Logic of First Order Structures

Piyush Patil

October 27, 2017

The bulk of this chapter is devoted to building up to Godel's completeness theorem and the necessary theory.

1 The Notion of Proof

We'll begin by building on our understanding of first-order logic and the associated structures, and going up a level of abstraction by using these ideas to cement our notions of proofs, validity, and truth. First, we define the first-order analog to a tautology, regarded as an absolute first-order truth.

> **(Definition) Validity**: *An $\mathcal{L}$-formula ϕ is a **validity** if it's satisfied in every interpretation, i.e. for every $\mathcal{L}$-structure $\mathcal{M}$ and $\mathcal{M}$-assignment ν, $(\mathcal{M}, \nu) \vDash \phi$.*

Though validities are quite uninteresting first-order formulas, the set of validities does have some illuminating properties; we'll show later tha a formula is valid if and only if it can be proven. The idea of proof, however, first requires us to speak of logical axioms. We'll port over our logical axoims from propositional logic, but in order to add axioms that make use of quantifiers, we'll need to first extend the idea of substitution and the substitution theorem from first-order logic.

> **(Definition) Substitutable**: *Let ϕ be an $\mathcal{L}$-formula τ be a term. For any free variable x_i in ϕ, τ is **substitutable** for x_i in ϕ if every variable of τ is free for x_i in ϕ (recall that this means, for variable x_j of τ, that none of the occurrences of x_i in ϕ are in the scope of $\forall x_j$). Analogously, for constant c_i in ϕ, τ is **substitutable** for c_i in ϕ if, for every variable x_j of τ, no occurrence of c_i is within the scope of $\forall x_j$.*

This is simply stating that if we were to replace all the x_i's in ϕ with τ, none of the newly introduced variables from τ interfere with copies in ϕ, which we obviously don't want since the meanings variables carry in a term or formula is local to that formula, and they ought not to clash. We use

$$\phi(x_1, \cdots, x_n; \tau_1, \cdots, \tau_n)$$

to denote the substitution of term τ_i for variable x_i in the formula ϕ in which x_i is free (assuming every term is substitutable for its corresponding variable). We use analogous notation for the substitution of terms for constants.

> **(Definition) Logical axioms**: *The set of logical axioms is denoted Δ and is defined as the smallest set of $\mathcal{L}$-formulas which satisfy the following closure conditions*
>
> 1. *Δ contains propositional tautologies: The following are logical axioms*
>   - (a) *Group I axioms*
>   - i. *$((\phi_1 \rightarrow (\phi_2 \rightarrow \phi_3)) \rightarrow ((\phi_1 \rightarrow \phi_2) \rightarrow (\phi_1 \rightarrow \phi_3)))$*
>   - ii. *$(\phi_1 \rightarrow \phi_1)$*
>   - iii. *$(\phi_1 \rightarrow (\phi_2 \rightarrow \phi_1))$*
>   - (b) *Group II axioms*
>   - i. *$(\phi_1 \rightarrow ((\neg \phi_1) \rightarrow \phi_2))$*
>   - (c) *Group III axioms*
>   - i. *$(((\neg \phi_1) \rightarrow \phi_1) \rightarrow \phi_1)$*
>   - (d) *Group IV axioms*
>   - i. *$((\neg \phi_1) \rightarrow (\phi_1 \rightarrow \phi_2))$*
>   - ii. *$(\phi_1 \rightarrow ((\neg \phi_2) \rightarrow (\neg(\phi_1 \rightarrow \phi_2))))$*
> 2. *Formulas that are true over a domain can be substituted: For $\mathcal{L}$-formula ϕ and term τ which is substitutable for x_i in ϕ,*
> $$((\forall x_i \phi) \rightarrow \phi(x_i; \tau)) \in \Delta$$

3. *Quantifiers split over implication: For $\mathcal{L}$-formulas ϕ_1, ϕ_2,*

$$((\forall x_i(\phi_1 \to \phi_2)) \to ((\forall x_i\phi_1) \to (\forall x_i\phi_2))) \in \Delta$$

4. *Quantifiers are vacuous on bound variables: For $\mathcal{L}$-formula ϕ and bound variable x_i of ϕ,*

$$(\phi \to (\forall x_i\phi)) \in \Delta$$

5. *Reflexivity axioms: For variable x_i,*
$$(x_i \hat{=} x_i) \in \Delta$$

6. *Changing variables doesn't change formulas: For $\mathcal{L}$-formulas ϕ_1, ϕ_2, and variable x_j that's substitutable for x_i in both formulas,*

$$\text{if } \phi_1(x_i; x_j) = \phi_2(x_i; x_j) \text{ then } ((x_i \hat{=} x_j) \to (\phi_1 \to \phi_2)) \in \Delta$$

7. *Quantifiers are degenerate on true formulas: For $\mathcal{L}$-formula ϕ,*

$$(\phi \to (\forall x_i\phi)) \in \Delta$$

We can now define an extended (from propositional logic) notion of proof. As before, we'll work over sets Γ of formulas which will serve as specific axioms, in addition to the logical axioms above.

(Definition) Γ-proof: *Let $\Gamma \subseteq \mathcal{L}$ be a set of $\mathcal{L}$-formulas and $\phi \in \mathcal{L}$. Then Γ **proves** ϕ, denoted*
$$\Gamma \vdash \phi$$

if there exists a sequence of formulas $(\phi_1, \cdots, \phi_n)$ such that

1. *ϕ_1 is an axiom, i.e. $\phi_1 \in \Gamma \cup \Delta$*

2. *$\phi_n = \phi$*

3. *For $1 \leq i \leq n$, either $\phi_i \in \Gamma \cup \Delta$ or there exist $j, k < i$ in the sequence where $\phi_j = (\phi_k \to \phi)$.*

*We call the sequence $(\phi_1, \cdots, \phi_n)$ a **deduction** of ϕ_n from Γ.*

Intuitively, the sequence of formulas represent steps in the logical progression composing a proof of the truth of ϕ. The proof must start at an axiom, and end at the statement we want to prove (i.e. ϕ); intermediate steps of the proof must either by axioms themselves, or must follow, through modus ponens, from a previous statement (i.e. there's a statement in the proof and another statement which says the current statement follows from the former).

2 Soundness

First, we define analogous concepts to consistency and satisfiability, which we previously considered in propositional logic. These are the heart of the requisites to Godel's completeness theorem.

(Definition) Consistent: *Let Γ be a set of formulas. Γ is **consistent** if for every formula $\phi \in \mathcal{L}_0$, if $\Gamma \vdash \phi$ then $\Gamma \nvdash (\neg\phi)$.*

(Definition) Satisfiable: *Let Γ be a set of formulas. Γ is **satisfiable** if there exists a structure $\mathcal{M}$ and a $\mathcal{M}$-assignment ν such that $(\mathcal{M}, \nu) \vDash \Gamma$.*

The Godel's completeness theorem extends the result we proved in propositional logic, that a set of axioms being logically consistent - which we intuitively take to mean as axiom system having no holes or gaps in the logical progression that follows from it, so that it never doubles back on itself or proves contradictory things (which of course, from the theorem that inconsistent systems can prove any formula, completely destroys any notion of meaningfulness in the system) - and being satisfiable - intuitively meaning that there exists some concrete interpretation of logical symbols over a universe of object within which our axioms are all true, indicating that there is indeed a way for the arbitrarily chosen meaningless formulas to attain a notion of validity and become self-evident - are actually the same thing. We'll state the theorem below without proof, and work up to the proof throughout the chapter.

(Theorem) Godel's completeness theorem: *Let Γ be a set of formulas. Then Γ is consistent if and only if Γ is satisfiable.*

One direction of the lemma is simpler than the other; that consistency follows from satisfiability is evident from the intuition that if a structure (and assignment) satisfies a set of axioms, then because of the nature of the structure and our definition of satisfiability, which we designed to encode the logical laws of our language (such as the definition of implication, for example), the structure should satisfy every deductive consequence of Γ and only those consequences. This is precisely what the soundness lemma tells us:

> **(Theorem) Soundness lemma**: *Let Γ be a set of formulas and $\phi \in \mathcal{L}_0$ such that $\Gamma \vdash \phi$. Then for any structure $\mathcal{M}$ and $\mathcal{M}$-assignment ν for which $(\mathcal{M}, \nu) \vDash \Gamma$, we have $(\mathcal{M}, \nu) \vDash \phi$.*

> *Proof*: TODO tedious af $\square$

3 Deduction and Generalization Theorems

Let's first state the deduction theorem, which formalizes the intuition that saying ϕ_1 "implies" ϕ_2 or that ϕ_2 "follows from" ϕ_1 (for formulas ϕ_1, ϕ_2) simly means that if we assume ϕ_1 to be true, then ϕ_2 is forced to also be true. The idea of assuming a formula to be true is codified by simply taking it as an axiom.

> **(Theorem) Deduction lemma**: *Let Γ be a set of formulas and $\phi_1, \phi_2 \in \mathcal{L}_0$. Then*
> $$\Gamma \cup \{\phi_1\} \vdash \phi_2 \text{ if and only if } \Gamma \vdash (\phi_1 \to \phi_2)$$
>
> *Proof*: Let's first prove the necessary direction. Suppose that $\langle \theta_1, \cdots, \theta_n \rangle$ is a Γ-proof of $(\phi_1 \to \phi_2)$. Then
>
> $$\langle \theta_1, \cdots, \theta_{n-1}, (\phi_1 \to \phi_2), \phi_1, \phi_2 \rangle$$
>
> is a deduction from $\Gamma \cup \{\phi_1\}$ as required, and so $\Gamma \cup \{\phi_1\} \vdash \phi_2$.
> Conversely, suppose that $\Gamma \cup \{\phi_1\} \vdash \phi_2$, and let $\langle \theta_1, \cdots, \theta_n \rangle$ be a deduction from $\Gamma \cup \{\phi_1\}$ of ϕ_2. Let's prove this by induction - suppose that for any $j < n$, $\Gamma \vdash (\phi_1 \to \theta_i)$. We'll prove that $\Gamma \vdash (\phi_1 \to \theta_j)$. Because θ_j is part of a deduction, either $\theta = \phi_1$ or $\theta_j \in \Gamma \cup \Delta$ or $\theta_{i_1}, \theta_{i_2} := (\theta_{i_1} \to \theta_j)$ are in the deduction.
> In the first case, since $(\phi_1 \to \phi_1)$ is a propositional tautology, and hence we must have $\Gamma \vdash (\phi_1 \to \theta_j)$.
> In the second case, where $\theta_j \in \Gamma \cup \Delta$, we can use the fact that $(\theta_j \to (\phi_1 \to \theta_j)) \in \Delta$ is a tautology and so $\langle \theta_j, (\theta_j \to (\phi_1 \to \theta_j)), (\phi_1 \to \theta_j) \rangle$ is a deduction showing that $\Gamma \vdash (\phi_1 \to \theta_j)$.
> In the third case, where there are i_1, i_2 such that $\theta_{i_2} = (\theta_{i_1} \to \theta_j)$. By the inductive hypothesis, we can assume that $\Gamma \vdash (\phi_1 \to \theta_{i_1})$ and that $\Gamma \vdash (\phi_1 \to \theta_{i_2}) = (\phi_1 \to (\theta_{i_1} \to \theta_j))$. Because
>
> $$((\phi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\phi_1 \to \theta_{i_1}) \to (\phi_1 \to \theta_j))) \in \Delta$$
>
> intuitively we see that since $(\phi_1 \to \theta_{i_1}), (\phi_1 \to (\theta_{i_1} \to \theta_j))$ are both proven by Γ, θ_j should be too. This is made formal by the axiom above, which shows that the following
>
> $$\begin{aligned}\langle \alpha_1, \cdots, \alpha_{n-1}, \\ (\phi_1 \to \theta_j), \\ (\phi_1 \to (\theta_{i_1} \to \theta_j)), \\ \beta_1, \cdots, \beta_{m-1}, \\ ((\phi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\phi_1 \to \theta_{i_1}) \to (\phi_1 \to \theta_j))), \\ ((\phi_1 \to \theta_{i_1}) \to (\phi_1 \to \theta_j)), \\ (\phi_1 \to \theta_j) \rangle\end{aligned}$$
>
> is a Γ-deduction for $(\phi_1 \to \theta_j)$, where $\langle \alpha_1, \cdots, \alpha_{n-1} \rangle$ is a deduction for $(\phi_1 \to \theta_{i_1})$ and $\langle \beta_1, \cdots, \beta_{m-1} \rangle$ is a deduction for $(\phi_1 \to (\theta_{i_1} \to \theta_j))$.
> Thus, the theorem follows by induction. $\square$

Next, let's investigate the conditions under which we can generalize a proof, in the sense of extending the result $\Gamma \vdash \phi$ to $\Gamma \vdash (\forall x_i \phi)$. That is, if we can prove ϕ then under what conditions can we prove that ϕ holds for every variable x_i? This is the case when the deduction proving ϕ doesn't depend on or assert any statements about x_i; if none of our axioms have any statements about or requirements on x_i, and ϕ contains x_i, then proving ϕ from these axioms is as good as proving ϕ for any arbitrary value of x_i, and so we ought to be able to prove $(\forall x_i \phi)$. In other words, if none of the axioms on which the

Γ-proof for ϕ relies make any statement on x_i, then ϕ's dependence on x_i is completely arbitrary. We can start the proof by selecting any arbitrary x_i and proving that ϕ holds for that value, and the result is equivalent to proving ϕ for every possible value of x_i. The theorem therefore codifies the intuition that proving a statement for an arbitrary member of some domain is equivalent to proving the statement over the entire domain.

> **(Theorem) Generalization theorem**: *Let Γ be a set of formulas and $\phi \in \mathcal{L}_0$ such that $\Gamma \vdash \phi$. If x_i is a variable not appearing in any formula of Γ, then $\Gamma \vdash (\forall x_i \phi)$.*
>
> *Proof*: We prove this by induction on the length of the Γ-proof. Let $\langle \theta_1, \cdots, \theta_n \rangle$ be a Γ-proof of ϕ, and suppose as the inductive hypothesis that for $j < n$, $\Gamma \vdash (\forall x_i \theta_j)$.
>
> If $\phi \in \Gamma$ then because $(\phi \to (\forall x_i \phi)) \in \Delta$, $\langle \phi, (\phi \to (\forall x_i \phi)), (\forall x_i \phi) \rangle$ is a Γ-proof of $(\forall x_i \phi)$.
>
> If instead there are i_1, i_2 such that $\theta_{i_2} = (\theta_{i_1} \to \phi)$ then by the inductive hypothesis $\Gamma \vdash (\forall x_i \theta_{i_1})$ and $\Gamma \vdash (\forall x_i (\theta_{i_1} \to \phi))$, and since quantifiers split over implications, $\Gamma \vdash ((\forall x_i \theta_{i_1}) \to (\forall x_i \phi))$. Hence, we can concatenate the deductions for $(\forall x_i \theta_{i_1})$, $(\forall x_i (\theta_{i_1} \to \phi))$, $((\forall x_i (\theta_{i_1} \to \phi)) \to ((\forall x_i \theta_{i_1}) \to (\forall x_i \phi)))$ to obtain a deduction for $(\forall x_i \phi)$. $\square$

Finally, we wrap up this section by proving a last theorem related to the previous generalization theorem. Whereas the generalization theorem cements the equivalence between proofs containing arbitrary members of a domain and proofs that make use of ranges over the whole domain, the theorem of constants below makes the related statement that when we prove a formula regarding a constant from a domain, then as long as the constant was arbitrary, it's equivalent to proving the statement for arbitrary values in the domain as well. We can swap out the constant for a variable, and since the constant was arbitrary to begin with we can prove the resulting substituted formula for all values of the variable.

> **(Theorem) Theorem of constants**: *Let Γ be a set of formulas and ϕ be a formula such that $\Gamma \vdash \phi$. If c_i is a constant not appearing in any formula of Γ, and x_j is a variable substitutable for c_i in ϕ, then*
> 1. *$\Gamma \vdash (\forall x_j \phi(c_i; x_j))$*
> 2. *We can find a Γ-proof $\langle \phi_1, \cdots, \phi_n \rangle$ of the above formula such that none of the ϕ_m contain c_i, and further if ϕ_m does contain some other constant c_k then either c_k occurs in $(\forall x_j \phi(c_i; x_j))$ or it occurs in some formula of Γ.*
>
> *Proof*: TODO $\square$

The second part of the theorem tells us that the proof of the substitution $(\forall x_j \phi(c_i; x_j))$ doesn't contain any instances of c_i; of course, why would it, since c_i doesn't show up in the axioms or in the conclusion? Further, it tells us that the only constants the proof does make use of are those that indeed can be found in the axioms or in the conclusion. All other constants are redundant and hence unnecessary in the proof.

4   The Henkin Property

5   Extensions of Consistent Sets of $\mathcal{L}$ Formulas

6   The Godel Completeness Theorem

7   The Craig Interpolation Theorem