(a) Construct a bond graph model incorporating the dynamic elements shown and assign a power convention.

(b) State your assumptions about more realistic dynamic effects and include them in your model.

4-23 Install the shock absorber model from Problem 4-22 into the quarter car model of Problem 4-20. Replace the damper with the shock model.

5

STATE-SPACE EQUATIONS

5.1 STANDARD FORM FOR SYSTEM EQUATIONS

One of the most remarkable features of bond graphs is that a study of equation formulation may be carried out prior to writing any equations. To understand how this can be so, we first consider some particular forms of equations that are used to represent a system, and select one form—the state-space type—as our goal.

Briefly, we may state that there are two limit forms for systems of differential equations, plus a wide range of possibilities within these limits. An nth-order system may be represented by

1. a single nth-order equation in terms of one unknown variable;
2. n first-order coupled equations in terms of n unknown variables; or
3. various combinations of unknowns and equations of appropriate orders (not necessarily equal).

Many important mathematical problems, methods, and results are organized in terms of the first form. At first, almost all engineering mathematics was cast in that form. The second form has certain advantages that have recommended it to mathematicians* for development of theory, but, even more important from our point of view, it is a convenient form for use by engineering system analysts, control engineers, and people conducting digital and analog computer studies. An interesting example of the third form is to be found in the sets of second-order equations generated by the Lagrangian approach to system analysis.

*See, for example, G. D. Birkhoff, *Dynamical Systems*, Providence: Amer. Math. Soc., 1966.

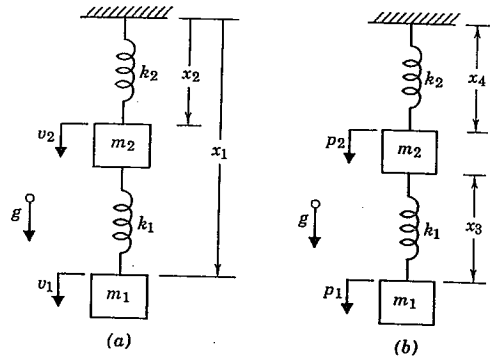

FIGURE 5.1. Mechanical double-oscillator example.

To illustrate each of the forms, and to give an appreciation of some of the differences, let us consider the mechanical double-oscillator example shown in Figure 5.1. There are several choices of unknowns available. Let us choose x_2 and express the system behavior in terms of that single displacement. The system equation in terms of x_2 turns out to be

$$\dddot{x}_2\left[\frac{k_2}{m_2}+k_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)\right]\ddot{x}_2+\frac{k_1k_2}{m_1M_2}x_2=k_1\left(\frac{q}{m_1}+\frac{1}{m_2}\right)g. \tag{5.1}$$

If a Lagrangian approach is used, with x_3 and x_4 (the spring extensions) shown in Figure 5.1*b* as the generalized coordinates, the following pair of coupled second-order equations may be developed:

$$\begin{aligned}\ddot{x}_3+k_1\left(\frac{1}{m_1}+\frac{1}{m_2}\right)x_3-\frac{k_2}{m_2}x_4&=0,\\ \ddot{x}_4-\frac{k_1}{m_2}x_3+\frac{k_2}{m_2}x_4&=g.\end{aligned} \tag{5.2}$$

Finally, if one concentrates closely upon the energy in the system, associating a momentum or displacement variable with each distinct energy element, a set of four coupled first-order equations in terms of p_1, p_2, x_3, *and* x_4 may be found, as shown in Eq. (5.3). Clearly, this set could also be converted to a geometric-variable form by replacing the momentum variables by the velocities to which they are related:

$$\begin{aligned}\dot{p}&=-k_1x_3+m_1g, \qquad \dot{p}_2=k_1x_3-k_2x_4+m_2g,\\ \dot{x}_3&=\frac{p_1}{m_1}-\frac{p_2}{m_2}, \qquad \dot{x}_4=\frac{p_2}{m_2}.\end{aligned} \tag{5.3}$$

It is possible, in principle, to transform from one of the given forms to any other, whether the system is linear or nonlinear. However, for nonlinear systems the desired transformations may be very difficult to achieve. Furthermore, if one starts with the form (5.1), for example, choosing additional unknowns is a rather haphazard process unless one has considerable insight into the system being studied. On the other hand, if a physically meaningful set of variables is already available [e.g., as displayed by (5.3)], elimination of unwanted variables can be carried out with considerable insight.

In studying engineering systems using bond graphs, there is an ideal opportunity to start the formulation in terms of significant physical variables and to generate simultaneous sets of first-order equations from the bond graphs. This is the approach we shall follow. When the system being studied is nonlinear, the form we seek is given by

$$\begin{aligned}\dot{x}_1(t)&=\phi_1(x_1,x_2,\ldots,x_n;u_1,u_2,\ldots,u_r),\\ \dot{x}_2(t)&=\phi_2(x_1,x_2,\ldots,x_n;u_1,u_2,\ldots,u_r),\\ &\vdots\\ \dot{x}_n(t)&=\phi_n(x_1,x_2,\ldots,x_n;u_1,u_2,\ldots,u_r),\end{aligned} \tag{5.4}$$

where the x_i are the *state* variables, the $\dot{x}_i$ are the time derivatives of the x_i, the u_i are *inputs* to the system, and the ϕ_i are a set of static (or algebraic) functions.*

If the system is linear, Eqs. (5.4) take on a simpler form, as shown below:

$$\begin{aligned}\dot{x}_1(t)&=a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n+b_{11}u_1+b_{12}u_2+\cdots+b_{1r}u_r,\\ \dot{x}_2(t)&=a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n+b_{21}u_1+b_{22}u_2+\cdots+b_{2r}u_r,\\ &\vdots\\ \dot{x}(t)&=a_{n1}x_1+a_{n2}x_2+\cdots+a_{nn}x_n+b_{n1}u_1+b_{n2}u_2+\cdots+b_{nr}u_r,\end{aligned} \tag{5.5}$$

where the a_{ij} and b_{ij} are constants in most cases. For a linear *time-varying* system the a_{ij} and b_{ij} may vary with time, but they must not depend on the x-variables.

Our task in the rest of this chapter is to learn how to select significant system variables from among the many possible ones available and to organize them in relations of the form (5.4) or (5.5), as appropriate.

5.2 AUGMENTING THE BOND GRAPH

Before writing any equations it is desirable to prepare the bond graph with additional information that will make the writing of equations take on a very orderly pattern. The major steps are threefold:

*This simply means that, given the values of the arguments on the right-hand side of Eq. (5.4), a set of values for the derivatives may be found by algebraic means.

1. name all the bonds in the graph (i.e., by numbering them consecutively);
2. assign to each bond a reference power direction; and
3. assign to each bond a causal sense for the pair of variables e and f.

Naming the bonds means that it is possible to refer to every variable in the system directly and unambiguously (e.g, e_4, f_7, p_3, q_{11}).

Assigning power directions, if not done in the modeling process itself, may be done in one of two basic ways: either choose a reference direction in the original system for every power variable (e.g., voltage and current) and transfer to the bond graph the reference power directions thereby implied or put the directions on the bond graph directly and interpret the implications in the original system as the need arises. Experience will show that often the latter approach is to be preferred for its ease and simplicity, as well as its high correspondence with typical engineering practice in selecting reference directions.

As an illustration of steps 1 and 2, we shall label and assign reference powers to the bond graph model of the circuit in Figure 5.2. Part a of the figure shows an unlabeled circuit, and part b shows a corresponding bond graph model obtained by inspection. In Figure 5.2c the bonds have been labeled by numbering them from 1 to 6 in arbitrary order. Now we may conveniently refer to resistance element 3, for example, or the indicator current i_2. We may also refer to the charge q_5, which is the charge associated with capacitance element 5.

In Figure 5.2d a set of power reference directions has been chosen directly on the bond graph. They seem reasonable in the sense that, if all powers are positive at some instant of time, then energy is flowing from the voltage source into each of

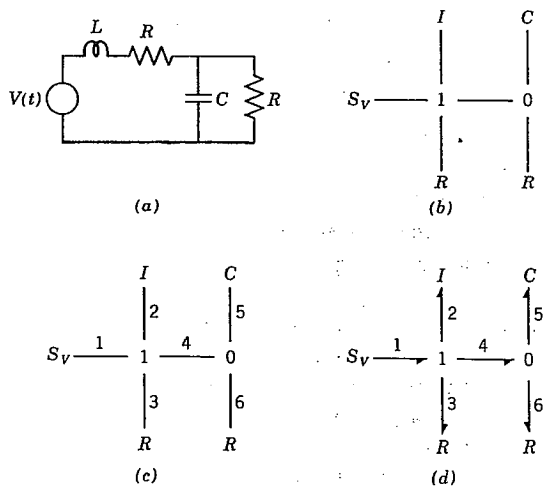

FIGURE 5.2. Partially augmented bond graph.

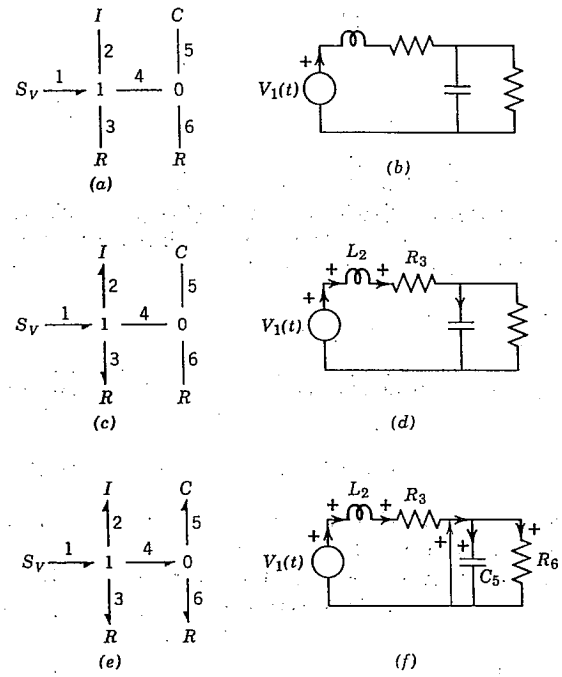

FIGURE 5.3. Correspondence between a directed bond graph and an oriented circuit.

the storage elements ($I2$ and $C5$), out of the system through resistance elements ($R3$ and $R6$), and generally "down the system from left to right."

To give further insight into the correlation between the physical variables and their bond graph representation, Figure 5.3 develops a set of oriented voltages and currents corresponding to the directed bond graph of Figure 5.2d. In part a of Figure 5.3 the bond graph is undirected except for bond 1, which shows power taken positive from the source into the system. Part b indicates a corresponding choice of voltage and current orientations for the source branch. In part c bonds 2 and 3 have been directed; part d gives corresponding branch orientations for $I2$ and $R3$. It is important to note that the branch current directions must be the same as that for $V1$ because the 1-junction connection among the several elements implies that they have a common current. In Figure 5.3e the remaining bonds are directed. Given the current direction associated with the 1-junction, we choose a voltage orientations, as shown in part f by the arrow, to correspond to the bond 4 power. The 0-junction then indicates the branch voltage orientation for branches 5 and 6. Finally, the directions on bonds 5 and 6 lead directly to the current orientations on the branches $C5$ and $R6$.

It is rarely necessary to transfer every single bond direction to the circuit. Usually

a few key orientations are sufficient. Furthermore, common patterns of bond direction and circuit branch orientation occur repeatedly, making the transfer very routine with practice. Similar remarks and interpretations may be made for bond graph directions and mechanical variable (i.e., force and velocity, torque and angular velocity) orientation, as well as all other power variable types. To a certain extent the visualization and selection of variable orientations is a personal matter, and we shall not pursue the matter further in a formal fashion.*

At this point, it should be noted that system equations can be written for a bond graph that has been labeled and directed. For a graph with N bonds there are $2N$ bond variables (N efforts and N flows). Each n-port implies n constraints among its associated bond variables. Each bond is adjacent to two multiports. From basic graph theory we can find that the number of bond variables and constraints are always equal (provided there are no open bonds). It is possible to write down all the constraints for each multiport and arrive at a pile of system equations, essentially unsorted and unorganized, but correct. Manipulation of the resulting equations to obtain insight and to achieve a useful form is very difficult in a complicated system in the absence of some algorithm. One reason is the excessive number of variables used in the initial formulation.

For example, in Figure 5.3*e* the bond graph has 6 bonds, hence 12 bond variables ($v_1, \ldots, v_6, \ldots, i_1, \ldots, i_6$). The number of constraints imposed by the elements is as follows: $SE1$ imposes one, $I2$ imposes one (dynamic), $R3$ imposes one, **1** 1, 2, 3, 4 imposes four (three identities and one summation), **0** 4, 5, 6 imposes three (two identities and one summation), $C5$ imposes one (dynamic), and $R6$ imposes one. Try writing 12 relations among the 12 bond variables and then reducing them to a pair of coupled first-order differential equations. There must be a better way! There is, and we will discuss it next.

The third step in the augmentation process is the *assignment of causality*. Basic considerations of causality for the various elements were introduced earlier (Section 3.4). It is now appropriate to apply such information to the entire system in a orderly fashion.

In a causal sense there are two distinct types of bond graph elements. Source (elements S_e and S_f) and junction elements (0, 1, TF, and GY) must meet certain causal conditions or their basic definitions are no longer valid. As an obvious example, if a source of force does not have a causality showing that it defines a force on the system to which it is connected, then it has no meaning. We generalize this observation and assert that *every source element must have its appropriate causal form assigned to it.*

For each of the 2-port junction elements, TF and GY, there are two possible causal forms that preserve the basic definition of the element. If neither of these forms can be assigned, the concept of input and output associated with the element is not valid. Consequently we assert that *every TF and GY must have one of its two allowable causal forms assigned to it.* The choice of form will generally be indicated by the adjoining system on the basis of other considerations to be discussed.

In a similar vein we argue with respect to the ideal multiport junctions 0 and 1 that *each 0 and 1 must have one of its appropriate causal forms assigned to it* or the basic definition of the particular element will not be valid. The selection of particular causal form will be motivated by other system considerations in general.

If a system cannot meet the causal conditions outlined above, the basic physical model on which the bond graph is based must be restudied. The indications are that an impossible situation has been created that is not capable of sensible mathematical resolution. Two such examples are shown in Figure 5.4 In part *a*, source element 2 is defined invalidly. Inspection of an electrical circuit interpretation indicates the nature of the difficulty; obviously, a modeling error has been made in putting two supposedly independent current sources in series. In part *c* of Figure 5.4, a TF is found to have invalid causality. If this bond graph were derived from a fluid circuit, as shown in part *d*, the interpretation would be that two independent pressure sources has been joined by an ideal transformer of pressure (the TF), leading to a physically incompatible situation. The next step is up to the system modeler, who must correct the model in an appropriate way.

Continuing our discussion of assigning causality, we come to the C- and I-elements. The energy variables on these elements (p on I, q on C) are the basis of the system state variables. That is, a knowledge of the values of a necessary and sufficient set (i.e., just enough and not too many), together with the inputs, will enable us to predict the system response in time. In an intuitive way we observe that the energy variables can be used to determine the system energy and its distribution, and the bond graph structural constraints serve to define the powers that cause the energy to flow subject to input conditions, resulting in the particular system dynamics.

As we shall soon see, it is not always possible to make all C- and I-elements have integration causality. If a storage element has differentiation causality forced

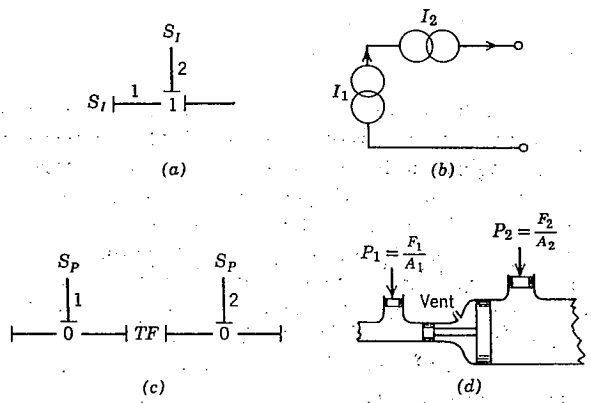

FIGURE 5.4. Two examples of invalid causality and physical interpretations.

*For complex systems, the assignment of sign conventions is only partly arbitrary, with certain forms being impossible for a given physical system. If a complete set of effort and flow conventions is chosen for the physical system and then the information is transferred to the bond graph, no ambiguities will arise.

upon it, the meaning is that its energy variable is not algebraically independent of other energy variables and source constraints. Hence, it is not an independent state variable and can be eliminated from the final state equations.

At the system level the causality associated with R elements is largely a matter of indifference. The major exception is in the case of a nonlinear constitutive law which is not bi-unique (e.g., Coulomb friction). Then the causality associated with a unique input–output relation for the element should be used. Otherwise, R-elements accept whatever causality they are assigned (and are grateful for it).

The basic causality assignment procedure is summarized below.

Sequential Causal Assignment Procedure

1. Choose any source (SE, SF), and assign its required causality. Immediately extend the causal implications through the graph as far as possible, using the constraint elements (0, 1, GY, TF).
2. Repeat step 1 until all sources have been used.
3. Choose any storage element (C or I), and assign its preferred (integration) causality. Immediately extend the causal implications through the graph as far as possible, using the constraint elements (0, 1, GY, TF).
4. Repeat step 3 until all storage elements have been assigned a causality. In many practical cases all bonds will be causally oriented after this stage. In some cases, however, certain bonds will not yet have been assigned. We then complete the causal assignment as follows:
5. Choose any unassigned R-element and assign a causality to it (basically arbitrary). Immediately extend the causal implications through the graph as far as possible, using the constraint elements (0, 1, GY, TF).
6. Repeat step 5 until all R-elements have been used.
7. Choose any remaining unassigned bond (joined to two constraint elements), and assign a causality to it arbitrarily. Immediately extend the causal implications through the graph as far as possible, using the constraint elements (0, 1, TF, GY).
8. Repeat step 7 until all remaining bonds have been assigned.

The procedure is straightforward and orderly. Some practice on examples will convince you of the ease and rapidity with which causality can be assigned. It is important to recognize that the constraint elements represent the physical structural ties in the system (e.g., Kirchhoff's voltage and current laws; Newton's law and geometric compatibility), and assigning causality to them means that they will be used correctly in a particular input–output fashion. Further discussion of the use and interpretation of causality in bond graphs is given in Chapter 7.

There are several situations that can arise when applying causality according to the procedure given. They are:

1. All storage elements have integration causality, and the graph is complete after step 4. This simple, common case is discussed in Section 5.3.
2. Causality is completed by using R-elements or bonds, as indicated in steps 5–8. This situation is discussed in Section 5.4.
3. Some storage elements are forced into differentiation causality at step 3. This case is discussed in Section 5.5.

Now let us consider several examples of using the procedure.

Assignment of causality is carried out step by step in the example in Figure 5.5. Part *a* shows a labeled bond graph without causality. A graph without causality is sometimes said to be *acausal*. In part *b* bond 1 is directed according to the meaning of the source element (a source of flow). Since the 0-junction has only one flow variable determined, the other bonds cannot yet be assigned a causality. There are no more sources, so we turn to step 3. In part *c* bond 2 is causally directed in view of the integration causality for the C-element. Immediately, bond 3 may be directed because of the 0-junction, which can have only one effort input. However, bonds 4 and 5 are as yet undirected. Bond 5 is directed, as shown in Figure 5.5*d*, to give integration causality to the I-element. Immediately bond 4 can (nay, must) be causally directed as shown, due to the 1-junction, which can have only one flow input. In the grand finale of part *e* we have added a set of power directions to the graph; the result is a completely augmented bond graph. That is, the bonds are labeled, power direc-

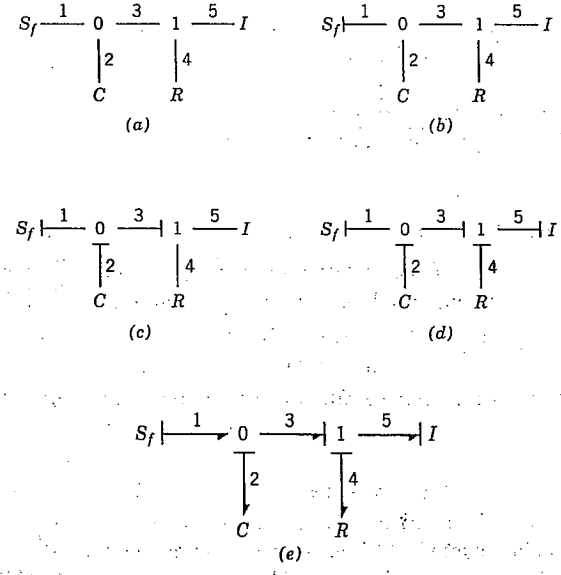

FIGURE 5.5. Causality assignment and complete augmentation of a bond graph. Example 1.

tions have been chosen, and causality has been assigned. Such a graph will yield its state equations to us with very little resistance, as we shall show in the next section. However, let us first augment another graph or two to gain some experience.

Figure 5.6 shows a bond graph derived from a fluid example involving pipes and reservoirs. There are three pressure sources ($SE1$, $SE2$, and $SE3$) feeding through three pipes (roughly, the 1-junction complexes) into a tank (0 and $C10$). In part a the graph is labeled. In part b, bonds 1, 2, and 3 associated with the sources have been assigned causality. In each case no extension of causality is possible. Also, bond 10 has been directed according to integration causality for the element $C10$, and causality has been extended to bonds 11, 12, and 13 by the 0-junction, using the effort identity condition. In part c bond 4 has been directed; consequently, so has bond 7 due to the 1-junction, and so on for bonds 5 and 8 and bonds 6 and 9. Finally, power directions have been added to produce the fully augmented graph shown in part d. In the next section, we shall show how this graph yields four first-order equations with three inputs to the system.

As a final example in this section, consider the acausal bond graph in Figure 5.7a. The model is derived from a study of a pressure-controlled valve and includes both mechanical and fluid mechanical power. The elements $SE1$ and $SE2$ represent sources of pressure and force, respectively, and the element TF couples the two power domains. In part b bonds 1 and 2 are causally assigned, one at a time. The causal information cannot be extended using constraint elements at this point. Bonds 3 and 4 are assigned next and causality is extended as shown in part c. Bond 4 causal-

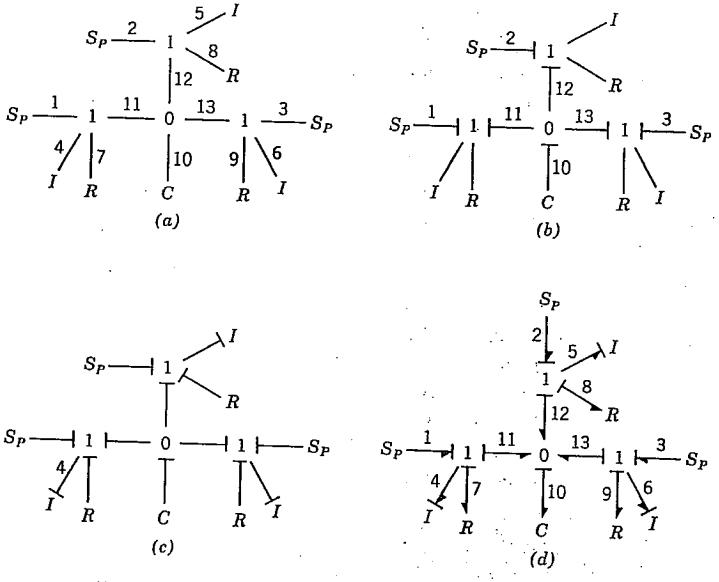

FIGURE 5.6. Augmentation of a bond graph. Example 2.

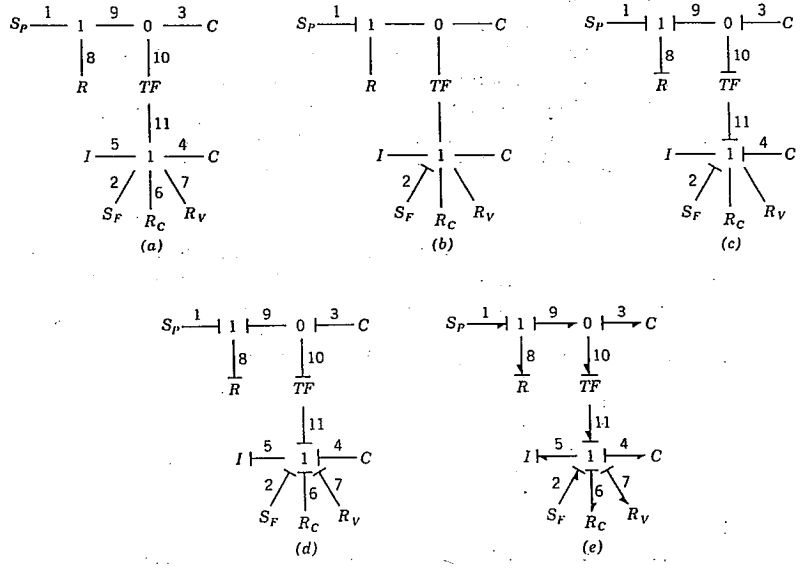

FIGURE 5.7. Augmentation of a bond graph. Example 3.

ity does not extend to other bonds, but bond 3 has implications for bonds 8 and 9, and also for bonds 10 and 11. Inspection of Figure 5.7d reveals that assigning causality to bond 5 (associated with the element $I5$) determines the causality on bonds 6 and 7. With powers assigned as shown in part e, the bond graph is completely augmented.

Two further points may be made with respect to augmentation. The first is that, in assigning causality, the results do not depend on the order of bonds chosen except in special circumstances. These will be discussed in Section 5.4. The second point is that assignment of causality and assignment of power directions are two entirely independent operations. Either one may be performed first. Typically, power directions will be first, but on occasion one may not bother to assign powers in studying aspects of system structure.

5.3 BASIC FORMULATION AND REDUCTION

Once a fully augmented bond graph model is available, the equations for the system can be developed in a very orderly fashion. Frequently, when the system is small or uncomplicated in structure, state-space equations can be written down directly. However, as system size and complexity grow, the need for an organized procedure for equation generation becomes apparent.

Very general and powerful procedures are available for producing sets of system equations. In this section we shall concentrate on a basic pattern which is applicable

in a large majority of cases encountered in engineering practice. There are three simple steps to be followed:

1. select input, energy, and co-energy variables;
2. formulate the initial set of system equations; and
3. reduce the initial equations to state-space form.

Selection of *inputs* is straightforward. For each source element write on the graph the input variable to the system. These variables will appear in the final state-space equations if they have any effect on system behavior. The list of input variables will be called U.

Selection of *state variables* is accomplished by choosing the energy variable for each storage element in the graph. When integration causality can be assigned on all C- and I-elements, we know that each energy variable is statically independent of all of the others.* We choose as our state variables the p-variable on every I-element and the q-variable on every C-element. The list of state variables will be called X. On the bond graph we write $\dot{p}$ and $\dot{q}$ on the appropriate bonds, representing the effort and flow corresponding to each p and q.

In addition to writing input and energy variables, it will prove useful to write one other set on the graph. This is the co-energy set, consisting of the f on each I and the e on each C. These variables will appear in the initial formulation and then be eliminated in the reduction process.

In Figure 5.8*a* the augmented bond graph given in Figure 5.5*e* is repeated. In part *b* of Figure 5.8 the input flow variable $F_1(t)$ is introduced. The energy variables for the examples are q_2 and p_5. Their derivatives are written on bonds 2 and 5, respectively, as shown in part *c*. Finally, the co-energy variables, e_2 and f_5, are also written on bonds 2 and 5, respectively, resulting in the graph of part *d*. From this graph, the equations will be written.

The lists are identified as follows:

$$X = \begin{bmatrix} q_2 \\ p_5 \end{bmatrix} \quad \text{and} \quad U = [F_1].$$

The formulation result we seek is, for a linear system,

$$\dot{q}_2 = a_{11}q_2 + a_{12}p_5 + b_1F_1, \qquad \dot{p}_5 = a_{21}q_2 + a_{22}p_5 + b_2F_1, \tag{5.6}$$

or, if the system is nonlinear,

$$\dot{q}_2 = \phi_1(q_2, p_5; F_1), \qquad \dot{p}_5 = \phi_2(q_2, p_5; F_1), \tag{5.7}$$

as indicated by Eq. (5.5) or (5.4) with $n = 2$ and $r = 1$.

*The case when some storage elements have differentiation causality is discussed in Section 5.5.

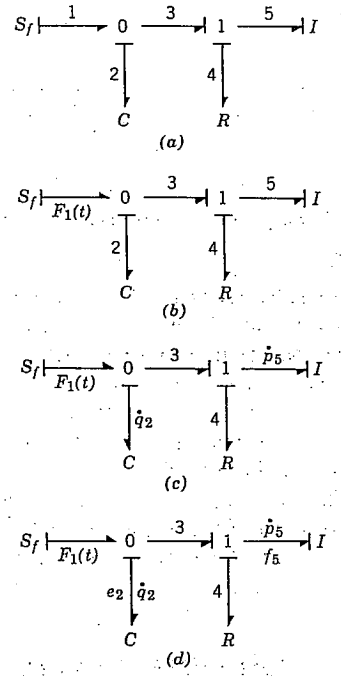

FIGURE 5.8. Identification of key variables (input, energy, and co-energy). Example 1.

Our plan is first to write the constitutive relations for the storage elements in appropriate form (i.e., for $C2$ and $I5$). Then we shall write equations for the derivatives of the energy variables, which are bond variables, and try to express them in terms of inputs and co-energy variables. These relations represent the structural constraints of the system, as well as dissipation. This will yield an initial set of equations, which we shall then reduce to state-space form.

Consider element $C2$ of Figure 5.8*d*. Its constitutive equation in linear form is

$$e_2 = \frac{1}{C_2}q_2. \tag{5.8a}$$

For element $I5$ the linear constitutive equation is

$$f_5 = \frac{1}{I_5}p_5 \tag{5.8b}$$

Now the co-energy variables are expressed in terms of the energy variables and subsequently may be replaced by them.

Turning now to the structure, we seek an expression for $\dot{q}_2$, which is the flow on bond 2. Using the 0-junction constraint and *following the causality assigned*, we get

$$\dot{q}_2 = F_1(t) - f_3. \tag{5.9a}$$

[Look at the causal strokes. They indicate that $\dot{q}_2$ is an output of the 0-junction and F_1 and f_3 are inputs. The signs in Eq. (5.9a) follow from the sign half arrows.] But the causality further indicates that f_3 is expressible in terms of the co-energy variable f_5 using the 1-junction. So

$$\dot{q}_2 = F_1(t) - f_5. \tag{5.9b}$$

To eliminate f_5 in favor of a state variable, we can use Eq. (5.8b). The result is

$$\dot{q}_2 = F_1(t) - \frac{1}{I_5} p_5, \tag{5.9c}$$

which involves only X- and U-variables.

Let us develop an equation for the other state variable, p_5, in terms of its structural relations. Starting with bond 5 in Figure 5.8*d*, we identify $\dot{p}_5$ as the effort. So, following causality,

$$\dot{p}_5 = e_3 - e_4. \tag{5.9d}$$

using the effort summation relation of the 1-junction and the sign convention shown. But the causal strokes also show that e_3 may be replaced by e_2 using the 0-junction. This yields

$$\dot{p}_5 = e_2 - e_4. \tag{5.9e}$$

Referring to the causality on bond 4, we see that e_4 may be replaced by its f_4 equivalent, as indicated by the R-causality. Where does f_4 come from? The 1-junction causality indicated f_5 as the determiner of f_4. This sequence is

$$\dot{p}_5 = e_2 - R_4 f_4 = e_2 - R_4 f_5. \tag{5.9f}$$

Since e_2 and f_5 are both co-energy variables (this is no accident), they may be replaced in Eq. (5.9f) by their equivalents in Eqs. (5.8a) and (5.8b). Thus,

$$\dot{p}_5 = \frac{1}{C} q_2 - R_4 \frac{1}{I_5} p_5. \tag{5.9g}$$

Now $\dot{p}_5$ is expressed in terms of X- and U-variables only.

As a final step, it is useful to order the terms of Eqs. (5.9c) and (5.9g) according to the order of the elements X and U. This is good practice and leads to a natural representation by matrices in the linear case. Then the desired state equations become

$$\begin{aligned} \dot{q} &= -\frac{1}{I_5} p_5 + F_1(t), \\ \dot{p}_5 &= \frac{1}{C_2} q_2 - \frac{R_4}{I_5} p_5. \end{aligned} \tag{5.10}$$

To emphasize the pattern, all zero terms have been implied by the spacing. Compare Eqs. (5.10) to Eqs. (5.5) with $n = 2$ and $r = 1$ to see that the basic information we have derived is the sets of coefficients a_{ij} and b_{ij}.

At this point, you might well ask yourself what would happen to the formulation if some constitutive laws were nonlinear. Retrace the formulation steps to see that the same pattern holds but the manipulation is not as simple as for the linear system. For example, let us assume that elements $C2$ and $I5$ are nonlinear, with relations

$$e_2 = \phi_2^{-1}(q_2), \tag{5.11a}$$

$$f_5 = \phi_5^{-1}(p_5). \tag{5.11b}$$

Since the structural equations are unchanged, we can turn directly to Eqs. (5.9b) and (5.9f) and substitute Eqs. (5.11). The result is

$$\begin{aligned} \dot{q}_2 &= F_1(t) - \phi_5^{-1}(p_5), \\ \dot{p}_5 &= \phi_2^{-1}(q_2) - R_4 \phi_5^{-1}(p_5). \end{aligned} \tag{5.12}$$

The equations (5.12) may be compared with (5.4) to see that we have attained our objective.

A second example is shown in Figure 5.9*a*, which is the augmented bond graph of Figure 5.3*e*. The graph with inputs, energy variables, and co-energy variables identified is shown in Figure 5.9*b*. The input list U contains $V_1(t)$; the energy variable list X contains λ_2; and the co-energy variables are i_2 and v_5.

For convenience, let us assume that the storage and dissipation elements are linear with constant coefficients. Then we may write the constitutive laws for $I2$ and $C5$ as

$$i_2 = \frac{1}{L_2} \lambda_2, \qquad v_5 = \frac{1}{C_5} q_5, \tag{5.13}$$

respectively, where L_2 is the inductance and C_5 the capacitance in the electric circuit in Figure 5.3*f*.

The derivatives of the energy variables are given by the structural equations (5.14c) and (5.14f), which are written by following the implications of the causal strokes and the sign convention until the state-variable derivatives are given in terms of co-energy and input variables:

$$\dot{\lambda}_2 = V_1(t) - v_3 - v_4, \tag{5.14a}$$

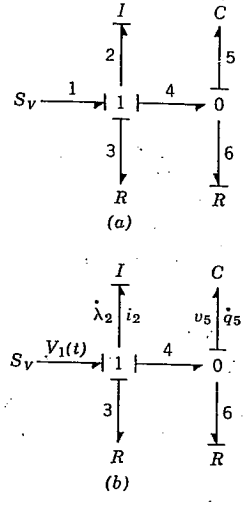

FIGURE 5.9. Identification of key variables. Example 2.

or

$$\dot{\lambda}_2 = V_1(t) - R_3 i_3 - v_5, \tag{5.14b}$$

or

$$\dot{\lambda}_2 = V_1(t) - R_3 i_2 - v_5, \tag{5.14c}$$

and

$$\dot{q}_5 = i_4 - i_6, \tag{5.14d}$$

or

$$\dot{q}_5 = i_2 - \frac{1}{R_6} v_6, \tag{5.14e}$$

or

$$\dot{q}_5 = i_2 - \frac{1}{R_6} v_5. \tag{5.14f}$$

Following the typical reduction pattern, we use (5.13) to eliminate i_2 and v_5 in Eqs. (5.14c) and (5.14f) to obtain unordered state-space equations in terms of the energy variables, given by

$$\dot{\lambda}_2 = V_1(t) - \frac{R_3}{L_2}\lambda_2 - \frac{1}{C_5} q_5, \qquad \dot{q}_5 = \frac{1}{L_2}\lambda_2 - \frac{1}{R_6 C_5} q_5. \tag{5.15}$$

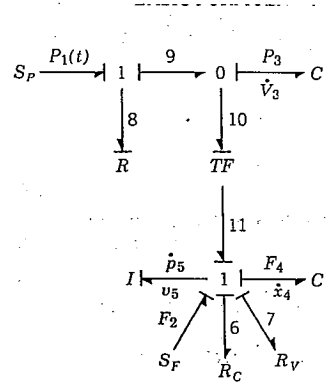

FIGURE 5.10. Identification of key variables. Example 3.

When the terms in Eqs. (5.15) are ordered according to X and U, with zero terms implied, we get

$$\dot{\lambda}_2 = -\frac{R_3}{L_2}\lambda_2 - \frac{1}{C_5} q_5 + V_1(t), \qquad \dot{q}_5 = \frac{1}{L_2}\lambda_2 - \frac{1}{R_6 C_5} q_5. \tag{5.16}$$

One more example should be sufficient to demonstrate the power and consistency of the formulation and reduction pattern in terms of energy variables. Consider the augmented bond graph for a pressure-controlled valve shown in Figure 5.7*e*. Bonds 1, 3, 8, 9, and 10 represent fluid power and have P, Q variables. Bonds 2, 4, 5, 6, 7, and 11 represent mechanical power and have F, V variables.* The TF has an area modulus A that couples the two power domains. Figure 5.10 shows the augmented bond graph of Figure 5.7*e* with input, energy, and co-energy variables identified. We anticipate a third-order system in terms of the X-variables: V_3, a fluid volume; x_4, a spring extension; and p_5, a momentum. The inputs U are $P_1(t)$, a control pressure, and $F_2(t)$, a loading force.

The X and U lists are

$$X = \begin{bmatrix} V_3 \\ X_4 \\ p_5 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} P_1 \\ F_2 \end{bmatrix}.$$

To begin, we write the constitutive equations in the order of X for $C3$,

$$P_3 = \phi_3(V_3); \tag{5.17a}$$

for $C4$,

$$F_4 = \phi_4(x_4); \tag{5.17b}$$

*Note that V_3 is a volume and v_5 is a velocity. If you do not like using the letter V for voltage, volume, and velocity, try using f for all flows and e for all efforts.

and for $I5$,

$$V_5 = \frac{1}{m_5} p_5. \tag{5.17c}$$

In the above expressions both $C3$ and $C4$ are assumed to be nonlinear, but the mass is taken as linear.

Next we write the X-derivative equations for the structure, of which there are three, by following the causal strokes and the sign convention. We always follow the causal marks until $\dot{X}$ is expressed in terms of input or co-energy variables:

$$\dot{V}_3 = Q_9 - Q_{10}, \tag{5.18a}$$

or

$$\dot{V}_3 = Q_8 - Av_{11}, \tag{5.18b}$$

or

$$\dot{V}_3 = \frac{1}{R_8}(P_1(t) - P_3) + Av_5; \tag{5.18c}$$

$$\dot{x}_4 = v_5, \tag{5.18d}$$

$$\dot{p}_5 = F_2(t) + F_{11} - F_6 - F_7 - F_4, \tag{5.18e}$$

$$\dot{p}_5 = F_2(t) + AP_{10} - R_6 v_6 - R_7 v_7 - F_4, \tag{5.18f}$$

$$\dot{p}_5 = F_2(t) + AP_3 - (R_6 + R_7)v_5 - F_4. \tag{5.18g}$$

Finally, using Eqs. (5.17), we obtain

$$\dot{V}_3 = \frac{1}{R_8} P_1(t) - \frac{1}{R_8}\phi_3(V_3) - \frac{A}{m_5} p_5, \tag{5.19a}$$

$$\dot{x}_4 = \frac{1}{m_5} p_5, \tag{5.19b}$$

$$\dot{p}_5 = F_2(t) + A\phi_3(V_3) - \phi_4(x_4) - \frac{R_6 + R_7}{m_5}\dot{p}_5. \tag{5.19c}$$

Inspection of these state-space equations shows that Eq. (5.19a) is a volume flow equation (in terms of its physical dimension), Eq. (5.19b) is a velocity equation, and Eq. (5.19c) is a force equation. As a check on the correctness of the final equations, it is useful to work out the dimensions of all the terms to make sure that they are consistent.

Summarizing the contents of this section, we recall that by starting with an augmented bond graph with the inputs, energy variables, and co-energy variables identified, it is possible to write two sets of equations initially—the constitutive relations and the structural relations—and then to reduce them to state-space form in terms of the energy variables.

The balance of this chapter is directed to exploring further aspects of formulation and reduction, including such questions as the following:

1. What if causality is not completed by source and storage elements alone?
2. What if differentiation causality shows up on certain storage elements?
3. What if we need expressions for variables other than energy variables?

5.4 EXTENDED FORMULATION METHODS—PART 1

As an example of a system in which causality is not completed by source and storage element causal assignment, consider the electric circuit and its bond graph shown in Figures 5.11*a* and *b*, respectively. The two diagrams have been labeled in direct

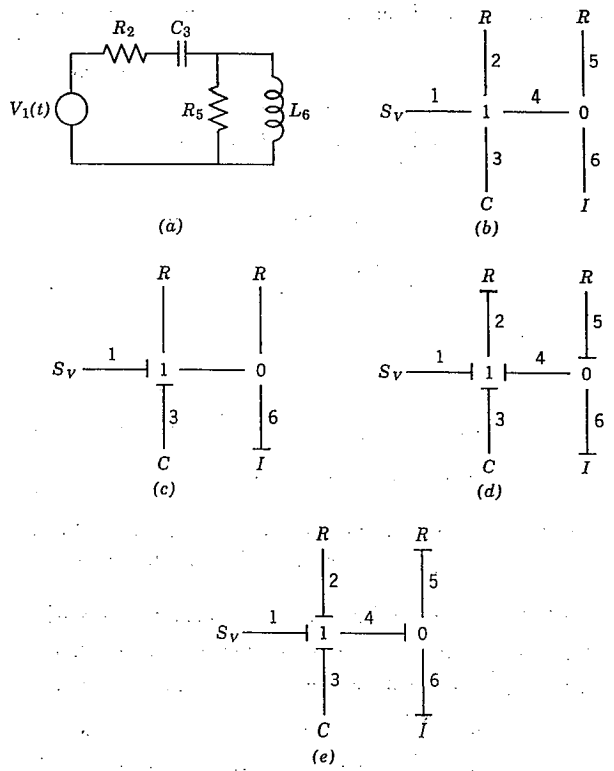

FIGURE 5.11. An example of completing causality.

correspondence. In part c causality has been assigned to bonds 1, 3, and 6. Bond 1 is determined by $SE1$, and bonds 3 and 6 by $C3$ and $I6$, respectively. No further causal extensions are possible based on source and storage elements. In order to obtain a complete causal assignment, we must choose another bond and make it causal. In part d bond 2 has been causally directed as shown. When this choice is extended, using the 1- and 0-junctions, bonds 4 and 5 are directed, and the graph is completed. To show that the causal choice is not unique, in Figure 5.11e bond 2 was directed the other way causally, yielding the graph indicated. Each of these cases, parts d and e, leads to a particular formulation of equations. The two sets are similar, but not identical, and reduce to the same state-space set as we shall show.

Referring to Figure 5.12a, which is the graph of Figure 5.11d with powers assigned and key variables labeled, including i_2, we see that

$$X = \begin{bmatrix} q_3 \\ \lambda_6 \end{bmatrix} \quad \text{and} \quad U = [V_1].$$

The set of constitutive relations is

$$v_3 = \frac{1}{C_3} q_3, \qquad i_6 = \frac{a}{L_6} \lambda_6, \tag{5.20}$$

for $C3$ and $I6$, respectively.

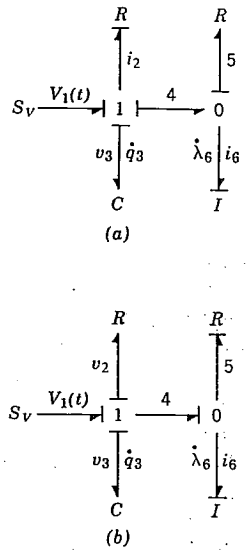

FIGURE 5.12. Two variations of an augmented bond graph.

The energy-variable derivatives are given by

$$\dot{q}_3 = i_2 \tag{5.21a}$$

and

$$\dot{\lambda}_6 = v_5 = R_5 i_5 = R_5(i_4 - i_6), \tag{5.21b}$$

or

$$\dot{\lambda}_6 = R_5(i_2 - i_6). \tag{5.21c}$$

The reason we stop at this point is that we anticipate that i_2 will be related to itself by an algebraic equation. This insight was gained in the process of assigning causality. Recall that the causal assignment of i_2 on bond 2 finally determined all other variables.

Let us see, by writing equations for i_2, that this is so. By reading the causal strokes in Figure 5.12a, the following sequence of equations is found:

$$i_2 = \frac{1}{R_2} v_2 \tag{5.22a}$$

$$= \frac{1}{R_2}[V_1(t) - v_3 - v_4], \tag{5.22b}$$

$$i_2 = \frac{1}{R_2}[V_1(t) - v_3 - v_5]. \tag{5.22c}$$

But v_3 is a co-energy variable, and v_5 is given by

$$v_5 = R_5 i_5 \tag{5.23a}$$

$$= R_5(i_4 - i_6), \tag{5.23b}$$

$$v_5 - R_5(i_2 - i_6), \tag{5.23c}$$

and i_6 is a co-energy variable. Examining Eqs. (5.22c) and (5.23c), we see that i_2 may be found in terms of the input $V_1(t)$ and the co-energy variables: v_3 and i_6 and i_2 itself. The equation is

$$i_2 = \frac{1}{R_2}[V_1(t) - v_3 - R_5(i_2 - i_6)]. \tag{5.24a}$$

Solving this algebraic equation for i_2 yields

$$i_2 = \frac{a}{R_2 + R_5}[V_1(t) - v_3 + R_5 i_6]. \tag{5.24b}$$

To complete the reduction, we put the results given by Eq. (5.24b) into Eqs. (5.21a) and (5.21c) and also use (5.20) to eliminate the co-energy variables, as we have done

before. The grand and glorious state-space results are given by

$$\dot{q}_3 = -\frac{1}{(R_2+R_5)C_3}q_3 + \frac{R_5}{(R_2+R_5)L_6}\lambda_6 + \frac{1}{R_2+R_5}V_1(t),$$
$$\dot{\lambda}_6 = -\frac{R_5}{(R_2+R_5)C_3}q_3 - \frac{R_2R_5}{(R_2+R_5)L_6}\lambda_6 + \frac{R_5}{R_2+R_5}V_1(t). \quad (5.25)$$

Now let us consider the other causal variation, as shown in the augmented graph of Figure 5.12*b*. The constitutive equations are identical to those derived before, namely, (5.20). The equations for structure will be slightly different, and again we anticipate simultaneous dependence involving an internal variable. For the state-variable derivatives we write, following the causality of Figure 5.12*b*,

$$\dot{q}_3 = i_4, \quad (5.26a)$$

or

$$\dot{q}_3 = i_5 + i_6, \quad (5.26b)$$

or

$$\dot{q}_3 = \frac{1}{R_5}v_5 + i_6 = \frac{1}{R_5}(V_1 - v_2 - v_3) + i_6, \quad (5.26c)$$

and

$$\dot{\lambda}_6 = v_4, \quad (5.26d)$$

or

$$\dot{\lambda}_6 = V_1(t) - v_2 - v_3. \quad (5.26e)$$

At this point, we recognize that an expression must be found for v_2 in terms of key system variables [$V_1(t)$, v_3, and i_6]. This is because when we chose the causality on R_2 such that v_2 was an input to the remainder of the system, all causality was thereby determined. We have

$$v_2 = R_2 i_2 = R_2 i_4 = R_2(i_5 + i_6), \quad (5.27)$$

but

$$i_5 = \frac{1}{R_5}v_5, \quad (5.28a)$$

or

$$i_5 = \frac{1}{R_5}v_4, \quad (5.28b)$$

or

$$i_5 = \frac{1}{R_5}(V_1 - v_2 - v_3). \quad (5.28c)$$

Putting Eq. (5.28) into Eq. (5.27) and solving for v_2, we get

$$v_2 = \frac{R_2}{R_2+R_5}[V_1(t) - v_3 + R_5 i_6]. \quad (5.29)$$

To eliminate the intermediate variable v_2, from Eqs. (5.26), we use the results of Eq. (5.29). This gives the following pair of equations:

$$\dot{q}_3 = \frac{1}{R_2+R_5}V_1(t) - \frac{1}{R_2+R_5}v_3 + \frac{R_5}{R_2+R_5}i_6,$$
$$\dot{\lambda}_6 = \frac{R_5}{R_2+R_5}V_1(t) - \frac{R_5}{R_2+R_5}v_3 - \frac{R_2R_5}{R_2+R_5}i_6. \quad (5.30)$$

When v_3 and i_6 are eliminated from (5.30) by use of (5.20), and the terms are re-ordered on the right-hand side of the resulting equations, the equations (5.25) are obtained, thus demonstrating that the same state-space equations in terms of energy variables may be derived in more than one way.

What we have done in these two variations of the same example is to use a single intermediate variable (i.e., i_2 in the first case and v_2 in the second) to assist in the formulation. It is possible to use more than one intermediate variable and obtain more than one algebraic equation to be solved. For example, both i_2 and v_5 could have been used in the first case. Then a pair of algebraic equations involving both variables could have been solved in the reduction to state-space form. With experience you can learn to introduce as many or few intermediate variables as you find convenient. *The minimum number of intermediate variables which must be algebraically related is equal to the number of bonds which must be causally assigned after all source and storage element causality has been completed and the resulting causal implications have been extended throughout the graph.*

At this stage in our development we defer further consideration and interpretation of the physical meaning of the causal situation with which we have been dealing. In Section 7.2 a general discussion of the subject of R-fields is to be found, and some of the reasons for the pattern described here will be put in more profound perspective.

We shall study one other variation, again by example, to further our understanding of the process of completing causality and deriving state-space equations. Consider the bond graph of Figure 5.13*a*, in which bonds are labeled. In part *b* causality is assigned to bonds 1, 4, and 7 and extended as far as possible. In order to complete causality for the graph, we arbitrarily choose to direct bond 3 as shown in part *c*. The consequence of this choice is that causality is completed on the graph, as shown in part *d*. Part *e* shows key variables labeled and powers directed so that the graph is

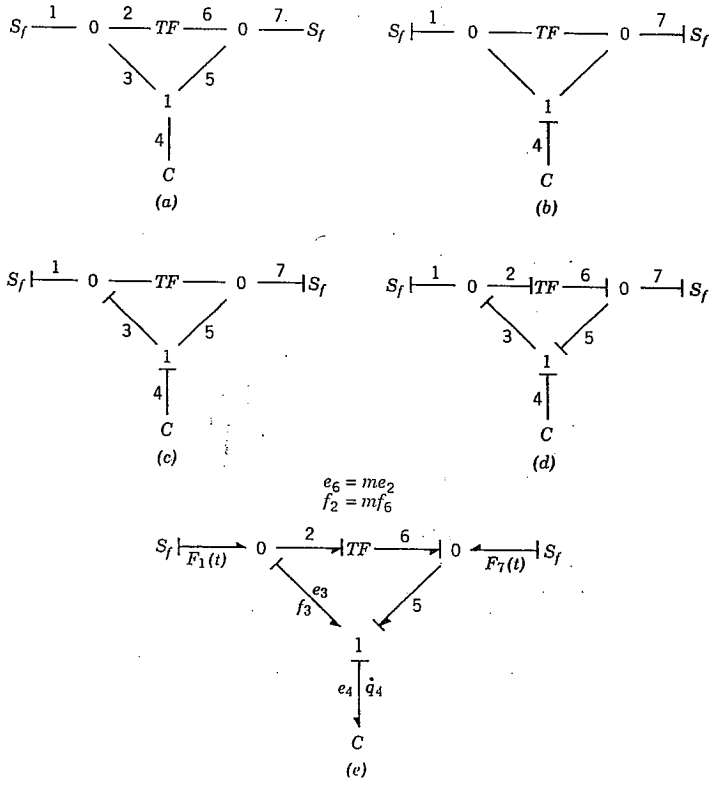

FIGURE 5.13. Completing causality by arbitrary bond assignment. An example.

augmented.* Note that e_3 and f_3 also have been written on the graph, for convenience in formulation.

The formulation begins with the constitutive equations, of which there is one, for $C4$, namely,

$$e_4 = \frac{1}{C4} q_4. \tag{5.31}$$

It continues with the state-variable derivative equations for the structure, of which there is one, namely,

$$\dot{q}_4 = f_3. \tag{5.32}$$

*Since this graph contains a loop, the sign convention is important and should really be determined from the orientations of the physical system variables.

The reason we stop here is that we anticipate an algebraic equation relating f_3 to itself, as indicated by the causality assignment process, and f_3 has been written as an intermediate variable.

Now let us seek an expression for f_3 in terms of key variables [i.e., $F_1(t)$, $F_7(t)$, e_4, f_3]:

$$f_3 = F_1(t) - f_2 \tag{5.33a}$$

$$= F_1(t) - mf_6 \tag{5.33b}$$

$$= F_1(t) - m[-F_7(t) + f_5], \tag{5.33c}$$

$$f_3 = F_1(t) - m[-F_7(t) + f_3]. \tag{5.33d}$$

Note that m is the modulus of the transformer element. In this case, Eq. (5.33d) shows that f_3 depends on itself, and we solve it explicitly to find

$$f_3 = \frac{1}{1+m} F_1(t) + \frac{m}{1+m} F_7(t). \tag{5.34}$$

The state equation is found by using Eq. (5.34) in Eq. (5.32), yielding

$$\dot{q}_4 = \frac{1}{1+m} F_1(t) + \frac{m}{1+m} F_7(t). \tag{5.35}$$

The interested reader is encouraged to experiment with other causal choices in the graph of Figure 5.13*b* and other intermediate bond variables (e.g., e_5, f_5, or e_2). The final result should be Eq. (5.35), of course.

In summarizing this section, we observe that if causality is not completed by source and storage element assignments, it may be completed by selecting appropriate bonds and causally directing them to complete the graph. In this process we may need to use one or more intermediate variables in the initial formulation of equations. The reduction to final state-space form follows an orderly pattern of elimination, as shown by several examples. Chapter 7 considers a physical interpretation of completing causality in the broader framework of R-fields and junction structures, and Reference [1] describes a formal vector-field method for generating state-space equations.

5.5 EXTENDED FORMULATION METHODS—PART 2

In this section, we study systems in which one or more storage elements have differentiation causality. This situation arises for systems in which storage elements are not dynamically independent. For example, consider the electrical circuit of Figure 5.14*a*. The labeled bond graph is shown in part *b*, and causality has been assigned and extended in part *c*. Observe that integration causality on bond 2 has led to differentiation causality on bond 3. In part *d* key variables have been identified, and

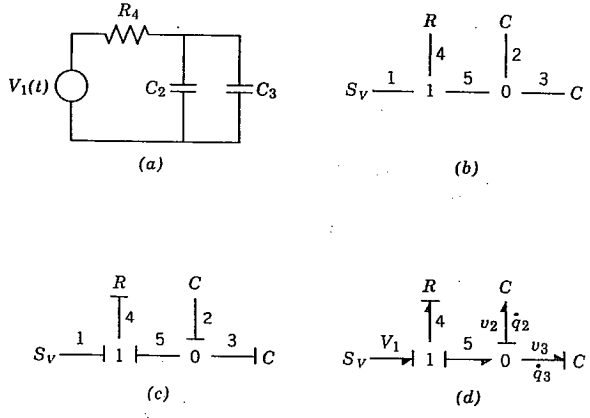

FIGURE 5.14. Differentiation causality in a bond graph. An example.

powers have been directed, so that the graph is augmented. The energy variables are q_2 and q_3, the co-energy variables are v_2 and v_3, and input is $V_1(t)$.

Because of the differentiation causality on bond 3, we anticipate only one state variable (i.e., q_2), and we expect to eliminate q_3 from the dynamic equations.* The general rule is that the energy variables for the storage elements in integration causality in a completely augmented bond graph can serve as state variables. *Any storage elements with differentiation causality do not contribute state variables.* (In the foregoing statements it is assumed that causality is assigned according to the procedure outlined in Section 5.2.)

The constitutive equations for the example are

$$v_2 = \frac{1}{C_2} q_2 \tag{5.36a}$$

and

$$q_3 = C_3 v_3. \tag{5.36b}$$

Observe that the relation for the element $C3$, Eq. (5.36b), has been written in inverse form because in differentiation causality v_3 is the input and $\dot{q}_3$ is the output from the point of view of $C3$.

An expression for the derivative of q_2 is

$$\dot{q}_2 = i_5 - \dot{q}_3, \tag{5.37a}$$

*Causality suggests the following: if q_2 is given, then e_2 is specified. But e_2 is related to e_3, and e_3 is related to q_3, both in an algebraic way. Therefore q_2 algebraically determines q_3.

or

$$\dot{q}_2 = i_4 - \dot{q}_3, \tag{5.37b}$$

or

$$\dot{q}_2 = \frac{1}{R_4}[V_1(t) - v_2] - \dot{q}_3. \tag{5.37c}$$

The input to the element $C3$ is v_3, and this is the other structure relation:

$$v_3 = v_2. \tag{5.37d}$$

Equations (5.36a), (5.36b), (5.37c), and (5.37d) contain four unknowns: q_2, q_3, v_2, and v_3. However, careful inspection of those equations reveals that they include three algebraic equations in terms of four unknowns, and q_3 may be expressed directly in terms of q_2. The result is

$$q_3 = C_3 v_3,$$

or

$$q_3 = C_3 v_2,$$

or

$$q_3 = C_3 \frac{1}{C_2} q_2. \tag{5.38}$$

Therefore, the derivative of q_3 may be found in terms of q_2 from Eq. (5.38) as follows:

$$\dot{q}_3 = \frac{C_3}{C_2} \dot{q}_2. \tag{5.39}$$

Now Eq. (5.37c) may be rewritten more nearly in state-space form by using Eqs. (5.36a) and (5.39) to eliminate v_2 and q_3, respectively. The result is

$$\dot{q}_2 = \frac{1}{R_4} V_1(t) - \frac{1}{R_4 C_2} q_2 - \frac{C_3}{C_2} \dot{q}_2. \tag{5.40}$$

It is typical of systems having differentiation causality that they give rise to equations of the form of Eq. (5.40), in which derivatives appear on both sides of the equation. The final step in reduction is to solve Eq. (5.40) for $\dot{q}_2$ explicitly, obtaining

$$\dot{q}_2 = \frac{1}{R_4(C_2 + C_3)} q_2 + \frac{C_2}{R_4(C_2 + C_3)} V_1(t). \tag{5.41}$$

Just as we anticipated, Eq. (5.41) shows that a first-order system describes the original circuit or bond graph. A very important additional equation for the system is Eq. (5.38), which expresses q_3, the other energy variable, in terms of q_2.

It should be noted that another acceptable causal form exists for the graph in Figure 5.14*b*. That form would have integration causality on the element $C3$ and differentiation causality on the element $C2$. In that case, q_3 would be the state variable, and an additional relation [the inverse of Eq. (5.38), in fact] would be obtained, relating q_2 to q_3 algebraically. With some experience in dealing with such situations, the bond graph analyst is able to select the state variables desired from among the energy variables and to organize the equations in an appropriate fashion. In every case involving differentiation causality, knowledge of the state variables and inputs implies knowledge of the remaining energy variables through algebraic relations, as in Eq. (5.38).

One more example involving differentiation causality may serve to illustrate sufficiently the pattern of formulation and reduction. Consider the mass–spring–lever mechanism depicted in Figure 5.15*a*. A labeled bond graph model is shown in part *b*, in which the lever–arm ratio is specified as a/b. The element parameters are the masses m_1, m_2 and the spring constant k_3 for the elements $I1$, $I2$, and $C3$, respectively; $F_4(t)$ is a force input.

When causality is assigned to the graph, the result is that the element $I1$ has differentiation causality imposed, as Figure 5.15*c* indicates. Physically, only one of the momenta p_1 and p_2 is independent. For example, if p_2 is given, then v_2 is algebraically fixed. But v_1 is algebraically determined by v_2, due to the lever arrangement. Now, v_1 algebraically specifies p_1, so we conclude that p_2 algebraically determines p_1, just as causality indicates. The equations will subsequently bear this out.

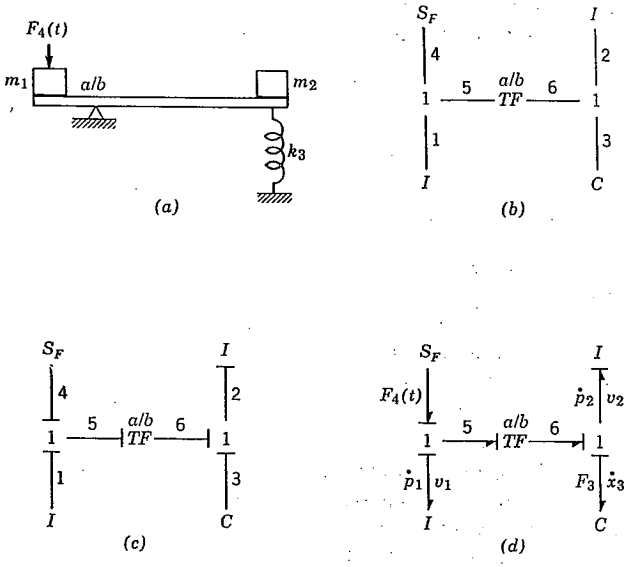

FIGURE 5.15. Augmented bond graph for a lever mechanism.

After key variables are identified and power directions are specified, a completely augmented graph is available from which equations may be written. We anticipate a second-order system with the state variables p_2 and x_3 plus an additional algebraic relation specifying p_1 in terms of p_2.

The key lists are

$$X = \begin{bmatrix} p_2 \\ x_3 \end{bmatrix} \quad \text{and} \quad U = [F_4(t)].$$

We begin the formulation with the constitutive relations

$$p_1 = m_1 v_1, \tag{5.42a}$$

$$v_2 = \frac{1}{m_2} p_2, \tag{5.42b}$$

$$F_3 = k_3 x_3. \tag{5.42c}$$

Equation (5.42a) is written in the form shown because of the causality on the element $I1$.

The next step is to write an expression for the input to each storage element in terms of the key variables. That is, we need equations for v_1, $\dot{p}_2$, and $\dot{x}_3$. Following the causality shown on the graph, we have

$$v_1 = v_5,$$

or

$$v_1 = \frac{a}{b} v_6.$$

or

$$v_1 = \frac{a}{b} v_2, \qquad \dot{p}_2 = F_6 - F_3, \tag{5.43a}$$

or

$$\dot{p}_2 = \frac{a}{b} F_5 - F_3,$$

or

$$\dot{p}_2 = \frac{a}{b}[F_4(t) - \dot{p}_1] - F_3 \tag{5.43b}$$

and

$$\dot{x}_3 = v_2. \tag{5.43c}$$

Since we expect to be able to relate p_1 to p_2 directly, we examine the equations and find that Eqs. (5.42a), (5.43a), and (5.42b) may be used to derive the result:

$$p_1 = m_1 \frac{a}{b} \frac{1}{m_2} p_2. \tag{5.44}$$

From Eq. (5.44) an expression for $\dot{p}_1$ may be found in terms of the variable $\dot{p}_2$. This expression together with Eq. (5.42c) may be put into Eq. (5.43b), and also Eq. (5.43c) may be simplified, to yield

$$\dot{p}_2 = \frac{a}{b} F_4(t) - \frac{a}{b} m_1 \frac{a}{b} \frac{1}{m_2} \dot{p}_2 - k_3 x_3, \tag{5.45a}$$

$$\dot{x}_3 = \frac{1}{m_2} p_2. \tag{5.45b}$$

As the final step in reduction we solve for $\dot{p}_2$ explicitly in Eq. (5.45a), giving the state-space equations

$$\dot{p} = -\frac{k_3}{1 + (m_1/m_2)(a/b)^2} x_3 + \frac{a/b}{1 + (m_1/m_2)(a/b)^2} F_4(t) \tag{5.46a}$$

and

$$\dot{x}_3 = \frac{1}{m_2} p_2. \tag{5.46b}$$

We have indeed obtained the equations for a second-order system, in fact those of a forced oscillator, plus an additional equation specifying the energy variable p_1 directly in terms of the state variable p_2 [Eq. (5.44)]. The reader may wish to try an alternative causality pattern, in which differentiation causality is imposed upon the element *I*2. The state equations will be somewhat different but the behavior of the system will be predicted equally well by either set of state equations.

Further discussion of systems with differentiation causality is to be found in Section 7.1, where a physical point of view is taken, and also in Reference [1], where formal methods of equating generation and reduction are presented.

5.6 OUTPUT-VARIABLE FORMULATION

So far we have seen how state variables can be chosen from the set of energy variables and how state-space equations governing the system dynamics can be developed. Frequently in engineering work the need arises to find expressions for particular output variables which may not have been chosen as state variables. Once a graph has been augmented and the key variables defined, it is a straightforward matter to derive expressions for any other system variables in terms of the state and input variables.

The list of output variables is typically denoted by Y. For linear systems the output equations are usually written as

$$\begin{aligned} y_1 &= c_{11}x_1 + c_{12}x_2 + \cdots + c_{1n}x_n + d_{11}u_1 + \cdots + d_{1r}u_r, \\ y_2 &= c_{21}x_1 + \cdots \qquad\quad + c_{2n}x_n + d_{21}u_1 + \cdots + d_{2r}u_r, \\ &\vdots \\ y_m &= c_{m1}x_1 + \cdots \qquad\quad + c_{mn}x_n + d_{m1}u_1 + \cdots + d_{mr}u_r, \end{aligned} \tag{5.47}$$

where the are n state variables, r input variables, and m output variables. In other words, each output variable is a linear combination of state and input variables.*

As an example, consider the problem of finding the volume flow drawn from the pressure supply in the bond graph model of the pressure-control valve shown in Figure 5.10. We wish to express Q_1 in terms of state variables V_3, x_4, and p_5, plus the inputs $P_1(t)$ and $F_2(t)$. We have, writing equations implied by the causal strokes,

$$Q_1 = Q_8,$$

or

$$Q_1 = \frac{1}{R_8}[P_1(t) - P_9],$$

or

$$Q_1 = \frac{1}{R_8}[P_1(t) - P_3],$$

or

$$Q_1 = \frac{1}{R_8}\left(P_1(t) - \frac{1}{C_3} V_3\right), \tag{5.48}$$

provided the constitutive laws of both R_8 and C_3 are constant-coefficient linear.

Let us now assume that R_8 is characterized by

$$P_8 = \phi_8(Q_8) \tag{5.49}$$

*It is possible for outputs to depend upon the time derivatives of input variables, so that an extra set of coefficients f_{ij} may also be necessary. This is not a common case. By a mathematical redefinition of input and state variables, it is always possible to relate output variables as indicated in Eq. (5.47), but in this text we prefer to use state variables closely related to the physics of the system. One should recognize that there are also variables which are not *linearly* related to state variables; examples are power flows and stored energies. If these output variables are of interest, the system of output equations is not linear even if the state equations are linear.

and C_3 is characterized by

$$P_3 = \phi_3^{-1}(V_3), \tag{5.50}$$

where ϕ_8 is a resistance function and ϕ_3 is a capacitance relation. An expression for Q_1 may be found, using indicated causality as follows:

$$\begin{aligned} Q_1 &= Q_8, \\ Q_1 &= \phi_8^{-1}(P_8) \qquad (\phi_8^{-1} \text{ is the inverse of } \phi_8), \\ Q_1 &= \phi_8^{-1}(P_1 - P_9), \\ Q_1 &= \phi_8^{-1}(P_1 - P_3), \\ Q_1 &= \phi_8^{-1}[P_1 - \phi_3^{-1}(V_3)]. \end{aligned} \tag{5.51}$$

Equations (5.51) give the desired result, namely Q_1 as a function of the state variables (V_3) and inputs (P_1).

Another common problem is that of estimating the effective power delivered to a resistive load. A mechanical example is shown in Figure 5.16*a*, in which an angular velocity source feeds through a compliant shaft supported by two bearings to drive a resistive load. We wish to estimate the power delivered to the load under various conditions.

A bond graph model is given in Figure 5.16*b*, in which R_{13} is the load. We see expressions for the load variables τ_{13} and ω_{13} in terms of the state variables θ_2, θ_7, θ_{12}, h_5, and h_{10} and the input ω_1, where θ denotes angular displacement and h denotes angular momentum. The augmented graph is given in Figure 5.16*c*. The key lists are

$$X = \begin{bmatrix} \theta_2 \\ h_5 \\ \theta_7 \\ h_{10} \\ \theta_{12} \end{bmatrix} \quad \text{and} \quad U = [\omega_1(t)].$$

The linear constitutive relations are

$$\tau_2 = k_2\theta_2, \tag{5.52a}$$

$$\omega_5 = \frac{1}{J_5}h_5, \tag{5.52b}$$

$$\tau_7 = k_y\theta_7, \tag{5.52c}$$

$$\omega_{10} = \frac{1}{J_{10}}h_{10}, \tag{5.52d}$$

$$\tau_{12} = k_{12}\theta_{12}. \tag{5.52e}$$

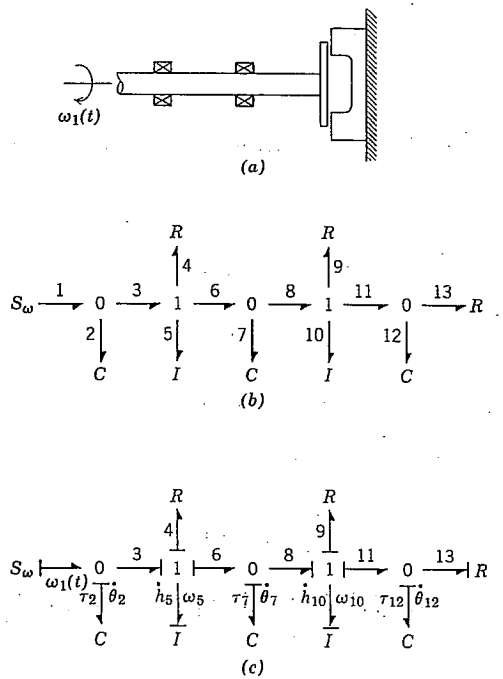

FIGURE 5.16. Augmented bond graph for a shaft-driven load.

The structure equations are

$$\dot{\theta}_2 = \omega_1(t) - \omega_5, \tag{5.53a}$$

$$\dot{h}_5 = \tau_2 - R_4\omega_5 - \tau_7, \tag{5.53b}$$

$$\dot{\theta}_7 = \omega_5 - \omega_{10}, \tag{5.53c}$$

$$\dot{h}_{10} = \tau_7 - R_9\omega_{10} - \tau_{12}, \tag{5.53d}$$

$$\dot{\theta}_{12} = \omega_{10} - \frac{1}{R_{13}}\tau_{12}. \tag{5.53e}$$

Substituting Eqs. (5.52) into Eqs. (5.53), we arrive at the state-space equations (in unordered form),

$$\dot{\theta}_2 = \omega_1(t) - \frac{1}{J_5}h_5, \tag{5.54a}$$

$$\dot{h}_5 = k_2\theta_2 - \frac{R_4}{J_5}h_5 - k_7\theta_7, \tag{5.54b}$$

$$\dot{\theta}_7 = \frac{1}{J_5}h_5 - \frac{1}{J_{10}}h_{10}, \tag{5.54c}$$

$$\dot{h}_{10} = k_7\theta_7 - \frac{R_9}{J_{10}}h_{10} - k_{12}\theta_{12}, \tag{5.54d}$$

$$\dot{\theta}_{12} = \frac{1}{J_{10}}h_{10} - \frac{k_{12}}{R_{13}}\theta_{12}. \tag{5.54e}$$

To estimate the power delivered to the load, we need values for τ_{13} and ω_{13}. For τ_{13} we have

$$\tau_{13} = \tau_{12},$$

or

$$\tau_{13} = k_{12}\theta_{12}. \tag{5.55}$$

For ω_{13} we have

$$\omega_{13} = \frac{1}{R_{13}}\tau_{13},$$

or

$$\omega_{13} = \frac{1}{R_{13}}k_{12}\theta_{12}. \tag{5.56}$$

Thus, the power on bond 13 is given by

$$\mathsf{P}_{13}(t) = \tau_{13}\omega_{13} = \frac{1}{R_{13}}(k_{12}\theta_{12})^2. \tag{5.57}$$

Although the state equations are linear in this case, the second-order term in the expression for P_{13} indicates that the problem of predicting the power flow is not strictly linear. Thus, the relation between inputs and state variables is linear, but the relation between inputs and the output P_{13} is not.

5.7 NONCONDENSED EQUATIONS FOR NONLINEAR SIMULATION

In this section we develop a more general concept of causal system equations and indicate how they can serve a useful purpose in support of simulation. The concepts are easily illustrated by means of examples. Consider the bond graph, fully augmented, shown in Figure 5.9*b*. A complete set of system equations in noncondensed form is given below. The bond graph causality is followed strictly in writing each equation:

Node $R3$:

$$v_3 = \phi_3(i_3). \tag{5.58a}$$

Node $R6$:

$$i_6 = \phi_6(v_6). \tag{5.58b}$$

Node I:

$$i_2 = \phi_2(\lambda_2), \tag{5.58c}$$

$$\frac{d\lambda_2}{dt} = v_2. \tag{5.58d}$$

Node C:

$$v_5 = \phi_5(q_5), \tag{5.58e}$$

$$\frac{dq_5}{dt} = i_5. \tag{5.58f}$$

Node S_v:

$$v_1 = V_1(t). \tag{5.58g}$$

Node 1:

$$i_1 = i_2, \tag{5.58h}$$

$$i_3 = i_2, \tag{5.58i}$$

$$i_4 = i_2, \tag{5.58j}$$

$$v_2 = v_1 - v_3 - v_4. \tag{5.58k}$$

Node 0:

$$v_4 = v_5, \tag{5.58l}$$

$$v_6 = v_5, \tag{5.58m}$$

$$i_5 = i_4 - i_6. \tag{5.58n}$$

A careful count shows that we have 14 variables (six v, six i, one λ, one q) and 14 equations, some of which are nonlinear. Therefore the system should be well posed.

The systems we are considering are of the initial value type. This means that we know the state (λ_2 and q_5) and can evaluate the inputs (V_1) at the initial time. The task is to find the state derivatives at that time. If the system equations (5.58) were to be recorded as

c, e, g;
i, j, a, m, b, n, f;
l, k, d;
h,

then a top-down evaluation could be made. That is, given λ_2 and q_5 at any time, start with Eq. (5.58c), then continue with (5.58e), then (5.58g), and so on. After Eq. (5.58f) has been used, we know dq_5/dt; after Eq. (5.58d) has been used, we know $d\lambda_2/dt$; and after the last equation has been used, we know the values of all system variables. The ordering given above is one of several satisfactory possibilities.

Now consider the fully augmented bond graph model in Figure 5.12*b*. A set of noncondensed but causal system equations can be written for this example. They are similar in many respects to the set (5.58). In particular they may be written as follows:

Node $R2$:

$$v_2 = \phi_2(i_2). \tag{5.59a}$$

Node $R5$:

$$i_5 = \phi_5(v_5). \tag{5.59b}$$

Node I:

$$i_6 = \phi_6(\lambda_6), \tag{5.59c}$$

$$\frac{d\lambda_6}{dt} = v_6. \tag{5.59d}$$

Node C:

$$v_3 = \phi_3(q_3), \tag{5.59e}$$

$$\frac{dq_3}{dt} = i_3. \tag{5.59f}$$

Node S_v:

$$v_1 = V_1(t). \tag{5.59g}$$

Node 1:

$$i_1 = i_4, \tag{5.59h}$$

$$i_2 = i_4, \tag{5.59i}$$

$$i_3 = i_4, \tag{5.59j}$$

$$v_4 = v_1 - v_2 - v_3. \tag{5.59k}$$

Node 0:

$$v_5 = v_4, \tag{5.59l}$$

$$v_6 = v_4, \tag{5.59m}$$

$$i_4 = i_5 + i_6. \tag{5.59n}$$

Once again there are 14 variables and 14 equations. However, an effort to order the equations for top-down evaluation runs into some difficulty. Equations (5.59c,d,g) come quite easily. Careful inspection (or extensive trial and error) shows that Eqs. (5.59a,b,i,k,l,n) are interdependent. They form an algebraic loop of equations. This type of problem, coupled nonlinear algebraic equations, normally must be solved by iterative methods. Had we observed the causality pattern carefully as it was assigned using the sequential causality assignment procedure, then we could have predicted the equation structure.

From these two nonlinear examples we observe that causality can be put to good use in organizing a compatible set of noncondensed system equations. If a suitable sorting procedure is applied to the equations, then they can be ordered in a top-down evaluation pattern. Causality properly interpreted also indicates when the evaluation can proceed explicitly and when (and where) algebraic coupling exists.

Many commercial simulation programs have sorting routines built in so that a set of equations such as Eqs. (5.58) written in arbitrary order can be integrated. Thus there is no need to process the component equations into the explicit state-space form of Eq. (5.4). This is a great convenience when complex nonlinear constitutive laws are to be included in the bond graph model.

REFERENCE

1. R. C. Rosenberg, "State Space Formulation for Bond Graph Models of Multiport Systems," *Trans. ASME J. Dyn. Syst. Meas. Control*, **93**, Ser. G, No. 1, 35–40 (Mar. 1971).

PROBLEMS

5-1 For each of the following bond graphs, assign causality, predict the number of state variables, and write a set of state equations. All elements may be assumed to be linear with constant coefficients.