Solving recurrences

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Three ways to solve recurrences

- Substitution
- Recursion tree
- Master theorem

Use whichever way you want (but if you use MT, you must quote it).

Master theorem (to solve recurrences)

Consider the recurrence

$$T(n) = aT(n/b) + f(n)$$

- If $f(n) \in O(n^{\log_b a-\epsilon})$ then $T(n) = \Theta(n^{log_b a})$
- If $f(n) \in \Theta(n^{\log_b a})$ then $T(n) = \Theta(n^{log_b a} \log n)$
- If $f(n) \in \Omega(n^{\log_b a+\epsilon})$ and $af(n/b) < (1-\epsilon)f(n)$ then $T(n) = \Theta(f(n))$

Solve

- $T(n) = 9T(n/3) + n$

$T(n) = 9T(n/3) + n$ by MT

We identify $a, b, f(n)$

$$\begin{aligned} T(n) &= aT(n/b) + f(n) \\ &= 9T(n/3) + n \end{aligned}$$

We compute

$$\begin{aligned} n^{\log_b a} &= n^{\log_3 9} \\ &= n^2 \end{aligned}$$

We compare $f(n)$ to $n^{\log_b a - \epsilon}, n^{\log_b a}, n^{\log_b a + \epsilon}$

$$n \in O(n^{1.9})$$

We conclude first case: $T(n) \in \Theta(n^2)$.

$T(n) = 9T(n/3) + n$ by substitution

$$
\begin{aligned}
T(n) &= 9T(n/3) + n \\
&= 9[9T(n/3^2) + n/3] + n && = 9^2T(n/3^2) + 4n \\
&= 9[9^2T(n/3^3) + n/3^2] + 4n && = 9^3T(n/3^3) + 13n \\
&= \dots \\
&= 9^{\log_3 n}T(1) + cn \\
&\in \Theta(n^2)
\end{aligned}
$$

Solve

- $T(n) = T(2n/3) + 1$

$T(n) = T(2n/3) + 1$ by MT

- We identify $a = 1, b = 3/2, f(n) = 1$
- Compute $n^{\log_b a} = n^{\log_{3/2} 1} = n^0 = 1$
- We classify $1 \in \Theta(1)$
- Therefore second case $T(n) = \Theta(\log n)$

$T(n) = T(2n/3) + 1$ by substitution

$$T(n) = T(2n/3) + 1 \qquad = T((2/3)n) + 1$$
$$= T((2/3)^2 n) + 1 + 1$$
$$= T((2/3)^3 n) + 1 + 1 + 1$$

We will do this until

$$(2/3)^k n \leq 1 \Leftrightarrow \log_{3/2} n \leq k$$

$$(2/3)^k n \leq 1 \Leftrightarrow \log_{3/2} n \leq k$$

Therefore

$$T(n) = C + \log_{3/2} n \in \Theta(\log_{3/2} n)$$

Notice that the base of the log is different from the MT result.

Solve

- $T(n) = 3T(n/4) + n \log n$

$T(n) = 3T(n/4) + n \log n$ by the MT

We identify $a, b, f(n)$

$$\begin{aligned} T(n) &= 3T(n/4) + n \log n \\ &= aT(n/b) + f(n) \end{aligned}$$

We compute

$$\begin{aligned} n^{\log_b a} &= n^{\log_4 3} \\ &= n^{\frac{\log 3}{\log 4}} \\ &\approx n^{0.79} \end{aligned}$$

We compare $f(n)$ to $n^{\log_b a - \epsilon}, n^{\log_b a}, n^{\log_b a + \epsilon}$

$$n \log n \in \Omega(n^{0.8})$$

Therefore we are in the third case and $T(n) \in \Theta(n \log n)$.
This one is hard to do by substitution.

Problems

Solve

- $T(n) = 2T(n/4) + 1$
- $T(n) = 2T(n/4) + \sqrt{n}$
- $T(n) = 2T(n/4) + n$
- $T(n) = 2T(n/4) + n^2$

Problem with solutions

Solve

- $T(n) = 2T(n/4) + 1$ (Case 1)

$$a = 2, b = 4, \log_4(2) = \frac{1}{2}, f(n) = 1 \in O(n^{\frac{1}{2}-\epsilon}),\ \text{hence}\ T(n) \in \Theta(\sqrt{n})$$

- $T(n) = 2T(n/4) + \sqrt{n}$ (Case 2)

$$a = 2, b = 4, \log_4(2) = \frac{1}{2}, f(n) = \sqrt{n} \in \Theta(n^{\frac{1}{2}}),\ \text{hence}\ T(n) \in \Theta(\sqrt{n})$$

- $T(n) = 2T(n/4) + n$ (Case 3)

$$a = 2, b = 4, \log_4(2) = \frac{1}{2}, f(n) = n \in \Omega(n^{\frac{1}{2}+\epsilon}),\ \text{hence}\ T(n) \in \Theta(n)$$

- $T(n) = 2T(n/4) + n^2$
  - n^2 (Obviously)