Notes and examples on the starred transform.
(open- and closed-loop systems)

The following figure represents a sampler and data hold.

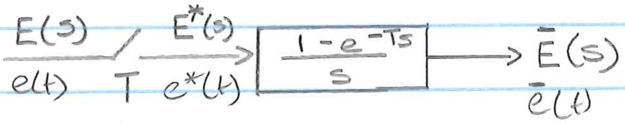


Fig 1.

Def: STARRED TRANSFORM

The output signal of an ideal sampler is defined as the signal whose Laplace transform is

$$E^*(s) = \sum_{n=0}^{\infty} e(nT)e^{-nTs}$$

where $e(t)$ is the input signal to the sampler

Let us consider different configurations of open-loop systems. In these systems there are 2 plants, and both $G_1(s)$ and $G_2(s)$ contain the transfer functions of the data holds

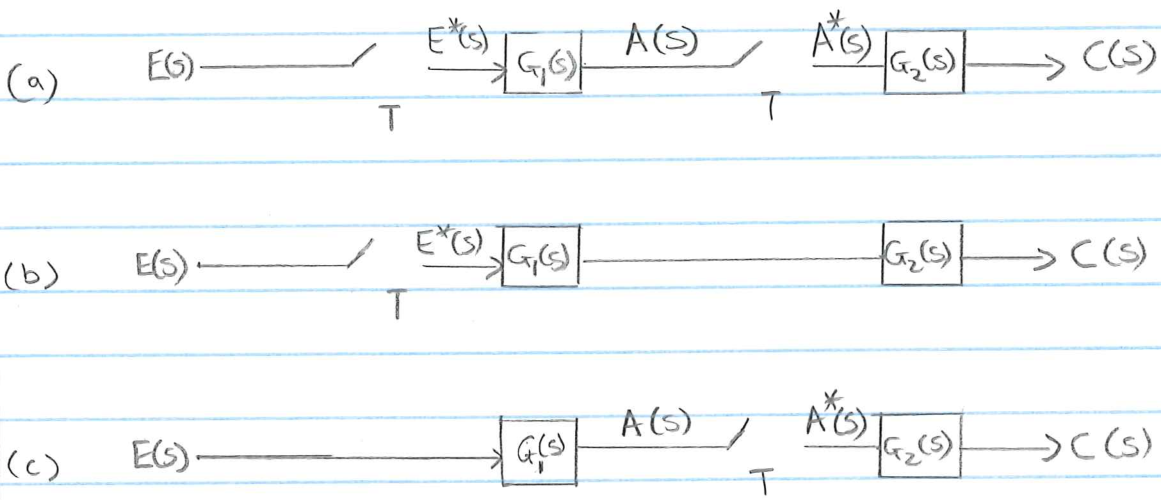


Fig 2.

Consider Fig 2 (a)

We have that $C(s) = G_2(s)\,A^*(s)$

and therefore $C(z) = G_2(z)\,A(z)$

Also $A(s) = G_1(s)\,E^*(s)$

and therefore $A(z) = G_1(z)\,E(z)$

$\therefore$
$$\begin{aligned} C(z) &= G_2(z)\,G_1(z)\,A(z) \\ &= G_1(z)\,G_2(z)\,A(z) \end{aligned}$$

Here the total transfer function is the product of the pulse transfer functions.

Consider Fig 2 (b)

Of course $G_2(s)$ in this case would not contain a data-hold transfer function. Then

$$C(s) = G_1(s)\,G_2(s)\,E^*(s)$$

and $C(z) = \overline{G_1 G_2}(z)\,E(z)$

where $\overline{G_1 G_2}(z) = \mathcal{Z}\left[\,G_1(s)\,G_2(s)\,\right]$

The bar above a product term indicates that the product must be performed in the s-domain before the z-transform is taken.

So in addition note that

$$\overline{G_1 G_2}(z) \neq G_1(z)\, G_2(z)$$

that is, the z-tranform of a product of functions is not equal to the product of the z-transforms of the functions.

Consider Fig 2 (c)

Here we have $\qquad C(s) = G_2(s)\, A^*(s)$

and $\qquad A(s) = G_1(s)\, E(s)$

so $\qquad A^*(s) = \overline{G_1 E}^*(s)$

Then $\qquad C(s) = G_2(s)\, \overline{G_1 E}^*(s)$

$\therefore \qquad C(z) = G_2(z)\, \overline{G_1 E}(z)$

For this system a transfer function cannot be written; that is, we cannot factor $E(z)$ from $\overline{G_1 E}(z)$.

$E(z)$ contains the values of $e(t)$ only at $t = kT$. But the signal $a(t)$ in Fig 2(c) is a function of all previous values of $e(t)$, not just the values at sampling instants. Since

$$a(t) = \int_0^t g_1(t-\tau)\, e(\tau)\, d\tau$$

and the dependency of $a(t)$ on all previous of $e(t)$ is seen, In general, if the input to a sampled-data system is applied directly to a continuous-time part of the system before being sampled, the z-transform of the output of the system cannot be expressed as a function of the z-transform of the input signal.

Examples

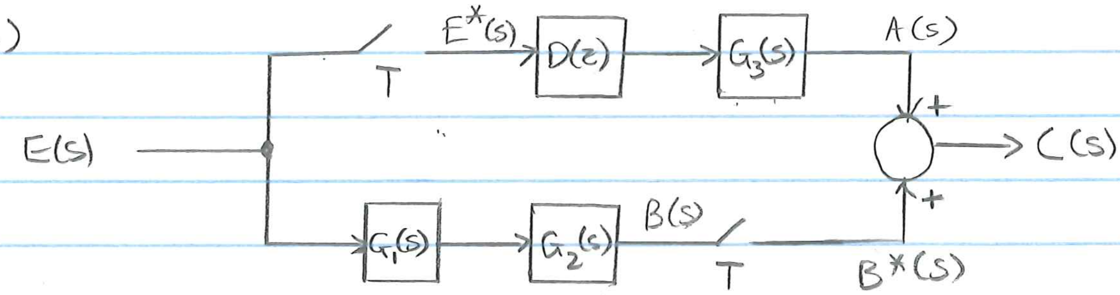
Ex 1

For the following systems

a) Express each output $C(z)$ as a function of the input

b) List those transfer functions in each system that contain the transfer function of a data hold.

(i)



a)

$$A(s) = G_3(s)\,D(z)\,E^*(s) = D(z)\,G_3(s)\,E^*(s)$$

$$A(z) = D(z)\,G_3(z)\,E(z)$$

$$B(s) = G_1(s)\,G_2(s)\,E(s)$$

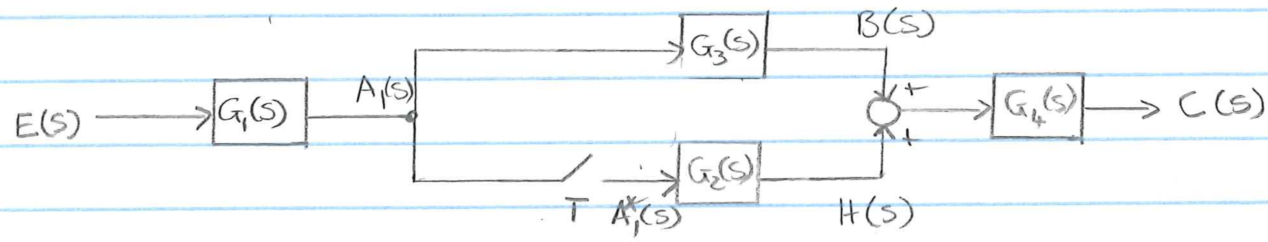
$$B^*(s) = \overline{G_1 G_2 E}^*(s)$$

$$B(z) = \overline{G_1 G_2 E}(z)$$

$$\therefore\ C(z) = A(z) + B(z) = D(z)\,G_3(z)\,E(z) + \overline{G_1 G_2 E}(z)$$

b) $G_3(s)$ contains a data hold.

ii)



a)

$$C(S) = G_4(S)\left[B(S) + H(S)\right]$$

① $B(S) = G_3(S)\,A_1(S)$ where $A_1(S) = G_1(S)\,E(S)$

$$= G_3(S)\,G_1(S)\,E(S)$$

$$B^*(S) = \overline{G_3 G_1 E}^*(S)$$

② $H(S) = G_2(S)A_1^*(S)$ wher $A_1(S) = G_1(S)\,E(S)$

$= G_2(S)\,\overline{G_1 E(S)}^*$ $A_1^*(S) = \overline{G_1 E}^*(S)$

$$H^*(S) = G_2^*(S)\,\overline{G_1 E}^*(S)$$

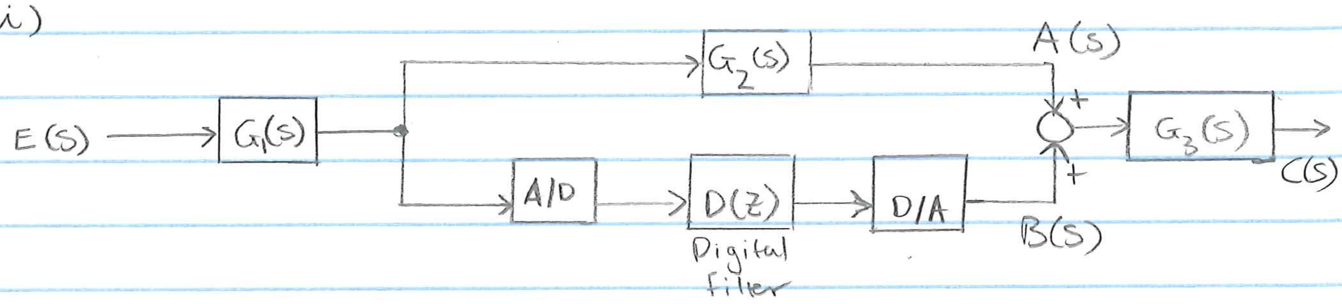
③ $C^*(S) = G_4^*\left[\overline{G_3 G_1 E}(S) + G_2^*(S)\,\overline{G_1 E}^*(S)\right]$

$$= \overline{G_1 G_3 G_4 E}(S) + \overline{G_2 G_4}^*(S)\,\overline{G_1 E}^*(S)$$

$$C(z) = \overline{G_1 G_3 G_4}(z) + \overline{G_2 G_4}(z)\,\overline{G_1 E}(z)$$

b) $G_2(S)$ will contain a data hold.

iii)



$A(S) = G_2(S)\,G_1(S)\,E(S)$

$A^*(s) = \overline{G_2\,G_1\,E}^*(s)$

D/A is a zero order hold.

∴ you need to group it wit $G_3(S)$

$\therefore\ G'_3(S) = \left[\dfrac{1-e^{-TS}}{s}\,G_3(s)\right]$

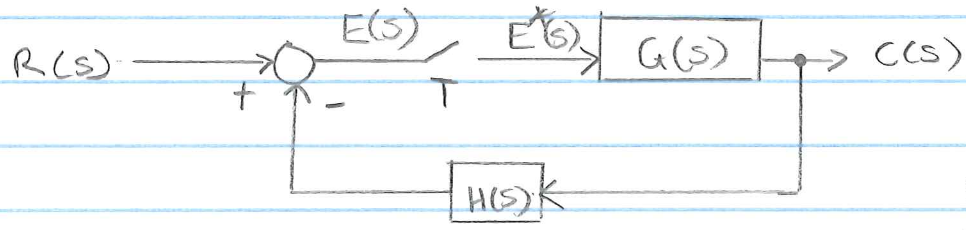
∴ $G_3(s)$ in effect contain a hold transfer function, but just for the bottom path

$$\therefore\ C^*(s) = \overline{G_1 G_2 G_3\ E}(s) + G_3'^*(s)\,D(z)\,\overline{G_1 E}^*(s)$$

$$C(z) = \overline{G_1 G_2 G_3 E}(z) + G'_3(z)\,D(z)\,\overline{G_1 E}(z)$$

Let us now consider closed-loop systems.

We now derive the output function for the system in Fig 3



E(S) = output
c(S) = output.

Fig 3

So $$C(s) = G(s)E^*(s) \qquad (1)$$

and $$E(s) = R(s) - H(s)\,C(s) \qquad (2)$$

Substituting (2) into (1), we obtain

$$E(s) = R(s) - H(s)G(s)E^*(s) \qquad (3)$$

and by taking the starred transform, we have

$$E^*(s) = R^*(s) - \overline{HG}^*(s)E^*(s) \qquad (4)$$

Then solving for $E^*(s)$:

$$E^*(s)\left[1 + \overline{HG}^*(s)\right] = R^*(s)$$

$$E^*(s) = \frac{R^*(s)}{1 + \overline{GH}^*(s)} \qquad (5)$$

Substituting (5) into (1) results in an expression for the continuous output.

$$C(s) = G(s) \frac{R^*(s)}{1 + \overline{GH}^*(s)}$$

The sampled output is then.

$$C^*(s) = G^*(s)\, E^*(s)$$
$$= \frac{G^*(s)\, R^*(s)}{1 + \overline{GH}^*(s)}$$

$$\therefore \quad C(z) = \frac{G(z)\, R(z)}{1 + \overline{GH}(z)}$$

OBSERVATIONS:

a) Since $E^*(s)$ is a series of impulse functions, we must assume that $G(s)$ receives the impulses through a zero-order hold.

b) This will always be the case in what follows unless specified otherwise.

c) Anytime a starred transform is applied to a Laplace transform function, assume that the transfer function contains a zero-order hold to process its starred-transform input.

Be careful! Problems can be encountered in deriving the output function of a closed-loop system.

This can be illustrated as follows:
Again consider Fig 3.

Suppose eq (2) $E(s) = R(s) - H(s)C(s)$ has been starred and then supstituted into (1) $C(s) = G(s)E^*(s)$, then

$$C(s) = G(s)\left[R^*(s) - \overline{HC}^*(s)\right]$$
$$= G(s)R^*(s) - G(s)\overline{HC}^*(s)$$

and then

$$C^*(s) = G^*(s)R^*(s) - G^*(s)\overline{HC}^*(s)$$

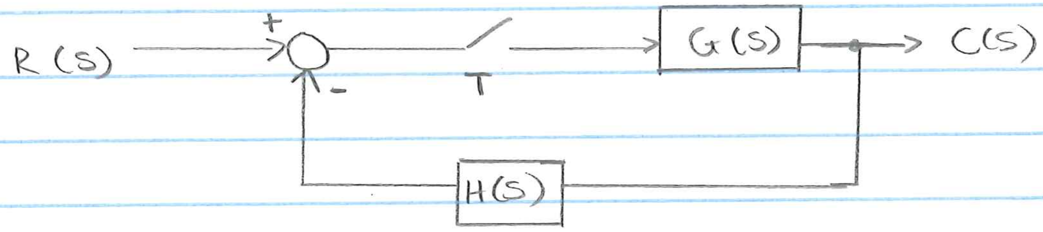
Now we have a problem, $C^*(s)$ cannot be factored from $\overline{HC}^*(s)$. Therefore we cannot solve for $C^*(s)$

NOTE: In general, in analyzing a system, an equation should not be starred if a system signal is lost as a factor, as shown above.

For more complex systems this becomes a difficult task so we need a procedure. This procedure will be explained next.

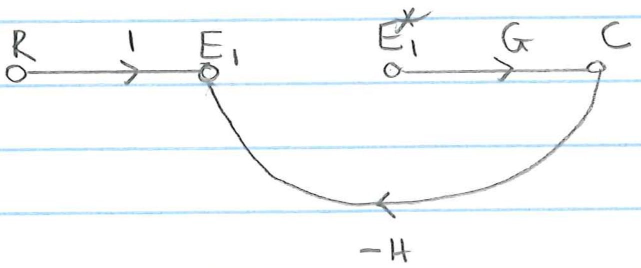
DERIVATION PROCEDURE

Consider the following system. The goal is to derive the sampled output $C^*(s)$ or $C(z)$.



Step 1. Construct the original signal flow graph. We omit the sampler from the system signal flow graph, since a transfer function cannot be written for the device.

Step 2. Assign a variable to each sampler input. Then the sampler output is this variable starred. For example, let E_1 be the input to the sampler and E_1^* is the sampler output.



Step 3 : ① Consider each sampler output to be a system input.

② Consider each sampler input to be a system output.

So we have R and E_1^* as system inputs, and E_1 and C as system outputs.

③ Write each output in terms of the system inputs.

So $E_1 = R - HC$, but C is an output so we need to get an eq for C.

so $C = GE_1^*$, this eq is ok, since the output is written only in terms of inputs

$\therefore$ $E_1 = R - HGE_1^*$, this eq is now ok since E_1 is written in terms of the system inputs R and E_1^*.

Finally
$$E_1 = R - HGE_1^* \quad \text{and}$$
$$C = GE_1^*$$

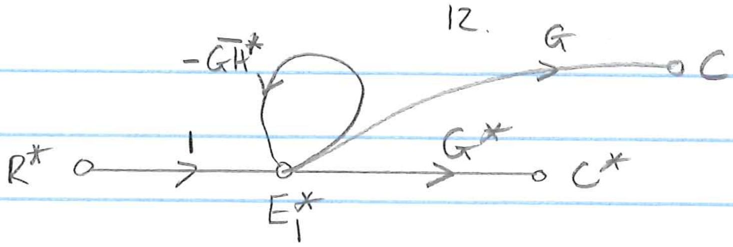
NB. For convenience, the dependency on s will not be shown, but remember $E_1 = E_1(s)$

Step 4: Take the starred transform of these equations and solve by any convenient method. Draw sampled signal-flow graph.

$$\therefore \quad E_1^* = R^* - \overline{GH}^* E_1^*$$
$$C^* = G^* E_1^*$$

Then
$$E_1^* = \frac{R^*}{1 + \overline{GH}^*}$$

so
$$C^* = \frac{G^* R^*}{1 + \overline{GH}^*} = \frac{G^*}{1 + \overline{GH}^*} R^*$$

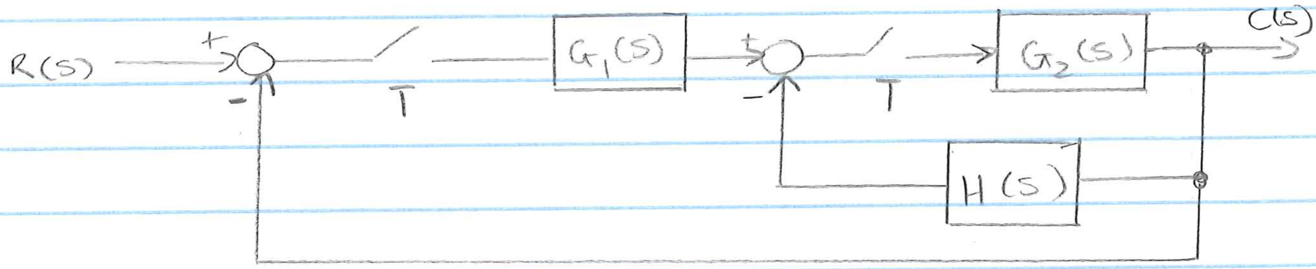


This signal-flow graph is drawn from

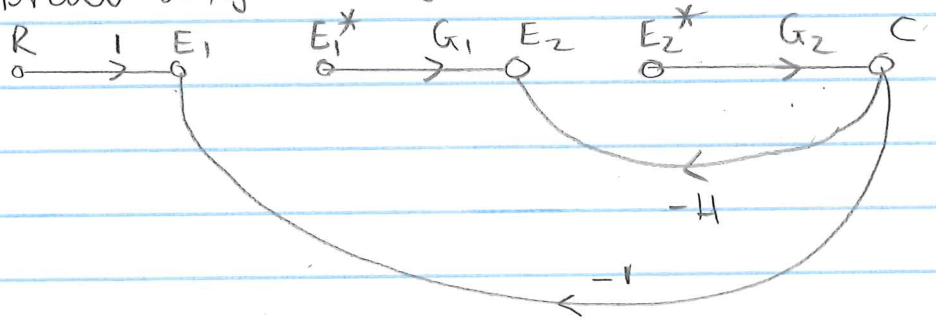
$$E_1^* = R^* - \overline{GH}^* E_1^*$$
$$C^* = G^* E_1^*$$
$$C = G E_1^*$$

Examples

Ex 1



① Draw original signal - flow.



Inputs = R, E_1^*, E_2^*

Outputs = E_1, E_2, C

②

output

$E_1 = R - C \Rightarrow E_1 = R - G_2 E_2^*$ ✓

input input

$E_2 = G_1 E_1^* - HC \Rightarrow E_2 = G_1 E_1^* - H G_2 E_2^*$ ✓

$C = G_2 E_2^*$ ✓

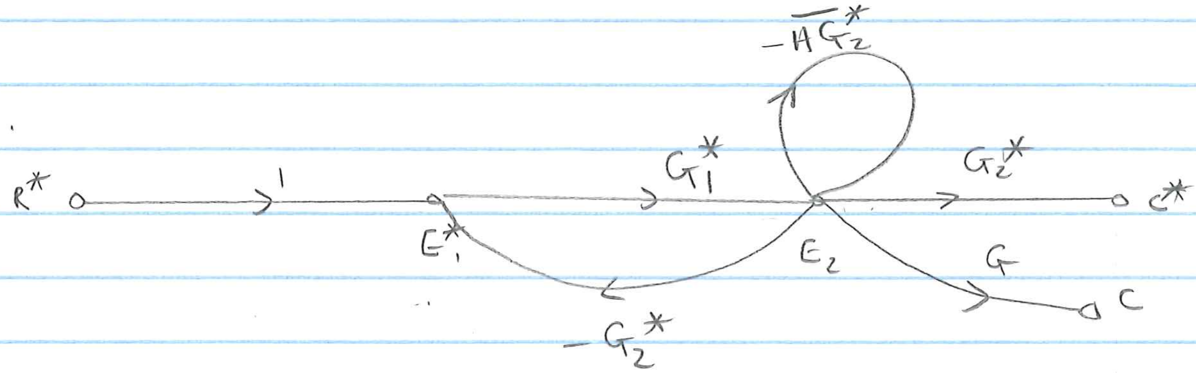
③

$E_1^* = R^* - G_2^* E_2^*$

$E_2^* = G_1^* E_1^* - \overline{HG_2}^* E_2^*$

$C^* = G_2^* E_2^*$

④ Draw sampled signal - flow and use Mason.



$$\frac{C^*}{R^*} = ?$$

$P_1 = G_1^* G_2^*$ (only one forward path)

$$\left.\begin{array}{l} L_1 = -G_1^* G_2^* \\ L_2 = -\overline{HG_2}^* \end{array}\right\} \text{only 2 loop}$$

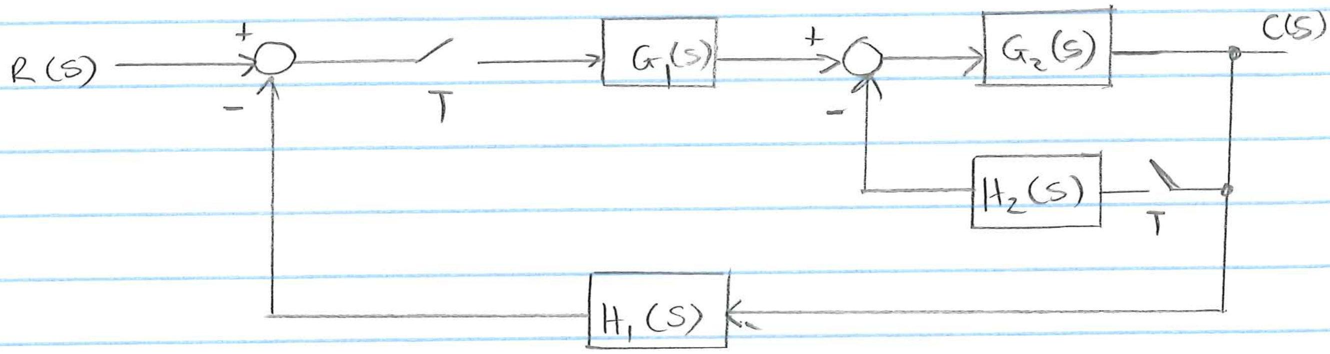
$$\Delta = 1 - \Sigma_{\text{Loop gain}}$$
$$= 1 + G_1^* G_2^* + \overline{G_2H}^*$$

$$\therefore \quad \frac{C^*}{R^*} = \frac{G_1^* G_2^*}{1 + G_1^* G_2^* + \overline{G_2H}^*}$$

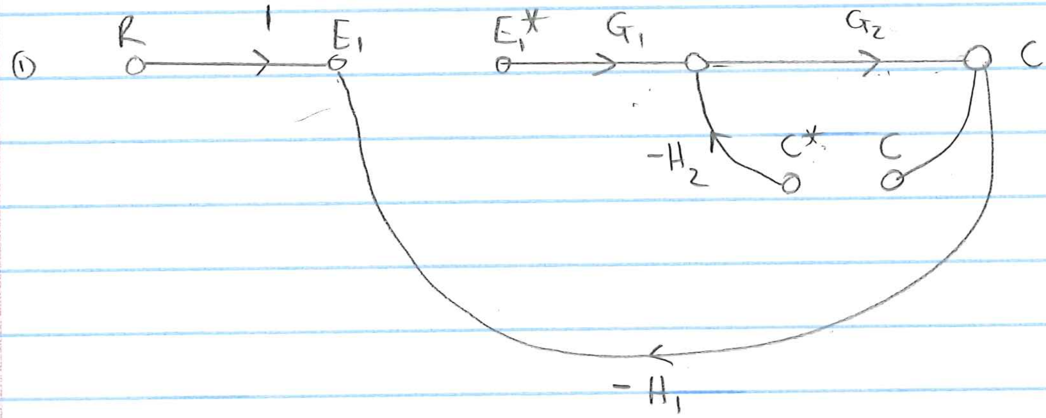
$$\therefore \quad C^* = \frac{G_1^* G_2^*}{1 + G_1^* G_2^* + \overline{G_2H}^*} R^*$$

$$C(z) = \frac{G_1(z) G_2(z)}{1 + G_1(z) G_2(z) + \overline{G_2H}(z)} R(z) \quad \square$$

Ex 2 (class test 17 March 2018)



(1)



Inputs : R, E_1^*, C^*

Outputs : E_1, C

(2)

$$E_1 = R - H_1 C \rightarrow \text{output}$$

$$C = G_2 (G_1 E_1^* - H_2 C^*)$$

$$= G_1 G_2 E_1^* - G_2 H_2 C^* \quad ✓$$

$$\therefore E_1 = R - H_1 [G_1 G_2 E_1^* - G_2 H_2 C^*]$$

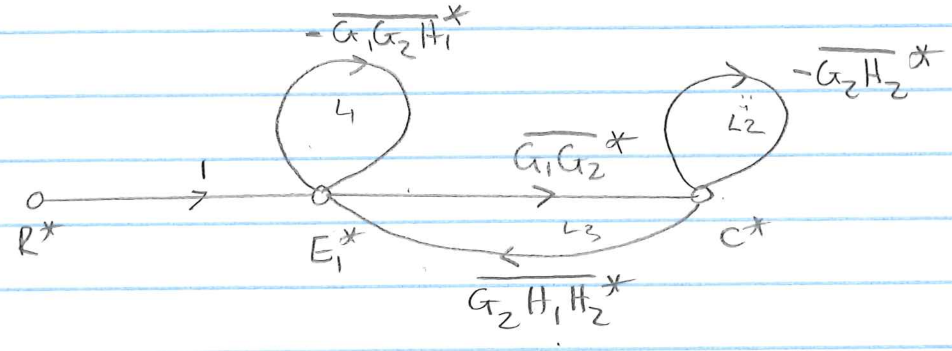
$$= R - G_1 G_2 H_1 E_1^* + G_2 H_1 H_2 C^* \quad ✓$$

(3)

$$E_1^* = R^* - \overline{G_1 G_2 H_1}^* E_1^* + \overline{G_2 H_1 H_2}^* C^*$$

$$C^* = \overline{G_1 G_2} E_1^* - \overline{G_2 H_2}^* C^*$$

④



Forward paths: $P_1 = \overline{G_1G_2}^*$

Loops: $L_1 = -\overline{G_1G_2H_1}^*$

$L_2 = -\overline{G_2H_2}^*$

$L_3 = \overline{G_1G_2}^* \, \overline{G_2H_1H_2}^*$

$$\Delta = 1 - (L_1 + L_2 + L_3) + (L_1 L_2)$$

$$\Delta_1 = 1$$

$$\frac{C^*}{R^*} = \frac{P_1 \Delta_1}{\Delta}$$

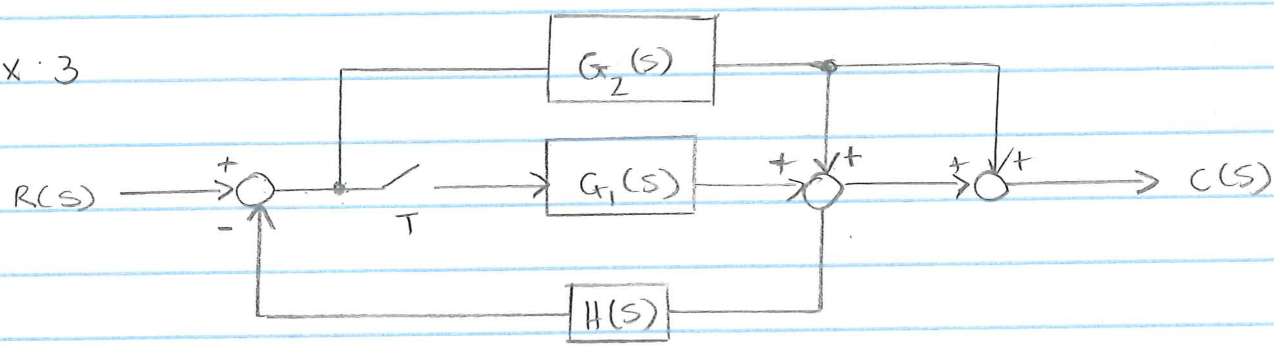
$$= \frac{\overline{G_1G_2}^*}{1 - \left(-\overline{G_1G_2H_1}^* - \overline{G_2H_2}^* + \overline{G_1G_2}^*\,\overline{G_2H_1H_2}^*\right) + \left(-\overline{G_1G_2H_1}^*\right)\left(-\overline{G_2H_2}^*\right)}$$

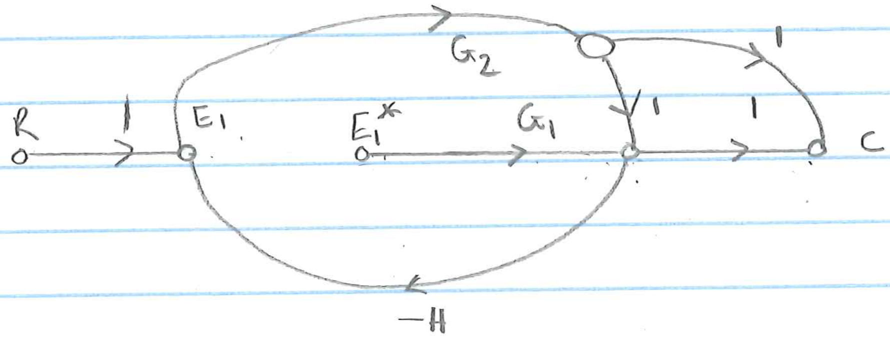
$$= \frac{\overline{G_1G_2}^*}{1 + \overline{G_1G_2H_1}^* + \overline{G_2H_2}^* - \overline{G_1G_2}^*\,\overline{G_2H_1H_2}^* + \overline{G_1G_2H_1}^*\,\overline{G_2H_2}^*}$$

$$\frac{C(z)}{R(z)} = \frac{\overline{G_1G_2}(z)}{1 + \overline{G_1G_2H_1}(z) + \overline{G_2H_2}(z) - \overline{G_1G_2}(z)\,\overline{G_2H_1H_2}(z) + \overline{G_1G_2H_1}(z)\,\overline{G_2H_2}(z)}$$

□

Ex · 3



① 

Inputs : R , E_1^*

Outputs : E_1 , C

Input Input output

② $$E_1 = R - H\left[G_1 E_1^* + G_2 E_1 \right]$$

output output

$$C = \left[G_1 E_1^* + G_2 E_1\right] + G_2 E_1$$

$$\therefore \quad E_1 = R - HG_1 E_1^* - HG_2 E_1$$

$$E_1 + HG_2 E_1 = R - HG_1 E_1^*$$

$$E_1\left[1 + HG_2\right] = R - HG_1 E_1^*$$

$$E_1 = \frac{R - HG_1 E_1^*}{1 + HG_2} = \left(\frac{1}{1+HG_2}\right) R - \left(\frac{HG_1}{1+HG_2}\right) E_1^*$$

$$C = G_1 E_1^* + 2G_2 E_1$$

$$= G_1 E_1^* + \left(\frac{2G_2}{1+HG_2}\right) R - \left(\frac{2G_1G_2H}{1+HG_2}\right) E_1^*$$

$$C = \left(\frac{G_1(1+HG_2) - 2G_1G_2H}{1+HG_2}\right) E_1^* + \left(\frac{2G_2}{1+HG_2}\right) R$$

$$C = \left(\frac{G_1 + HG_1G_2 - 2HG_1G_2}{1 + HG_2} \right) E_1^* + \left(\frac{2G_2}{1 + HG_2} \right) R$$

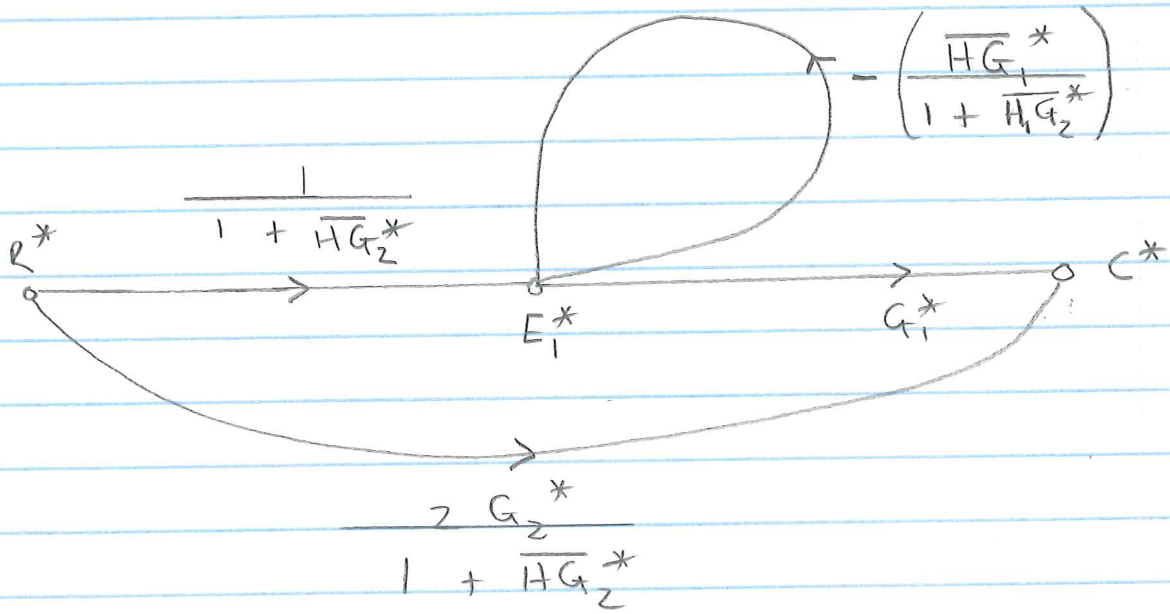
$$= \left(\frac{G_1 + G_1G_2H}{1 + HG_2} \right) E_1^* + \cdots$$

$$= \frac{G_1 \cancel{(1 + G_2H)}}{\cancel{(1 + HG_2)}} E_1^* + \left(\frac{2G_2}{1 + HG_2} \right) R$$

$$= G_1 E_1^* + \left(\frac{2G_2}{1 + HG_2} \right) R$$

(3) $$E_1^* = \left(\frac{1}{1 + \overline{HG_2}^*} \right) R^* - \left(\frac{\overline{HG_1}^*}{1 + \overline{HG_2}^*} \right) E_1^*$$

$$C^* = G_1^* E_1^* + \frac{2G_2^*}{1 + \overline{HG_2}^*} R^*$$



$$\frac{C^*}{R^*} = \frac{\dfrac{G_1^*}{1 + \overline{HG_2}^*} + \dfrac{z\,G_2^*}{1 + \overline{HG_2}^*}}{1 + \dfrac{\overline{HG_1}^*}{1 + \overline{H_1G_2}^*}}$$

$$= \frac{G_1^* + z\,G_2^*}{1 + \overline{H_1G_2}^* + \overline{HG_1}^*}$$

$$C(z) = \left[\frac{G(z) + z\,G_2(z)}{1 + \overline{H_1G_2}(z) + \overline{HG_1}(z)}\right] R(z)$$