30/09

1. เทคนิคการหาปริพันธ์

* อย่าลืมหน้านี้ *

Intro what is อินทิเกรต & ดิฟเฟอเรนเชี่ยล ?

$$\int x^5\,dx = \frac{x^6}{6} + c$$

อย่าลืม

$$\frac{dx^n}{dx} = nx^{n-1}$$

$$\frac{dc}{dx} = 0$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{da^x}{dx} = a^x \ln a$$

$$\frac{d\ln|x|}{dx} = \frac{1}{x}$$

$$\frac{d\log_a x}{dx} = \frac{1}{x\ln a}$$

$$\frac{d\sin x}{dx} = \cos x \cdot dx$$

$$\frac{d\cos x}{dx} = -\sin x \cdot dx$$

$$\frac{d\tan x}{dx} = \sec^2 x \cdot dx$$

$$\frac{d\csc x}{dx} = -\csc x\cot x \cdot dx$$

$$\frac{d\sec x}{dx} = \sec x\tan x \cdot dx$$

$$\frac{d\cot x}{dx} = -\csc^2 x \cdot dx$$

$$\int \frac{1}{x}\,dx = \ln|x| + c$$

$$\frac{df(u)}{dx} = f'(u)\cdot\frac{du}{dx}$$

$$\frac{d\,f(x)\cdot g(x)}{dx} = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

$$\frac{d\,f(x) \pm g(x)}{dx} = f'(x) \pm g'(x)$$

$$\frac{d}{dx}\frac{x^{n+1}}{n+1} = \frac{\cancel{n+1}\cdot x^n}{\cancel{n+1}} \rightarrow \int x^n\,dx = \frac{x^{n+1}}{n+1} + c$$

$$\frac{d}{dx}\frac{\ln|ax+b|}{a} = \frac{1}{|ax+b|}\cdot\frac{\overbrace{\cancel{a}}^{\frac{d(ax+b)}{dx}}}{\cancel{a}} \rightarrow \int \frac{1}{|ax+b|}\,dx = \frac{\ln|ax+b|}{a} + c$$

$$\int 5\,dx = 5x + c$$

$$\int k\,f(x)\,dx = k\int f(x)\,dx$$

$$\int f(x) \pm g(x) = \int f(x) \pm \int g(x)$$

ex $\displaystyle\int 2x^2 + \frac{3}{2x+1}\,dx = \int 2x^2\,dx + \int \frac{3}{2x+1}\,dx$

$$= \frac{2}{3}x^3 + 3\,\frac{\ln|2x+1|}{2} + C$$

ex $\displaystyle\int \left(2x^3 + \frac{1}{x^4}\right)^2 dx$

$$= \int (2x^3)^2 + 2(2x^3)\left(\frac{1}{x^4}\right) + \left(\frac{1}{x^4}\right)^2 dx$$

x^{-1} → เข้าสูตร $\int \frac{1}{x} = \ln|x|$

$$= \int 4x^6\,dx + 4\int \frac{1}{x}\,dx + \int x^{-8}\,dx$$

เข้า $\int$ ปกติ

$$= \frac{4x^7}{7} + 4\ln|x| + \frac{x^{-7}}{-7}$$

$$= \frac{4x^7}{7} + 4\ln|x| - \frac{1}{7x^7} + C$$

น³ + 3น²ล + 3นล² + ล³

$$\int \frac{(2x+1)^3}{\sqrt{x}}\,dx$$

$$= \int \frac{8x^3 + 12x^2 + 6x + 1}{x^{\frac{1}{2}}}\,dx$$

$$= \int 8x^{\frac{5}{2}} + 12x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + x^{-\frac{1}{2}}\,dx$$

$\frac{5}{2}+\frac{2}{2}$, $\frac{3}{2}+\frac{2}{2}$, $\frac{1}{2}+\frac{2}{2}$, $-\frac{1}{2}+\frac{2}{2}$

$$= 8 \cdot \frac{2}{7}x^{\frac{7}{2}} + 12 \times \frac{2}{5}x^{\frac{5}{2}} + \cancel{6}^{\,2} \times \frac{2}{\cancel{3}}x^{\frac{3}{2}} + 2x^{\frac{1}{2}}$$

$$= \frac{16}{7}x^{\frac{7}{2}} + \frac{24}{5}x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$$

ex $\displaystyle\int \frac{8x^3 + 4x^2 + 2x + 5}{x-1}\,dx$

$$\begin{array}{r|rrrr} 1 & 8 & 4 & 2 & 5 \\ & & 8 & 12 & 14 \\ \hline & 8 & 12 & 14 & 19 \end{array}$$

19 → เศษ

$$8x^2 + 12x + 14 + \frac{19}{x-1}$$

เศษ

$$\int \frac{\left(8x^2 + 12x + 14 + \frac{19}{x-1}\right)\cancel{(x-1)}}{\cancel{x-1}}\,dx$$

$$\int 8x^2\,dx + \int 12x\,dx + \int 14\,dx + \int \frac{19}{x-1}\,dx$$

$$= \frac{8}{3}x^3 + 6x^2 + 14x + 19\ln|x-1| + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + c$$

ex $$\int 5^{2x} \, dx = \int 25^x \, dx$$

$$= \frac{25^x}{\ln 25} + c$$

ex $$\int \frac{5^x}{3^{2x}} \, dx = \int \left(\frac{5}{9}\right)^x dx$$

$$= \frac{\left(\frac{5}{9}\right)^x}{\ln \frac{5}{9}} + c$$

ex $$\int \frac{\sqrt{5^x} + 1}{3^{2x}} = \int \frac{5^{\frac{1}{2}x} + 1}{9^x} \, dx$$

$$= \int \frac{\sqrt{5}^x + 1}{9^x} \, dx$$

$$= \int \left(\frac{\sqrt{5}}{9}\right)^x + \left(\frac{1}{9}\right)^x dx$$

$$= \frac{\left(\frac{\sqrt{5}}{9}\right)^x}{\ln\left(\frac{\sqrt{5}}{9}\right)} + \frac{\left(\frac{1}{9}\right)^x}{\ln\left(\frac{1}{9}\right)} + c$$

$$\frac{d \sin ax}{dx} = \cos ax \cdot \frac{d\, ax}{dx}$$

$$= a \cdot \cos ax$$

$$\int \cos ax \, dx = \frac{\sin ax}{a} + c$$

$$\int \sin ax \, dx = -\frac{\cos ax}{a} + c$$

$$\frac{d \cos ax}{dx} = -\sin ax \cdot \frac{d\, ax}{dx}$$

$$= -a \sin ax$$

$$\sin A \sin B = \frac{1}{2}(\sin(A-B) - \cos(A+B))$$

ex $$\int -\sin 2x \, dx = \frac{\cos 2x}{2} + c$$

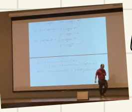
$$\sin A \cos B = \frac{1}{2}(\sin(A+B) + \sin(A-B))$$

ex $$\int \sin^2(3x) \, dx = \int \frac{1-\cos 6x}{2} \, dx$$

$$\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$$

$$= \frac{1}{2}x - \frac{\sin 6x}{12} + c$$

$$\sin^2 A = \frac{1-\cos 2A}{2} \qquad \cos^2 A = \frac{1+\cos 2A}{2}$$



1/12

$$\frac{1}{2} - \frac{\cos 6x}{2}$$

ex $\int \cos^2(5x)\,dx = \int \frac{1+\cos 10x}{2}\,dx$

$= \frac{1}{2}x + \frac{\sin 10x}{20} + c$

ex $\int (\sin 3x + \cos 5x)^2\,dx$

$= \int \sin^2 3x + 2\sin 3x\cos 5x + \cos^2 5x\,dx$

พิจารณา $\int 2\sin 3x\cos 5x\,dx = \int \cancel{2}\left(\frac{\sin 8x + \sin(-2x)}{\cancel{2}}\right)dx$

$$\boxed{\begin{aligned}
&\frac{d\sin^{-1}x}{dx} = \frac{1}{\sqrt{1-x^2}}\\
&y = \sin^{-1}x\\
&\sin y = x\\
&1 = \frac{d\sin y}{dx} \Rightarrow \frac{d\sin y}{dy}\cdot\frac{dy}{dx} = \cos y\,\frac{dy}{dx}\\
&\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}}\\
&\int \frac{1}{\sqrt{1-x^2}}\,dx = \sin^{-1}x + C\\
&\int \frac{1}{1+x^2}\,dx = \tan^{-1}x + C\\
&\int \frac{1}{x\sqrt{x^2-1}}\,dx = \sec^{-1}x + C
\end{aligned}}$$

* เปลี่ยนตัวแปร *

Ex $\int \frac{1-2x}{(x^2-x+9)^{5/2}}\,dx = \int \frac{1-2x}{u^{5/2}}\,dx = \int \frac{1-2x}{u^{5/2}}\cdot\frac{du}{2x-1}$

$= \int \frac{-1}{u^{5/2}}\,du$

$u = x^2 - x + 9$

$du = \frac{du}{dx}\cdot dx = (2x-1)dx$

$dx = \frac{du}{2x-1}$

ex $\int \frac{x}{\sqrt{4-x^2}}\,dx = -\int du = -u + c = -\sqrt{4-x^2} + c$

$u = \sqrt{4-x^2}$

$du = \frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)\,dx = \frac{-x\,dx}{\sqrt{4-x^2}}$

ex $\int \frac{x^2}{\sqrt{4-x^2}}\,dx = -\int x\,dx = -\int \sqrt{4-u^2}\,dx$

↓ ทำต่อไม่ได้

$u = \sqrt{4-x^2}$

$du = \frac{-x\,dx}{\sqrt{4-x^2}}$

ex $\int \frac{x^3}{\sqrt{4-x^2}}\,dx = -\int x^2\,du = \int u^2 - 4\,du$

$= \frac{u^3}{3} - 4u + c$

ex $\int \frac{1}{3^x+1}\,dx$

$u = 3^x + 1$

$du = 3^x \ln 3\,dx$

$= \int \frac{1}{u}\,\frac{du}{3^x\ln 3}$

$= \frac{1}{\ln 3}\int \frac{1}{u\cdot 3^x}\,du$

$= \frac{1}{\ln 3}\int \frac{1}{u\cdot(u-1)}\,du$

$\boxed{= \frac{1}{\ln 3}\int \frac{1}{u-1} - \frac{1}{u}\,du}$ ✓

อันนี้ ถ้าไม่รู้ ∫

$u = 1 + 3^{-x}$

$du = -3^{-x}\ln 3\,dx$

$\frac{-1}{\ln 3}\,du = 3^{-x}\,dx$

$= \frac{-1}{\ln 3}\ln|1+3^{-x}| + c$

Ex $\int \frac{\ln\sqrt{x}}{x}\,dx$

$\ln a^B = B\ln a \rightarrow \frac{1}{2}\int \frac{\ln x}{x}\,dx$

$u = \ln x$ $\quad = \frac{1}{2}\int u\cdot du$

$du = \frac{1}{x}\cdot dx$ $\quad = \frac{1}{2}\cdot\frac{u^2}{2} + c$

$= \frac{\ln^2 x}{4} + c$

$\neq \ln x^2$

ex $\int e^{\ln\cos x} \cancel{\tan x}\, dx = -\int e^{u}\, du$

$u = \ln\cos x \qquad = -e^{\ln\cos x} + c$

$du = \frac{1}{\cos x}(-\sin x)\, dx$

$= -\cancel{\tan x}\, dx$

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ex $\int \frac{\cos x}{1+\sin x} dx = \int \frac{1\,du}{u} = \ln|1+\sin x| + c$

$u = 1 + \sin x$

$du = \cos x\, dx$

---

ex $\int e^{\cos x} \sin x\, dx = -\int e^{u}\, du$

$= -e^{\cos x} + c$

$u = \cos x$

$du = -\sin x \cdot dx$

$\frac{du}{-\sin x} = -\sin x \cdot dx$

* ตรีโกณ *

$\sin dx = d\cos x$

ex $\int \cos^7 x \sin x\, dx = \int \cos^7 x\, d\cos x$

$= \frac{-\cos^8 x}{8}$

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ex $\int \sin^{2.3} x \cos x\, dx = \int \sin^{2.3} x\, d\sin x$

$= \frac{\sin^{3.3} x}{3.3} + c$

Ex $\int \sin^3 x\, dx = \int \sin^2 x \cdot \sin x dx = -\int \sin^2 x\, d\cos x$

$= \int \cos^2 x - 1\, d\cos x$

$= \frac{\cos^3 x}{3} - \cos x + c$

$\sin^2 x + \cos^2 x = 1$

$\sin^2 x = 1 - \cos^2 x$

$= -1 + \cos^2 x$

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ex $\int \sin^8 x \cos^5 x\, dx = \int \sin^8 x \cos^4 x \cos x\, dx$

$= \int \sin^8 x \cos^4 x\, d\sin x$

$= \int \sin^8 x (1 - \sin^2 x)^2\, d\sin x$

$\boxed{\Rightarrow \int \sin^m x \cos^n x\, dx \text{ เมื่อ } m \text{ หรือ } n \text{ เป็นจ.น.คี่บวก จะสามารถใช้เทคนิคนี้ได้เสมอ}}$

ex $\int \sin^2 x \, dx = \int \boxed{\frac{1-\cos 2x}{2}} \, dx$

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ex $\int \sin^4 x \, dx = \int \left(\frac{1-\cos 2x}{2}\right)^2 dx$

$= \int \frac{1 - 2\cos 2x + \cos^2 2x \, dx}{4}$

$= \frac{x}{4} - \frac{\sin 2x}{4} + \int \frac{1 + \cos 4x}{8} \, dx$ $= \frac{x}{8} + \frac{\sin^2 2x}{8}$

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ex $\int \tan^{10} x \sec^2 x \, dx = \int \tan^{10} x \, d\tan x$

$= \frac{\tan^{11} x}{11} + c$

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ex $\int \sec^4 x \, dx = \int \sec^2 x \sec^2 x \, dx$ $\tan^2 x + 1 = \sec^2 x$

$= \int \tan^2 x + 1 \quad d\tan x$

$= \frac{\tan^3 x}{3} + \tan x + c$

---

ex $\int \tan^{-7.8} x \sec^6 x \, dx = \int \tan^{-7.8} x \sec^4 x \, d\tan x$

$= \int \tan^{-7.8} x (\tan^2 x + 1)^2 \, d\tan x$

$\boxed{\Rightarrow \int \tan^m x \sec^n x \, dx \quad \text{เมื่อ } n \text{ เป็นจ.น. คู่บวก}}$

$\tan x = \frac{\sin x}{\cos x}$ | $\sec x = \frac{1}{\cos x}$

ex $\int \frac{1}{\frac{\sin^4 x}{\cos^4 x} \cos^2 x \cdot \cos^4 x} \, dx = \int \frac{1}{\tan^4 x \cdot \cos^6 x} \, dx$

$= \int \frac{\sec^6 x}{\tan^4 x} \, dx$

พยายามทำเป็น tan x ให้เข้าเทคนิค

$= \int \tan^{-4} x \cdot \sec^6 x \, dx$

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ex $\int \frac{1}{\sqrt[3]{\sin x \cos^{11}}} \, dx = \int \frac{1}{\frac{\sin^{\frac{1}{3}} x}{\cos^{\frac{1}{3}} x} \cdot \cos^{\frac{11}{3}} x \cdot \cos^{\frac{1}{3}} x} \, dx$ $\tan^{\frac{1}{3}} x$

$= \int \tan^{-\frac{1}{3}} x \cdot \sec^4 x \, dx$

มากไปกว่านั้น เอ่อ... เรายังใช้เทคนิคนี้ได้กับ $\int \cot^m x \csc^n x$ เมื่อ n เป็นจ.น. คู่ ⊕

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ex $\int \cot^{1.7} x \csc^4 x \, dx = \int \cot^{1.7} x \csc^2 x \csc^2 x \, dx$

$= -\int \cot^{1.7} x \csc^2 x \, d\cot x$

$= -\int \cot^{1.7} x (1 + \cot^2 x) \, d\cot x$

$d\sec x = \sec x \tan x \, dx$

Q1 $\int x^2 e^{x^3+1} \, dx$

$u = x^3 + 1$

$du = 3x^2 \, dx$

$\frac{du}{3x^2} = dx$

$\int \cancel{x^2} \cdot e^u \frac{du}{3\cancel{x^2}}$

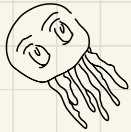
$\frac{1}{3} \int e^u \, du$

$\frac{1}{3} e^{x^3+1} + c$

ex $\displaystyle \int \sec^3 x \tan x \, dx = \int \sec^2 x \sec x \tan x \, dx$

$\displaystyle = \int \sec^2 x \, d\sec x$

$\displaystyle = \frac{\sec^3 x}{3} + c$



ex $\displaystyle \int \sec^{3.9} x \tan^3 x \, dx = \int \sec^{3.9} x \tan^2 x \underline{\sec x \tan x \, dx}$

$\displaystyle = \int \sec^{3.9} x (\sec^2 x - 1) \, d\sec x$

$\boxed{\Rightarrow \displaystyle \int \tan^m x \sec^n x \, dx \text{ และ } \int \cot^m x \csc^n x \, dx}$
เมื่อ m เป็น จ.น. คี่บวก

ex $\displaystyle \int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx = -\int \frac{d\cos x}{\cos x}$

$= -\ln|\cos x| + c$

$= \ln|\sec x| + c$

ex $\displaystyle \int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx$

$= \tan x - x + c$

ex $\displaystyle \int \tan^m x \, dx = \int \tan^{m-2} x \tan^2 x \, dx$

$\displaystyle = \int \tan^{m-2} x (\sec^2 x - 1) \, dx$

$\displaystyle = \int \tan^{m-2} x \sec^2 x - \tan^{m-2} x \, dx$

$\displaystyle = \frac{\tan^{m-2} x}{m-1} - \int \tan^{m-2} x \, dx$

$\displaystyle \int \tan^m x \, dx = \frac{\tan^{m-1} x}{m-1} - \int \tan^{m-2} x \, dx \quad m \neq 1$

ex $\displaystyle \int \tan^6 x \, dx = \frac{\tan^5 x}{5} - \int \tan^4 x \, dx$

$\displaystyle = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - \int 1 \, dx$

$\displaystyle = \frac{\tan^5 x}{5} - \frac{\tan^3 x}{3} + \tan x - x + c$

สอนที่มา ที่ไป

ex $\displaystyle \int \sec x \, dx = \ln|\sec x + \tan x| + c$

ex $\displaystyle \int \sec^2 x \, dx = \tan x + c$

ex $\displaystyle \int \sec^3 x \, dx = ?$

เทคนิคแยกส่วน (By part) หน้า 35

*** เน้นใช้ ตรีโกณ**

$$\int u\,dv = uv - \int v\,du$$

$\int v\,du \to 0\,dx$

csc ทำเหมือน sec

cot ~ — ~ tan

$\int e^x \tan x\,dx$ = อาจจะไม่ออก

↳ ไม่เหมือน $\int e^x \sin x\,dx$

$\int e^{x^2} dx$ = อาจต้องประมาณค่า

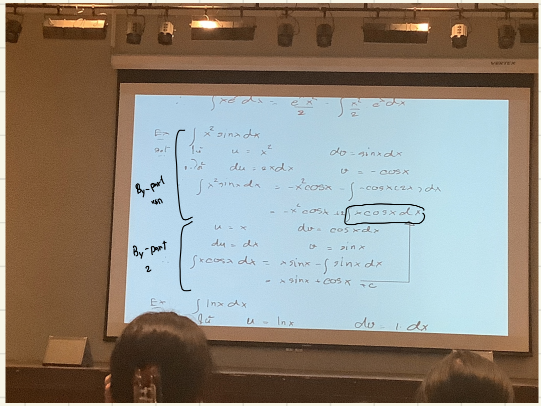
= อาจจะไม่ออก

Ex $\int x e^x dx = \int x\, de^x$

$= xe^x - \int e^x dx = xe^x - e^x + c$

$v = e^x$, $u = x$

$du = \frac{du}{dx}\,dx = dx$



Ex $\int x e^x\, dx$

① เลือก u และ dv

$u = x \qquad dv = e^x dx$

② เลือก

$dv = e^x$

$v = e^x$

$u =$

$du = \qquad dx$

$e^x = dv$ | $u = \cos x$

$v = e^x$ | $du = -\sin x\, dx$

* มักเลือก e^x มา diff เป็น u

* อย่าเลือก ln เป็น dv

$e^x \sin x - \int e^x \cdot \cos x\, dx$

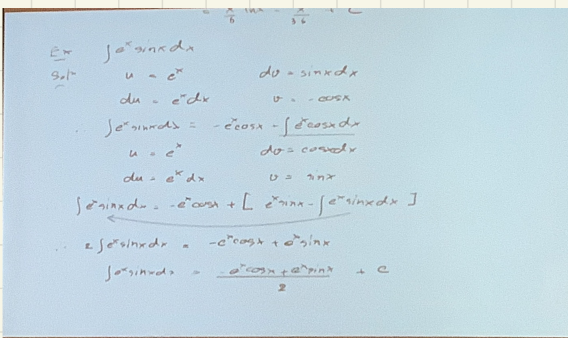
$\int e^x \sin x = e^x \sin x - e^x \cos x + \int e^x \sin x\, dx$

Ex $\int e^x \sin x\, dx$ $\int$

By - part 2 รอบ

เจอตัวเดิม → ย้ายไปรวมกัน* → จัดรูป

sol^n



Ex $\int \sec^3 x\ dx$

$u = \sec x$, $dv = \sec^2 x\ dx$

$du = \sec x \tan x$, $v = \tan x$

$u \cdot v = \int v\ d$

$\int \sec^3 x = \sec x \tan x - \int \sec x \tan^2 x\ dx$

$= \sec x \tan x - \int \sec^3 x - \sec x\ dx$

$2\int \sec^3 x\, dx = \sec x \tan x - \int \sec x\ dx$

$\int \sec^3 x\, dx =$

Ex $\int x e^x \, dx = \int x \, de^x = xe^x - \int e^x \, dx$

$= xe^x - e^x + c$

$\frac{d e^x}{dx} = e^x$

$\therefore \; de^x = e^x \, dx$

อีกวิธี

Ex $\int x e^x \, dx$

sol^n $\boxed{\text{1. เลือก } u \text{ และ } dv}$

$u = x \qquad dv = e^x \, dx$ มี dx ด้วย

$\boxed{\text{2. หา } du \text{ และ } v}$

$du = \frac{du}{dx} \cdot dx = 1 \cdot dx$ ($u = x$) $\qquad v = \int dv = \int e^x \, dx = e^x$

$\therefore \int xe^x \, dx = xe^x - \int e^x \, dx$

$= xe^x - e^x + c$

Ex $\int \sec^n x \, dx$ เมื่อ $n \geq 3$ $\int$ โจทย์ $= uv - \int du$

Ex $\int \cos^n x \, dx$

$u = \cos^{n-1} x$ $\quad dv = \cos x \, dx$

$du = (n-1)(\cos^{x-2} x)(-\sin x)\, dx$ $\quad v = \sin x$

$\int \frac{x^2}{\sqrt{4 - x^2}} \, dx = ?$

→ เทคนิคแทนค่าด้วยฟังก์ชันตรีโกณ

Ex $\int \frac{x^2}{\sqrt{4 - x^2}} \, dx = \int \frac{x^2}{\underbrace{2}_{2^2}\sqrt{1 - \left(\frac{x}{2}\right)^2}} dx = \int \frac{x^2}{\quad} dx = \int \frac{4\sin^2\theta \cdot \cancel{2}\cancel{\cos\theta}}{\cancel{2}\sqrt{1 - \sin^2\theta}} d\theta$ $\quad \cancel{\cos}$

$\cos^2\theta = 1 - \sin^2\theta$

ให้ $\frac{x}{2} = \sin\theta$

$x = 2\sin\theta$

$dx = \frac{dx}{d\theta} d\theta$ ↖ ดิฟ

$dx = 2\cos\theta \, d\theta$

$\sin^{-1}\left(\frac{x}{2}\right) = \theta$

$= 4\int \sin^2\theta \, d\theta$

$= 4\int \frac{1 - \cos 2\theta}{2} d\theta$

$= 4\left(\frac{\theta}{2} - \frac{\sin 2\theta}{2 \times 2}\right)$

$= 2\theta - \sin 2\theta + c$ #

↓ แปลงกลับเป็น x

$= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\underbrace{\left(\frac{x}{2}\right)}_{\sin\theta}\underbrace{\sqrt{1 - \left(\frac{x}{2}\right)^2}}_{\cos\theta} + c$

$\tan^2\theta + 1 = \sec^2\theta$

Ex $\displaystyle\int \frac{x^2}{\sqrt{4+x^2}}\,dx = \frac{1}{2}\int \frac{x^2}{\sqrt{1+\left(\frac{x}{2}\right)^2}}\,dx = \frac{1}{2}\int \frac{4\tan^2\theta \cdot \overbrace{2\sec^2\theta\, d\theta}^{dx}}{\sqrt{1+\tan^2\theta}}$

$\frac{x}{2} = \tan\theta$

$x = 2\tan\theta$

$dx = 2\sec^2\theta\, d\theta$

$= 4\displaystyle\int \tan^2\theta \sec\theta\, d\theta$

$= 4\displaystyle\int \sec^3\theta + \sec\theta \quad d\theta$

$= 4\left[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta| + \ln|\sec\theta + \tan\theta|\right] + C$

$\downarrow$

$\tan^2\theta = \sec^2\theta - 1$

Ex $\displaystyle\int \frac{x^2}{\sqrt{x^2-4}}\,dx = \frac{1}{2}\int \frac{x^2}{\sqrt{\left(\frac{x}{2}\right)^2-1}}\,dx = \cancel{\frac{1}{2}}\int \frac{4\sec^2\theta \cdot \cancel{2}\sec\theta\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta$

$\frac{x}{2} = \sec\theta$

$x = 2\sec\theta$

$dx = 2\sec\theta\tan\theta$

$= \displaystyle\int 4\sec^3\theta \quad d\theta$

$= 4\displaystyle\int \sec^3\theta \quad d\theta$

$\downarrow$

เปลี่ยนตรีโกณ

$\downarrow$

ทำ By part

$1 \pm (\quad)^2$

ตรีโกณ กำลัง 2

$\sin^2\theta + \cos^2\theta = 1$

$\sec^2\theta - \tan^2\theta = 1$

$1 - \triangle^2 \; ; \; \triangle = \sin\theta$

$1 + \square^2 \; ; \; \square = \tan\theta$

$\diamond^2 - 1 \; ; \; \diamond = \sec\theta$

$1 - \triangle^2 = \triangle\sin$

$1 + \square^2 = \tan$

$\diamond^2 - 1 = \sec\theta$

Ex $\displaystyle\int \frac{1}{x^2-4x-5}\,dx$

Sol^{n} พิจารณา $x^2 - 4x - 5 = x^2 - 2x(2) + 2^2 - 2^2 - 5$

$= (x-2)^2 - 9$

$\displaystyle\int \frac{1}{x^2-4x-5}\,dx = \int \frac{dx}{(x-2)^2-9}$

$= \displaystyle\frac{1}{9}\int \frac{dx}{\left(\frac{x-2}{3}\right)^2 - 1}$

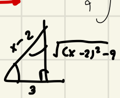
$= \frac{1}{3}\ln|\csc\theta - \cot\theta| + c$

$\frac{x-2}{3} = \sec\theta$

$x = 3\sec\theta + 2$

$dx = 3\sec\theta\tan\theta\, d\theta$

$\displaystyle\frac{1}{9}\int \frac{3\sec\theta\cancel{\tan\theta}}{\tan^{\cancel{2}}\theta}\,d\theta \qquad \frac{1}{\cancel{\cos\theta}},\ \frac{\cancel{\cos\theta}}{\sin\theta}$



$\displaystyle\frac{1}{9}(3)\int \frac{\sec\theta}{\tan\theta}\,d\theta$

$\displaystyle\int \csc\theta$

ex $\displaystyle\int \frac{1}{x^2 - 4x - 5}\, dx$

Sol^{n} พิจารณา $x^2 - 4x - 5 = x^2 - 2x(2) + 2^2 - 2^2 - 5$

$$= (x-2)^2 - 9$$

$$\therefore \int \frac{1}{x^2-4x-5}\, dx = \int \frac{dx}{(x-2)^2-9}$$

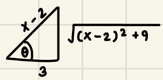
$$= \int \frac{dx}{9\left[\left(\frac{x-2}{9}\right)^2 - 1\right]}$$

$$= \frac{1}{9}\int \frac{dx}{\left(\frac{x-2}{9}\right)^2 - 1}$$

$\sec^2\theta - 1 = \tan^2\theta$

ให้ $\dfrac{x-2}{3} = \sec\theta$

$x = 3\sec\theta \rightarrow dx = 3\sec\theta\tan\theta\, d\theta$



$$= \frac{1}{9}\int \frac{3\sec\theta\, \cancel{\tan\theta}}{\underbrace{\sec^2\theta - 1}_{\tan^{\cancel{2}}\theta}}\, d\theta$$

$$= \frac{1}{3}\int \csc\theta \quad d\theta$$

$$= -\frac{1}{3}\ln\left|\overset{\text{ข/ฉ}}{\csc\theta} + \overset{\text{ข/ช}}{\cot\theta}\right| + c \ *$$

$\displaystyle\int \sqrt{3 - 4x^2 + 4x}\quad dx$

$$-4x^2 + 4x + 3 = -4(x^2 - x) + 3$$

$$= -4\left(x^2 - 2\left(\tfrac{1}{2}\right)(x) + \left(\tfrac{1}{2}\right)^2 - \left(\tfrac{1}{2}\right)^2\right) + 3$$

$$= -4\left[\left(x - \tfrac{1}{2}\right)^2 - \tfrac{1}{4}\right] + 3$$

$$= -4\left(x - \tfrac{1}{2}\right)^2 + 1 + 3$$

$$= -4\left(x - \tfrac{1}{2}\right)^2 + 4$$

$$\int\sqrt{3-4x^2+4x}\quad dx = \int\sqrt{-4\left(x-\tfrac{1}{2}\right)^2+4}\quad dx$$

$\sin^2\theta$ -

ให้ $x - \frac{1}{2} = \sin\theta$

$$= 2\int\sqrt{\underbrace{1-\left(x-\tfrac{1}{2}\right)^2}_{\rightarrow \cos^2\theta}}\quad dx$$

$x = \sin\theta + \frac{1}{2}$

$dx = \cos\theta\ d\theta$

$$= 2\int \cos\theta \cdot \cos\theta \quad d\theta$$

$$= 2\int \frac{1 + \cos 2\theta}{2}\quad d\theta$$

$$= \theta + \frac{\sin 2\theta}{2} + c$$

$$= \sin^{-1}\left(x - \tfrac{1}{2}\right) + \left(x - \tfrac{1}{2}\right)\sqrt{1 - \left(x-\tfrac{1}{2}\right)^2} + c$$

การแยกเศษส่วนย่อย

กรณี เศษเป็น 1

ex $\int \frac{dx}{x^2 - x} = \int \boxed{\frac{1}{x(x-1)}} dx \longrightarrow = \int \frac{1}{x+1} - \frac{1}{x} \, dx$

$= \ln|x+1| - \ln|x| + c$

สูตร $\boxed{\frac{1}{ab} = \frac{1}{b} - \frac{1}{a}}$

① $\frac{1}{x-1} - \frac{1}{x}$

② เทียบ สปส.

$\frac{1}{x(x-1)} = \frac{a}{x} + \frac{b}{x-1}$

$\frac{\boxed{1}}{x(x-1)} = \frac{\boxed{(x-1)a + bx}}{x(x-1)}$

① $1 + 0x = ax + bx - a$; $a = -1$, $b = 1$

② แทน $x = 0$

$1 = -1a$

$a = -1$

แทน $x = 1$

$1 = 0 + b$

$b = 1$

$= \frac{-1}{x} + \frac{1}{x+1}$

Ex $\frac{3x^2 + 9x + 6}{2x^3 - x^2 - 2x + 1}$

$P(x) = 2x^3 - x^2 - 2x + 1$

$P(1) = 0$ → ได้ $(x-1)$

$\therefore P(x) = (x-1)q(x)$; $\frac{p(x)}{x-1}$

$P(x) = (x-1)(2x^2 + x - 1)$

$P(x) = (x-1)(2x-1)(x+1)$

หารสังเคราะห์

$x - 1 \,\big)\, 2x^3 - x^2 - 2x + 1$

1	2	-1	-2	1
	0	+2	1	-1
	2	1	-1	0

$(x-1)(\underbrace{2x^2 + x - 1}_{q(x)})$

จะได้ $\frac{3x^2 + 9x + 6}{2x^3 - x^2 - 2x + 1} = \frac{A}{(x-1)} + \frac{B}{(2x-1)} + \frac{C}{(x+1)} \longrightarrow 9\ln|x-1| - \frac{15\ln|2x-1|}{2}$

$3x^2 + 9x + 6 = (2x-1)(x+1)A + B(x-1)(x+1) + C(2x-1)(x-1)$

แทน ① $x = \frac{1}{2}$; $\frac{3}{4} + \frac{9}{2} + 6 = (0)(x+1)A + B\left(-\frac{1}{2}\right)\left(\frac{3}{2}\right) + C(0)(x-1)$

$\frac{21}{4} + 6 = -\frac{3}{4}B$

$B = -\frac{\cancel{4}}{3}\left(\frac{21+24}{\cancel{4}}\right) = -\frac{45}{3} = -15$

แทน x ② $x = -1$; $3 - \cancel{9} + \cancel{6} = (0)A + (0)B + (-3)(-2)C$; $-6+6=0$

$C = 0$

แทน x ③ $x = 1$; $3 + 9 + 6 = (1)(2)A + (0)B + (0)C$

$18 = 2A$

$A = 9$

ex $\int \frac{x^3+3x}{x^2+2x-8}\,dx \longrightarrow \int (x-2) + \frac{15x-16}{x^2+2x-8}\,dx$

$= \frac{x^2}{2} - 2x + \int \boxed{\frac{15x-16}{x^2+2x-8}}\,dx$

$$\begin{array}{r} x-2 \\ x^2+2x-8 \enclose{longdiv}{x^3+0x^2+3x+0} \\ \underline{x^3+2x^2-8x} \\ -2x^2+11x+0 \\ \underline{-2x^2-4x+16} \\ 15x-16 \end{array}$$

$(x-2) + \left(\frac{15x-16}{x^2+2x-8}\right)$

$\frac{15x-16}{x^2+2x-8} = \frac{A}{x+4} + \frac{B}{x-2}$

$15x-16 = (x-2)A + (x+4)B$

① $x=2;\ 30-16 = 6B$ ② $x=-4;\ -60-16=-6A$

$\frac{\cancel{14}^{7}}{\cancel{6}_{3}} = B$ $\frac{-76}{-6} = A$

ได้ $\frac{14}{6(x+4)} + \frac{76}{6(x-2)}$

$= \frac{x^2}{2} - 2x + \frac{76}{6}\int \frac{1}{(x+4)} + \frac{7}{3}\int \frac{1}{(x-2)}$

$= \frac{x^2}{2} - 2x + \frac{76}{6}\ln|x+4| + 7\ln|x-2| + C$

Ex $\frac{x+1}{(x-1)^3(x+2)^2} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{(x-1)^3} + \frac{D}{x+2} + \frac{E}{(x+2)^2}$

$\sec^2\theta - \tan^2\theta = 1$

กรณีส่วนมีตัวปก. กำลังสอง

$x = \tan\theta$

$dx = \sec^2\theta\, d\theta$

Ex $\int \frac{x^2+x+1}{(x+1)(x^2+1)}dx = \int \frac{A}{(x+1)} + \frac{Bx+C}{(x^2+1)}$ บังคับ $\longrightarrow \int \frac{1}{2(x+1)} + \frac{x+1}{2(x^2+1)}$

$x^2+x+1 = (x^2+1)A + (Bx+C)(x+1)$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\int \frac{x+1}{x^2+1}\,dx$

$x^2+x+1 = Ax^2+Bx^2+Cx+Bx+C+A$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\int \frac{(\tan\theta+1)(\cancel{\sec^2\theta})}{\cancel{\sec^2\theta}}\,d\theta$

$1 = A+B$

$1 = B+C$

$1 = C+A$

$= \frac{1}{2}\ln|x+1| + \frac{1}{2}\int \tan\theta + 1\,d\theta$

or

$2A+2B+2C = 3$

$2(A+B+C) = 3$

$A+B+C = \frac{3}{2}$

① − ② $A - C = 0$

$A = C$

③ $2A = 1$

$C, A = \frac{1}{2}$

② $B = 1 - \frac{1}{2} = \frac{1}{2}$

Q 2 $\int \frac{1}{(2x^2+4x+7)^2}\,dx$

$2(x^2+2x)+7$

$2(x^2+2(x)(1)+1-1)+7$

$2(x+1)^2-2+7$

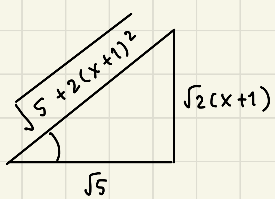
$[2(x+1)^2+5]^2$

$\frac{1}{25}\int \frac{dx}{\left(\left(\frac{\sqrt{2}}{\sqrt{5}}(x+1)\right)^2+1\right)^2}$

$\tan\theta = \frac{\sqrt{2}}{\sqrt{5}}(x+1)$; $\theta =$

$\frac{\sqrt{5}}{\sqrt{2}}(\tan\theta) = x+1$

$\frac{\sqrt{5}}{\sqrt{2}}\sec^2\theta \, d\theta = dx$



$$\frac{1}{25}\int \frac{\frac{\sqrt{5}}{\sqrt{2}}\sec^2\theta \, d\theta}{(\tan^2\theta+1)^2}$$

$$= \frac{1}{25}\int \frac{\sqrt{5}\sec^2\theta \, d\theta}{\sqrt{2}(\sec^2\theta)^2}$$

$$= \frac{1}{25}\times\frac{\sqrt{5}}{\sqrt{2}}\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= \frac{1}{25}\sqrt{\frac{5}{2}}\int \cos^2\theta \, d\theta$$

$$= \frac{1}{25}\sqrt{\frac{5}{2}}\int 1-\sin^2\theta \, d\theta$$

$$= \frac{1}{25}\sqrt{\frac{5}{2}}\int 1-\left(\frac{1-\cos(2\theta)}{2}\right) d\theta$$

$$= \frac{1}{25}\sqrt{\frac{5}{2}}\int \frac{2-1-\cos(2\theta)}{2}\,d\theta$$

$$= \frac{1}{25}\sqrt{\frac{5}{2}}\int 1-\cos(2\theta)\,d\theta$$

$$= \frac{1}{2}\cdot\frac{1}{25}\sqrt{\frac{5}{2}}\left(\theta-\frac{\sin(2\theta)}{2}+c\right)$$

$$= \frac{1}{50}\sqrt{\frac{5}{2}}\left(\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{5}}(x+1)\right)-\frac{\cancel{2}\sin\theta\cos\theta}{\cancel{2}}+c\right)$$

$$= \frac{1}{50}\sqrt{\frac{5}{2}}\left(\tan^{-1}\left(\frac{\sqrt{2}}{\sqrt{5}}(x+1)\right)-\frac{\sqrt{2}(x+1)\sqrt{5}}{5+2(x+1)^2}+c\right)$$

จิรพัชร ศิริสุวรรณ

653040441-7

Ex $\displaystyle\int \frac{5x+8}{x^2+4}\,dx$

$$\int \frac{5x}{x^2+4}\,dx \;+\; 8\int \frac{1}{x^2+4}\,dx$$

$u = x^2+4$

$du = 2x\,dx$

$$\int \frac{\frac{5\cancel{x}\,du}{2\cancel{x}}}{u} \;+\; \frac{8}{4}\int \frac{1}{\left(\frac{x}{2}\right)^2+1}\,dx$$

$\frac{x}{2} = \tan\theta$

$\frac{1}{2}dx = \sec^2\theta \; d\theta$

$$= \frac{5}{2}\ln|u| \;+\; 2\int \frac{2\sec^2\theta}{\sec^2\theta}\,d\theta$$

$$= \frac{5}{2}\ln|x^2+4| \;+\; 4\theta$$

$$= \frac{5}{2}\ln|x^2+4| \;+\; 4\left(\tan^{-1}\left(\frac{x}{2}\right)\right)$$

$\displaystyle\int \frac{dx}{\sqrt{1-\sin x}}$

* เทคนิคครึ่งมุม // เทคนิคฅารกรรยะ ~ *

$$\sin x = \sin\left(\frac{x}{2}+\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)$$

$$= \frac{2\sin\left(\frac{x}{2}\right)}{\cos\left(\frac{x}{2}\right)} = \cos^2\left(\frac{x}{2}\right)$$

$$= \frac{2\tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)} = \frac{2\tan\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

$$\cos x = \cos\left(\frac{x}{2}+\frac{x}{2}\right) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right)$$

$$= 1 - 2\sin^2\left(\frac{x}{2}\right)$$

$$= 1 + \frac{2\tan^2\left(\frac{x}{2}\right)}{\frac{1}{\cos^2\left(\frac{x}{2}\right)}}$$

$\sec^2\left(\frac{x}{2}\right)$

$$= \frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}$$

Jump !!

$$\boxed{\begin{aligned} u &= \tan\left(\tfrac{x}{2}\right) \\ \sin x &= \frac{2u}{1+u^2} \\ \cos x &= \frac{1-u^2}{1+u^2} \end{aligned}}$$

$$\boxed{\begin{aligned} du &= \frac{du}{dx}\cdot dx \\ du &= \frac{1}{2}\sec^2\left(\tfrac{x}{2}\right)dx \\ dx &= \frac{2du}{1+\tan^2\left(\frac{x}{2}\right)} \\ dx &= \frac{2\,du}{1+u^2} \end{aligned}}$$

$\sec^2\left(\frac{x}{2}\right)$

Ex $\displaystyle\int \frac{dx}{1-\sin x} = \int \frac{\frac{2\,dx}{1+u^2}}{1 - \frac{2u}{1+u^2}}$

$u = \tan\left(\frac{x}{2}\right)$

$dx = \frac{2\,du}{1+u^2}$

$\sin x = \frac{2u}{1+u^2}$

$$= \int \frac{\frac{2\,dx}{\cancel{1+u^2}}}{\frac{1+u^2-2u}{\cancel{1+u^2}}}$$

$$= \int$$

$$= \int \frac{2\,du}{u^2-2u+1}$$

$$= 2\int \frac{du}{(u-1)^2}$$

$w = u-1$

$dw = du$

$$= 2\int \frac{dw}{w^2} \quad w^{-2}$$

$$= \frac{-2}{w} + C$$

Ex $\displaystyle\int \frac{dx}{4\sin x + 3\cos x + 3} = \int \frac{\frac{2\,dx}{1+u^2}}{\frac{4\cdot 2u}{1+u^2} + \frac{3(1-u^2)}{1+u^2} + 3}$

$dx = \frac{2\,du}{1+u^2}$

$\sin x = \frac{2u}{1+u^2}$

$\cos x = \frac{1-u^2}{1+u^2}$

$$= \int \frac{\frac{2\,dx}{1+u^2}}{\frac{8u+3-3u^2+3+3u^2}{1+u^2}}$$

$$= \int \frac{2\,dx}{8u+3}$$

$$= \frac{2\ln|8u+3|}{8} + C$$

Ex $\displaystyle\int \frac{dx}{\sin x + \tan x}$ $= \frac{\sin x}{\cos x}$

$$= \int \frac{\frac{2\,du}{1+u^2}}{\frac{2u}{1+u^2} + \frac{\frac{2u}{\cancel{1+u^2}}}{\frac{1-u^2}{\cancel{1+u^2}}}}$$

$$= \int \frac{\frac{2\,du}{\cancel{1+u^2}}}{\frac{2u - \cancel{2u^3} + 2u + \cancel{2u^3}}{(\cancel{1+u^2})(1-u^2)}}$$

$$= \int \frac{2(1-u^2)}{4u}\,du$$

$$= \frac{1}{2}\int \frac{1}{u} - u \; du$$

$$= \frac{1}{2}\ln\left|\tan\left(\frac{x}{2}\right)\right| - \frac{\tan^2\left(\frac{x}{2}\right)}{2} + C$$

อินทิกรัลไม่ตรงแบบ

$\underline{\underline{Ex}}$ $\int \frac{dx}{1 - \sin x}$ dx

$u = \tan\left(\frac{x}{2}\right)$

$\frac{x}{2} = \arctan(u)$

$\frac{1}{2} \, dx = \frac{1}{u^2 + 1} \cdot u \, du$

$dx = \frac{2u}{u^2 + 1} \, du$

$\underline{\underline{Ex}}$ $\int \frac{dx}{4\sin x + 3\cos x + 3}$