational potential is defined as the convolution of the mass density function $dm(\boldsymbol{r})$ with the potential of a unit mass at distance $\boldsymbol{r}$: $G(\boldsymbol{r})$. I.e:

$$V(\boldsymbol{r}) = -\int_{\mathbb{R}^3} \frac{G}{|\mathrm{x} - \boldsymbol{r}|} dm(\mathrm{x}) = -\int_{\mathbb{R}^3} \frac{G}{|\mathrm{x} - \boldsymbol{r}|} \rho(\mathrm{x}) dv$$

We can then go about computing the potential in two ways. The first method is to use some properties of convolution in fourier space, i.e. mapping $\boldsymbol{r}$ to $\boldsymbol{k}$:

$$V(\boldsymbol{r}) = \mathcal{F}\{\frac{G}{r} * dm(\boldsymbol{r})\}(\boldsymbol{r}) \implies \hat{V}(\boldsymbol{k}) = \frac{4\pi G}{\boldsymbol{k}^2} \hat{dm}(\boldsymbol{k})$$

This method is good if we have a periodic mass density, and can allow inversion to go from a power spectrum for our device to a mass density function. This is quite complicated to implement, and will not be a one to one mapping. For this appendix I think it is better to compute the potential from the mass density function directly, integrating over the shape of our massive object or void. To make computing the volume integral cheaper it often makes more sense to examine the former case. This is because the outer bounds of the integral become the boundaries of our object. By modelling our seismometer as a dampened harmonic oscillator the following model can be used:

$$\ddot{x}_i - \frac{\omega_0}{Q}\dot{x}_i - \omega_0^2 x_i = F_i(t) \quad i = 1 \quad \text{or} \quad 2 \quad \textit{representing top or bottom.}$$

Where F_i is the vertical component of V_i. We are assuming the gradiometer moves over the massive object at a constant speed v. Hence we can parameterise $F(\boldsymbol{r}) = F(vt, 0, z_0 \pm d) = F_i(t)$. Where d is the distance between the top and bottom seismometer. The sensors sensitivity will depend on several factors:

- The speed over the massive object v.
- The density of the object / mass of the object.
- Length in the direction of travel of the object.

So experimentation with these values in the simulation is encouraged.

The mass density function I selected was a uniform cuboidal mass, that is sufficiently long in the x-direction that the effect of the boundary to one end of the mass have a negligible effect when sensing the other end.

It is computationally more efficient to define the mass density function as uniform everywhere, then set the region of integration to be by setting the curves that define the bounding surface of the object in the integrator. This will minimise the amount of points that scipy will integrate over. Integration is performed using scipy's `tplquad` function for performing volume integrals. We then parameterise the path on which we want to evaluate to find the gravitational potential. Now, it is important to note that in order to find the gravitational force at a paticular point on this path, we need to calculate the derivative in the vertical direction. We can approximate this by using the definition of the derivative. Since we expect that the gravitational potential should vary slowly ($\propto 1/y$) in the y direction, we can approximate the derivative by calculating two nearby(ish) points, and taking their difference. This means evaluating two paths, one a small displacement above the other. This gets computational very expensive quickly, the more points you evaluate the function at. For this reason we make the assumption that the gravitational potential should not vary in a piece-meal manner in the x direction, allowing us to use a method of interpolation of our choosing[4]. This allows us to approximate the computationally expensive triple integral with a cheap to evaluate spline function, which can be used in its place. It also allows us to run the integration once, and then test different configurations of seismometer on top of it, as the potential will be unaffected by the change of seismometer.

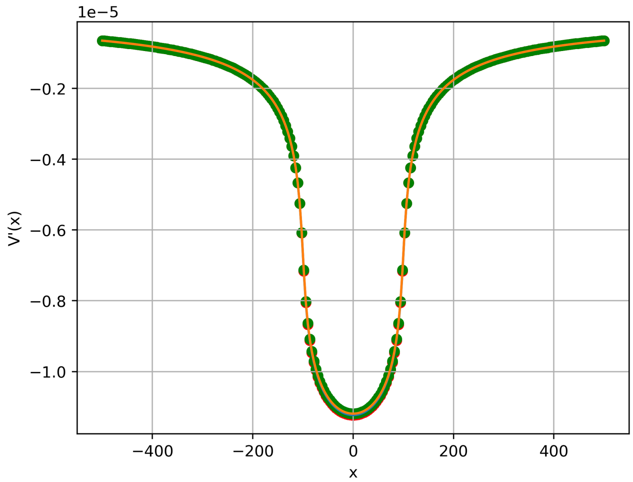


Figure 7: Gravitational potential measured at two points at different distances above a large cylindrical mass

I found the best results came from moving the seismometer at a speed of $1.0ms^{-1}$ over a mass 200m in length. This took 800s, and is not unrealistic for a drone to be able to move at this speed. What is a little unrealistic is a tunnel 100m wide. This generated the following velocity readout on our gradiometer.

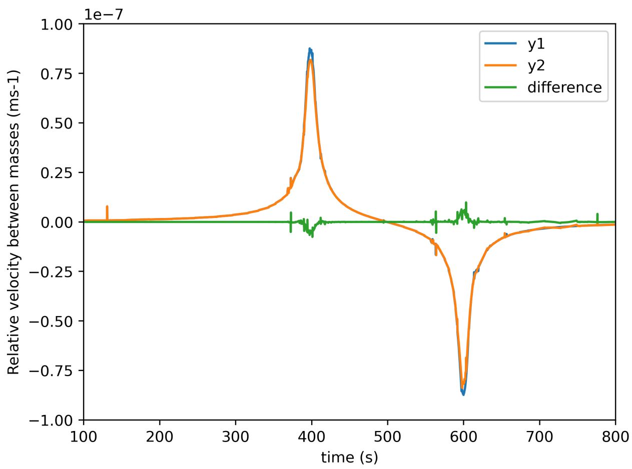


Figure 8: Difference in velocity readout between two seismometers.

There is a second method that can work to eliminate the numerical artifacts seen in these plots. Referring

[4] I chose to use cubic spline interpolation, but quadratic should work just as well. This is a standard technique when you would rather not make repeated calls to an expensive function

back to 1 we can find the expected acceleration undergone measured by our gradiometer as being equal to:

$$a = 2\frac{dg}{dz}|_{z=z_0}d$$

Since $mg = F = -\nabla V$ we find that:

$$a = -2\frac{d^2V}{dz^2}|_{z=z_0}d$$

In order to evaluate the second derivative at z_0 we need to compute at least two more points, one above and one below. This is not too expensive however, as in this approximation we are assuming that the seismometers are above resonance, hence we can treat them like free masses. As such we have another method to calculate what our readout might look like:

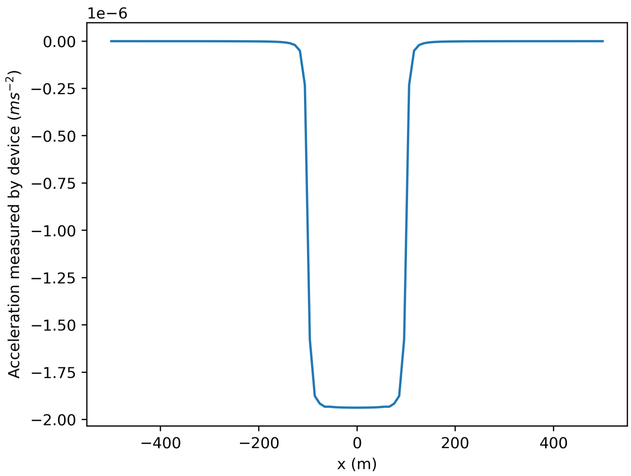


Figure 9: Acceleration plot generated using gravitational potential we found earlier.

Hopefully you have enjoyed this summary of my summer research project. I really enjoyed undertaking it, and I just want to thank a couple people, namely: My supervisor George Smetana, and Prof. Denis Martynov, as well as the rest of the QI group at the UOB QI group for being very supportive and giving me this wonderful opportunity.

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