MATHEMATICS

BO CDA : 101

Probability theory :-

Recap of Set theory :-

A set is a well defined collection of objects, which are the elements of set.

If S is a set & x is an element of S, we write $x \in S$.

If x is not an element of S, we write $x \notin S$.

If S contains infinitely many elements $x_1, x_2, \ldots$

$$S = \{x_1, x_2, \ldots\}$$

Complement of a set S, with respect to the universal set Ω, is the set $\{x \in \Omega \mid x \notin S\}$

Union of two set S & T

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

Intersection

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

$$\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup \ldots = \{x \mid x \in S_n \text{ for some } n\}$$

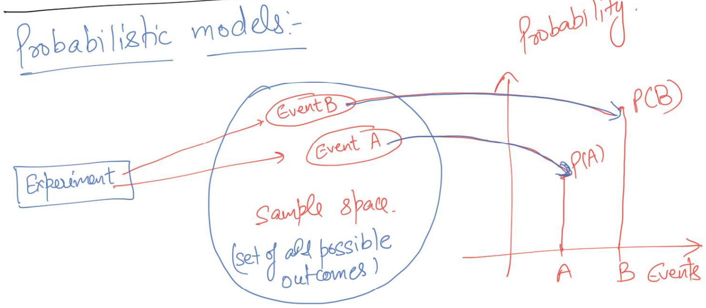
$$\bigcap_{n=1}^{\infty} S_n = S_1 \cap S_2 \cap \ldots = \{x \mid x \in S_n \text{ for all } n\}$$

De Morgan's law :-

$$\left(\bigcup_n S_n\right)^c = \bigcap_n S_n^c$$

$$\left(\bigcap_n S_n\right)^c = \bigcup_n S_n^c$$

Probabilistic models :-



Example :- Experiment : Rolling a die

Outcomes: 1, 2, 3, 4, 5, 6.

Sample space $(\Omega) = \{1, 2, 3, 4, 5, 6\}$.

Event A :- Getting an odd number

" B :- " " even "

C :- " a prime number

$P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$, $P(C) = \frac{1}{2}$ ✓

Every probabilistic model involves an underlying process. This process is called experiment. It produces exactly one out of several possible outcomes.

Set of all possible outcomes is called sample space of that experiment (Ω)

A subset of the sample space is called an event.

* Do it yourself :- Find an experiment where the sample space is infinite.

Definition :- The probability law assigns to every event A, a number $P(A)$, called the probability of A, satisfying the following axioms :-

1. (Non negativity) $P(A) \geq 0$ for every event A.
2. (Additivity) If A & B are two disjoint events, then $P(A \cup B) = P(A) + P(B)$
3. (Normalization) The probability of entire sample space is 1, $P(\Omega) = 1$

* Do it yourself :- Using above, prove that

$$\boxed{P(\emptyset) = 0.}$$

Properties of Probability laws :-

Let $A, B, \& C$ be events from an experiment :

(1) If $A \subset B$ then $P(A) \leq P(B)$

(2) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

(3) $P(A \cup B) \leq P(A) + P(B)$

Discrete model :-

Experiment :- tossing a coin three times

$\Omega = \{HHH, HHT, HTH, HTT, TTH, THT, THH, TTT\}$.

$A = \{\text{exactly 2 heads occur}\} = \{HHT, HTH, THH\}$

$P(A) = ?$ $\left(\frac{3}{8}\right)$

Example:-

Conditional Probability:- Consider the following examples:-

(1) Rolling a die twice, you are told that sum of the two rolls is 9. What is the probability that first roll was a '6'

(2) The medical report of a person about a decease is negative, how likely is it that the person has that desease.

Definition:- If all the outcomes are finitely many & equal likely.

$$P(A|B) = \frac{\text{number of elements of } A \cap B}{\text{number of elements of } B.} \quad — (*)$$

$$\boxed{P(A|B) = \frac{P(A\cap B)}{P(B)}} \quad — (**)$$

Example: Toss a fair coin three times. Consider the events $A = \{\text{more heads than tails come up}\}$

$B = \{1^{st} \text{ toss is a head}\}$.

Sample space $\Omega = \{HHH, HHT, HTT, HTH, TTT, THH, TTH, THT\}$.

$B = \{HHH, HHT, HTH, HTT\}$

$P(B) = \frac{4}{8} = \frac{1}{2}$

we want to find $P(A|B)$. ?

$A \cap B = \{HHH, HHT, HTH\}$

$P(A\cap B) = \frac{3}{8}$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{3/8}{4/8} = \frac{3}{4}.$$

Example 2:- Radar Detection:- If an aircraft is present in a certain area, the radar detects it. It sends an alarm with probability 0.99. If no aircraft is there, it generates a alarm (false alarm) with probability 0.10. It is assumed that an aircraft is present with probability 0.05.

$P(\text{no aircraft present and a false alarm}) = ?$

$P(\text{aircraft present and no detection}) = ?$

Solution:- Let A & B be the events

$A = \{\text{an aircraft is present}\}$

$B = \{\text{the radar generates an alarm}\}$

$P(A^c \cap B) = ?$

$P(A \cap B^c) = ?$

Recap:- $P(A|B) = \frac{P(A\cap B)}{P(B)} \Rightarrow P(A\cap B) = P(B) \cdot P(A|B)$

generates alarm.

$P(B\cap A^c) = P(A^c) \cdot P(B|A^c)$

aircraft not present

$= 0.95 * 0.1$

$= 0.095.$

no detection.

$P(B^c \cap A) = P(A)\, P(B^c | A)$

aircraft is present

$= 0.05 * 0.01$

$= 0.0005$

Multiplication Rule:- Suppose an event A occurs if and only if each one of several events $A_1, A_2, \ldots, A_n$ has occured, i.e.

$$A = A_1 \cap A_2 \cap \ldots \cap A_n.$$

Definition:-

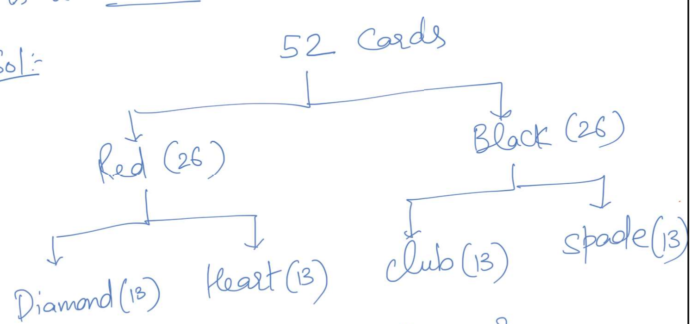
$$P(A) = P\left(\bigcap_{i=1}^{n} A_i\right)$$

$$= P(A_1) \cdot P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2) \cdot \ldots \cdot P\left(A_n \middle| \bigcap_{i=1}^{n-1} A_i\right) \quad \text{---} \circledast\circledast\circledast$$

assuming that all the conditioning events have positive probability.

Example:- Three cards are drawn from a deck of 52 cards without replacement. What is the probability that none of these three cards is a "heart"?

Sol:-



Ace, King, Queen, Jack, 10, 9, ..., 2.

Sol:- Let $A_1 = \{1^{st}$ card is not heart$\}$

$A_2 = \{2^{nd}$ " " " $\}$

$A_3 = \{3^{rd}$ " " " " $\}$

$$P(A_1 \cap A_2 \cap A_3) = ?$$

$$= P(A_1)\, P(A_2 | A_1) \cdot P(A_3 | A_1 \cap A_2)$$

Now $P(A_1) = P(1^{st}$ card is not a heart$)$

$$= \frac{39}{52}$$

$P(A_2 | A_1) = P(2^{nd}$ card is not a heart $|$ 1^{st} card is not a heart$)$

$$= \frac{38}{51}$$

$P(A_3 | A_1 \cap A_2) = P(3^{rd}$ card is not a heart $|$ 1^{st} two card are not heart$)$

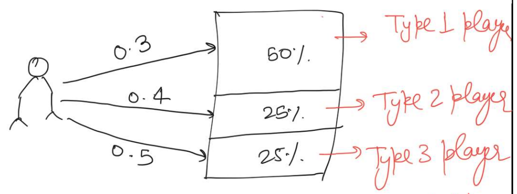
$$= \frac{37}{50}$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) = \frac{39}{52} \times \frac{38}{51} \times \frac{37}{50}}$$

Total probability theorem:- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of sample space. Assume that $P(A_i) > 0 \; \forall i$. Then for any event B, then

$$P(B) = P(A_1) \cdot P(B|A_1) + P(A_2) \cdot P(B|A_2) + \ldots + P(A_n) \cdot P(B|A_n)$$

Example:-



You are playing chess tournament. Your probability of winning a game is 0.3 against half of the players. The probability of winning is 0.4 against a quarter of the players. The probability of winning is 0.5 against a quarter of the players.

What is the probabity that you will win?

Solution:- A_1 = the event of playing with Type 1
A_2 = " " " " Type 2
A_3 = " " " " Type 3.

$P(A_1) = \frac{1}{2}$, $P(A_2) = \frac{1}{4}$, $P(A_3) = \frac{1}{4}$

Let B = event of winning.

$$P(B \mid A_1) = P(\text{you won} \mid \text{played against Type 1})$$
$$= 0.3$$

$$P(B \mid A_2) = 0.4$$

$$P(B \mid A_3) = 0.5$$

$$P(B) = P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + P(A_3)\cdot P(B|A_3)$$
$$= \frac{1}{2}\times 0.3 + \frac{1}{4}\times 0.4 + \frac{1}{4}\times 0.5$$
$$= 0.375$$

Baye's Rule:- It is used for inference. There are a number of "causes" that may result in certain effect. We observe the effect & we wish to infer the cause.

Definition:- Let $A_1, A_2, \ldots, A_n$ be disjoint events that form a partition of the sample space & assume that $P(A_i) > 0$. Then for any event B ($P(B) > 0$), we have

$$P(A_i|B) = \frac{P(A_i)\cdot P(B|A_i)}{P(B)}$$

$$= \frac{P(A_i)\cdot P(B|A_i)}{P(A_1)\cdot P(B|A_1) + P(A_2)\cdot P(B|A_2) + \ldots + P(A_n)P(B|A_n)}$$

Example:- Relook at the previous example of chess tournament.

Suppose you win. What is the probability that you played against type 1 player?

$$P(A_1 \mid B) = ?$$

(Do it yourself)

Tutorial:

Ques 1:- When we through a dart on a target, the probability of hitting the target is $\frac{1}{4}$. If the dart is thrown three times, what is the probability of obtaining atleast one hit?

Sol:- Approach 1:-

$P(\text{atleast one hit}) = 1 - P(\text{no hit})$

Let A_1 be the event that 1^{st} dart hits target.
A_2 " " " " 2^{nd} " " "
A_3 " " " " 3^{rd} " " "

$$P(\text{atleast one hit}) = 1 - P(A_1^c \cap A_2^c \cap A_3^c)$$
$$= 1 - P(A_1^c)\cdot P(A_2^c)\cdot P(A_3^c)$$
$$= 1 - \left(\frac{3}{4}\times\frac{3}{4}\times\frac{3}{4}\right) = \frac{37}{64}.$$

Approach 2:-

P(atleast one hit out of 3 throws)
$$= P(A_1 \cup A_2 \cup A_3)$$
$$= P(A_1) + P(A_2) + P(A_3) - P(A_1\cap A_2) - P(A_2\cap A_3) - P(A_1\cap A_3) + P(A_1\cap A_2\cap A_3)$$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4} - \left(\frac{1}{4} \times \frac{1}{4}\right) - \left(\frac{1}{4} \times \frac{1}{4}\right) - \left(\frac{1}{4} \times \frac{1}{4}\right) + \frac{1}{64}$$

$$= \frac{37}{64}$$

Qus:- Manufacturer X produces PC at two different location (Delhi & Bombay). 15% of PC produced at Delhi are defective & 5% of PC produced at Bombay are defective. If 10,00,000 PCs are produced at Delhi & 1,50,000 PCs are produced at Bombay in a year, find the probability of purchasing a defective PC.

Sol:- Let A_1 = PC is from Delhi

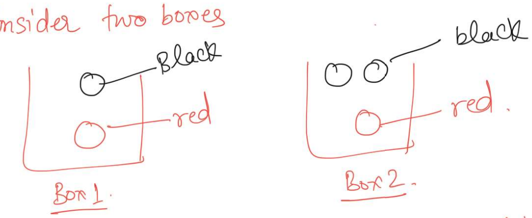
A_2 = PC " " Bombay.

$$P(\text{defective}) = P(A_1) \cdot P(\text{defective} \mid A_1) + P(A_2) \cdot P(\text{defective} \mid A_2)$$

$$= \left(\frac{1000000}{1150000}\right) \times \frac{15}{100} + \frac{150000}{11,50,000} \times \frac{5}{100}$$

$$= 0.137$$

Qus:- Consider two boxes



A box is selected at random and a marble is drawn at random from that box. What is the probability that marble is Black?

Sol:- $A_1 \longrightarrow$ box 1 is chosen

$A_2 \rightarrow$ box 2 is chosen

$B \Rightarrow$ the marble is Black

$$P(B) = P(A_1) \cdot P(B \mid A_1) + P(A_2) \cdot P(B \mid A_2)$$

$$= \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{2}{3} = \frac{7}{12}$$

Next:- Suppose the marble drawn is red, what is the probability that 1st box was selected?

Let R = event that marble is red.

$$P(A_1 \mid R) = ?$$

$$P(A_1 \mid R) = \frac{P(R \mid A_1)\, P(A_1)}{P(R \mid A_1) \cdot P(A_1) + P(R \mid A_2) \cdot P(A_2)}$$

$$= \frac{\frac{1}{2} \times \frac{1}{2}}{\frac{1}{2} \times \frac{1}{2} + \frac{1}{3} \times \frac{1}{2}}$$

$$= \frac{3}{5}.$$

Independence:- Let A & B be two events.
The interesting case is, when the occurence of B provides no information & does not alter the probability of occurence of A

$$P(A|B) = P(A) \quad \text{--- } (1)$$

when Eq (1) holds, then the events A & B are said to be independent.
From conditional probability, we know that

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{--- } (A)$$

putting in (1)

$$\boxed{P(A \cap B) = P(A) \cdot P(B)} \quad \text{--- } (2)$$

Eq^n (2) is adopted as the definition of independence of two events because it can be used even when $P(B) = 0$, in which case $P(A|B)$ is undefined.

Independence is a symmetric property:- If A is independent of B, then B is independent of A.

we say that "A & B are independent events"

Remark:- If two events are disjoint ⇓ they are independent — True or False?

Ans: FALSE

Two disjoint events A & B with $P(A) > 0$, $P(B) > 0$ &

$A \cap B = \phi$ ✓

$P(A \cap B) = 0$ ✓

$\because P(A)$ & $P(B) > 0 \Rightarrow P(A) \cdot P(B) > 0$ ✓

for independence, $P(A \cap B) = P(A) \cdot P(B)$ — (A)
which is not possible.

Example:- Consider an experiment involving two successive rolls of a 4-sided die. Assume that outcomes are equal likely.

$$\Omega = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) \\ (2,1) & (2,2) & (2,3) & (2,4) \\ (3,1) & (3,2) & (3,3) & (3,4) \\ (4,1) & (4,2) & (4,3) & (4,4) \end{Bmatrix}$$

Consider the following two events:-

A = {1st roll is 1} B = {sum of two rolls is 5}

Are A & B independent?

Solution:- we need to check $P(A \cap B) = P(A) \cdot P(B)$

$A \cap B = (1,4)$

$$P(A \cap B) = \frac{1}{16}$$

$$P(A) = \frac{4}{16}, \quad P(B) = \frac{4}{16}$$

$$\because P(A \cap B) = P(A) \cdot P(B) = \frac{1}{16}$$

⇒ A & B are independent.

Case II:- Let A = {maximum of the two rolls is 2}
B = {minimum " " " " 2}

Are A & B independent?

$A = \{(1,2) \ (2,1) \ (2,2)\}$

$B = \{(2,2) \ (2,3) \ (2,4), (3,2) \ (4,2)\}$

$$P(A) = \frac{3}{16}$$

$$P(B) = \frac{5}{16}$$

$A \cap B = \{(2,2)\}$

$$P(A \cap B) = \frac{1}{16}$$

$$P(A \cap B) \neq P(A) \cdot P(B)$$

⇒ A & B are not independent

Conditional Independence:- Given an event C, the events A & B are called conditionally independent if

$$\boxed{P(A \cap B \mid C) = P(A|C) \cdot P(B|C)}$$

Example:- Consider two independent fair coin tosses. (Assume outcomes are equal likely)

Let $A = \{1^{st} \text{ toss is a head}\}$

$B = \{2^{nd} \text{ " " " "}\}$

$C = \{\text{two tosses have different results}\}$.

A & B are conditionally independent or not?

we need - $P(A \cap B \mid C)$ $\Downarrow$ $P(A|C) \cdot P(B|C)$

Try yourself -

Continuing the previous example:- $\Omega = \{HH, HT, TH, TT\}$

$A = 1^{st}$ toss is head
$B = 2^{nd}$ " " "
$C =$ two tosses have different result.

$$P(A|C) = \frac{1}{2}$$

$$P(B|C) = \frac{1}{2}$$

$$P(A \cap B \mid C) = 0$$

$$\Rightarrow P(A \cap B \mid C) \neq P(A|C) \cdot P(B|C)$$

$\Rightarrow$ A & B are NOT conditionally independent

★ The independence of several events.

The events $A_1, A_2, \ldots, A_n$ are said to be independent if

$$P\left(\bigcap_{i \in S} A_i\right) = \underbrace{\prod_{i \in S}}_{\text{Product}} P(A_i) \quad \left\{\begin{array}{l}\text{for every subset } S \text{ of} \\ \{1, 2, \ldots, n\}\end{array}\right.$$

Example:- Let A_1, A_2 & A_3 be three ~~events~~ events. If

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) \cdot P(A_2) \cdot P(A_3)$$

pairwise independence
$$\left\{\begin{array}{l} P(A_1 \cap A_2) = P(A_1) \cdot P(A_2) \\ P(A_2 \cap A_3) = P(A_2) \cdot P(A_3) \\ P(A_1 \cap A_3) = P(A_1) \cdot P(A_3) \end{array}\right.$$

happens, then A_1, A_2, & A_3 are said to be independent.

Example:- Consider the previous example of tossing two independent fair coins &

$A_1 = \{1^{st} \text{ toss is head}\}$ $\quad \Omega = \{HH, HT, TH, TT\}$

$A_2 = \{2^{nd} \text{ " " "}\}$

$A_3 = \{\text{two tosses have different results}\}$

A_1 & A_2 are independent.

$$P(A_3 \mid A_1) = \frac{P(A_3 \cap A_1)}{P(A_1)} = \frac{1/4}{2/4}$$

$$= \frac{1}{2} = P(A_3)$$

$\Rightarrow A_1$ & A_3 are independent.

$$P(A_3 \mid A_2) = \frac{P(A_3 \cap A_2)}{P(A_2)} = \frac{1/4}{2/4} = \frac{1}{2} = P(A_3)$$

$\Rightarrow A_2$ & A_3 are independent.

$\Rightarrow A_1, A_2$ & A_3 are pairwise independent

Are they independent? NO

$$P(A_1 \cap A_2 \cap A_3) = 0$$

$$P(A_1) = \frac{1}{2}, \quad P(A_2) = \frac{1}{2}, \quad P(A_3) = \frac{1}{2}$$

$$\boxed{P(A_1 \cap A_2 \cap A_3) \neq P(A_1) P(A_2) P(A_3)}$$

$\Rightarrow A_1, A_2$ & A_3 are pairwise independent but they are not independent.

Example:- Consider two independent rolls of a fair six-sided die &

$A = \{1^{st} \text{ roll is } 1, 2, \text{ or } 3\}$

$B = \{1^{st} \text{ " " } 3, 4, \text{ or } 5\}$

$C = \{\text{the sum of two rolls is } 9\}$

$$P(A) = \frac{1}{2}, \quad P(B) = \frac{1}{2}, \quad P(C) = \frac{4}{36}$$

$$\Omega = \begin{Bmatrix} (1,1) & (1,2) & \dots & (1,6) \\ (2,1) & (2,2) & \dots & (2,6) \\ \vdots & & & \\ (6,1) & (6,2) & \dots & (6,6) \end{Bmatrix}$$

$$A \cap B = \{(3,1)\ (3,2)\ (3,3)\ (3,4)\ (3,5)\ (3,6)\}$$

$$P(A \cap B) = \frac{6}{36}$$

✓ $P(A\cap B) \neq P(A)\cdot P(B)$

$$P(B \cap C) = \frac{3}{36}$$

✓ $P(B\cap C) \neq P(B)P(C)$

$$P(A\cap C) = \frac{1}{36}$$

✓ $P(A\cap C) \neq P(A)P(C)$

$$A\cap B \cap C = (3,6)$$

$$P(A\cap B\cap C) = \frac{1}{36}$$

$$P(A\cap B\cap C) = P(A)P(B)P(C)$$

$$\frac{1}{36} = \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{4}{36}$$

$\Rightarrow$ 1st Condition is satisfied, however they are not pairwise independent.

Independent trials & Binomial probabilities:-

If an experiment involves a sequence of independent & identical stages, we say that we have a sequence of independent trials.

Suppose there are only two possible results at each stage, these trials are called Bernoulli trials.

Ques 4:- The events A & B are mutually exclusive. Can they be independent?

Solution:- If A & B are mutually exclusive

$$\Rightarrow \quad A\cap B = \phi$$

$$P(A\cap B) = 0$$

If A & B are independent then

$$P(A\cap B) = P(A)\cdot P(B)$$

Hence A & B are mutually exclusive & independent if

$$P(A)\cdot P(B) = 0$$

Ques 5:- How many equations do you need to establish the independence of n events.

Sol^n:-

$P(A_1\cap A_2) = P(A_1)\cdot P(A_2) \longrightarrow$ 2 events

$$\left.\begin{aligned} P(A_1\cap A_2\cap A_3) &= P(A_1)P(A_2)P(A_3) \\ P(A_1\cap A_2) &= P(A_1)\cdot P(A_2) \\ P(A_2\cap A_3) &= P(A_2)\cdot P(A_3) \\ P(A_3\cap A_1) &= P(A_3)\cdot P(A_1) \end{aligned}\right\} \text{3 events}$$

For 4 events:- ${}^4C_2 + {}^4C_3 + {}^4C_4 = 11$

No. of equations of the form $P(A_i\cap A_j) = P(A_i)P(A_j)$ equals ${}^nC_2$

No. of equation of the form $P(A_i\cap A_j\cap A_k) = P(A_i)P(A_j)P(A_k)$ equal ${}^nC_3$

$\vdots$

Total number of equations -

$$N = {}^nC_2 + {}^nC_3 + \dots + {}^nC_n$$

We know that

$${}^nC_0 + {}^nC_1 + \underbrace{{}^nC_2 + \dots + {}^nC_n} = 2^n$$

Total number of equations

$$N = 2^n - {}^nC_0 - {}^nC_1$$

$$\boxed{N = 2^n - 1 - n}$$

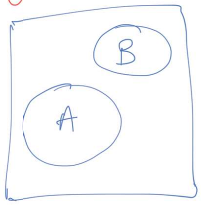
Ex:- How many eq^n do you need to show the independence of 10 events?

Ques 6:- If $A \cap B = \{\phi\}$, then what is the relation between the probabilities of A & B^c.

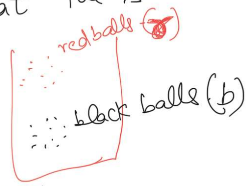
Solution:- If $A \cap B = \phi$, then

$$A \subset B^c$$

$$\boxed{P(A) \le P(B^c)}$$



Ques 7:- An urn contains 'b' black balls & 'r' red balls. One of the balls is drawn at random but when it is put back in the urn, 'C' additional balls of the same color are put in with it. Now suppose that we draw another ball. What is the probability that the first ball drawn was black given that the second ball drawn was red?



Solution:- $B_1 \to$ 1st ball is black.
$B_2 \to$ 2nd " " "
$R_1 \to$ 1st ball is red
$R_2 \to$ 2nd " " "

$$P(B_1 \mid R_2) = \frac{P(B_1)\,P(R_2 \mid B_1)}{P(B_1)\,P(R_2 \mid B_1) + P(R_1)\cdot P(R_2 \mid R_1)}$$

$$= \frac{\frac{b}{b+r}\cdot\frac{r}{b+r+C}}{\frac{b}{b+r}\cdot\frac{r}{b+r+C} + \frac{r}{b+r}\cdot\frac{r+C}{b+r+C}}$$

simplify it.

Ques 15:- Bill & George go together for target shooting. Both shoot at a target at the same time. Suppose Bill hits the target with probability 0.7 & whereas George, independently, hits the target with probability 0.4.

(a) Given that exactly one shot hit the target, what is the probability that it was George's shot?

(b) Given that the target is hit, what is the probability that George hit it?

Solu^n:-

$$P(\text{George} \mid \text{Exactly 1 hit}) =$$

$$= \frac{P(\text{George, not Bill})}{P(\text{Exactly 1 hit})}$$

$$= \frac{P(\text{George, not Bill})}{P(\text{George, not Bill}) + P(\text{Bill, not George})}$$

$$= \frac{0.4 \times 0.3}{0.4 \times 0.3 + 0.7 \times 0.6} = \frac{2}{9}.$$

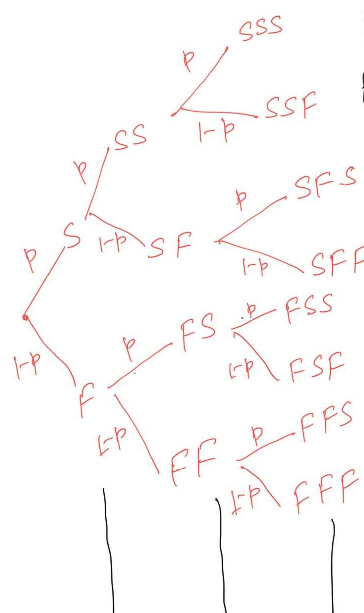
(b) Do it yourself:-

Independent trials & Binomial probabilities.

Recall the Bernoulli trials

Let Probability of Success is p, & probability of failure is $1-p$.

Success $\to S$
Failure $\to F$



Prob(3 success) $= p^3$

$P(2 \text{ success out of } 3 \text{ trials}) = p^2(1-p)$

$P(2 \text{ success out of } 3 \text{ trials}) = p^2(1-p)$

$P(1 \text{ success out of } 3 \text{ trials}) = p(1-p)^2$

$P(2 \text{ success out of } 3 \text{ trials}) = p^2(1-p)$

$P(1 \text{ success out of } 3 \text{ trials}) = p(1-p)^2$

$P(1 \text{ success out of } 3 \text{ trials}) = p(1-p)^2$

$P(0 \text{ success} \dots) = (1-p)^3$

The probability that we get k success in 'n' trials is

$$p(k) = \binom{n}{k} p^k (1-p)^{n-k}$$

$$\boxed{\binom{n}{k} = {}^nC_k}$$

The numbers $\binom{n}{k}$ are known as Binomial coefficients, the probabilities $p(k)$ are known as the Binomial probabilities

$$\binom{n}{k} = {}^nC_k = \frac{n!}{k!\,(n-k)!}$$

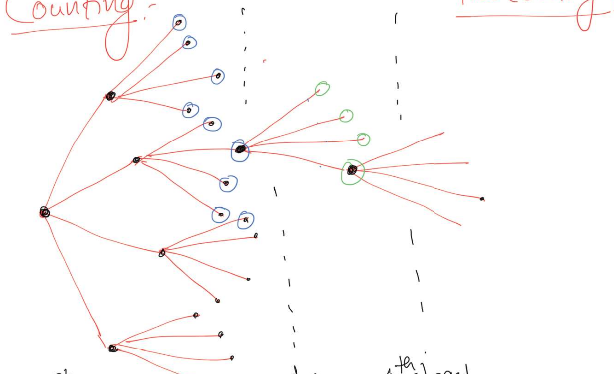
Remark:- The binomial probabilities $p(k)$ must add to 1, i.e.

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1 \quad \text{—} (\star)$$

The eq^n $(\star)$ is called the binomial formula.

Counting:-

The counting principle

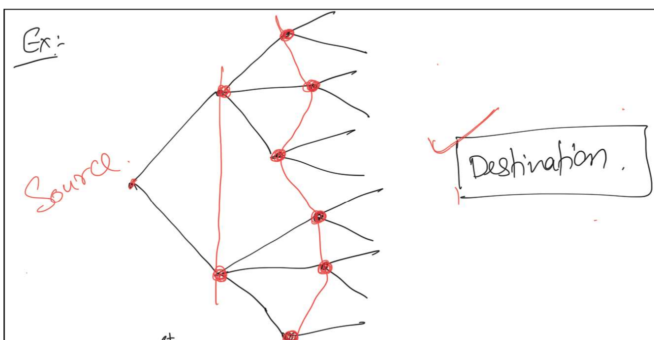


1st stage	2nd stage	3rd stage	4th stage	. . . . .
n_1 choices	n_2 choices	n_3 choices	n_4 choices.	

The 1st stage has n_1 possible results. For every possible result at the first $i-1$ stages, there are n_i possible result at the i^{th} stage. Then the total number of possible results of the k-stage process is

$$\boxed{n_1 \cdot n_2 \cdots n_k}$$

Ex:-



1st stage	2nd stage	3rd stage
2 choices	3 choices	2 choices

Example:- A telephone number is a 7-digit sequence, but the 1st digit should not be one or zero. How many distinct telephone numbers are there.

Solution:-

✓	✓	✓				
1st stage	2nd stage	3rd stage	4th stage	5th stage	6th stage	7th stage
8 choice	10 choices	10 choices	10 choices	10 choices	10 choices	10 choices

The answer is $= 8 \times 10 \times 10 \times 10 \times 10 \times 10 \times 10$
$= 8 \times 10^6$.

Permutation & Combination:-

Aim:- selection of k objects out of n objects.

- order matters — Selection is called a "permutation"
- order does not matter — Selection is called a "Combination"

Permutations:- the number of different ways to choose k out of these n objects?

Permutation:- We have n objects & K is a (+)ve integer $K \le n$.

Count the number of different ways that we can pick distinct K out of these n objects.

The number of ways $= \dfrac{n!}{(n-K)!} = {}^{n}P_{K}$

Special case:- $K = n$, the no. of possible ways $= n!$

Example:- How many distinct four letter words can be made from English alphabet?

Solution:- $n = 26$, $K = 4$

$\#$ of possible ways $= {}^{26}P_{4} = \dfrac{26!}{(26-4)!}$

$= \dfrac{26 \times 25 \times 24 \times 23 \times \cancel{22!}}{\cancel{22!}}$

$= 358800$

Combinations:- There are n people & we want to form a committee of K people.

for example:- We have 4 English alphabets A, B, C, D.

for $K=2$

Permutation: AB BA AC CA AD DA BC CB BD DB CD DC.

Combination: AB, AC, AD, BC, BD, CD

The number of possible combinations of K people from n people are ${}^{n}C_{K}$

$$= \frac{n!}{K!\,(n-K)!}$$

In above example, $n = 4$, $K = 2$, therefore the no. of combinations $= {}^{4}C_{2} = 6$.

Recall - Prove the following:-

$${}^{n}C_0 + {}^{n}C_1 + {}^{n}C_2 + \dots + {}^{n}C_n = 2^n \quad — (1)$$

We have binomial formula:-

$$\sum_{K=0}^{n} \binom{n}{K} p^K (1-p)^{n-K} = 1 \quad — (2)$$

put $p = \frac{1}{2}$ in (2)

$$\sum_{K=0}^{n} \binom{n}{K} \left(\frac{1}{2}\right)^K \left(1-\frac{1}{2}\right)^{n-K} = 1$$

$$\Rightarrow \sum_{K=0}^{n} \binom{n}{K} \left(\frac{1}{2}\right)^K \left(\frac{1}{2}\right)^{n-K} = 1$$

$$\Rightarrow \sum_{K=0}^{n} \binom{n}{K} \frac{1}{2^n} = 1$$

$$\Rightarrow \boxed{\sum_{K=0}^{n} \binom{n}{K} = 2^n}$$

Topics covered till now:-

(i) Sets
(ii) Probability models
(iii) Conditional probability
(iv) Total probabilty theorem, Bayes theorem.
(v) Independence (pairwise independence etc.)
(vi) Counting, Permutations & Combinations.

Random Variable

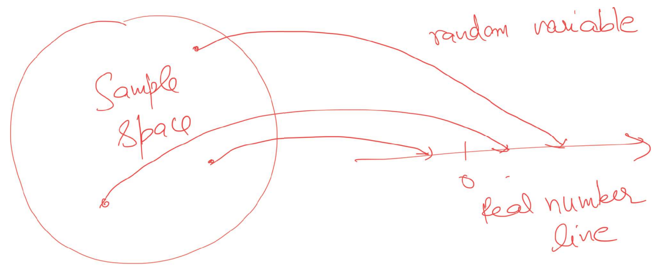
Experiment

↓

Outcomes

Numerical ✓ | Not numerical

Idea:- Given an experiment & the possible outcomes, a random variable associates a particular number with each outcome

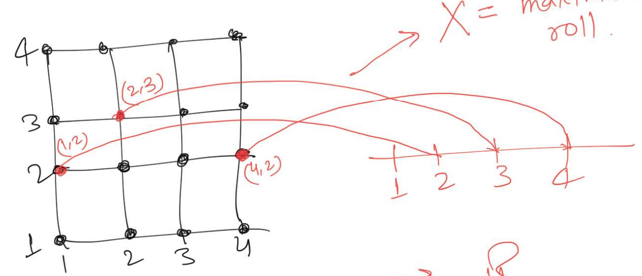


Random variable:- A random variable is a real valued function of the experimental outcome. i.e $X : \Omega \longrightarrow \mathbb{R}$

X — random variable

Ω — sample space

$\mathbb{R}$ — set of real numbers.



$X : \Omega \longrightarrow \mathbb{R}$

Examples 1 → Experiment: rolling two 6-sided die.

Out comes: (1,1) . . . . (6,6)

the following random variables can be defined

(i) Sum of the two rolls.

(ii) number of 6 in the two rolls.

: and so on.

Suppose X = Sum of the two rolls

Then possible values of X are:- 2, 3, ... 12

Suppose Y = number of 6 in the two rolls

The possible values of Y are: 0, 1, 2.

what we learned →

1. random variables
2. what values a random variable can have?
3. In which situation, random variable is taking that values?

Recall:- A random variable X

$X : \Omega \longrightarrow \mathbb{R}$

Definition:- A random variable is called discrete if its range is either finite or countably infinite. The random variables defined in example 1 are discrete random variables.

Definition:- A random variable that can take an uncountably infinite number of values is

NOT discrete.

Example:- Experiment:- choosing a point 'a' from the interval $[-1, 1]$

random variable is giving a^2 to the outcome a

A Concepts related to Discrete random variables

Probability mass function (PMF) →

If x is any possible value of random variable X, the probability mass of x is the probability of event $\{X = x\}$. i.e

Probability mass of $x = p_X(x) = P\{X = x\}$

Example:- Experiment: tossing a fair coin twice

random variable X: no. of heads obtained.

$\Omega = \{HH, HT, TH, TT\}$

$X = 0, 1, 2$ $\qquad x = 0, 1, 2.$

PMF of random variable X

$$p_X(x) = \begin{cases} 1/4 & x = 0 \\ 1/2 & x = 1 \\ 1/4 & x = 2. \end{cases}$$

Tutorial sheet 2

Qu1 ⇒ Prove the following identities :-

(a) $\binom{n}{k} = \binom{n}{n-k}$

L.H.S ⇒ $\binom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$

R.H.S → $\binom{n}{n-k} = \dfrac{n!}{(n-k)!\,(n-(n-k))!}$

$= \dfrac{n!}{(n-k)!\,k!} = \underline{L.H.S.}$

(b) $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

L.H.S. $\dfrac{n!}{k!\,(n-k)!} + \dfrac{n!}{(k+1)!\,(n-k-1)!}$

$= \dfrac{n!}{k!\,(n-k)\cdot(n-k-1)!} + \dfrac{n!}{(k+1)\cdot k!\,(n-k-1)!}$

$= \dfrac{n!}{k!\,(n-k-1)!}\cdot\left[\dfrac{1}{n-k} + \dfrac{1}{k+1}\right]$

$= \dfrac{n!}{k!\,(n-k-1)!}\cdot\left[\dfrac{(k+1)+(n-k)}{(n-k)(k+1)}\right]$

$= \dfrac{(n+1)\,n!}{\underbrace{(k+1)\,k!}\;\underbrace{(n-k)\,(n-k-1)!}}$

$= \dfrac{(n+1)!}{(k+1)!\,(n-k)!} = \binom{n+1}{k+1} = \underline{R.H.S.}$

(d) $\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0.$, Consider Binomial expansion.

$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}$ —— ☆

take $p = -1$ & $q = 1$ in eq^n ☆, we get

$0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k (1)^{n-k}$

$\Rightarrow \boxed{\sum_{k=0}^{n} \binom{n}{k} (-1)^k = 0}$

(e) $\sum_{k=1}^{n} \binom{n}{k} k = n 2^{n-1}$

Consider the Binomial expansion

$(p+q)^n = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}.$

Differentiate with respect to p.

$$\frac{d}{dp}(p+q)^n = \frac{d}{dp}\left(\sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\right)$$

$$\Rightarrow \boxed{n(p+q)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}k p^{k-1} q^{(n-k)}} \quad (2)$$

Take $p=q=1$

$$n\cdot 2^{n-1} = \sum_{k=0}^{n}\binom{n}{k}k\cdot(1)^{k-1}(1)^{n-k}$$

$$= \sum_{k=0}^{n}\binom{n}{k}\cdot k = \sum_{k=1}^{n}\binom{n}{k}k$$

(f) $\sum_{k=0}^{n}\binom{n}{k}k(-1)^k = 0$

Take Binomial expansion & differentiate with respect to p, we get the following eqn (Eq. 2)

$$n(p+q)^{n-1} = \sum_{k=0}^{n}\binom{n}{k}\cdot k p^{k-1} q^{n-k}$$

Put $p=-1$ & $q=1$

$$0 = \sum_{k=0}^{n}\binom{n}{k}k(-1)^{k-1}(1)^{n-1}$$

multiply with (-1), we get

$$\boxed{0 = \sum_{k=0}^{n}\binom{n}{k}\cdot k(-1)^k}$$

Ques 3:- Cards are drawn from a standard 52-Card deck untill the third club is drawn. After each card is drawn, it is put back in the deck & the Cards are resuffled so that each card drawn is independent of all others.

(1) Find the probability that the 3rd club is drawn on the 8th selection.

Sol:- probability of k success in n trials

$$p(k) = \binom{n}{k}p^k(1-p)^{n-k}, \quad p = \text{probability of success}$$

probabity of 2 clubs in 7 selections

$$= \binom{7}{2}\left(\frac{13}{52}\right)^2\left(1-\frac{13}{52}\right)^{7-2}$$

P(3rd club is drawn on the 8th selection)

= P(2 clubs in 7 trials) P(club on 8th trial)

$$= \binom{7}{2}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{7-2}\times\left(\frac{1}{4}\right) = \underline{\underline{0.0779}}$$

(b) Find the probability that atleast 8 cards are drawn before the 3rd club appears.

Sol:- P(atleast 8 cards drawn before 3rd club)

= P(we draw either zero, one or two clubs in 8 trials)

$$= \binom{8}{0}\left(\frac{1}{4}\right)^0\left(1-\frac{1}{4}\right)^{8-0} + \binom{8}{1}\left(\frac{1}{4}\right)^1\left(1-\frac{1}{4}\right)^{8-1}$$

$$+\binom{8}{2}\left(\frac{1}{4}\right)^2\left(1-\frac{1}{4}\right)^{8-2} = 0.7119$$

check

Note that

$$\boxed{\sum_x p_X(x) = 1}$$

In the previous example of tossing coin 2 times, the PMF is

$$p_X(x) = \begin{cases} 1/4 & x=0 \\ 1/2 & x=1 \\ 1/4 & x=2 \\ 0 & \text{otherwise} \end{cases}$$

X = # of heads

$X = 0, 1, 2.$

$$P(\text{at least one head}) = P(X=1) + P(X=2) = \sum_{x=1}^{2} p_X(x)$$

$$P(X>0) = \frac{1}{2}+\frac{1}{4} = \frac{3}{4}$$

Bernoulli Random variable :-

X is said to be Bernoulli random variable if

$$X = \begin{cases} 1, & \text{if success} \\ 0, & \text{if fail} \end{cases}$$

Suppose probability of success is p

Its PMF is

$$p_X(k) = \begin{cases} p & \text{if } k=1 \\ 1-p & \text{if } k=0 \end{cases}$$

Example:- ① A telephone at a given time can be either free or busy.
② A person can be healthy or sick.

Binomial random variable:- Let X be the number of success in a n-trials. X is called Binomial random variable with parameters n & p & the PMF of X is

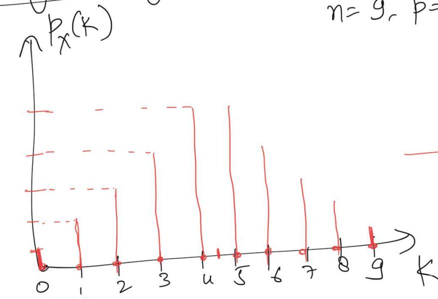
$$p_X(k) = P(X=k)$$
$$= \binom{n}{k} p^k (1-p)^{n-k}, \quad k=0,1,\ldots,n \quad \text{—} ⊛$$

The normalization property give

$$\sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} = 1$$

Plot of PMF of Binomial random variable:-

$n=9$, $p=\frac{1}{2}$



— Fig ①

① Homework:- We know that

$$p_X(k) = \binom{n}{k} p^k (1-p)^{n-k} \qquad n=9, \; p=\frac{1}{2}$$

Put $k=0,1,2,\ldots,9$ find $p_X(k)$ from above eqⁿ & plot.

② Home work:- Think about the pattern of Fig ① when n is large & p is small.

The Geometric random variable:-

Geometric Random variable:- The geometric random variable is the number X of trials to get the first success. Its PMF is given by $X = 1, 2, 3, \ldots\ldots$

Suppose probability of success is p.

$$P(X=1) = p$$

$$P(X=2) = (1-p)\cdot p$$

$$\vdots$$

$$P(X=k) = \boxed{p_X(k) = (1-p)^{k-1}\cdot p, \quad k=1,2,3\ldots}$$

Prove that:

$$\sum_{k=1}^{\infty} p_X(k) = 1.$$

Proof:

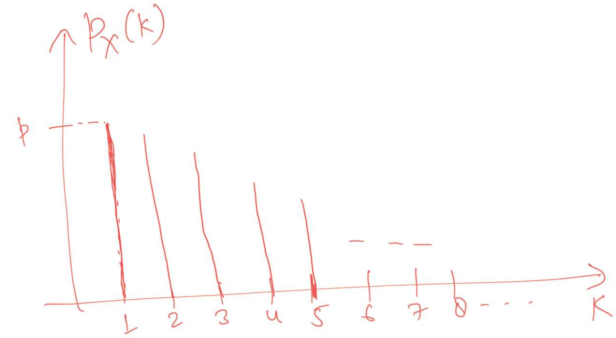
$$\sum_{k=1}^{\infty} p_X(k) = \sum_{k=1}^{\infty} (1-p)^{k-1}\cdot p$$

$$= p \sum_{k=1}^{\infty} (1-p)^{k-1}$$

$$= p\left[(1-p)^0 + (1-p)^1 + (1-p)^2 + \ldots\ldots\right]$$

$$= p\left[1 + (1-p) + (1-p)^2 + \ldots\ldots\right]$$

$$= p\cdot\left[\frac{1}{1-(1-p)}\right] = 1$$



The Poisson Random variable:- A Poisson random variable has the following PMF

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots\ldots \quad (\star)$$

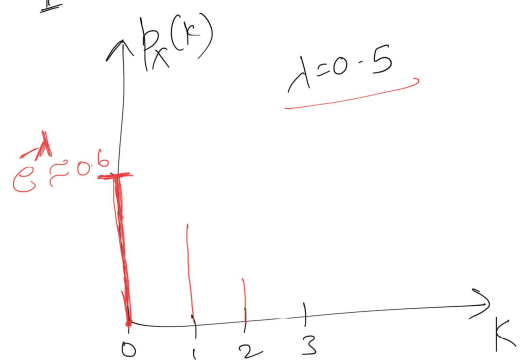
& $\lambda > 0$.

Prove that: $\displaystyle\sum_{k=0}^{\infty} \frac{e^{-\lambda}\lambda^k}{k!}$

$$= e^{-\lambda}\underbrace{\left(1 + \lambda + \frac{\lambda^2}{\lfloor 2} + \ldots\ldots\right)} = e^{-\lambda}\cdot e^{\lambda}$$

$$= 1$$

Ex:- $\lambda = 0.5$



Home Work:- Plot PMF of Poisson for $\lambda = 3$.

Poisson random variable (Continued).

Take a binomial random variable with very small p & very large n.

For example:- Let X be the number of typos in a book with a total of n words. Then X is binomial random variable.

The Poisson PMF with parameter λ is a good approximation for a binomial PMF with parameters n & p, i.e.

$$\frac{e^{-\lambda}\lambda^{k}}{k!} \approx \binom{n}{k} p^{k}(1-p)^{n-k} \quad \text{— (1)}$$

$k = 0, 1, \ldots, n$

provided $\lambda = np$.

Example:- $n = 100$, $p = 0.01$, then probability of 5 successes in 100 trials is

binomial $\rightarrow$ $\binom{100}{5}(0.01)^{5}(1-0.01)^{100-5}$

$= 0.0029$

Using Poisson PMF with $\lambda = np = 100 \times 0.01 = 1$

$$P(X=5) = \frac{e^{-\lambda}\lambda^{k}}{k!} = \frac{e^{-1}}{5!} = 0.0030$$

Function of random variables:- Let X be a random variable. If $Y = g(X)$ is a function of X, then Y is also a random variable, since it provides a numerical value for each possible outcome.

If X is discrete the Y is also discrete random variable. The PMF of Y is calculated from PMF of X.

The PMF of Y is

$$\boxed{p_Y(y) = \sum_{\{x \mid g(x)=y\}} p_X(x)} \quad \text{— (2)}$$

Now, we will find the PMF of $Y = g(X)$ as follows:

Example:- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{1}{9} & \text{if } x \text{ is an integer in } [-4, 4] \\ 0 & \text{otherwise.} \end{cases}$$

Let $Y = |X|$ $\Rightarrow$ Y is a discrete random variable.

Possible value of $Y = 0, 1, 2, 3, 4$.

$P(Y=0) = P(X=0) = \frac{1}{9}$

$P(Y=1) = P(X=1) + P(X=-1) = \frac{2}{9}$

$P(Y=2) = P(X=2) + P(X=-2) = \frac{2}{9}$

$P(Y=3) = P(X=3) + P(X=-3) = \frac{2}{9}$

$P(Y=4) = P(X=4) + P(X=-4) = \frac{2}{9}$

The PMF of Y is

$$p_Y(y) = \begin{cases} \frac{1}{9} & y=0 \\ 2/9 & y=1,2,3,4 \\ 0 & \text{otherwise.} \end{cases}$$

Home work :- Find the PMF of

$$Z = X^2.$$

Tutorial class

Qu 8:- A sofware manufacturer knows that one out of 10 software games that the company markets will be a financial success. The manufacturer selects 10 new games to market. what is the probability that exactly one game will be a financial success. what is the probability that atleast 2-games will be a success.

Solu^n:- $p = P(\text{success}) = \frac{1}{10}$

ⓐ $P(1 \text{ success}) = \binom{10}{1} p^1 (1-p)^{10-1} = 0.38$

ⓑ $P(\text{atleast 2 success}) = P(2 \text{ success}) + P(3 \text{ success}) + \dots + P(10 \text{ success})$

$= 1 - P(0 \text{ success}) - P(1 \text{ success})$

$P(0 \text{ success}) = \binom{10}{0} p^0 (1-p)^{10-0} = 0.34$

$P(1 \text{ success}) = 0.38$

$$\boxed{P(\text{atleast 2 success}) = 0.28}$$

Qus 9:- In pulse code modulation (PCM), a PCM word consists of a sequence of binary digits of 1^s & 0^s.

ⓐ Suppose the PCM word length is n bits long. How many distinct words are there?

Solu^n:-

2bits long → 10 01 00 11 → 2^2

3bits long → $\underbrace{100 \quad 101 \quad 110 \quad 111 \quad 001 \quad 011 \quad 010, \quad 000}$ → 2^3

n bits long → 2^n.

ⓑ If each PCM word, three bits long, is equal likely to occur what is the probability of a word with exactly two 1^s occurring?

Method 1

$P(\text{two } 1^s) = P(110) + P(101) + P(011)$

$= \frac{1}{8} + \frac{1}{8} + \frac{1}{8} = \frac{3}{8}$

Method 2.

$P(\text{success}) = P(\text{getting one}) = \frac{1}{2}$

$P(2 \text{ ones}) = \binom{3}{2} \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}$

Ques 11:- A balanced coin is tossed nine times.

(a) $P(\text{exactly } 3 \text{ heads}) = \binom{9}{3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{9-3} = 0.164$

(b) $P(\text{atleast } 3 \text{ heads}) = P(3 \text{ heads}) + P(4 \text{ heads}) + \cdots + P(9 \text{ heads})$

$$= \sum_{i=3}^{9}\binom{9}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{9-i}$$

$= 1 - P(0 \text{ head}) - P(1 \text{ head}) - P(2 \text{ head})$

$= 0.91$

(c) $P(\text{atleast } 3 \text{ heads } \& \text{ atleast } 2 \text{ tails})$

$= P(3 \leq \text{no. of heads} \leq 7)$

$$= \sum_{i=3}^{7}\binom{9}{i}\left(\frac{1}{2}\right)^i\left(\frac{1}{2}\right)^{9-i} = 0.89$$

Expectation or Mean of a random variable:

We define the expected value of a random variable X as follows:-

$$\boxed{E[X] = \sum_x x\, p_X(x)}$$

where p_X is PMF of X.

Example:- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{1}{9} & x \text{ is an integer and } x \in [-4, 4] \\ 0 & \text{otherwise.} \end{cases}$$

Find $E[X]$?

Solⁿ:- Possible values of $X = -4, -3, -2, -1, 0, 1, 2, 3, 4 = x$

$E[X] = \left(-4\times\frac{1}{9}\right) + \left(-3\times\frac{1}{9}\right) + \left(-2\times\frac{1}{9}\right) + \left(-1\times\frac{1}{9}\right) + \left(0\times\frac{1}{9}\right) + \left(1\times\frac{1}{9}\right) + \left(2\times\frac{1}{9}\right) + \left(3\times\frac{1}{9}\right) + \left(4\times\frac{1}{9}\right)$

$= 0$

Expectation of Random variable X is zero

Example:-2 A coin is tossed two times, each with a probability 3/4 for head.

$X =$ number of heads.

Possible values of $X = 0, 1, 2$

$$p_X(x) = \begin{cases} \frac{1}{4}\times\frac{1}{4} & x=0 \\ \frac{3}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{3}{4} & x=1 \\ \frac{3}{4}\times\frac{3}{4} & x=2 \end{cases} \quad (A)$$

$$E[X] = 0\times\left(\frac{1}{4}\right)^2 + 1\cdot\left(\frac{6}{16}\right) + 2\times\frac{9}{16}$$

$$= \frac{24}{16} = \frac{3}{2} = 1.5 \quad (AA)$$

Remark:- Average $= \dfrac{\text{Sum of observations}}{\text{total no. of observations.}}$

If $X = 1, 2, 3, 4, 5$ with probability $\frac{1}{5}$ each

$$E[X] = 1\times\frac{1}{5} + 2\times\frac{1}{5} + \frac{3\times 1}{5} + 4\times\frac{1}{5} + 5\times\frac{1}{5} = 3$$

Average of $(1, 2, 3, 4, 5) = 3$

Variance:- Variance of random variable X is defined as

$$Var(X) = E\left[X - E[X]\right]^2 \quad (3)$$

Example:- Let X be a random variable with PMF

$$p_X(x) = \begin{cases} \frac{1}{9} & x \text{ is integer } \& \ x\in[-4,4] \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = 0$$

$$Var(X) = E\left[X - E[X]\right]^2$$

$$= E\left[X - 0\right]^2 = E[X^2]$$

So we need PMF of X^2. Let $Z = X^2$

$$p_Z(z) = \begin{cases} \frac{1}{9} & z=0 \\ \frac{2}{9} & z = 1, 4, 9, 16 \\ 0 & \text{otherwise} \end{cases}$$

$$E[Z] = 0\times\frac{1}{9} + 1\times\frac{2}{9} + 4\times\frac{2}{9} + 9\times\frac{2}{9} + 16\times\frac{2}{9}$$

$$= \frac{60}{9}.$$

$$\Rightarrow \boxed{Var(X) = \frac{60}{9}.}$$

Variance (Continued): we know that

$$Var(X) = E\left[\left(X - E[X]\right)^2\right] \quad — (*)$$

(#) Variance is always non-negative.

Explanation:- Since $\left(X - E[X]\right)^2$ is non-negative

Standard deviation:- It is square root of variance & denoted by σ_X.

$$\boxed{\sigma_X = \sqrt{Var(X)}}$$

Remark:- To find variance

compute $E[X]$

compute $E\left[\left(X - E[X]\right)^2\right]$

For this we need PMF of $\left(X - E[X]\right)^2$

Another method to compute variance will not require the PMF of $\left(X - E[X]\right)^2$

Expectation of function of random variable:

Let X be a random variable with PMF p_X & let $g(X)$ be the function of X. The

$$\boxed{E[g(X)] = \sum_x g(x)\, p_X(x)} \quad — (1)$$

Example:- Consider the previous example:-

$$p_X(x) = \begin{cases} \frac{1}{9} & x \in [-4, 4] \\ 0 & \text{otherwise} \end{cases}$$

$$\boxed{Var(X) = E\Big[\underbrace{\left(X - E[X]\right)^2}_{= g(X)}\Big] = E[g(X)]}$$

$$= \sum_x g(x)\, p_X(x)$$

we know that $E[X] = 0$

$$= \sum_x (x - 0)^2 p_X(x) = \sum_x x^2 p_X(x)$$

$$= (-4)^2 \times \frac{1}{9} + (-3)^2 \times \frac{1}{9} + (-2)^2 \times \frac{1}{9} + (-1)^2 \times \frac{1}{9}$$
$$+ (0)^2 \times \frac{1}{9} + (1)^2 \times \frac{1}{9} + (2)^2 \times \frac{1}{9} + (3)^2 \times \frac{1}{9} + (4)^2 \times \frac{1}{9}$$

$$= \frac{60}{9}$$

Properties of Mean & variance:-

Let X be a random variable & let

$$Y = aX + b = g(X) \quad (a, b \text{ are scalars})$$

$$E[Y] = E[g(X)] = \sum_x (ax + b)\, p_X(x)$$

$$= a \sum_x x\, p_X(x) + b \sum_x p_X(x)$$

$$= a\, E[X] + b \cdot 1.$$

$$\Rightarrow \boxed{E[Y] = a\, E[X] + b} \quad — (2)$$

Prove that $Var(Y) = a^2\, Var(X)$

$$Var(Y) = \sum_x \left(ax + b - E[aX + b]\right)^2 p_X(x)$$

$$\because Var(g(X)) = E\left[\left(g(X) - E[g(X)]\right)^2\right]$$

$$Var(Y) = \sum_x \left(ax + \cancel{b} - aE[X] - \cancel{b}\right)^2 p_X(x)$$

$$= a^2 \underbrace{\sum_x \left(x - E[X]\right)^2 p_X(x)}$$

$$= a^2\, E\left[\left(X - E[X]\right)^2\right]$$

$$\boxed{Var(Y) = a^2\, Var(X)}$$

Alternative formula for variance:-

$$Var(X) = E[X^2] - (E[X])^2 \quad \text{—— ①}$$

$$Var(X) = \sum_x (x - E[X])^2 \, p_X(x)$$

$$= \sum_x \left(x^2 - 2x\,E[X] + (E[X])^2 \right) p_X(x)$$

$$= \underbrace{\sum_x x^2 p_X(x)} - 2E[X]\underbrace{\sum_x x\,p_X(x)} + (E[X])^2 \underbrace{\sum_x p_X(x)}$$

$$= E[X^2] - 2E[X]\,E[X] + (E[X])^2 \cdot 1$$

$$= E[X^2] - 2(E[X])^2 + (E[X])^2$$

$$= E[X^2] - (E[X])^2$$

Random variables $\longrightarrow$ PMF $\longrightarrow$ 4 types of random variables

$\longrightarrow$ Function of random variables $\longrightarrow$ Mean & Variance.

Mean & Variance of Bernoulli random variable

Let X be a Bernoulli random variable

$$p_X(x) = \begin{cases} p & x = 1 \\ 1-p & x = 0 \end{cases}$$

Mean: $E[X] = 1 \cdot p + 0 \cdot (1-p) = p$

Variance: $\boxed{E[X^2] = \sum_x x^2 p_X(x)}$

$$= (1)^2 \times p + (0)^2 (1-p) = p$$

$$\therefore \boxed{Var(X) = E[X^2] - (E[X])^2}$$

$$= p - p^2 = p(1-p)$$

Ex:- Mean and variance of Poisson random variable:- The PMF of Poisson random variable is

$$p_X(k) = \frac{e^{-\lambda}\lambda^k}{k!} \qquad k = 0, 1, 2, \ldots$$

Mean:- $E[X] = \sum_{k=0}^{\infty} k \frac{e^{-\lambda}\lambda^k}{k!}$

for $k=0$ 1st term is zero

$$= \sum_{k=1}^{\infty} \frac{k\, e^{-\lambda}\lambda^k}{k!}$$

$k! = k \cdot (k-1)!$

$$= \lambda \underbrace{\sum_{k=1}^{\infty} e^{-\lambda}\frac{\lambda^{k-1}}{(k-1)!}} \quad \text{—— ③}$$

$i = k-1$

$$= \lambda \underbrace{\sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^{i}}{i!}} \longrightarrow \text{summation of PMF is 1}$$

$$= \lambda \cdot 1 = \lambda.$$

Find $Var(X)$ $\longrightarrow$ Do it yourself.

(*) Joint PMF of Multiple random variables

Consider two discrete random variables X & Y associated with same experiment

Then joint PMF of X & Y is

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

Ex:- $X = 1,2,3$

$Y = 1,2,3$

$$\underbrace{p_{X,Y}(x,y)}_{\text{Joint PMF}} = \begin{cases} \frac{1}{9} & x=1, y=1 \\ \frac{1}{9} & x=2, y=1 \\ \frac{1}{9} & x=3, y=1 \\ \frac{1}{9} & x=1, y=2 \\ \frac{1}{9} & x=2, y=2 \\ \frac{1}{9} & x=3, y=2 \\ \frac{1}{9} & x=1, y=3 \\ \frac{1}{9} & x=2, y=3 \\ \frac{1}{9} & x=3, y=3 \end{cases}$$

Ques:- Can we get PMF of X & Y from this?

Ex:- $X = 1, 2, 3, 4$, $Y = 1, 2, 3, 4$ } Suppose we have joint PMF of X & Y as follows:-

(PMF of Y Row Sum)

y \ x	1	2	3	4	Row Sum	
4	0	1/20	1/20	1/20	3/20	$P(Y=4)$
3	1/20	2/20	3/20	1/20	7/20	$P(Y=3)$
2	1/20	2/20	3/20	1/20	7/20	$P(Y=2)$
1	1/20	1/20	1/20	0	3/20	$P(Y=1)$

sum = 1

(PMF of X) Column Sum

	$P(X=1)$	$P(X=2)$	$P(X=3)$	$P(X=4)$
Column Sum	3/20	6/20	8/20	3/20

sum → 1

$$p_X(x) = \sum_y p_{X,Y}(x,y) \quad \text{— (7)}$$

$$p_Y(y) = \sum_x p_{X,Y}(x,y) \quad \text{— (8)}$$

The PMF $p_X(x)$ & $p_Y(y)$ are reffered to as marginal PMFs of X & Y respectively.

Function of multiple random variable:-

Let $Z = g(X,Y)$

$$p_Z(z) = \sum_{\{(x,y) \mid g(x,y)=z\}} p_{X,Y}(x,y) \quad \text{— (1)}$$

Example:- Consider the previous example of joint PMF given to us:-

$X = 1,2,3,4$

$Y = 1,2,3,4$.

y \ x	1	2	3	4
4	0	1/20	1/20	1/20
3	1/20	2/20	3/20	1/20
2	1/20	2/20	3/20	1/20
1	1/20	1/20	1/20	0

$\boxed{\text{Find the PMF of } Z = X + 2Y \text{ ?}}$ — (A)

Possible values of $Z = 3, 4, \ldots, 12$

$P(Z=3) = \frac{1}{20}$ $(x=1, y=1)$

$P(Z=4) = \frac{1}{20}$ $(x=2, y=1)$

$P(Z=5) = \frac{1}{20} + \frac{1}{20}$ $(x=3, y=1)$ $(x=1, y=2)$

$P(Z=6) = 2/20$

$P(Z=7) = 4/20$

$P(Z=8) = 3/20$

$P(Z=9) = 3/20$

$P(Z=10) = 2/20$

$P(Z=11) = 1/20$

$P(Z=12) = \frac{1}{20}$

⇒ Sum is 1 PMF.

$$E[Z] = 3 \times \frac{1}{20} + 4 \times \frac{1}{20} + 5 \times \frac{2}{20} + 6 \times \frac{2}{20} + 7 \times \frac{4}{20} + 8 \times \frac{3}{20} + 9 \times \frac{3}{20} + 10 \times \frac{2}{20}$$

$$+ 11 \times \frac{1}{20} + 12 \times \frac{1}{20} = 7.55$$

Ques? Can we directly find the $E[Z]$ without calculating PMF of Z? Yes.

$$Z = X + 2Y \quad (3)$$

$$\boxed{E[Z] = E[X] + 2E[Y]}$$

we know

$$p_X(x) = \begin{cases} 3/20 & x=1 \\ 6/20 & x=2 \\ 8/20 & x=3 \\ 3/20 & x=4 \end{cases}$$

$$p_Y(y) = \begin{cases} 3/20 & y=1 \\ 7/20 & y=2 \\ 7/20 & y=3 \\ 3/20 & y=4 \end{cases}$$

$$E[X] = \frac{51}{20} \quad \left(1\times\frac{3}{20} + 2\times\frac{6}{20} + 3\times\frac{8}{20} + 4\times\frac{3}{20}\right)$$

$$E[Y] = \frac{50}{20} \quad \left(1\times\frac{3}{20} + 2\times\frac{7}{20} + 3\times\frac{7}{20} + 4\times\frac{3}{20}\right)$$

$$E[Z] = E[X] + 2E[Y]$$

$$= \frac{51}{20} + 2 \times \frac{50}{20} = 7.55.$$

In general :- $X_1 \ldots X_n$ are random variables

$$E[a_1 X_1 + a_2 X_2 + \ldots + a_n X_n]$$

$$= a_1 E[X_1] + a_2 E[X_2] + \ldots + a_n E[X_n]$$

Where $a_1, a_2, \ldots, a_n$ are constants.

Mean of Binomial :-

Example:- Your CDA101 class has 300 students, each student has probability $\frac{1}{3}$ of getting an 'A', independent of any other student. Let X be the random variable defined as

X = number of students that get an 'A', Find $E[X]$.

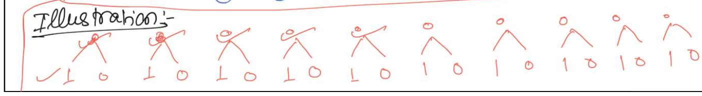
Let $X_1, X_2, \ldots, X_{300}$ represents the students $1, 2, \ldots 300$ respectively. Define.

$$X_i = \begin{cases} 1 & \text{getting A grade} \\ 0 & \text{otherwise} \end{cases}$$

Each X_i is a Bernoulli random variable.

$$E[X_i] = p = \frac{1}{3}$$

Illustration:-



total A grade $= X_1 + X_2 + \cdots + X_{10}$

$= 1+0+0+1+0+0+0+1+1+0$

$= \underline{4}$.

we have

$$X = X_1 + X_2 + \cdots + X_{300}$$

Since X is the number of success in 300 independent trials, it is a binomial random variable

$$E[X] = E[X_1] + E[X_2] + \cdots + E[X_{300}]$$
$$= \frac{1}{3} + \frac{1}{3} + \cdots + \frac{1}{3} = 100$$

In general, if we have n bernoulli random varibles having probability of Success p, then

$$\boxed{E[X] = \sum_{i=1}^{n} E[X_i] = np}$$

The mean of binomial random variable is np.

Conditioning:- Let X & Y be two random variables associated with same experiment.

The conditional probability

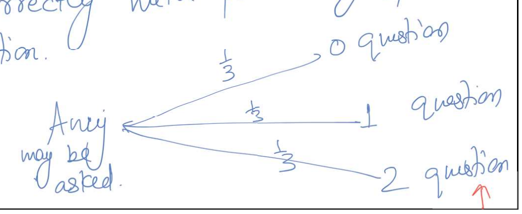
$$p_{X|Y}(x|y) = P(X=x \mid Y=y) \quad \text{—(1)}$$

From conditional probability definition.

$$p_{X|Y}(x|y) = \frac{P(X=x, Y=y)}{P(Y=y)}$$

$$\boxed{p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}} \quad \text{—(2)}$$

Ques:- Prof. Anuj answer each of his students question incorrectly with probability $\frac{1}{4}$ independent of other question.



X = number of questions he is asked

Y = " " " he answered incorrectly.

Joint PMF of X & Y.

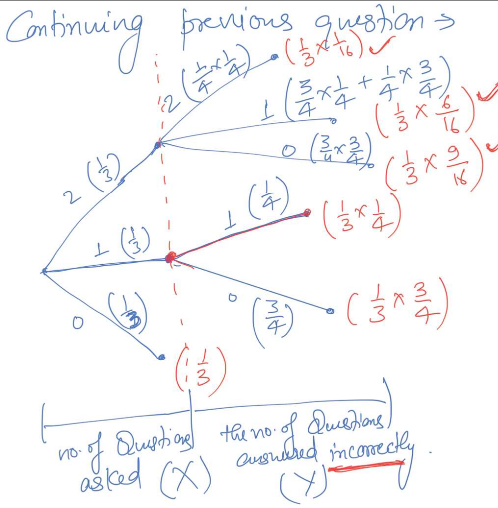
$y \backslash x$	0	1	2
2	0	0	
1	0	$1/12$	
0			

$$p_{X,Y}(1,1) = P(X=1, Y=1)$$
$$= p_{Y|X}(y|x)\, p_X(x)$$
$$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$$

Do it yourself:

Complete the above table.

Continuing previous question →



$y \backslash x$	0	1	2
2	0	0	$\frac{1}{48}$
1	0	$\frac{1}{12}$	$\frac{2}{16}$
0	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{3}{16}$

Ans:- $P(\text{at least one wrong answer})$

$$= P(X=1, Y=1) + P(X=2, Y=1) + P(X=2, Y=2) = \frac{1}{12} + \frac{2}{16} + \frac{1}{48} = \frac{11}{48}.$$

Conditional Expectation:-

Let X & Y be two random variables with the same experiment. Then

$$\boxed{E[X \mid Y=y] = \sum_x x\, p_{X|Y}(x|y)} \quad (1)$$

Result:- If $A_1, A_2, \dots A_n$ be disjoint events that form a partition of the sample space with $P(A_i) > 0$ for all i, then

$$\boxed{E[X] = \sum_{i=1}^{n} P(A_i)\, E[X \mid A_i]} \quad (2)$$

$\Omega = \{HH, HT, TH, TT\}$
$A_1 = \{HH\}$
$A_2 = \{HT, TH\}$
$A_3 = \{TT\}$.

Mean of Geometric Random variable:-

Example:- Mr. Pawan is writing a code for some software. The probability that it works correctly is p (independent of previous attempts)

Let X = number of attempts until the code works correctly

$\Rightarrow$ X is a geometric random variable with PMF

$$P(X=k) = p_X(k) = (1-p)^{k-1} \cdot p \qquad k = 1, 2, \dots$$

The mean of random variable X

$$E[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p \longrightarrow$$

(which is difficult to evaluate)

Define:- $A_1 = \{X=1\} = \{\text{first attempt is a success}\}$

$A_2 = \{X>1\} = \{\text{first attempt is a failure}\}$

from previous result, we can write

$$E[X] = P(A_1) \cdot E[X \mid A_1] + P(A_2)\, E[X \mid A_2] \quad (1)$$

$$= P(X=1)\, E[X \mid X=1] + P(X>1)\, E[X \mid X>1] \quad (2)$$

$$E[X] = p\, \underbrace{E[X \mid X=1]} + (1-p)\, \underbrace{E[X \mid X>1]} \quad (3)$$

$E[X \mid X=1] = ?$

If the 1st try is successful, we have $X=1$.

$$E[X \mid X=1] = 1 \quad (*)$$

If the 1st try fails $(X>1)$, we have wasted one try & we are back where we started. So the expected number of remaining tries is $E[X]$ &

$$E[X \mid X>1] = 1 + E[X] \quad (**)$$

From Eq (3)

$$E[X] = p\, E[X \mid X=1] + (1-p)\, E[X \mid X>1] \quad (4)$$

$$= p \cdot 1 + (1-p)(1 + E[X])$$

$$\cancel{E[X]} = p + (1 + \cancel{E[X]}) - p(1 + E[X])$$

$$\boxed{E[X] = \frac{1}{p}}$$

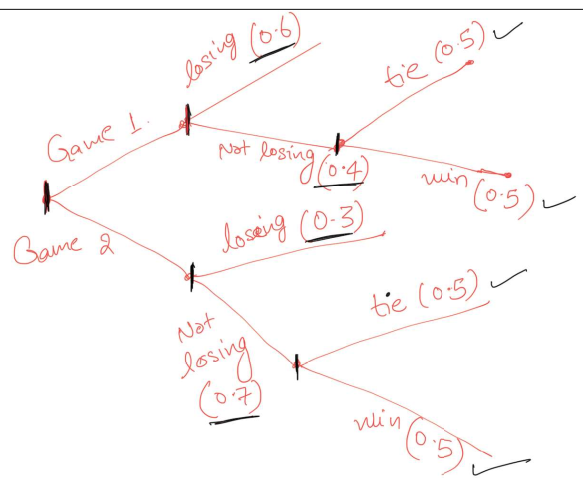
5

Ques 13 :- The MIT soccer team has 2 games scheduled for one weekend. It has a 0.4 probability of not losing 1st game & 0.7 probability of not losing the second game independent of 1st. If it does not lose a particular game, it is equal likely to win or tie, independent of what happens in other games. The team will receive 2 points for a win, 1 for a tie & 0 for a loss. Find the PMF of random variable X where X is the number of points that the team earns over the weekend.

Sol: Possible values of X

$= 0, 1, 2, 3, 4$

PMF. $P(X = 0, 1, 2, 3, 4) = ?$



$P(X=0)$ = losing both the games

$= 0.6 \times 0.3 = 0.18$

$P(X=1)$ = losing 1st & 2nd tie or losing 2nd & 1st tie

$= 0.6 \times 0.7 \times 0.5 + 0.3 \times 0.4 \times 0.5$

$= 0.27$

$P(X=2)$ = 1st win 2nd loss or 2nd win 1st loss or Both tie.

$= 0.4 \times 0.5 \times 0.3 + 0.7 \times 0.5 \times 0.6 + 0.4 \times 0.5 \times 0.7 \times 0.5$

$= 0.34$

$P(X=3)$ = (1st tie & 2nd win) or (2nd tie & 1st win)

$= 0.4 \times 0.5 \times 0.7 \times 0.5 + 0.7 \times 0.5 \times 0.4 \times 0.5 = 0.14$

$P(X=4)$ = Both win

$= 0.4 \times 0.5 \times 0.7 \times 0.5 = 0.07$

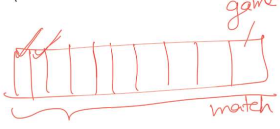
$P(X>4) = 0$

$$P_X(x) = \begin{cases} 0.18 & x=0 \\ 0.27 & x=1 \\ 0.34 & x=2 \\ 0.14 & x=3 \\ 0.07 & x=4 \\ 0 & \text{otherwise} \end{cases}$$

Ques 14 :-
(Do it yourself)

Ques 14 :- Fischer & Spassky play a chess match in which the first player to win a game wins the match. After 10 successive draws, the match is declared drawn. Each game is won by Fischer with proba. 0.4, is won by Spassky with probability 0.3 & is a draw with 0.3 probability.

① What is the probability that Fischer wins the match?



Solu^n :-

$$P(\text{Fischer wins}) = \underset{W}{0.4} + \underset{D}{0.3} \times \underset{W}{0.4} + \underset{D}{0.3} \times \underset{D}{0.3} \times \underset{W}{0.4} + \dots$$

$$= \sum_{n=1}^{10} (0.3)^{n-1} (0.4)$$

② What is the PMF of the duration of the match?

X = no. of games played. $\Rightarrow$ possible values of X = 1, 2, --- 10.

$$P(X=x) = p_X(x) = \begin{cases} 0.7 & x=1 \\ 0.3 \times 0.7 & x=2 \\ (0.3)^2 \times 0.7 & x=3 \\ (0.3)^3 \times 0.7 & x=4 \\ (0.3)^4 \times 0.7 & x=5 \\ (0.3)^5 \times 0.7 & x=6 \\ (0.3)^6 \times 0.7 & x=7 \\ (0.3)^7 \times 0.7 & x=8 \\ (0.3)^8 \times 0.7 & x=9 \\ (0.3)^9 & x=10 \end{cases}$$

Independence of random variables :-

Two random variables are independent if

$$\boxed{p_{X,Y}(x,y) = p_X(x) \cdot p_Y(y) \quad \forall\, x, y}$$

Result :- If X & Y are independent then

$$\boxed{E[XY] = E[X]\cdot E[Y]}$$

Proof:

$$E[XY] = \sum_x \sum_y xy\, p_{X,Y}(x,y) \quad \text{--- (1)}$$

$$= \sum_x \sum_y xy\, p_X(x)\, p_Y(y)$$

$$= \sum_x x\, p_X(x) \sum_y y\, p_Y(y)$$

$$\boxed{E[XY] = E[X]\, E[Y]} \quad \text{--- (2)}$$

Result:- If $X_1, X_2, \ldots, X_n$ are independent random variables then

$$Var(X_1 + X_2 + \ldots + X_n)$$

$$= Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

Remark:- Take two independent random variable X & Y

$$\boxed{Var(X+Y) = Var(X) + Var(Y)} \quad \text{--- (5)}$$

Tutorial - 4

Ques 4:-

Prove that $E[X^2] \geq (E[X])^2$. When do we have equality?

Solu:- $Var(X) = E[X^2] - (E[X])^2$

we know that $Var(X) \geq 0$

$$E[X^2] - (E[X])^2 \geq 0$$

$$\Rightarrow \quad E[X^2] \geq (E[X])^2$$

we have $E[X^2] = (E[X])^2$ when $Var(X) = 0$.

Ques 7:- A coin is biased such that a head is three times as likely to occur as a tail. Find the expected number of tails when this coin is tossed twice.

Sol:- X = # of tails that occur in two tosses. $= 0, 1, 2$

$P(H) = \frac{3}{4}$ $\quad P(T) = \frac{1}{4}$

we need to find $E[X] = ?$

$\downarrow$

PMF of X.

$$p_X(x) = \begin{cases} \frac{3}{4} \times \frac{3}{4} & x = 0 \\ \frac{3}{4}\times\frac{1}{4} + \frac{1}{4}\times\frac{3}{4} & x = 1 \\ \frac{1}{4}\times\frac{1}{4} & x = 2 \end{cases}$$

$$E[X] = 0 \times \frac{9}{16} + 1 \times \left(\frac{6}{16}\right) + 2 \times \frac{1}{16}$$

$$\boxed{E[X] = \frac{1}{2}}$$

Ques 9:- Suppose an antique Jewellery dealer is interested in purchasing a gold necklace for which the probabilities are 0.22, 0.36, 0.28 & 0.14 respectively, that she will be able to sell it for profit of 250\$, sell it for profit of 150\$, break even, or sell it for a loss of \$150. What is her expected profit?

Sol:- X = profit she makes

$$E[X] = 250 \times 0.22 + 150 \times 0.36 + 0 \times 0.28 + (-150) \times 0.14$$

$$= 88\$.$$

Ques 10:- A lot containing 7 components is sampled by a quality inspector, the lot contains 4 good components & 3 defective components. A sample of 3 is taken by the inspecter. Find the expected value of the number of good components in this sample.

Sol:- X = number of good components in the sample.

$X = 0, 1, 2, 3$

$$p_X(x) = \begin{cases} {}^4C_0 \cdot {}^3C_3 / {}^7C_3 & x = 0 \\ {}^4C_1 \cdot {}^3C_2 / {}^7C_3 & x = 1 \\ {}^4C_2 \cdot {}^3C_1 / {}^7C_3 & x = 2 \\ {}^4C_3 \cdot {}^3C_0 / {}^7C_3 & x = 3 \end{cases}$$

$P(X=0) = \frac{1}{35}$, $P(X=1) = \frac{12}{35}$, $P(X=2) = \frac{18}{35}$

$P(X=3) = 4/35$

$$E[X] = 0 \times \frac{1}{35} + 1 \times \frac{12}{35} + 2 \times \frac{18}{35} + 3 \times \frac{4}{35} = \frac{12}{7} = 1.7$$

Ques 13:- An industrial process manufatures items that can be classified as either defective or not defective. The probability that item is defective is 0.1. An experiment is conducted in which 5 items are drawn randomly from the process. Let X be the number of defectives in this sample of 5. What is PMF of X.

Sol:- $X = 0, 1, 2, 3, 4, 5$

$$P(X=0) = \binom{5}{0} (0.1)^0 (1-0.1)^{5-0}$$

$$P(X=1) = \binom{5}{1} (0.1)^1 (1-0.1)^{5-1}$$

$\vdots$

Continuous random variables:

A random variable X is called continuous if there is a $\boxed{\text{non negative function}}$ f_X, such that

$$P(X \in B) = \int_B f_X(x)\,dx \qquad \text{— } \textcircled{\star}$$

subset of real line.

or

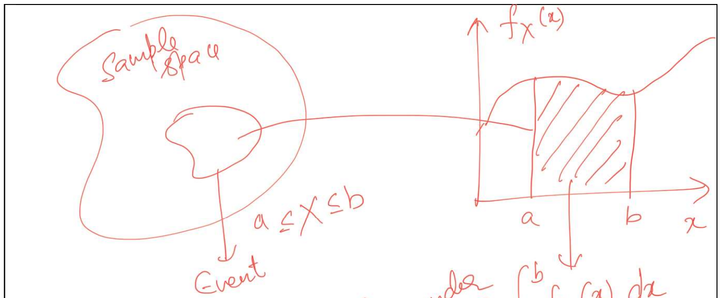
$$P(a \le X \le b) = \int_a^b f_X(x)\,dx.$$

The function $f_X(x)$ is called the probability density function (PDF) of X.

$$P(1 \le X \le 2) = \int_1^2 f_X(x)\,dx$$

$$P(2 \le X \le 3) = \int_2^3 f_X(x)\,dx$$

$\vdots$



Area under the curve $= \int_a^b f_X(x)\,dx$

Ex:- X = speed of car

$$P(30 \le X \le 40) = \int_{30}^{40} f_X(x)\,dx$$

$$P(70 \le X \le 80) = \int_{70}^{80} f_X(x)\,dx$$

$\vdots$

$$P(a \le X \le b) = P(a < X < b) = P(a \le X < b)$$
$$= P(a < X \le b)$$

$$\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1} = P(-\infty < X < \infty)$$

↳ This is known as normalization property.

Uniform random variable:-

Consider a random variable X that takes values in an interval $[a, b]$, assuming that any two subintervals of same length have same probability. Such random variables are called uniform random variable.

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases} \qquad \textcircled{\star\star}$$

(i) Non-negative.

(ii) $$\int_{-\infty}^{\infty} f_X(x)\,dx = \underbrace{\int_{-\infty}^{a} 0\,dx}_{=0} + \int_a^b \frac{1}{b-a}\,dx + \underbrace{\int_b^{\infty} 0\,dx}_{=0}$$

normalization

$$= \frac{1}{b-a}\left[b-a\right] = 1.$$

(3) Suppose X takes values in $[0, 10]$

$a = 0,\ b = 10$

subintervals

$$P(5 < X \le 6) = \frac{1}{10}$$

$$P(7 \le X \le 10) = \frac{3}{10}$$

$$= \int_5^6 f_X(x)\,dx = \int_5^6 \frac{1}{(10-0)}\,dx = \frac{1}{10}$$

$$= \int_7^{10} f_X(x)\,dx = \int_7^{10} \frac{1}{10-0}\,dx = \frac{3}{10}$$

⇒ The function $f_X(x)$ given in eqn $\textcircled{\star\star}$ is PDF for random variable X.

A PDF can take arbitrary large values:-

Let X be a random variable with PDF

$$f_X(x) = \begin{cases} \dfrac{1}{2\sqrt{x}} & 0 < x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Non (-)ve

normalization property.

$$\boxed{\int_{-\infty}^{\infty} f_X(x) = \int_0^1 \frac{1}{2\sqrt{x}}\, dx = 1}$$

Note that $f_X(x)$ can be infinitely large.

Expectation:- The expectation of a continuous random variable X is defined as

$$\boxed{E[X] = \int_{-\infty}^{\infty} x\, f_X(x)\, dx} \quad \text{— (1)}$$

continuous

If $g(X)$ is a function of random variable X then

$$E[g(X)] = \int_{-\infty}^{\infty} g(x)\, f_X(x)\, dx \quad \text{— (☆)}$$

for Example: $g(X) = X^2$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx \quad \text{— (2)}$$

$$\mathrm{Var}(X) = E[X^2] - \left(E[X]\right)^2 \quad \text{— (3)}$$

$$\mathrm{Var}(X) = E\left[\underbrace{\left(X - E[X]\right)^2}_{g(X)}\right] \quad \text{— (3.1)}$$

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} \left(x - E[X]\right)^2 f_X(x)\, dx \quad \text{— (4)}$$

Mean & Variance of Uniform Random Variable:- Let X be a uniform random variable then its PDF is

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$= \int_a^b x \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^2}{2}\right]_a^b$$

$$= \frac{a+b}{2}$$

$\mathrm{Var}(X)$

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x)\, dx$$

$$= \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right]$$

$$= \frac{a^2 + b^2 + ab}{3}$$

$$Var(X) = E[X^2] - (E[X])^2$$

$$= \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}$$

Exponential Random variable:-

An exponential random variable has PDF of the form

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

where λ is (+)ve.

(1) Non (-)ve

(2) $$\int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx = \left[-e^{-\lambda x}\right]_0^{\infty} = 1$$

An exponential random variable is used to model the amount of time until an incident of interest takes place. For example:-

1. Breaking down of some equipment
2. Burning out of a light bulb.
3. An accident occuring.

Exponential random variable (Continued)

If X is an exponential random variable,

$$P(X \geq a) = \int_a^{\infty} f_X(x)\,dx = \int_a^{\infty} \lambda e^{-\lambda x}\,dx$$

$$\boxed{P(X \geq a) = e^{-\lambda a}}$$

The mean & variance of X:-

$$E[X] = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\,dx$$

$$= -x e^{-\lambda x}\Big|_0^{\infty} + \int_0^{\infty} e^{-\lambda x}\,dx$$

$$= 0 - \frac{e^{-\lambda x}}{\lambda}\Big|_0^{\infty}$$

$$= \frac{1}{\lambda}$$

$$E[X^2] = \int_0^{\infty} x^2 \lambda e^{-\lambda x}\,dx$$

$$= (-x^2 e^{-\lambda x})\Big|_0^{\infty} + \int_0^{\infty} 2x e^{-\lambda x}\,dx$$

$$= 0 + \frac{2}{\lambda} E[X]$$

$$= \frac{2}{\lambda^2}$$

$$Var(X) = E[X^2] - (E[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2} \quad (5)$$

Example:- The time untill a small meteorite first lands anywhere in Sahara desert is modeled as an exponential random variable with mean of 10 days. The time is currently mid-night. What is the probability that a meteorite first lands some time between 6 AM to 6 PM of the first day?

Sol:- Let X be the time elapsed until the event of interest. It is given that

$$E[X] = 10$$

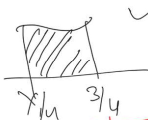
we know $E[X] = \frac{1}{\lambda} = 10 \Rightarrow \lambda = \frac{1}{10}$

$$P\left(\frac{1}{4} \leq X \leq \frac{3}{4}\right) = P\left(X \geq \frac{1}{4}\right) - P\left(X \geq \frac{3}{4}\right)$$

$$= e^{-\frac{1}{40}} - e^{-\frac{3}{40}}$$

$$= 0.0476$$

$$P(X \geq a) = e^{-\lambda a}$$



Cumulative Distribution function (CDF).

The CDF of a random variable X is denoted by F_X & provides the probability $P(X \le x)$. For every x, we have

$$F_X(x) = P(X \le x) = \begin{cases} \sum_{k \le x} p_X(k) & \text{If } X \text{ is discrete} \\ \int_{-\infty}^{x} f_X(t)\,dt & \text{If } X \text{ is Continuous} \end{cases}$$

Area

Example: X is discrete & $X = 1, 2, \ldots 10$

$$F_X(5) = P(X \le 5)$$

$P(X=1)$

$$= p_X(1) + p_X(2) + p_X(3) + p_X(4) + p_X(5)$$

$$= \sum_{k \le 5} p_X(k)$$

$k = 1,2,3,4,5$

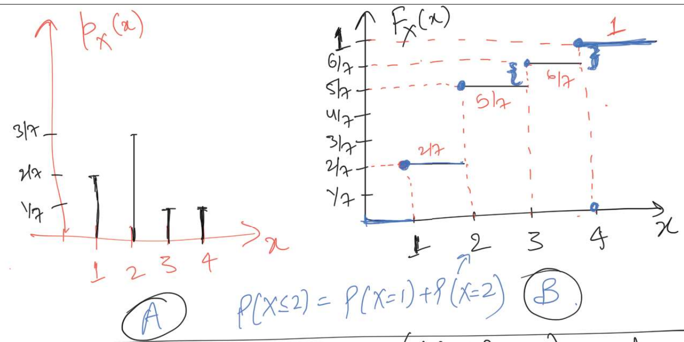
$F_X(8)$ $F_X(10)$ $F_X(1)$

Let X be a discrete random variable

Is it a PMF?

$$p_X(x) = \begin{cases} 2/7 & x = 1 \\ 3/7 & x = 2 \\ 1/7 & x = 3 \\ 1/7 & x = 4 \end{cases}$$

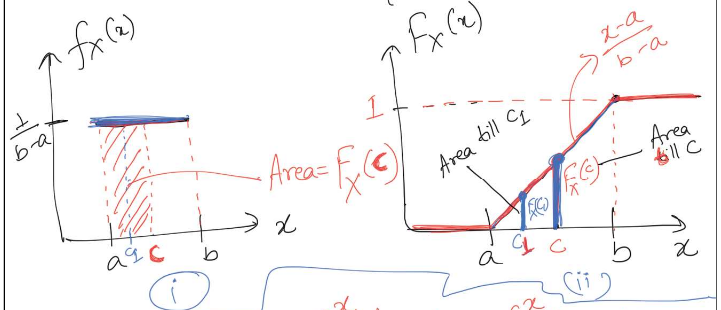
Assume $X = 1,2,3,4$



$P(X \le 2) = P(X=1) + P(X=2)$

Let X be a continuous (Uniform) random variable with PDF

$$f_X(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$



$$F_X(x) = \int_{-\infty}^{x} \frac{1}{b-a}\,dt = \int_{a}^{x} \frac{1}{b-a}\,dt$$

$$= \frac{x-a}{b-a} \quad \text{(A)}$$

Properties of CDF:-

1. If $x \le y$ $\quad F_X(x) \le F_Y(y)$
2. $F_X(x)$ tends to 0 as $x \to -\infty$ and tends to 1 as $x \to \infty$
3. If X is discrete, then $F_X(x)$ is a piecewise constant function.
4. If X is continuous then $F_X(x)$ is a continuous function.
5. If X is Continuous then

$$F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt \quad \& \quad \boxed{f_X(x) = \frac{dF_X}{dx}(x)}$$

CDF

PDF

(A)

DIY:- How to get CDF $\longrightarrow$ PMF

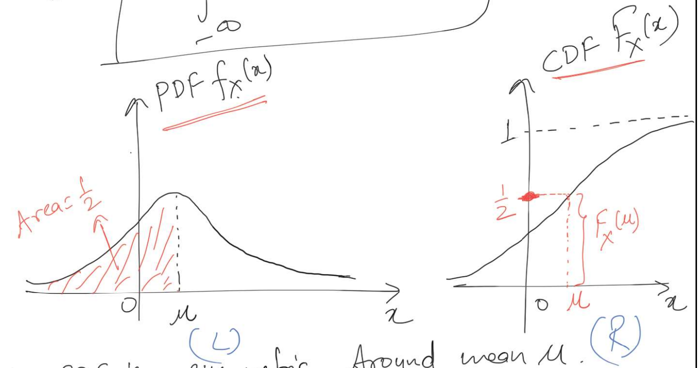
Normal random variable:-

A continuous random variable X is said to be normal (or Gaussian) if it has a PDF of the form

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad -\infty < x < \infty \qquad \text{(I)}$$

where μ & σ are two scalar parameters with $\sigma > 0$.

Verify? $\boxed{\int_{-\infty}^{\infty} f_X(x)\,dx = 1}$ → Home work.



the PDF is symmetric Around mean μ.

The mean & variance

$$E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu$$

$$Var(X) = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \sigma^2$$

Home work?

Standard normal random variable:-

A normal random variable with $\mu = 0$ & $\sigma^2 = 1$ is said to be standard normal random variable (denoted by Y). It's PDF will be

$$\boxed{f_Y(y) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}} \quad -(2)$$

Find CDF of standard normal random variable:-

$$\boxed{\Phi(y) = P(Y \le y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt} \quad -(\star)$$

Phi

Example:- we know that this integral can be evaluated for all y.

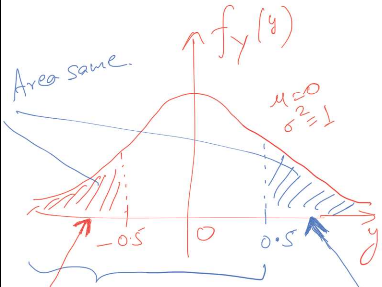
Let Y be a standard normal random variable taking values in $[1, 100]$

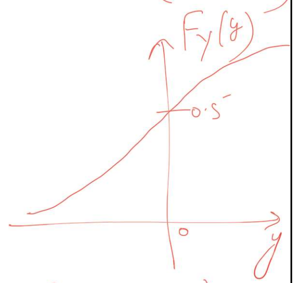
$$F_Y(5) = P(Y \le 5) = \int_{-\infty}^{5} f_Y(y)\,dy$$

$$F_Y(8) = P(Y \le 8) = \int_{-\infty}^{8} f_Y(y)\,dy$$

People have already calculated the integration in $(\star)$ for various values of y & put it in table. This table is called Standard normal table.

Standard normal random variable (Continued)





$$\Phi(-0.5) = P(Y \le -0.5) = P(Y \ge 0.5)$$

$$= 1 - P(Y \le 0.5)$$

$$= 1 - \Phi(0.5)$$

In general

$$\boxed{\Phi(-y) = 1 - \Phi(y)}$$

Let X be a normal random with mean μ and variance σ^2. Define another random variable

$$Y = \frac{X-\mu}{\sigma}$$

Y is a linear function of X, hence it is also a Normal random variable.

$$E[Y] = E\left[\frac{X-\mu}{\sigma}\right] = \frac{1}{\sigma}\left[E[X] - \mu\right]$$

$$\left(E[aX+b] = a\,E[X] + b\right)$$

$$= \frac{1}{\sigma}\left[\cancel{\mu} - \cancel{\mu}\right] = 0. \quad (\because E[X] = \mu)$$

$$\mathrm{Var}(Y) = \mathrm{Var}\left(\frac{X-\mu}{\sigma}\right) = \mathrm{Var}\left(\frac{X}{\sigma} - \frac{\mu}{\sigma}\right)$$

$$\boxed{\mathrm{Var}(aX+b) = a^2\,\mathrm{Var}(X)}$$

$$= \frac{1}{\sigma^2}\,\mathrm{Var}(X) \quad \text{——} \quad (2)$$

$$\boxed{\mathrm{Var}(Y) = \frac{1}{\cancel{\sigma^2}} \cdot \cancel{\sigma^2} = 1}$$

Thus, Y is a standard normal random variable.

$\Rightarrow$ Prob. of Y can be found from table.

Example:- The annual snowfall at a particular location is modeled using normal random variable with mean 60 inches and standard deviation of $\sigma = 20$. What is the probability that this year's snowfall will be atleast 80 inches?

Sol^n:- Let X be the snowfall. It is given that X is normal random variable with $\mu = 60$ & $\sigma = 20$. Define

$$Y = \frac{X-60}{20}, \quad \left(Y = \frac{X-\mu}{\sigma}\right)$$

We know that Y is a standard normal random variable.

$$P(X \ge 80) = P\left(\frac{X-60}{20} \ge \frac{80-60}{20}\right)$$

$$= P(Y \ge 1) = 1 - P(Y \le 1)$$

(From Table)

$$= 1 - 0.8413$$

$$\boxed{P(X \ge 80) = 0.1587}$$

Joint PDFs of Multiple random variables:-

Let X & Y be continuous random variables associated with same experiment. Then X & Y are jointly continuous if there is a non-ve function $f_{X,Y}(x,y)$ such that

$$P(a \le X \le b,\ c \le Y \le d) = \int_c^d \int_a^b f_{X,Y}(x,y)\,dx\,dy$$

Function $f_{X,Y}$ satisfies following property:-

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx\,dy = 1$$

This $f_{X,Y}(x,y)$ is joint PDF of X & Y

Joint CDFs:-

Let X & Y be two random variables associated with same experiment, their joint CDF is given by

$$F_{X,Y}(x,y) = P(X \le x, Y \le y)$$

If X & Y are continuous

$$F_{X,Y}(x,y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(s,t)\,dt\,ds$$

we have

$$\boxed{\frac{\partial^2 F_{X,Y}}{\partial x\,\partial y}(x,y) = f_{X,Y}(x,y)}$$

The marginal PDFs of X & Y can be obtained from the joint PDF as follows:-

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx$$

Expectation:- If X & Y continuous r.v.

$$E[aX+bY+c] = aE[X]+bE[Y]+c$$

Conditioning one random variable on another:-

Let X & Y be continuous random variables with joint PDF f_{XY}. The conditional PDF of X given $Y=y$ is defined as

$$\boxed{f_{X|Y}(x|y) = \frac{f_{XY}(x,y)}{f_Y(y)}}$$

Example:- The speed of a typical vehicle that drives past a police radar is modeled as an exponential random variable X with mean 50 miles/hour. The police radar's measurement Y of the vehicle's speed has an error which is modeled as a normal random variable with zero mean & standard deviation equal to one-tenth of the vehicle's speed. What is the joint PDF of X & Y.

Sol:- X is exponential random variable & $\lambda = \frac{1}{50}$ $\left(\text{mean} = 50 = \frac{1}{\lambda}\right)$

Given that $X=x$, the measurement Y has a normal PDF with mean x & variance $\frac{x^2}{100}$

$$f_{Y|X}(y|x) = \frac{1}{\sqrt{2\pi}\left(\frac{x}{10}\right)} e^{\left(-\frac{(y-x)^2}{2\left(\frac{x}{10}\right)^2}\right)}$$

$$f_{Y|X}(x,y) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

$$\Rightarrow f_{X,Y}(x,y) = f_{Y|X}(x,y)\cdot f_X(x)$$

$$f_{X,Y}(x,y) = \lambda e^{-\lambda x} \cdot \frac{1}{\sqrt{2\pi}\left(\frac{x}{10}\right)} e^{-\frac{(y-x)^2}{2x^2/100}}$$

$$\boxed{f_{X,Y} = \frac{1}{50} e^{-\frac{x}{50}} \cdot \frac{1}{\sqrt{2\pi}\,\frac{x}{10}} e^{-50(y-x)^2/x^2}}$$

Do it yourself:- Plz read above carefully

Independence:- Two continuous random variables X & Y are independent if their joint PDF is the product of the marginal PDFs.

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y) \quad \text{for all } x,y. \quad \text{☆}$$

Note that,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

If random variable are independent then

$$f_{X|Y}(x|y) = \frac{f_X(x)\, f_Y(y)}{f_Y(y)}$$

$$\Rightarrow \boxed{f_{X|Y}(x|y) = f_X(x)} \quad \text{—(2)}$$

for all y with $f_Y(y) > 0$ and all x.

In general, if we have three random variables X, Y & Z. These are independent if

$$f_{X,Y,Z}(x,y,z) = f_X(x)\, f_Y(y)\, f_Z(z)$$

Relation between independence & CDF.

Two random variables X & Y are independent then

$$\boxed{F_{X,Y}(x,y) = F_X(x)\, F_Y(y)}$$

Relation Between expectation & independence

If X & Y are independent, then

$$E[XY] = E[X]\, E[Y] \quad \text{—(3)}$$

Furthermore, for any functions g & h, the random variables $g(X)$ & $h(Y)$ are independent & , we have

$$E\left[g(X)\cdot h(Y)\right] = E\left[g(X)\right]\, E\left[h(Y)\right]$$

Independence & Variance?

If X & Y are independent, then

$$\boxed{Var(X+Y) = var(X) + var(Y)}$$

Example:- The waiting time, in hours, between successive speeders spotted by a radar is a continuous random variable with CDF

$$P(X<x) = F(x) = \begin{cases} 1-e^{-8x} & x \geq 0 \\ 0 & x<0 \end{cases}$$

Find the probability of waiting less than 12 minutes between successive speeders.

Sol:- Let random variable be X representing the waiting time.

$$P(X<0.2) = F(0.2)$$

$$= 1-e^{-8(0.2)} = 0.7981$$

Another way, The PDF will be

$$f_X(x) = F'(x) = 8e^{-8x}$$

$$P(X<0.2) = \int_{-\infty}^{0.2} f_X(x)\,dx$$

$$= \int_0^{0.2} 8e^{-8x}\,dx$$

$$= 0.7981.$$

Ques 1:- Find the CDF corresponding to the following PDFs

(i) $f(x) = \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$

Sol:-

CDF $F_X(x) = \int_{-\infty}^{x} f_X(t)\,dt$

$$= \int_{-\infty}^{x} \frac{1}{b-a}\,dt$$

$$= \int_{-\infty}^{a} f_X(t)\,dt + \int_a^x f_X(t)\,dt$$

$\Downarrow$ 0

$$= \int_a^x \frac{1}{b-a}\,dt = \frac{x-a}{b-a}$$

$$F_X(x) = \begin{cases} 0 & x \leq a \\ \frac{x-a}{b-a} & a < x \leq b \\ 1 & x \geq b \end{cases}$$

$$f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2-x & 1 \leq x \leq 2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx$$

$$= \begin{cases} 0 & x<0 \\ \int_0^x x\,dx & 0 \leq x < 1 \\ \int_0^1 x\,dx + \int_1^x (2-x)\,dx & 1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Ques:- The CDF is given as

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{2} & 0 \le x < 1 \\ 3/5 & 1 \le x < 2 \\ 4/5 & 2 \le x < 3 \\ 9/10 & 3 \le x < 3.5 \\ 1 & x \geq 3.5 \end{cases}$$

Find the PMF : $X = 0, 1, 2, 3, 3.5$

$$\begin{cases} p_X(0) = P(X=0) = \frac{1}{2} \\ p_X(1) = P(X=1) = \frac{3}{5} - \frac{1}{2} = \frac{1}{10} \\ p_X(2) = P(X=2) = \frac{4}{5} - \frac{3}{5} = \frac{1}{5} \\ p_X(3) = P(X=3) = \frac{9}{10} - \frac{4}{5} = \frac{1}{10} \\ p_X(3.5) = P(X=3.5) = 1 - \frac{9}{10} = \frac{1}{10} \end{cases}$$

Recall :- CDF $\longrightarrow$ PDF

$\downarrow$

PMF $\longleftarrow$ CDF

Ques:- The Department of Energy puts projects out on bid & generally estimates what a reasonable bid should be. We call the estimate '**b**'. The Department of Energy has determined that the density function of the winning (low) bid is

$$f(y) = \begin{cases} \frac{5}{8b} & \frac{2b}{5} < y < 2b \\ 0 & \text{otherwise} \end{cases}$$

1

Find $F(y)$?

$$F(y) = \int_{2b/5}^{y} \frac{5}{8b}\,dt = \frac{5y}{8b} - \frac{1}{4}$$

$$F(y) = \begin{cases} 0 & y < \frac{2b}{5} \\ \frac{5y}{8b} - \frac{1}{4} & \frac{2b}{5} \le y < 2b \\ 1 & y \geq 2b \end{cases}$$

what is the probability that the winning bid is less than the estimate 'b'.

$$P(Y \le b) = F(b)$$

$\because b \in \left(\frac{2b}{5}, 2b\right)$ therefore,

$$\boxed{F(b) = \frac{5}{8b}(b) - \frac{1}{4} = \frac{3}{8}}$$

Ques:- The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$, where X has the following PDF

$$f_X(x) = \begin{cases} 2(x-1) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

3

Find the expected value of the weekly demand for the drink?

$$E(g(X)) = ?$$

Do it yourself:

Derived distributions:- Calculating the PDF of $Y = g(X)$, where X is a continuous random variable:-

Ex:- Let X be uniform on $[0,1]$ & let $Y = \sqrt{X}$. Find PDF of Y.

Sol:- PDF of X

$$f_X(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

The rang of Y will be $[0,1]$

First we will find CDF then PDF

$$F_Y(y) = P(Y \le y)$$
$$= P(\sqrt{X} \le y)$$
$$= P(X \le y^2)$$
$$= \int_0^{y^2} f_X(x)\, dx = \int_0^{y^2} 1\, dx$$

$$\boxed{F_Y(y) = y^2}$$

Then PDF of Y

$$f_Y(y) = \frac{d F_Y(y)}{dy} = \frac{d(y^2)}{dy} = 2y \quad 0 \le y \le 1$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Covariance & Correlation: we will study a quantitative measure of the strength & direction of the relationship b/w two random variables:-

Let X & Y be two random variables. The Covariance of X & Y denoted by $\mathrm{Cov}(X,Y)$ is defined as

$$\mathrm{Cov}(X,Y) = E\big[(X - E(X))\,(Y - E[Y])\big]$$

A simpler formula is

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]\,E[Y] \quad (3)$$

1) when $\mathrm{Cov}(X,Y) = 0$, we say that X & Y are uncorrelated.

2) $\mathrm{Cov}(X,X) = E[X^2] - E[X]E[X]$
$= \mathrm{Var}(X)$

3) $\mathrm{Cov}(X, aY + b) = a\, \mathrm{Cov}(X,Y)$

4) $\mathrm{Cov}(X, Y + Z) = \mathrm{Cov}(X,Y) + \mathrm{Cov}(X,Z)$

Que:- If X & Y are independent then

$$\mathrm{Cov}(X,Y) = E[XY] - E[X]E[Y]$$
$$= E[X]\cdot E[Y] - E[X]E[Y]$$

$$\boxed{\mathrm{Cov}(X,Y) = 0}$$

Independent random variable

$\Downarrow$ ~~$\Uparrow$~~

$\mathrm{Cov}(X,Y) = 0$

Think one example?

The correlation coefficient $\rho(X,Y)$ of two random variables X & Y that have non-zero variances is defined as

$$\rho = \rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\ var(Y)}} \quad \text{— } (*)$$

ρ → rho

Do it yourself: Prove that $\boxed{-1 \leq \rho \leq 1}$

Variance of the sum of random variables

$$Var(X+Y) = Var(X) + var(Y) + 2\,Cov(X,Y)$$

Moment Generating function:

Let X be a random variable, then n^{th}-order moment of X is defined as $E[X^n]$

The moment generating function (MGF) associated with random variable X is defined as

$$M_X(s) = E[e^{sX}] \quad \text{— } (1)$$

$X \to e^X \to e^{sX}$ ✓

where s is a scalar parameter.

For a discrete random variable X

$$M(s) = \sum_x e^{sx}\,p_X(x)$$

For a continuous random variable X

$$M(s) = \int_{-\infty}^{\infty} e^{sx} f_X(x)\,dx \quad (*)$$

Eg:- MGF of Poisson random variable:-

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}, \quad x = 0,1,2,\ldots$$

$$\text{MGF of } X = M(s) = \sum_{x=0}^{\infty} e^{sx}\,\frac{e^{-\lambda}\lambda^x}{x!}$$

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{e^{sx}\cdot\lambda^x}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^s\cdot\lambda)^x}{x!}$$

Let $\boxed{a = e^s\cdot\lambda}$, we have

$$= e^{-\lambda}\sum_{x=0}^{\infty}\frac{a^x}{x!}$$

$$(\phi^m)^n = \phi^{mn}$$

$$(e^s\cdot\lambda)^x = e^{sx}\cdot\lambda^x$$

$$e^a = \frac{1}{\underline{|0}} + \frac{a}{\underline{|1}} + \frac{a^2}{\underline{|2}} + \frac{a^3}{\underline{|3}} + \cdots$$

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \cdots$$

$$\boxed{e^a = \sum_{x=0}^{\infty}\frac{a^x}{x!}} \quad \text{— } (3)$$

$$\text{MGF of } X = M(s) = e^{-\lambda}\cdot e^a = e^{a-\lambda}$$

$$\boxed{M(s) = e^{(e^s\cdot\lambda - \lambda)} = e^{\lambda(e^s - 1)}} \quad (4)$$

MGF of Exponential random variable:-

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

$$M(s) = \int_0^\infty e^{sx} \cdot \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^\infty e^{(s-\lambda)x}\, dx$$

$$= \lambda \cdot \left[\frac{e^{(s-\lambda)x}}{s-\lambda} \right]_0^\infty$$

if $s < \lambda$

$$= \lambda \left[0 - \frac{1}{s-\lambda} \right]$$

(otherwise integral is infinite)

$$\boxed{M(s) = \frac{\lambda}{\lambda - s}}$$

From MGF to $E[x^n] = ?$

From MGF to moments:-

Let X be a continuous random variable then MGF of X is defined as

$$M(s) = \int_{-\infty}^\infty e^{sx} f_X(x)\, dx$$

Differentiate both side w.r.t. s

$$\frac{d}{ds} M(s) = \frac{d}{ds} \int_{-\infty}^\infty e^{sx} f_X(x)\, dx$$

$$= \int_{-\infty}^\infty \frac{d}{ds} e^{sx} f_X(x)\, dx$$

when $\int$ & $\frac{d}{dx}$ can be interchanged

$$\frac{d}{ds}(M(s)) = \int_{-\infty}^\infty x e^{sx} f_X(x)\, dx \quad — (*)$$

Consider the special case when $s=0$. we get

$$\left.\frac{d}{ds} M(s)\right|_{s=0} = \int_{-\infty}^\infty x f_X(x)\, dx \quad — (2)$$

$$\boxed{\left.\frac{d}{ds} M(s)\right|_{s=0} = E[X]} \quad — (3)$$

In general, we have

$$\frac{d^n}{ds^n} M(s) = \int_{-\infty}^\infty x^n e^{sx} f_X(x)\, dx$$

$$\Rightarrow \boxed{\left.\frac{d^n}{ds^n} M(s)\right|_{s=0} = \int_{-\infty}^\infty x^n f_X(x)\, dx = E[X^n]}$$

$$\text{Random variable } X \xrightarrow{(1)} \text{MGF of } X$$

$$\downarrow (2)$$

n^{th} ~~Moment:~~ $E[X^n] \longleftarrow$ Differentiate and put $s=0$

Example: Let X be a random variable

$$p_X(x) = \begin{cases} 1/2 & x=2 \\ 1/6 & x=3 \\ 1/3 & x=5 \end{cases}$$

$$E[X] = 2\times\frac{1}{2} + 3\times\frac{1}{6} + 5\times\frac{1}{3} = \frac{19}{6}$$

$$\text{MGF of } X = M_X(s) = \sum_x e^{sx} p_X(x) \quad (4)$$

$$= e^{s\cdot 2}\cdot\frac{1}{2} + e^{s\cdot 3}\cdot\frac{1}{6} + e^{s\cdot 5}\cdot\frac{1}{3}$$

$$\Rightarrow \boxed{M(s) = \frac{e^{2s}}{2} + \frac{e^{3s}}{6} + \frac{e^{5s}}{3}}$$

$$\frac{d}{ds}M(s) = \frac{1}{2}\cdot\left(2e^{2s}\right) + \frac{1}{6}\left(3e^{3s}\right) + \frac{1}{3}\left(5e^{5s}\right)$$

$$\left.\frac{d}{ds}M(s)\right|_{s=0} = \frac{1}{2}\times 2 + \frac{1}{6}\times 3 + \frac{1}{3}\times 5$$

$$= \frac{19}{6} = E[X].$$

Example: For exponential random variable

$$f_X(x) = \lambda e^{-\lambda x} \quad x \geq 0$$

$$M(s) = \frac{\lambda}{\lambda - s} \quad (s < \lambda)$$

$$\frac{d}{ds}M(s) = \frac{\lambda}{(\lambda - s)^2}$$

$$\boxed{\left.\frac{d}{ds}M(s)\right|_{s=0} = \frac{1}{\lambda} = E[X]}$$

$$\frac{d^2}{ds^2}M(s) = \frac{2\lambda}{(\lambda - s)^3}$$

$$\left.\frac{d^2}{ds^2}M(s)\right|_{s=0} = \frac{2}{\lambda^2} = E[X^2]$$

Do it yourself: $M_X(0) = ?$

Markov Inequality: If a random variable X can take only non-negative values, then

$$\boxed{P(X \geq a) \leq \frac{E[X]}{a}} \quad \text{for all } a > 0$$

Chebyshev Inequality: If X is a random variable with mean μ & variance σ^2, then

$$\boxed{P\left(|X - \mu| \geq c\right) \leq \frac{\sigma^2}{c^2}} \quad (1) \quad \forall c > 0$$

Ex: Let X be uniform random variable in the interval $[0, 4]$.

$$f_X(x) = \begin{cases} \frac{1}{4} & 0 \leq x \leq 4 \\ 0 & \text{otherwise} \end{cases}$$

$$E[X] = \int_0^4 x\cdot f_X(x)\,dx = \frac{1}{4}\int_0^4 x\,dx = 2$$

Recall the Markov Inequality.

$$\boxed{P(X \geq a) \leq \frac{E[X]}{a}}$$

PDF

$a=2$

$P(X \geq 2) \leq \frac{2}{2} = 1.$

$a=3$

$P(X \geq 3) \leq \frac{2}{3} = 0.67$

$a=4$

$P(X \geq 4) \leq \frac{2}{4} = 0.5$

$P(X \geq 2) = \int_2^4 f_X(x)\,dx = 0.5$

$P(X \geq 3) = 0.25$

$P(X \geq 4) = 0$

Using chebyshev inequality

In previous example $E[X] = 2$, $Var(X) = \frac{(b-a)^2}{12} = \frac{16}{12} = \frac{4}{3}$

Let $c = 1$

$$\boxed{P(|X-2| \geq 1) \leq \frac{4}{3}} \quad \text{—}\star\star$$

The Weak law of large numbers:-

If $X_1, X_2, \ldots$ be independent identically distributed random variables with mean μ

For every $\varepsilon > 0$ (epsilon), we have —$\star$

$$P\left(\left|\frac{X_1 + X_2 + \ldots + X_n}{n} - \mu\right| \geq \varepsilon\right) \to 0 \quad \text{as } n \to \infty$$

Explanation:- Let $X_1, X_2, \ldots$ are independent r.v. have mean μ and variance σ^2. Define a new random variable

$$M_n = \frac{X_1 + X_2 + \ldots + X_n}{n} \quad \text{—}(2)$$

$$E[M_n] = \frac{1}{n}\left[E[X_1] + E[X_2] + \ldots + E[X_n]\right]$$

$$= \frac{1}{n}\left[\mu + \mu + \ldots + \mu\right]$$

$$= \frac{1}{n}[n\mu] = \mu$$

$$Var(M_n) = \frac{1}{n^2}\left[Var(X_1) + Var(X_2) + \ldots + Var(X_n)\right]$$

$$= \frac{1}{n^2}\left[\sigma^2 + \sigma^2 + \ldots + \sigma^2\right]$$

$$= \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

Recall Chebyshev inequality

$c = \varepsilon$

$$P(|X - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{c^2}$$

$E[X] = \mu$, $Var(X) = \mu$

Take $X = M_n$ in above, for any $\varepsilon > 0$.

$$P(|M_n - \mu| \geq \varepsilon) \leq \frac{\sigma^2}{n\varepsilon^2} \quad \text{—}(3)$$

as $n \to \infty$

$$P(|M_n - \mu| \geq \varepsilon) \to 0$$

The Central limit theorem:- (CLT)

Let $X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with common mean μ & variance σ^2. Define

$$\boxed{Z_n = \frac{X_1 + X_2 + \ldots + X_n - n\mu}{\sigma\sqrt{n}}}$$

Then the CDF of Z_n converges to standard normal CDF.

Explanation:- $E[Z_n] = \frac{1}{\sigma\sqrt{n}}\left[E[\underbrace{X_1 + \ldots + X_n}] - n\mu\right]$

$$\boxed{E[Z_n] = 0}$$

$$Var(Z_n) = \frac{1}{\sigma^2 n}\left[\overset{\sigma^2}{Var(X_1)} + \ldots + \overset{\sigma^2}{V(X_n)} - 0\right]$$

$$= \frac{n\sigma^2}{\sigma^2 n} = 1$$

Example: We load on a plane 100 packages whose weights are independent random variables that are uniformly distributed between 5 and 50 kg. What is the probability that the total weight will exceed 3000 kg.?

Sol: we want to calculate

$$\boxed{P(S_{100} > 3000) = ?}$$

S_{100} = sum of the weights of 100 packages

$$= \underbrace{X_1}_{\text{weight of } 1^{st} \text{ package}} + \underbrace{X_2}_{\text{weight of } 2^{nd} \text{ package}} + \cdots + \underbrace{X_{100}}_{\text{weight of } 100^{th} \text{ package}}$$

For all i

$$E[X_i] = \frac{5+50}{2} = 27.5 = \mu$$

$$Var(X_i) = \frac{(50-5)^2}{12} = 168.75 = \sigma^2$$

Define:

$$Z = \frac{X_1 + X_2 + \cdots + X_{100} - 100 \times 27.5}{\sqrt{168.75} \times \sqrt{100}}$$

follows standard normal random variable.

$$\boxed{Z = \frac{S_{100} - 100 \times 27.5}{\sqrt{168.75} \times \sqrt{100}}}$$

Now, we need to fined

$$\boxed{a > b \qquad \frac{a-\mu}{\sigma\sqrt{n}} > \frac{b-\mu}{\sigma\sqrt{n}}}$$

$$P(S_{100} > 3000)$$

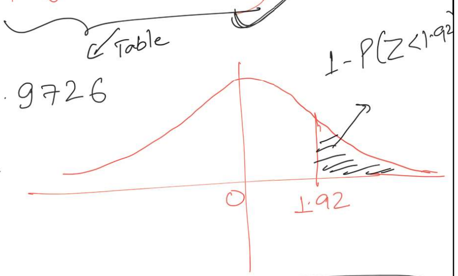
$$= P\left(\frac{S_{100} - 100 \times 27.5}{\sqrt{168.75} \times 10} > \frac{3000 - 100 \times 27.5}{\sqrt{168.75} \times 10}\right)$$

$$= P(Z > 1.92)$$

$$= 1 - \underbrace{P(Z \le 1.92)}_{\text{Table}}$$

$$= 1 - 0.9726$$

$$= 0.0274$$

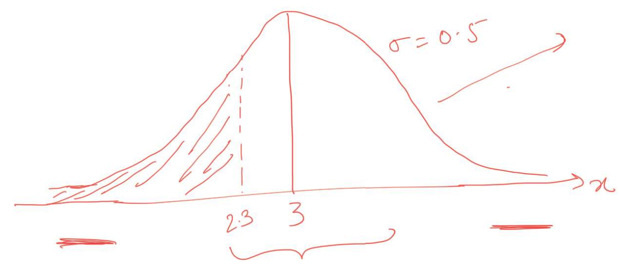


$$\boxed{P(S_{100} > 3000) = 0.0274}$$

Ques: A certain type of storage Battery lasts, on average, 3 years with a standard deviation of 0.5 years. Assuming that battery life is normally distributed, find the probability that a given battery will last less that 2.3 years.

Sol: Let X be the random variable representing life time of battery.

$$P(X < 2.3) = ?$$



Given

$$E[X] = 3$$

Standard deviation of $X = 0.5 = \sigma$

Define

$$Z = \frac{X-3}{0.5}$$

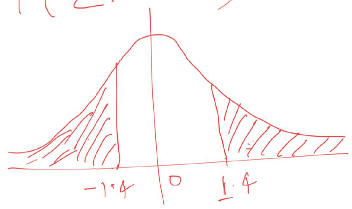
we know that Z follows standard normal distribution. Now

$$P(X < 2.3)$$

$$= P\left(\frac{X-3}{0.5} < \frac{2.3-3}{0.5}\right)$$

$= P(Z < -1.4)$

$= P(Z > 1.4)$



$= 1 - P(Z < 1.4)$

$$\boxed{P(X < 2.3) = 0.0808}$$

Relation & function :-

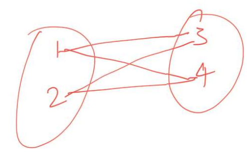
Cartesion product of sets :-

$A = \{red, blue\}$ $B = \{b, c, s\}$

$A \times B = \{(red, b)\ (red, c)\ (red, s)\ (blue, b)\ (blue, c)\ (blue, s)\}$

Relations :- A relation R from a non-empty set A to a non-empty set B is a subset of the Cartesian product $A \times B$.

Example: $A = \{1, 2\}$, $B = \{3, 4\}$



$A \times B = \{(1,3)\ (1,4)\ (2,3)\ (2,4)\}$

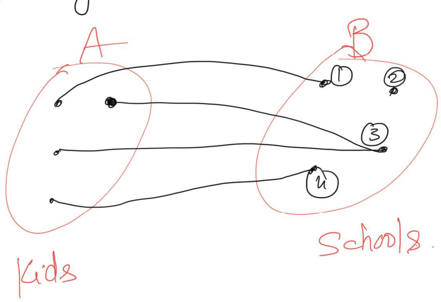
The number of relations from A to B ?
= no. of subsets of $A \times B$.

no. of elements in $A \times B = 4$

no. of subsets of $A \times B = 2^4$

no. of relations from A into B $= 2^4$.

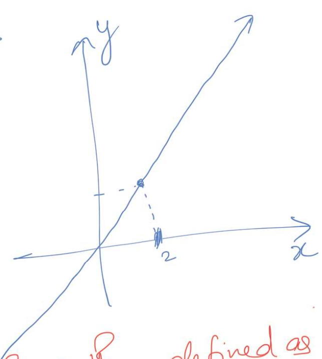
Functions :- A relation f from a set A to a set B is said to be a function if every element of set A has one & only only one image in set B.



⊛ Some functions & their graphs:-

① Identity function:-

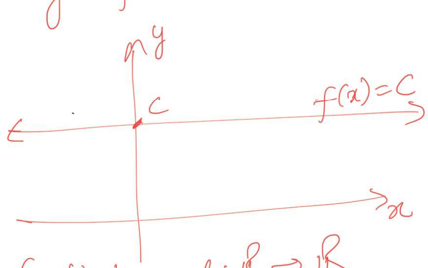
$$f: \mathbb{R} \longrightarrow \mathbb{R}$$
$$f(x) = x$$



② Constant function:-

$f: \mathbb{R} \longrightarrow \mathbb{R}$, defined as

$$y = f(x) = c \quad , \; c \text{ is constant } \& \; x \in \mathbb{R}$$



③ Polynomial function:-

$f: \mathbb{R} \to \mathbb{R}$

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n,$$

where $a_0, a_1, \dots, a_n \in \mathbb{R}$.

④ Rational function:-

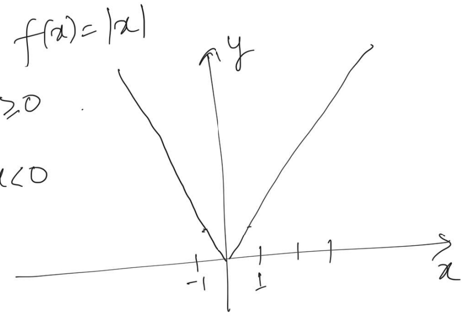
These are function of the type $\frac{f(x)}{g(x)}$ where $f(x)$ & $g(x)$ are polynomial functions of x, $g(x) \neq 0$.

⑤ Modulus function:-

$f: \mathbb{R} \to \mathbb{R}$ &

$$f(x) = |x|$$

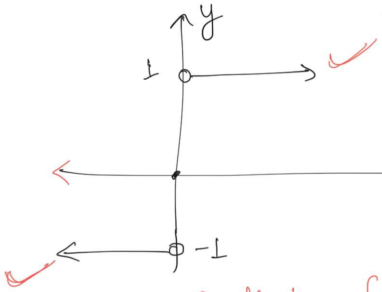
$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$



⑥ Signum function:-

$f: \mathbb{R} \to \mathbb{R}$

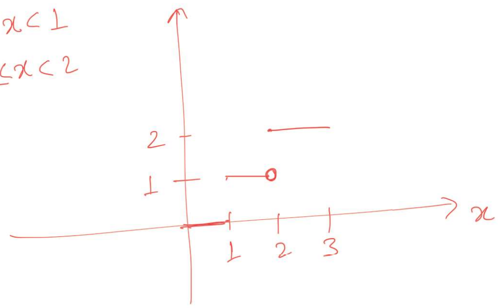
$$f(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$



(7) Greatest integer function:- $f: \mathbb{R} \to \mathbb{R}$

$f(x) = [x]$ (Greatest integer less than or equal to x)

$$f(x) = \begin{cases} 0 & 0 \le x < 1 \\ 1 & 1 \le x < 2 \\ \vdots \end{cases}$$



Algebra of real functions:-

(1) $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$ where $X \subset \mathbb{R}$

Then $(f+g): X \to \mathbb{R}$ defined as

$$(f+g)(x) = f(x) + g(x) \quad \forall x \in X$$

Similar to above

(2) $(f-g): X \to \mathbb{R}$ defined as

$$(f-g)(x) = f(x) - g(x) \quad \forall x \in X$$

(3) If $f: X \to \mathbb{R}$ then

$$(\alpha f)(x) = \alpha f(x), \quad \alpha \text{ is a scalar}$$

(4) $f: X \to \mathbb{R}$, $g: X \to \mathbb{R}$ (real no.)

$fg: X \to \mathbb{R}$

$fg(x) = f(x)\, g(x)$ (Pointwise multiplication)

(5) $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$, $g(x) \neq 0$.

Definitions:- A relation R in a set A is called

(i) reflexive, if $(a,a) \in R$ for every $a \in R$

(ii) Symmetric, if $(a,b) \in R$ implies the $(b,a) \in R$.

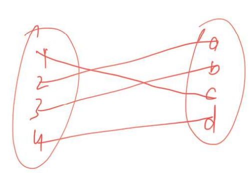
(iii) Transitive; if $(a,b) \in R$ & $(b,c) \in R$ then $(a,c) \in R$.

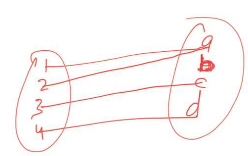
If a relation is reflexive, symmetric & transitive, it is called an equivalence relation.

Definition:- A function $f: X \to Y$ is defined to be one-one (or injective), if the images of distinct elements of X under f are distinct.

For every $x_1, x_2 \in X$

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$





Definition:- A function $f: X \to Y$ is said to be onto (or surjective) if every element of Y is the image of some element of X under f.

Remark:- $f: X \to Y$ is onto if and only if

$$\boxed{\text{Range of } f = Y}$$

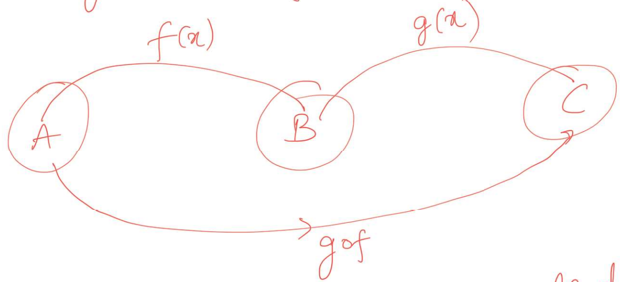
Definition: A function $f : X \to Y$ is said to be bijective if f is both one-one & onto.

Composition of functions:-

Definition:- $f : A \to B$ & $g : B \to C$ Then

$gof : A \to C$ defined as

$$gof(x) = g(f(x)) \quad \forall x \in A$$



Definition:- A function $f : X \to Y$ is called invertible, if $\exists$ a function $g : Y \to X$ such that

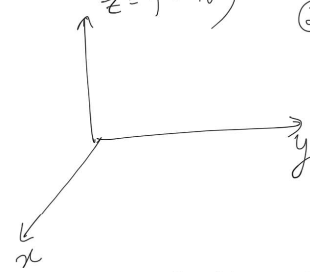
$$gof = I_X \quad \& \quad fog = I_Y$$

The function g is called inverse of f & denoted by f^{-1}.

$$\boxed{\begin{array}{c} f \text{ is invertible} \\ \Downarrow \ \Uparrow \\ f \text{ is one-one \& onto} \end{array}}$$

Function of two variables:-

$z = f(x,y)$



Examples

① $\sqrt{y-x^2}$

② $\frac{1}{xy}$

③ $\sin xy$

Limit & Continuity for functions of one variable (Please Recall from your earlier classes)

Continuity of functions in two variables:-

A function $f(x,y)$ is continuous at the point (x_0, y_0) if

① f is defined at (x_0, y_0)

② $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

③ $\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)$

Partial derivatives:-

The partial derivative of $f(x,y)$ with respect to x at the point (x_0,y_0) is

$$\left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0+h, y_0) - f(x_0,y_0)}{h} \quad (1)$$

Similarly

$$\left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} = \lim_{h\to 0} \frac{f(x_0, y_0+h) - f(x_0,y_0)}{h} \quad (2)$$

Example:- Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ for $f(x,y) = y\sin xy$

$$\frac{\partial f}{\partial x} = y^2 \cos xy$$

$$\frac{\partial f}{\partial y} = \sin xy + xy\cos xy.$$

check?

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(y\sin xy) = y\frac{\partial}{\partial x}(\sin xy)$$
$$= y[\cos xy \times y] = y^2\cos xy$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(y\sin xy) = \frac{\partial y}{\partial y}\cdot \sin xy + y\cdot\frac{\partial}{\partial y}\sin xy$$
$$= \sin xy + y\cdot[\cos xy \cdot x]$$
$$= \sin xy + xy\cos xy.$$

Partial derivatives & Continuity:-

(A) For function of one variable:-

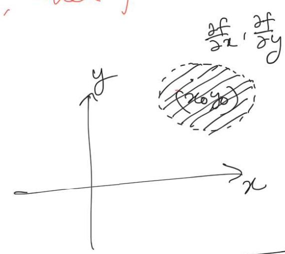
Continuity $\cancel{\Longrightarrow}$ Differentiability

Continuity $\Longleftarrow$ Differentiability

(Do it yourself)

(B) For function of two variables:- A function $f(x,y)$ can have partial derivatives with respect to x & y at a point without the function being continuous there.

Remark:- If the partial derivatives of $f(x,y)$ exist & are continuous throughout the disk centered at (x_0,y_0), then f is continuous at (x_0,y_0).



Second order partial derivatives:-

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right), \quad \frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right), \quad \frac{\partial^2 f}{\partial y\partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$$

The mixed derivative theorem:-

If $f(x,y)$ & its partial derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x\partial y}, \frac{\partial^2 f}{\partial y\partial x}$ are defined

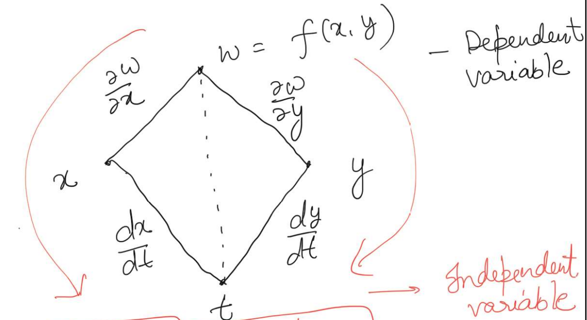
(a,b)

through out an open region containing a point (a,b) & all are continuous at (a,b), then

$$\frac{\partial^2 f}{\partial x \partial y}(a,b) = \frac{\partial^2 f}{\partial y \partial x}(a,b)$$

The chain Rule: If $w = f(x,y)$ has continuous partial derivatives $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ & if $x = x(t)$, $y = (t)$ are differentiable functions of t, then the composite function $w = f(x(t), y(t))$ is a differentiable function of t &

$$\frac{df}{dt} = \frac{\partial f}{\partial x}(x(t), y(t)) \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y}(x(t), y(t)) \cdot \frac{dy}{dt}$$



$$\boxed{\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}}$$

Example: Find the derivative of $w = xy$ w.r.t. t along the path $x = \cos t$, $y = \sin t$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt}$$

$$= y(-\sin t) + x(\cos t)$$

$$= \sin t(-\sin t) + \cos t(\cos t)$$

$$= \cos^2 t - \sin^2 t$$

$$\boxed{\frac{dw}{dt} = \cos 2t}$$

Gradient vector: The gradient of $f(x,y)$ at a point (x_0, y_0) is the vector

grad f, Del f $\leftarrow$ $\boxed{\nabla f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}}$

obtained by evaluating the partial derivatives of f at point P_0.

Example: Find ∇f for $f(x,y) = xe^y + \cos(xy)$ at point $(2,0)$

Sol:
$$\frac{\partial f}{\partial x} =$$

Do it yourself!

$$\frac{\partial f}{\partial y} =$$

then find $\left.\frac{\partial f}{\partial x}\right|_{(2,0)} = ?$ & $\left.\frac{\partial f}{\partial y}\right|_{(2,0)} = ?$

Then
$$\boxed{\left.\nabla f\right|_{(2,0)} = \left.\frac{\partial f}{\partial x}\right|_{(2,0)}\hat{i} + \left.\frac{\partial f}{\partial y}\right|_{(2,0)}\hat{j}}$$

You found that, $f = xe^y + \cos(xy)$

$$\frac{\partial f}{\partial x} = e^y - y\sin(xy)$$

$$\left.\frac{\partial f}{\partial x}\right|_{(2,0)} = 1 - 0 = 1$$

$$\frac{\partial f}{\partial y} = xe^y - x\sin(xy)$$

$$\left.\frac{\partial f}{\partial y}\right|_{(2,0)} = 2 - 0 = 2. \quad (\sin 0 = 0)$$

then

$$\left.\nabla f\right|_{(2,0)} = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

$$\boxed{\left.\nabla f\right|_{(2,0)} = i + 2j}$$

Directional derivatives:- we know that

$\frac{\partial f}{\partial x}$ = rate of change of f in the direction of $\hat{i}$

$\frac{\partial f}{\partial y}$ = rate of change of f in the direction of $\hat{j}$.

Ques? How to find the rate of change of f in any direction.

Definition: The derivative of f at $P(x_0, y_0)$ in the direction of the unit vector $u = u_1\hat{i} + u_2\hat{j}$ is given as

$$\left(D_u f\right)_P = \lim_{s \to 0} \frac{f(x_0 + s u_1, y_0 + s u_2) - f(x_0, y_0)}{s}$$

Relation between directional derivative & gradient :-

$$\boxed{\left(D_u f\right)_P = \left(\nabla f\right)_P \cdot \hat{u}} \quad ☆$$

Example: Find the derivative of $f(x,y) = xe^y + \cos(xy)$ at the point $(2,0)$ in the direction $v = 3i - 4j$

Sol^n:-

$$\left.\nabla f\right|_{(2,0)} = i + 2j$$

$$|v| = \sqrt{3^2 + 4^2} = 5$$

Since v is not a unit vector, first convert it to unit vector.

$$\hat{u} = \frac{v}{|v|} = \frac{3i - 4j}{5}$$

Therefor

$$\left(D_u f\right)_P = \nabla f \cdot \hat{u} = (i + 2j) \cdot \left(\frac{3}{5} i - \frac{4}{5} j\right)$$
$$= -1$$

Properties:-

$$D_u f = \nabla f \cdot \hat{u} = |\nabla f|\,|\hat{u}| \cos\theta$$
$$= \underbrace{|\nabla f| \cos\theta}_{\text{max} \qquad \text{min}}$$

① The function will increase most rapidly when $\cos\theta = 1$, i.e. $\hat{u}$ is in same direction as ∇f

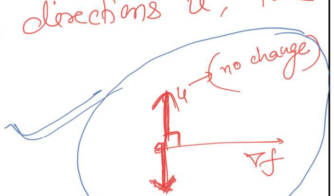
$$D_u f = |\nabla f|$$

② The function will decrease most rapidly in the direction of $-\nabla f$.

$$D_u f = |\nabla f| \cos\pi = -|\nabla f|$$

③ If $\theta = \pi/2$ then $D_u f = |\nabla f| \cos\frac{\pi}{2} = 0$

If u is orthogonal to ∇f then $\theta = \pi/2$ & $D_u f = 0$. In such directions u, the change in zero.



Example:- Find the direction in which

$$f(x,y) = \frac{x^2}{2} + \frac{y^2}{2}$$

① increases most rapidly at $(1,1)$

Sol^n: function increases most rapidly in the direction of ∇f.

$$\nabla f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = 1$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 1$$

$$\left.\nabla f\right|_{(1,1)} = \left.\frac{\partial f}{\partial x}\right|_{(1,1)} i + \left.\frac{\partial f}{\partial y}\right|_{(1,1)} j$$

$$\boxed{\left.\nabla f\right|_{(1,1)} = i + j}$$

f will increase maximum in the direction of u.

$$u = \frac{i + j}{|i + j|} = \frac{i}{\sqrt{2}} + \frac{j}{\sqrt{2}}.$$

(ii) The function will decrease most rapidly in the direction of $-u$, where

$$\boxed{-\hat{u} = -\frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}}$$

(iii) The direction of zero change at $(1,1)$

$\Downarrow$

which are orthogonal to $u = \frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$

the 1^{st} vector $p = -\frac{\hat{i}}{\sqrt{2}} + \frac{\hat{j}}{\sqrt{2}}$ $\quad p \cdot u = 0$

2^{nd} vector $q = \frac{\hat{i}}{\sqrt{2}} - \frac{\hat{j}}{\sqrt{2}}$ $\Rightarrow$ $q \cdot u = 0$

In these direction, there will be zero change in f.

Recap the concept of Equation of tangent & normal lines from your class 12^{th}

Tangent :- $\quad y - y_0 = \left(\frac{dy}{dx}\right)_{(x_0, y_0)} (x - x_0)$

Normal :- $\quad y - y_0 = -\frac{1}{\left(\frac{dy}{dx}\right)_{(x_0, y_0)}} (x - x_0)$

Tangent planes & Normal lines :-

Let $P(x_0, y_0, z_0)$ be a point on the level surface of function $f(x, y, z)$.

The equation of tangent plane to $f(x,y,z) = c$ at P is

$$\boxed{\left.\frac{\partial f}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_P (y - y_0) + \left.\frac{\partial f}{\partial z}\right|_P (z - z_0) = 0}$$

The equation of Normal line to $f(x,y,z) = c$ at P is

$$\boxed{\begin{aligned} x &= x_0 + \left.\frac{\partial f}{\partial x}\right|_P t \\ y &= y_0 + \left.\frac{\partial f}{\partial y}\right|_P t \\ z &= z_0 + \left.\frac{\partial f}{\partial z}\right|_P t \end{aligned}}$$

Example :- Find the equation of tangent plane & normal lines for

$$f(x, y, z) = x^2 + y^2 + z - 9$$ at the point $(1, 2, 4)$.

$x_0 = 1$, $y_0 = 2$, $z_0 = 4$

Soln :- $\frac{\partial f}{\partial x} = 2x \qquad \left.\frac{\partial f}{\partial x}\right|_P = 2$

$\frac{\partial f}{\partial y} = 2y \qquad \left.\frac{\partial f}{\partial y}\right|_P = 4$

$\frac{\partial f}{\partial z} = 1 \qquad \left.\frac{\partial f}{\partial z}\right|_P = 1$

Equation of tangent plane

$$\left.\frac{\partial f}{\partial x}\right|_P (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_P (y - y_0) + \left.\frac{\partial f}{\partial y}\right|_P (z - z_0) = 0$$

$$2(x - 1) + 4(y - 2) + 1 \cdot (z - 4) = 0$$

$$2x + 4y + z = 14.$$

Equation of Normal lines :-

$$x = x_0 + \left.\frac{\partial f}{\partial x}\right|_P t = 1 + 2t$$

$$y = y_0 + \left.\frac{\partial f}{\partial y}\right|_P t = 2 + 4t$$

$$z = z_0 + \left.\frac{\partial f}{\partial z}\right|_P t = 4 + t$$

Maxima & Minima for function of 2-variables

Recall the maxima & minima concept for function of one variable from your previous class. You are given $y = f(x)$

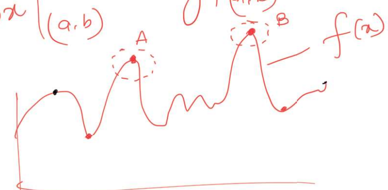
then put $\frac{dy}{dx} = 0$, find x

Find $\frac{d^2y}{dx^2} > 0$ — minima

< 0 — maxima.

Theorem :- If $f(x, y)$ has a local maximum or minimum value at an interior point (a, b) of its domain & if the first partial derivative exist, then

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = \left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$$



This is known as First derivative test.

Definition: Critical point : An interior point of the domain of a function $f(x,y)$ where both $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ are zero or where one or both of $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$ do not exist, is called critical point of f.

Remark: Not every critical point is a point of maxima or a point of minima. A function of single variable may have point of inflection. Similarly, a function of two variable may have saddle points.

Example: Find the points where maximum & minimum may occur for $f(x,y) = x^2+y^2$.

Sol:

$$\frac{\partial f}{\partial x} = 0 \Rightarrow 2x = 0 \Rightarrow x = 0$$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow 2y = 0 \Rightarrow y = 0$$

The critical point is $(0,0)$ may be point of maxima or minima.

Conclusion: The $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$ at an interior point (a,b) does not guarantee that $f(x,y)$ has local maxima or minima.

Can we take help of second derivative test?

Theorem: (Second derivative test)

Suppose $f(x,y)$ & its first/second partial derivatives are continuous & $\left.\frac{\partial f}{\partial x}\right|_{(a,b)} = 0$, $\left.\frac{\partial f}{\partial y}\right|_{(a,b)} = 0$

① If $\frac{\partial^2 f}{\partial x^2} < 0$ & $\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (a,b) then f has local maximum at (a,b)

② If $\frac{\partial^2 f}{\partial x^2} > 0$ & $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 > 0$ at (a,b) then f has local minimum at (a,b)

③ If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 < 0$ at (a,b) then f has a saddle point at (a,b)

④ If $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 0$ at (a,b) then second derivative test is inconclusive

The expression $\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$ is called discriminant or Hessian of $f(x,y)$ and is equal to

$$= \begin{vmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial x \partial y} & \frac{\partial^2 f}{\partial y^2} \end{vmatrix}$$

Discussion:

$\frac{\partial^2 f}{\partial x^2} < 0$ $\Rightarrow$ $\frac{\partial^2 f}{\partial y^2} < 0$

$$\underbrace{\frac{\partial^2 f}{\partial x^2}}_{(-)ve}\cdot\underbrace{\frac{\partial^2 f}{\partial y^2}}_{(-)ve} - \underbrace{\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2}_{(+)ve} > 0$$

Example: Find local maxima or minima for

$$f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$$

Sol: Find critical points

$$\frac{\partial f}{\partial x} = y - 2x - 2 \qquad \frac{\partial^2 f}{\partial x^2} = -2$$

$$\frac{\partial f}{\partial y} = x - 2y - 2 \qquad \frac{\partial^2 f}{\partial y^2} = -2$$

Put $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

$$\left.\begin{aligned} y - 2x &= 2 \\ x - 2y &= 2 \end{aligned}\right\} \qquad \begin{aligned} x &= -2 \\ y &= -2 \end{aligned}$$

critical point

$(-2,-2)$ is the only point where $f(x,y)$ may have maximum or minimum value

Now, we need to use 2nd derivative test

Find $\frac{\partial^2 f}{\partial x^2} = -2$

$\frac{\partial^2 f}{\partial y^2} = -2$

$\frac{\partial^2 f}{\partial x \partial y} = 1$

$\Rightarrow \frac{\partial^2 f}{\partial x^2} < 0$ & Hessian?

$\frac{\partial^2 f}{\partial x^2}\cdot\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 3 > 0$

It gives the $f(x,y)$ has local maximum at $(-2,-2)$. The value of f at $(-2,-2)$ is 8.

Qus:- Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$, $f(x,y) = (xy-1)^2$

$$\frac{\partial f}{\partial x} = 2y(xy-1)$$

$$\frac{\partial f}{\partial y} = 2x(xy-1)$$

Qus:- $f(x,y) = \cos^2(3x-y^2)$

$$\frac{\partial f}{\partial x} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial x}\cos(3x-y^2)$$

$$= -2\cos(3x-y^2) \cdot \sin(3x-y^2) \cdot \frac{\partial}{\partial x}(3x-y^2)$$

$$= -6\cos(3x-y^2)\sin(3x-y^2)$$

$$\frac{\partial f}{\partial y} = 2\cos(3x-y^2) \cdot \frac{\partial}{\partial y}\cos(3x-y^2)$$

$$= -2\cos(3x-y^2) \cdot \sin(3x-y^2) \cdot \frac{\partial}{\partial y}(3x-y^2)$$

$$= 4y\cos(3x-y^2)\sin(3x-y^2)$$

Qus:- Second order partial derivatives

$f(x,y) = \tan^{-1}(y/x)$, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}$

Sol:- $$\frac{\partial f}{\partial x} = \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{\partial}{\partial x}\left[\frac{y}{x}\right] = \left(-\frac{y}{x^2}\right)\left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right]$$

$$= \frac{-y}{x^2+y^2}$$

$$\frac{\partial f}{\partial y} = \left[\frac{1}{1+\left(\frac{y}{x}\right)^2}\right]\frac{\partial}{\partial y}\left[\frac{y}{x}\right] = \frac{1}{x}\left\{\frac{1}{1+\left(\frac{y}{x}\right)^2}\right\}$$

$$= \frac{x}{x^2+y^2}$$ $\frac{\partial}{\partial y}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{y(2x)}{(x^2+y^2)^2} = \frac{2xy}{(x^2+y^2)^2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-x(2y)}{(x^2+y^2)^2} = \frac{-2xy}{(x^2+y^2)^2}$$

formula:- $$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{\frac{du}{dx}\cdot v - \frac{dv}{dx}\cdot u}{v^2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y^2)(-1) + y(2y)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}$$

Qus:- $f = e^x + x\ln y + y\ln x$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{y} + \frac{1}{x}$$

$$\frac{\partial f}{\partial y} = 0 + \frac{x}{y} + \ln x$$

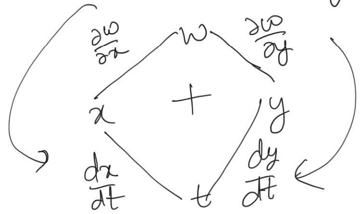
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{x}{y} + \ln x\right)$$

$$= \frac{1}{y} + \frac{1}{x}$$

Chain Rule:- $w = f(x,y)$, $x = g(t)$, $y = h(t)$

Find $\frac{dw}{dt} = ?$

$$= \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt}$$



Qus:- $w = x^2+y^2$ & $x = \cos t$, $y = \sin t$

$\frac{dw}{dt} = ?$

$\frac{\partial w}{\partial x} = 2x$, $\frac{\partial w}{\partial y} = 2y$, $\frac{dx}{dt} = -\sin t$, $\frac{dy}{dt} = \cos t$

$$\frac{dw}{dt} = 2x\cdot(-\sin t) + 2y(\cos t)$$

put $x = \cos t$, $y = \sin t$

$$= -2\cancel{\cos t \sin t} + 2\cancel{\sin t \cos t}$$

$$= 0$$

If $w = f(x,y,z)$, $x = g(t)$, $y = h(t)$, $z = k(t)$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt}$$

Qus:- $w = \ln(x^2+y^2+z^2)$, $x = \cos t$, $y = \sin t$, $z = 4\sqrt{t}$

$\frac{\partial w}{\partial x} = \frac{2x}{x^2+y^2+z^2}$, $\frac{dx}{dt} = -\sin t$

$\frac{\partial w}{\partial y} = \frac{2y}{x^2+y^2+z^2}$, $\frac{dy}{dt} = \cos t$

$\frac{\partial w}{\partial z} = \frac{2z}{x^2+y^2+z^2}$, $\frac{dz}{dt} = 2t^{-1/2}$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} =$$

$$= \frac{-2x\sin t}{x^2+y^2+z^2} + \frac{2y\cos t}{x^2+y^2+z^2} + \frac{4zt^{-1/2}}{x^2+y^2+z^2}$$

$$= \frac{-2\cos t\cancel{\sin t} + 2\cancel{\sin t\cos t} + 4(4t^{\frac{1}{2}})t^{-1/2}}{\cos^2 t + \sin^2 t + 16t}$$

$$= \frac{16}{1+16t}$$

Ques:- $w = 2ye^x - \ln z$ $\quad x = \ln(t^2+1)$, $y = \tan^{-1} t$, $z = e^t$

$\frac{dw}{dt} = ?$

Sol:- $\frac{\partial w}{\partial x} = 2ye^x$ $\qquad \frac{dx}{dt} = \frac{2t}{1+t^2}$

$\frac{\partial w}{\partial y} = 2e^x$ $\qquad \frac{dy}{dt} = \frac{1}{1+t^2}$

$\frac{\partial w}{\partial z} = -\frac{1}{z}$ $\qquad \frac{dz}{dt} = e^t$

$$\frac{dw}{dt} = \frac{4yte^x}{t^2+1} + \frac{2e^x}{t^2+1} - \frac{e^t}{z}$$

$$= \frac{4t(\tan^{-1}t)\cdot\cancel{(t^2+1)}}{\cancel{(t^2+1)}} + \frac{2\cancel{(t^2+1)}}{\cancel{(t^2+1)}} - \frac{\cancel{e^t}}{\cancel{e^t}}$$

$$\boxed{\frac{dw}{dt} = 1 + 4t\tan^{-1}t}$$

Directional Derivatives & Gradient vector:-

Gradient vector $= \nabla f$

$$= \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k.$$

Ques:- Find Gradient of function $f(x,y) = \ln(x^2+y^2)$ at $(1,1)$

Sol:-

$\frac{\partial f}{\partial x} = \frac{2x}{x^2+y^2} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(1,1)} = \frac{2}{2} = 1$

$\frac{\partial f}{\partial y} = \frac{2y}{x^2+y^2} \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(1,1)} = 1$

$$\boxed{\nabla f = \left.\frac{\partial f}{\partial x}\right|_{(1,1)} i + \left.\frac{\partial f}{\partial y}\right|_{(1,1)} j = i + j}$$

Ques:- $f(x,y,z) = e^{x+y}\cos z + (y+1)\sin^{-1}x$ at $(0, 0, \pi/6)$

$\frac{\partial f}{\partial x} = e^{x+y}\cos z + \frac{y+1}{\sqrt{1-x^2}} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0,\pi/6)} = \frac{\sqrt{3}}{2} + 1$

$\frac{\partial f}{\partial y} = e^{x+y}\cos z + \sin^{-1}x \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(0,0,\pi/6)} = \frac{\sqrt{3}}{2}$

$\frac{\partial f}{\partial z} = -e^{x+y}\sin z \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{(0,0,\pi/6)} = -\frac{1}{2}$

$$\boxed{\nabla f = \left(\frac{\sqrt{3}}{2}+1\right) i + \frac{\sqrt{3}}{2} j - \frac{1}{2} k}$$

Directional Derivatives:-

$$(D_u f)_{P_0} = \nabla f\Big|_{P_0} \cdot \hat{u}$$

Ques:- Find directional derivative of $f(x,y) = 2x^2 + y^2$ at $(-1,1)$ & $u = 3i - 4j$

Sol:-

$\frac{\partial f}{\partial x} = 4x$ $\qquad \left.\frac{\partial f}{\partial x}\right|_{(-1,1)} = -4$

$\frac{\partial f}{\partial y} = 2y$ $\qquad \left.\frac{\partial f}{\partial y}\right|_{(-1,1)} = 2$

$\nabla f = -4i + 2j$

$$\hat{u} = \frac{\vec{u}}{|\vec{u}|} = \frac{3i - 4j}{\sqrt{3^2+(-4)^2}} = \frac{3}{5}i - \frac{4}{5}j$$

$$\Rightarrow \nabla f \cdot \hat{u} = (-4 + 2j)\cdot\left(\frac{3}{5}i - \frac{4}{5}j\right)$$

$$= -\frac{12}{5} - \frac{8}{5} = -4.$$

Qus:- $f(x,y,z) = \cos xy + e^{yz} + \ln zx$

at $(1, 0, \frac{1}{2})$ & $u = i + 2j + 2k$.

Sol:- $\hat{u} = \dfrac{i+2j+2k}{3} = \frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k$.

$\dfrac{\partial f}{\partial x} = -y\sin xy + \dfrac{1}{x} \Rightarrow \dfrac{\partial f}{\partial x}\Big|_{(1,0,\frac{1}{2})} = 1$

$\dfrac{\partial f}{\partial y} = -x\sin xy + ze^{yz} \Rightarrow \dfrac{\partial f}{\partial y}\Big|_{(1,0,\frac{1}{2})} = \dfrac{1}{2}$

$\dfrac{\partial f}{\partial z} = ye^{yz} + \dfrac{1}{z} \Rightarrow \dfrac{\partial f}{\partial z}\Big|_{1,0,\frac{1}{2}} = 2$

$\Rightarrow \nabla f = i + \frac{1}{2}j + 2k$

$\Rightarrow (D_u f)_{P_0} = \left(i + \frac{1}{2}j + 2k\right) \cdot \left(\frac{1}{3}i + \frac{2}{3}j + \frac{2}{3}k\right)$

$= 2.$

Find Maxima & Minima, or saddle point ÷

Qus:- $f(x,y) = x^2 + xy + y^2 + 3x - 3y + 4$

Sol:

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + y + 3 = 0 \\ \frac{\partial f}{\partial y} &= x + 2y - 3 = 0 \end{aligned}\right\} \quad \begin{aligned} x &= -3 \\ y &= 3 \end{aligned}$$

$\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y^2} = 2, \quad \dfrac{\partial^2 f}{\partial x \partial y} = 1$

$$\& \begin{cases} \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 3 > 0 \\ \dfrac{\partial^2 f}{\partial x^2} > 0 \end{cases}$$

$\rightarrow$ f has local minimum at $(-3, 3)$

$\boxed{\text{Minimum value of } f(-3,3) = -5}$

Qus:- $f(x,y) = x^2 + 3xy + 3y^2 - 6x + 3y - 6$

$$\left.\begin{aligned} \frac{\partial f}{\partial x} &= 2x + 3y - 6 = 0 \\ \frac{\partial f}{\partial y} &= 3x + 6y + 3 = 0 \end{aligned}\right\} \text{solve} \quad \begin{aligned} x &= 15 \\ y &= -8 \end{aligned}$$

$\dfrac{\partial^2 f}{\partial x^2} = 2, \quad \dfrac{\partial^2 f}{\partial y^2} = 6, \quad \dfrac{\partial^2 f}{\partial x \partial y} = 3$

$$\begin{cases} \dfrac{\partial^2 f}{\partial x^2} \cdot \dfrac{\partial^2 f}{\partial y^2} - \left(\dfrac{\partial^2 f}{\partial x \partial y}\right)^2 = 3 > 0 \\ \dfrac{\partial^2 f}{\partial x^2} = 2 > 0 \end{cases}$$

f has local minimum at $(15, -8)$.

the minimum value is $= -63$.

Qus:- $f(x,y) = x^3 - y^3 - 2xy + 6$

$\dfrac{\partial f}{\partial x} = \dfrac{\partial}{\partial x}\left(x^3 - y^3 - 2xy + 6\right)$

$= 3x^2 - 0 - 2y + 0$

$= 3x^2 - 2y$

$\dfrac{\partial f}{\partial y} = \dfrac{\partial}{\partial y}\left(x^3 - y^3 - 2xy + 6\right)$

$= 0 - 3y^2 - 2x + 0$

$= -3y^2 - 2x$

Put

$3x^2 - 2y = 0 \Rightarrow \underline{y = 3x^2/2}$

$-3y^2 - 2x = 0$

$-3\left(\frac{3x^2}{2}\right)^2 - 2x = 0$

$-\frac{27x^4}{4} - 2x = 0$

$\Rightarrow \quad -x\left(\frac{27x^3}{\cancel{4}} + 2\right) = 0$

$\Rightarrow \boxed{x = 0}$ or $\frac{27x^3}{4} + 2 = 0$

then $\boxed{y = 0}$

$\Downarrow$

$27x^3 = -8$

$x^3 = -\frac{8}{27}$

$\boxed{x = -\frac{2}{3}}$

then

$y = \frac{3x^2}{2} = \frac{3}{2}\left(\frac{-2}{3}\right)^2 = \frac{3 \times 4}{2 \times 9}$

$\boxed{y = \frac{2}{3}}$

$(0,0)$ and $\left(-\frac{2}{3}, \frac{2}{3}\right)$ are critical point

$(0,0)$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(3x^2 - 2y)$

$= 6x$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} = 0$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-3y^2 - 2x)$

$= -6y$

$\left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} = 0$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-3y^2 - 2x)$

$= -2$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{(0,0)} \cdot \left.\frac{\partial^2 f}{\partial y^2}\right|_{(0,0)} - \left.\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right|_{(0,0)} = 0 \cdot 0 - 4$

$= -4$

< 0

$\Rightarrow (0,0)$ is a $\underline{\text{saddle point}}$

$\left(-\frac{2}{3}, \frac{2}{3}\right)$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} = 6 \times \left(-\frac{2}{3}\right) = -4$

$\left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} = -6 \times \left(\frac{2}{3}\right) = -4$

$\left.\frac{\partial^2 f}{\partial x \partial y}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} = -2$

$\left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} \left.\frac{\partial^2 f}{\partial y^2}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} - \left.\left(\frac{\partial^2 f}{\partial x \partial y}\right)^2\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)}$

$= 16 - 4 = 12 > 0$

& $\left.\frac{\partial^2 f}{\partial x^2}\right|_{\left(-\frac{2}{3}, \frac{2}{3}\right)} < 0$

$\Rightarrow \left(-\frac{2}{3}, \frac{2}{3}\right)$ is point of maximum.

& the maximum value of $f(x,y)$ at $\left(-\frac{2}{3}, \frac{2}{3}\right)$

$= x^3 - y^3 - 2xy + 6$

$= \left(\frac{-2}{3}\right)^3 - \left(\frac{2}{3}\right)^3 - 2\left(\frac{-2}{3}\right)\left(\frac{2}{3}\right) + 6$

$= \frac{170}{27}$

Equation of tangent plane & normal lines

Ques: Surface: $x^2 + y^2 + z^2 = 3$ & $P_0\,(1,1,1)$

Eqn of tangent plane

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{P_0}(y - y_0) + \left.\frac{\partial f}{\partial z}\right|_{P_0}(z - z_0) = 0 \quad \text{— (A)}$$

Sol:

$\frac{\partial f}{\partial x} = 2x \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{P_0} = 2$

$\frac{\partial f}{\partial y} = 2y \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{P_0} = 2$

$\frac{\partial f}{\partial z} = 2z \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{P_0} = 2$

$2(x-1) + 2(y-1) + 2(z-1) = 0$

$2x - 2 + 2y - 2 + 2z - 2 = 0$

$\boxed{x + y + z = 3}$

Eqn of Normal lines

$x = x_0 + \left.\frac{\partial f}{\partial x}\right|_{P_0} t = 1 + 2t$

$$y = y_0 + \left.\frac{\partial f}{\partial y}\right|_{P_0} t = 1+2t$$

$$z = z_0 + \left.\frac{\partial f}{\partial y}\right|_{P_0} t = 1+2t$$

Ques:- $\underbrace{x^2+y^2-2xy-x+3y-z}=-4$

$P_0\ (2,-3,18)$

$$\frac{\partial f}{\partial x} = 2x-2y-1 \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{P_0} = 9$$

$$\frac{\partial f}{\partial y} = 2y-2x+3 \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{P_0} = -7$$

$$\frac{\partial f}{\partial z} = -1 \Rightarrow \left.\frac{\partial f}{\partial z}\right|_{P_0} = -1$$

Equation tangent plane:-

$$\left.\frac{\partial f}{\partial x}\right|_{P_0}(x-x_0) + \left.\frac{\partial f}{\partial y}\right|_{P_0}(y-y_0) + \left.\frac{\partial f}{\partial z}\right|_{P_0}(z-z_0)=0$$

$$9(x-2)-7(y+3)-1(z-18)=0$$

$$\Rightarrow \boxed{9x-7y-z=21}$$

Equation of normal lines:-

$$x = x_0 + \left.\frac{\partial f}{\partial x}\right|_{P_0} t = 2+9t$$

$$y = y_0 + \left.\frac{\partial f}{\partial y}\right|_{P_0} t = -3-7t$$

$$z = z_0 + \left.\frac{\partial f}{\partial z}\right|_{P_0} t = 18-t$$

Ques:- Tangent plane for

$$z = e^{-(x^2+y^2)} \qquad (0,0,1)$$

$$\frac{\partial f}{\partial x} = -2x\,e^{-(x^2+y^2)} \Rightarrow \left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0$$

$$\frac{\partial f}{\partial y} = -2y\,e^{-(x^2+y^2)} \Rightarrow \left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$$

Eq^n of tangent plane:-

$$0+0+(z-1)=0 \Rightarrow \boxed{z=1}$$

Do it yourself:-

Ques:- $z = \ln(x^2+y^2) \qquad (1,0,0)$

Ans:- $2x-z=2$.

Ques:- $\cos\pi x - x^2y + e^{xz} + yz = 4$ at $(0,1,2)$

find tangent plane & normal line.

Ques 1:- A privately owned business operates both a drive in facility & a walk in facility. On a randomly selected day, let X & Y, respectively be the proportions of the time that the drive-in & the walk in facilities are in use and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y) & 0 \le x \le 1,\ 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$P[(X,Y) \in A]$, $A = \{(x,y) \mid 0 \le x \le \frac{1}{2},\ \frac{1}{4} \le y \le \frac{1}{2}\}$

Sol:- $P\left(0 < X < \frac{1}{2},\ \frac{1}{4} < Y < \frac{1}{2}\right) =$

$$= \int_{1/4}^{1/2} \int_0^{1/2} \frac{2}{5}(2x + 3y)\, dx\, dy$$

$\int 3y\, dx = 3xy.$

$$= \frac{2}{5} \int_{1/4}^{1/2} \left[\cancel{2} \cdot \frac{x^2}{\cancel{2}} + 3xy \right]_0^{1/2} dy$$

$$= \frac{2}{5} \int_{1/4}^{1/2} \left(\left(\frac{1}{2}\right)^2 + \frac{3}{2} y - 0 \right) dy$$

$$= \frac{2}{5} \left[\frac{1}{4} y + \frac{3}{2} \cdot \frac{y^2}{2} \right]_{1/4}^{1/2}$$

$$= \frac{2}{5} \left[\left(\frac{1}{4} \times \frac{1}{2} + \frac{3}{4} \times \left(\frac{1}{2}\right)^2 \right) - \left(\frac{1}{4} \times \frac{1}{4} + \frac{3}{4}\left(\frac{1}{4}\right)^2 \right) \right]$$

$$= \frac{13}{160}$$

Ques:- Find the marginal PDFs for random variable X & Y in above problem.

Sol:- Marginal PDF of X

$$= \int_{-\infty}^{\infty} f(x,y)\, dy$$

$$= \int_0^1 \frac{2}{5}(2x + 3y)\, dy$$

$$= \frac{2}{5} \left[2 \cdot xy + 3 \frac{y^2}{2} \right]_0^1$$

$$= \frac{4x + 3}{5}$$

Marginal PDF of Y

$$= \int_{-\infty}^{\infty} f(x,y)\, dx = \int_0^1 \frac{2}{5}(2x + 3y)\, dx$$

$$= \frac{2}{5} \left[2 \cdot \frac{x^2}{2} + 3xy \right]_0^1 = \frac{2 + 6y}{5}$$

Ques:- The joint density for the random variables (X, Y), where X is the unit temperature change & Y is the proportion of spectrum shift that a certain atomic particle produces, is

$$f(x,y) = \begin{cases} 10xy^2 & \underbrace{0 < x < y < 1} \\ 0 & \text{otherwise.} \end{cases}$$

(a) find Marginal PDF of X, & Y & find conditional PDF $f_{Y|X}(y|x)$

Sol:- Marginal PDF of X

$$= \int_{-\infty}^{\infty} f(x,y)\, dy = \int_x^1 10xy^2\, dy$$

$$= \frac{10}{3} x (1 - x^3) \Leftarrow \left[10x \cdot \frac{y^3}{3} \right]_x^1$$

Marginal PDF of Y

$$= \int_0^y 10xy^2\, dx$$

$$= 10y^2 \left[\frac{x^2}{2}\right]_0^y = 5y^4$$

Conditional PDF $\boxed{f(y|x) = \frac{f(x,y)}{f(x)}} = \frac{10xy^2}{\frac{10}{3}x(1-x^3)}$

$$= \frac{3y^2}{1 - x^3}$$

(b) Find the probability that the spectrum shifts more than half of the total observations, given that temperature is increased by 0.25 unit.

$$P\left(Y > \frac{1}{2} \,\middle|\, X = 0.25\right) =$$

$$= \int_{1/2}^{1} \frac{3y^2}{1 - (0.25)^3}\, dy = \frac{1}{1 - (0.25)^3} \left[y^3 \right]_{1/2}^{1}$$

$$= \frac{8}{9}.$$

Ques:- The length of time, in minutes, for an airplane to obtain clearance for take off at a certain airport is a random variable

$$Y = 3X - 2, \text{ where } X \text{ has}$$

density function

$$f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & : x > 0 \\ 0 : & \text{otherwise} \end{cases}$$

Find mean & variance of Y.

Sol:- $E[Y] = E[3\cancel{X}-2]$

$$= \int_0^\infty (3x-2) \cdot \frac{1}{4} e^{-x/4} dx$$

$$= \frac{1}{4}\left[\int_0^\infty 3x e^{-x/4} dx - \int_0^\infty 2 e^{-x/4} dx\right]$$

$$= 10 \quad ?$$

Do it yourself

Continued previous question

$$Var(Y) = E\left[(Y - E(Y))^2\right]$$

put Y & E(Y) = 10

$$= E\left[\left((3X-2) - 10\right)^2\right]$$

$$= E\left[(3\cancel{X}-12)^2\right]$$

$$= \int_0^\infty (3x-12)^2 \cdot \underbrace{\frac{1}{4} e^{-x/4}}_{\text{PDF of } X} dx$$

$$= 144 \quad \text{(Do it yourself)}$$

Ques:- The fraction X of male runners & fraction Y of female runners who compete in marathon races are described by joint density function

$$f(x,y) = \begin{cases} 8xy & 0 < y < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Find Covariance of X & Y

Sol:- $\boxed{Cov(X,Y) = E[XY] - E[X]E[Y]}$

$$E[XY] = \int_0^1 \int_y^1 (xy) \cdot 8xy \, dx \, dy = \frac{4}{9}$$

To find $E[X]$, first we find Marginal PDF of X

$$f_X(x) = \int_0^x 8xy \, dy = 8x\left[\frac{y^2}{2}\right]_0^x = 4x^3 \quad 0<x<1$$

$$E[X] = \int_0^1 x \cdot \underbrace{4x^3}_{\text{PDF of } X} dx = \frac{4}{5}$$

Similarly, Marginal PDF of Y

$$f_Y = \int_y^1 8xy \, dx = 4y(1-y^2)$$

$$E[Y] = \int_0^1 y \cdot 4y(1-y^2) \, dy$$

$$= \frac{8}{15}$$

Therefore,

$$\text{Covariance } (X,Y) = E[XY] - E[X]E[Y]$$

$$= \frac{4}{9} - \frac{4}{5} \times \frac{8}{15}$$

$$\boxed{Cov(X,Y) = \frac{4}{225}}$$