Complexity

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You do not have to type this homework (though I would appreciate it if your handwriting is not ideal). You can hand-write it and scan into a pdf (there are lots of scanner apps out there).

1 DNF is easy

Show that the problem of determining the satisfiability of boolean formulas in disjunctive normal form (DNF) is polytime solvable. (As opposed to CNF, which is NPC.)

1.1 Solution

A formula in DNF is $\phi = S_1 \vee S_2 \vee \ldots \vee S_k$. For ϕ to be true, we need only one of the clause s_i to be true. Each clause is of the form $S_i = x_1 \wedge x_2 \wedge \ldots \wedge x_n$. For a clause to be true, it needs to be free of a variable and its negation.

So a general algorithm is to scan each clause, in turn, until we can detect one true or end the list. For a clause, we scan each variable and verify if we have seen its negation before. If so, we exit as the clause is a contradiction. If not, we add it to our hash table.

We either run out of clause ot find one that we can make true. The algorithm is polytime. It is actually linear.

2 2-CNF is easy

2-CNF is the problem of determining satisfiability of a formula in CNF but with exactly two terms per clause. (For example $(x_1 \vee x_3) \wedge (\neg x_2 \vee x_4) \ldots)$. Show that this can be solved in polytime by describing an algorithm to solve the problem.

2.1 Solution

The key is to recognize that each clause is of the form $x \vee y$ which is equivalent to $\neg x \Rightarrow y$ and also equivalent to $\neg y \Rightarrow x$. So we can construct a directed graph from the implications with a node for every variable and an arc corresponding to each implication.

One of two possibilities will occur, either the graph will have a path from some variable x to its negation $\neg x$, in which case there can be no truth assignment making the whole formula true. Or there is no such path, in which case, there is a truth assignment that will work.

3 Clique

This is to help you understand the proof that Clique is NPC. Consider the 3-CNF problem: Is there an assignment to

$$\phi = (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (x_1 \vee x2 \vee x_3)$$

that satisfies ϕ? Convert this problem into a graph problem of the form : "Here is a graph. Does it have a clique of size 3?" I want to see the graph. Draw it as cleanly as possible.

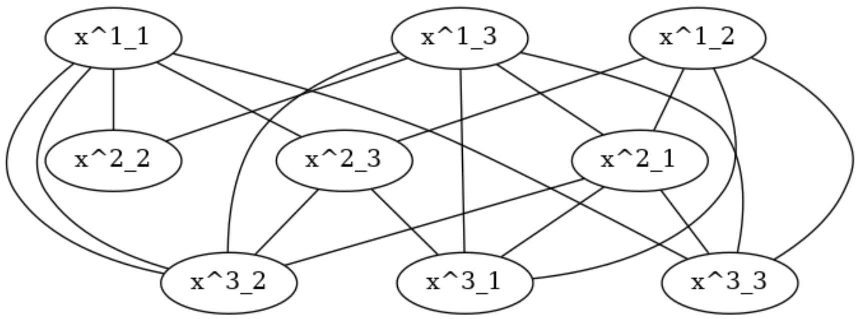
3.1 Solution

We need to construct a graph. The vertex set has a vertex for each variable.

- $V = \{x_1^1, x_2^1, x_3^1, x_1^2, x_2^2, x_3^2, x_1^3, x_2^3, x_3^3\}$

There is an edge for pairs of nodes not in the same clause and not the negation of each other.

- $E = \{(x_1^1, x_2^2), (x_1^1, x_3^2), (x_1^1, x_2^3), (x_1^1, x_2^3), (x_1^1, x_3^3),$
  $(x_2^1, x_1^2), (x_2^1, x_3^2), (x_2^1, x_1^3), (x_2^1, x_3^3),$
  $(x_3^1, x_1^2), (x_3^1, x_2^2), (x_3^1, x_1^3), (x_3^1, x_2^3), (x_3^1, x_3^3),$
  $(x_1^2, x_1^3), (x_1^2, x_2^3), (x_1^2, x_3^3),$
  $(x_3^2, x_1^3), (x_3^2, x_2^3)\}$



4 Vertex Cover

This is to help you understand the proof that Vertex Cover is NPC. Consider the Clique problem: Is there a clique of size 4 in the following graph

$G = (\{0, 1, 2, 3, 4, 5\}, \{(0, 1), (0, 2), (0, 3), (0, 4), (1, 2), (1, 3), (1, 5), (2, 3), (2, 4), (4, 5)\})$

Convert this problem to a Vertex Cover problem. Give me the graph and the exact question.

4.1 Solution

We construct the complement graph $\overline{G} = (V, \overline{E})$

- $V = \{0, 1, 2, 3, 4, 5\}$
- $\overline{E} = \{(0, 5), (1, 4), (2, 5), (3, 4), (3, 5)\}$
- The question is "Is there a vertex cover in $\overline{G}$ of size $6 - 4 = 2$"?

5 Subset sum

Construct the subset sum matrix corresponding to the Vertex Cover of the preceeding question. We need the incidence matrix first. Assuming the node and edge order as above, this is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Which we will embed into a larger matrix. Note that $|E| = 5$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & = & 1280 \\ 1 & 0 & 1 & 0 & 0 & 0 & = & 1088 \\ 1 & 0 & 0 & 1 & 0 & 0 & = & 1040 \\ 1 & 0 & 0 & 0 & 1 & 1 & = & 1029 \\ 1 & 0 & 1 & 0 & 1 & 0 & = & 1092 \\ 1 & 1 & 0 & 1 & 0 & 1 & = & 1297 \\ 0 & 1 & 0 & 0 & 0 & 0 & = & 256 \\ 0 & 0 & 1 & 0 & 0 & 0 & = & 64 \\ 0 & 0 & 0 & 1 & 0 & 0 & = & 16 \\ 0 & 0 & 0 & 0 & 1 & 0 & = & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & = & 1 \\ 2 & 2 & 2 & 2 & 2 & 2 & = & 2730 \end{bmatrix}$$

And the question is "Is there a subset of the set $\{1, 4, 16, \ldots, 1088, 1280\}$ summing to 2730?