Implementing Gaussian Basis Functions for Predicting a Function

Objective

In this assignment, students will learn how to implement a **Linear Regression model** using **Gaussian Basis Functions**. They are required to build the model using two different approaches:

1. Direct computation of weights (w)

2. Using Gradient Descent

Instructions

1. Choose a target function

- Select a function to predict (e.g., a sine function or a polynomial).
- Generate input and output data points (optionally, add noise to the data).

2. Design the Gaussian Basis Functions

- Define the Gaussian Basis Function as follows:

$$\phi_j(x) = \exp\left(-\frac{(x-\mu_j)^2}{2\sigma^2}\right)$$

where μ_j is the center of the basis function and σ is the standard deviation.

3. Method 1: Direct computation of weights (w)

- Construct the matrix Φ using Gaussian Basis Functions.
- Compute the weights (w) using the following formula:

$$w = (\Phi^T \Phi)^{-1} \Phi^T t$$

where y is the vector of target values.

4. Method 2: Using Gradient Descent

- Define a cost function $J(w)$ (e.g., Mean Squared Error).

$$w^{(t+1)} = w^{(t)} - \eta \nabla J(w^{(t)})$$

where η is the learning rate.

5. Adjusting the Mean and Variance

To adjust the mean and variance in practice, follow these steps:

1. Data Analysis

Identify the range of the input data and determine the number and locations of the basis functions based on the data distribution.

2. Adjusting the Mean (μ)

- **Choosing Centers (μ_j)**: Select the centers such that they adequately cover the data range. For example:
  - If the data range is from 0 to 10, the centers can be evenly distributed (e.g., $[0, 2, 4, 6, 8, 10]$).
  - Alternatively, use clustering methods to determine the centers based on specific patterns in the data.
- **Alternative Approach**: Divide the data range into equal segments and place the basis functions at the midpoints of these segments.

3. Adjusting the Variance (σ)

- **Determining the Width of Basis Functions**: The standard deviation (σ) defines the width of each basis function. To select an appropriate value:
  - Ensure sufficient overlap between basis functions to avoid gaps in the feature space.
  - Experiment with different σ values or use empirical rules. For instance:

$$\sigma = 0.5 \times (\mu_{j+1} - \mu_j)$$

6. Compare the results

- Compare the predicted values with the actual target values.
- Calculate the error for each method and analyze the results.

Submission

- Complete code for both methods.
- A report that includes:

  1. Description of the chosen function.

  2. Predictions from each method.

  3. Comparison of errors between the two methods.

  4. Explanation of how you selected the mean (μ) and variance (σ) for the Gaussian Basis Functions:

  - Describe the method you used to choose the centers (μ_j) for the basis functions.
  - Discuss how you determined the variance (σ), including any experimentation or rules you applied (e.g., based on the distance between centers or the data range).
  - Include any observations on how adjusting the mean and variance affected the model's performance.