Solutions to selected problems of Phillips Chapter 1

1.1-1

(a) Show that the transfer function of two systems in parallel, as shown in Figure P1.1-1(a), is equal to the sum of the transfer functions.

(b) Show that the transfer function of two systems in series (cascade), as shown in Figure P1.1-1(b), is equal to the product of the transfer functions.

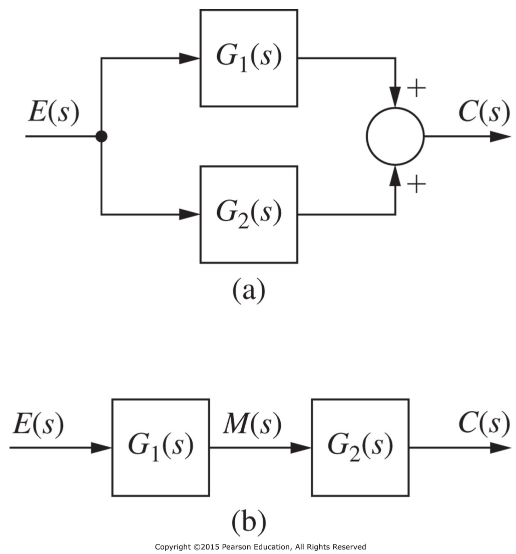


Figure P1.1-1 (a)(b)

Solution:

(a) $C(s) = G_1(s)\,E(s) + G_2(s)\,E(s) = [G_1(s) + G_2(s)]\,E(s)$

$\therefore \dfrac{C(s)}{E(s)} = \underline{G_1(s) + G_2(s)}$

(b) $C(s) = G_2(s)\,M(s) = G_1(s)\,G_2(s) \quad \therefore \dfrac{C(s)}{E(s)} = \underline{G_1(s)\,G_2(s)}$

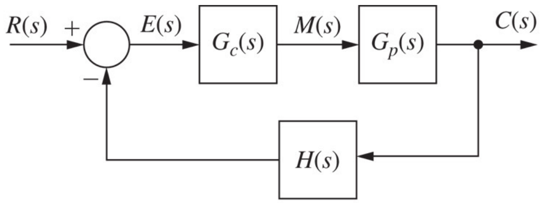
1.1-2

By writing algebraic equations and eliminating variables, calculate the transfer function $C(s)/R(s)$ for the system of:

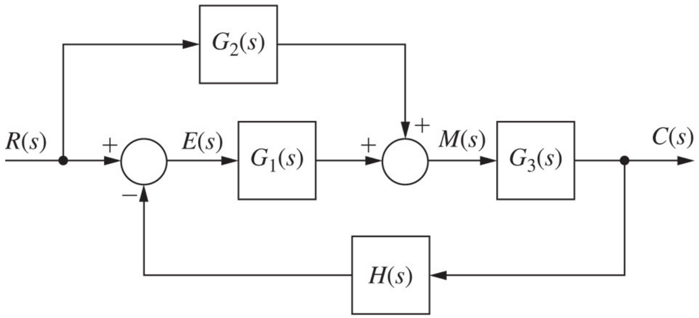
(a) Figure P1.l-2(a).

(b) Figure P1.l-2(b).

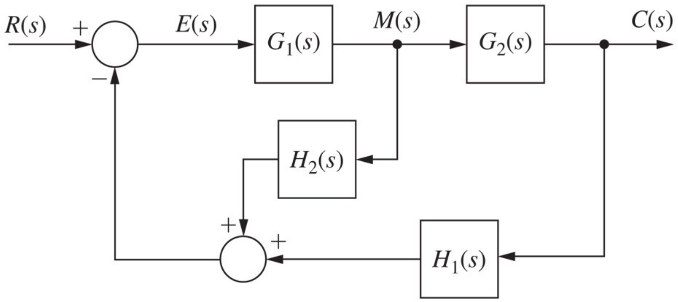
(c) Figure P1.l-2(c).



(a)



(b)



(c)

Figure P1.1-2 (a)(b)(c)

Solution:

1-2.(a) $C(s) = G_p(s) M(s) = G_c(s) G_p(s) E(s) = G_c(s) G_p(s) [R(s) - H(s) C(s)]$

$$[1 + G_c(s) G_p(s) H(s)] C(s) = G_c(s) G_p(s) R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_c(s) G_p(s)}{1 + G_c(s) G_p(s) H(s)}$$

(b) $C(s) = G_3(s) M(s) = G_3(s)[G_1(s) E(s) + G_2(s) R(s)]$

$$= G_3(s) G_1(s) [R(s) - H(s) C(s)] + G_2(s) G_3(s) R(s)$$

$$[1 + G_1(s) G_3(s) H(s)] C(s) = [G_1(s) + G_2(s)] G_3(s) R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{[G_1(s) + G_2(s)] G_3(s)}{1 + G_1(s) G_3(s) H(s)}$$

(c) $C(s) = G_2(s) M(s) = G_1(s) G_2(s) E(s) = G_1(s) G_2(s) [R(s) - H_2(s) M(s)$

$- H_1(s) C(s)]$ and $M(s) = C(s)/G_2(s)$

$$\therefore \left[1 + \frac{G_1 G_2 H_2}{G_2} + G_1 G_2 H_1\right] C(s) = G_1 G_2 R(s)$$

$$\therefore \frac{C(s)}{R(s)} = \frac{G_1(s) G_2(s)}{1 + G_1(s) H_2(s) + G_1(s) G_2(s) H_1(s)}$$

1.1-3

Use Mason's gain formula of Appendix II to verify the results of Problem P1.1-2 for the system of:

(a) Figure P1.l-2(a).

(b) Figure P1.l-2(b).

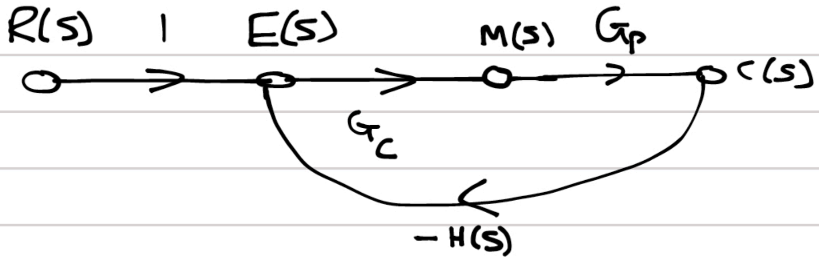
(c) Figure P1.l-2(c).

Solution:

Remember Mason's Rule:

$$T(s) = \frac{\sum_i P_i \Delta_i}{\Delta} \tag{1}$$

(a)



$$L_1 = -G_c G_p H$$

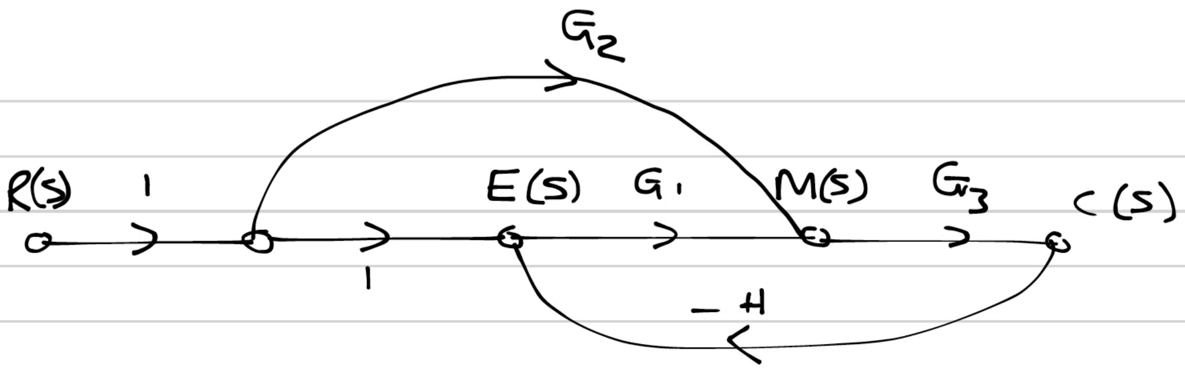
$$P_1 = G_c G_p$$

$$\Delta = 1 - L_1 \qquad \Delta_1 = 1$$

$$T(s) = \frac{P_1 \Delta_1}{\Delta}$$

$$= \frac{G_c G_p}{1 + G_c G_p H}$$

(b)



$P_1 = G_1 G_3$

$L_1 = -G_1 G_3 H$

$P_2 = G_2 G_3$

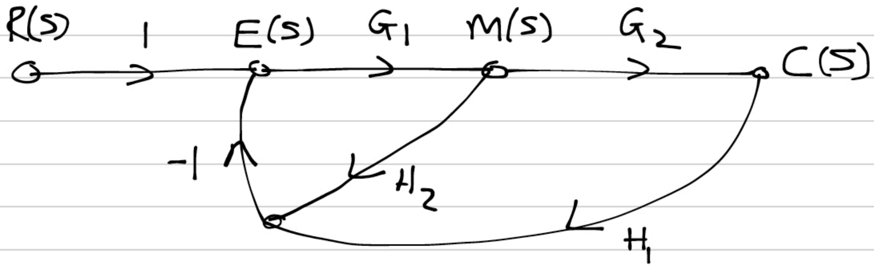
$\Delta = 1 - L_1$

$\Delta_1 = 1$

$\Delta_2 = 1$

$$T(s) = \frac{P_1 \Delta_1 + P_2 \Delta_2}{1 - \Delta} = \frac{G_1 G_3 + G_2 G_3}{1 + G_1 G_3 H}$$

(c)



$P_1 = G_1 G_2$ $\quad L_1 = -G_1 H_2$

$L_2 = -G_1 G_2 H_1$

$\Delta = 1 - (L_1 + L_2)$

$\Delta_1 = 1$

$$T(s) = \frac{P_1 \Delta_1}{1 + L_1 + L_2} = \frac{G_1 G_2}{1 + G_1 H_2 + G_1 G_2 H_1}$$

1.1-4

A feedback control system is illustrated in Figure P1.1-4. The plant transfer function is given by

$$G_p(s) = \frac{4}{0.3s + 1}$$

(a) Write the differential equation of the plant. This equation relates $c(t)$ and $m(t)$.

(b) Modify the equation of part (a) to yield the system differential equation; this equation relates $c(t)$ and $r(t)$. The compensator and sensor transfer functions are given by

$$G_c(s) = 10, \quad H(s) = 1$$

(c) Derive the system transfer function from the results of part (b).

(d) It is shown in Problem 1.1-2(a) that the closed-loop transfer function of the system of Figure P1.1-4 is given by

$$\frac{C(s)}{R(s)} = \frac{G_c(s)G_p(s)}{1 + G_c(s)G_p(s)H(s)}$$

Use this relationship to verify the results of part (c).

(e) Recall that the transfer-function pole term $(s+a)$ yields a time constant $\tau = 1/a$, where a is real. Find the time constants for both the open-loop and closed-loop systems.

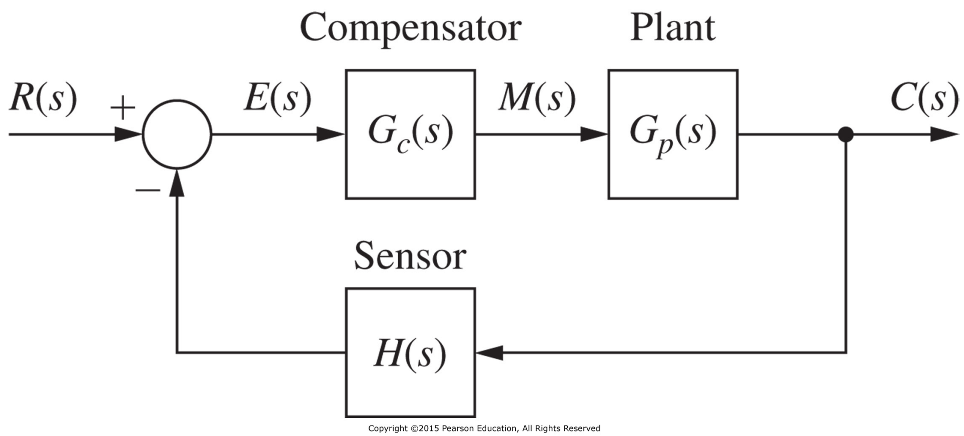


Figure P1.1-4

Solution:

a) $C(s) = G_p(s)M(s) = \left(\frac{4}{0{,}3s+1}\right)M(s)$

$\Rightarrow \quad C(s)[0{,}3s+1] = 4\,M(s)$

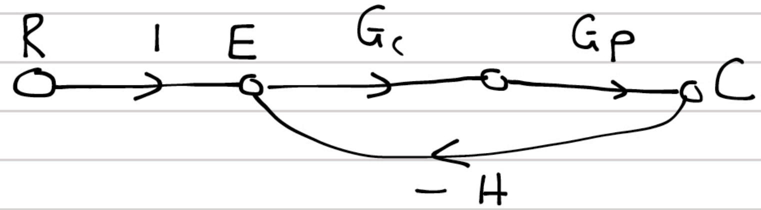
$0{,}3sC(s) + C(s) = 4\,M(s)$

Take the inverse Laplace:

$$0{,}3\frac{dc(t)}{dt} + c(t) = 4\,m(t)$$

$$\frac{dc(t)}{dt} + 3{,}33c(t) = 13{,}33$$

b)



$$\frac{C}{R} = \frac{G_c G_p}{1 + G_c G_p H}$$

$$= \frac{20 \frac{4}{0{,}3s+1}}{1 + 20 \cdot \frac{4}{0{,}3s+1} \cdot 1}$$

$$= \frac{20 \cdot 4}{(0{,}3s+1) + 20 \cdot 4}$$

$$= \frac{80}{0{,}3s + 1 + 80}$$

$$= \frac{266{,}67}{s + 270} = \frac{C(s)}{R(s)} = T(s) =$$ system transfer functi

$$c[s+270] = 266{,}67R$$

$$C(s)\,s + C(s)\,270 = 266{,}67\,R(s)$$

$$\Rightarrow \quad \dot{c}(t) + 270\,c(t) = 266{,}67\,r(t)$$

c) See (b)

d) $$\frac{C(s)}{R(s)} = \frac{(20)\frac{4}{0{,}3s+1}}{1+(20)\frac{4}{0{,}3s+1}\cdot 1} = \frac{266{,}67}{s+270}$$

e) Open-loop transfer function

$$G(s) = G_c G_p H$$

$$= 20\,\frac{4}{0{,}3s+1}\cdot 1$$

$$= \frac{80/0{,}3}{s + \underbrace{1/0{,}3}_{a}} \Rightarrow \tau = 1/a = 1/(1/0{,}3)$$

$$= 0{,}3\ \text{s}$$

Closed-loop transfer function

$$T(s) = \frac{266{,}67}{\underbrace{s+270}_{a}} \Rightarrow \tau = 1/a$$

$$= 1/270$$

$$= 0{,}0037\ \text{s}$$

1.1-4

Repeat Problem 1.1-4 with the transfer functions given by

$$G_c(s) = 2, \quad G_p(s) = \frac{5}{s^2 + 2s + 2}, \quad H(s) = 2s + 1$$

Solution:

(a) $G(s) = \dfrac{C(s)}{M(s)} = \dfrac{5}{s^2 + 2s + 2} \Rightarrow (s^2 + 2s + 2)C(s) = 5M(s)$

$$\therefore \ddot{c}(t) + 2\dot{c}(t) + 2c(t) = 5m(t)$$

(b) $E(s) = R(s) - (3s + 1)C(s)$

$\therefore e(t) = r(t) - 3\dot{c}(t) - c(t)$

$\therefore \ddot{c}(t) + 2\dot{c}(t) + 2c(t) = 5[2e(t)] = 10r(t) - 30\dot{c}(t) - 10c(t)$

$\therefore \ddot{c}(t) + 32\dot{c}(t) + 12c(t) = 10r(t)$

(c) $(s^2 + 32s + 12)C(s) = 10R(s)$

$\therefore \dfrac{C(s)}{R(s)} = \dfrac{10}{s^2 + 32s + 12}$

(d) $$\frac{C(s)}{R(s)} = \frac{(2)\dfrac{5}{s^2+2s+2}}{1+(2)\dfrac{5}{s^2+2s+2}(3s+1)} = \frac{10}{s^2+32s+12}$$

(e) open-loop: $\tau = 1s$, from Problem 1.1-5(e)

closed-loop: poles $= -16 \pm \sqrt{244} = -31.62, -0.38$

$\therefore \tau_1 = 1/31.62 = 0.0316s;\ \tau_2 = 1/0.38 = 2.63s$

1.5-1

The antenna positioning system described in Section 1.5 is shown in Fig. P1.51. In this problem we consider the yaw angle control system, where $\theta(t)$ is the yaw angle. Suppose that the gain of the power amplifier is 10 V/V, and that the gear ratio and the angle sensor (the shaft encoder and the data hold) are such that

$$v_0(t) = 0.04 \cdot \theta(t)$$

where the units of $v_0(t)$ are volts and of $\theta(t)$ are degrees. Let $e(t)$ be the input voltage to the motor; the transfer function of the motor pedestal is given as

$$\frac{\Theta(s)}{E(s)} = \frac{20}{s(s+6)}$$

(a) With the system open loop [$v_0(t)$ is always zero], a unit step function of voltage is applied to the motor [$E(s) = 1/s$] Consider only the *steady-state response*. Find the output angle $\theta(t)$ in degrees, and the angular velocity of the antenna pedestal, $\theta(t)$, in both degrees per second and rpm.

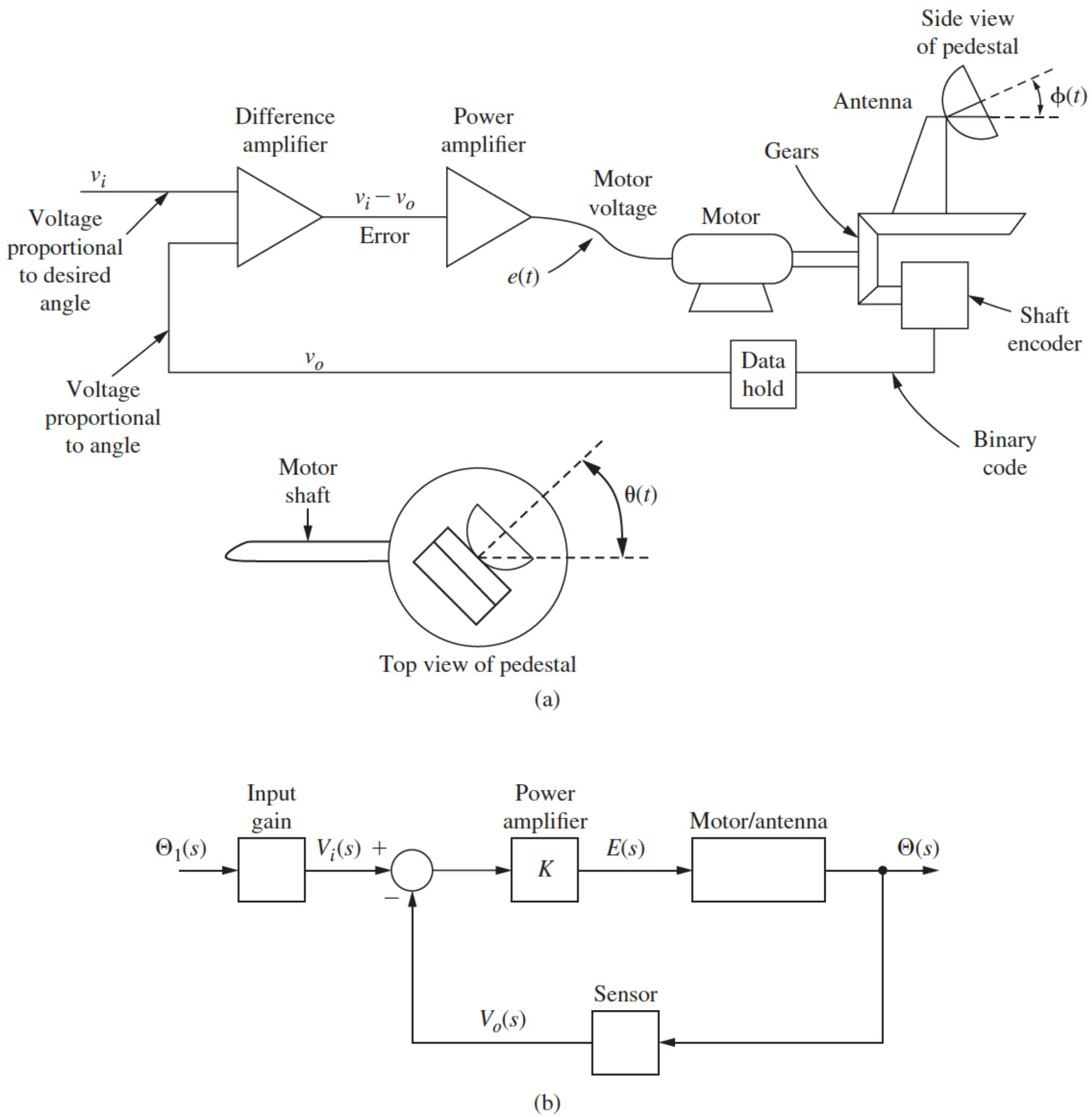

Side view
of pedestal
Antenna
$\phi(t)$
Difference
amplifier
Power
amplifier
Gears
v_i
Motor
voltage
$v_i - v_o$
Error
Motor
Voltage
proportional
to desired
angle
$e(t)$
Shaft
encoder
v_o
Data
hold
Voltage
proportional
to angle
Binary
code
Motor
shaft
$\theta(t)$
Top view of pedestal
(a)
Input
gain
Power
amplifier
Motor/antenna
$\Theta_1(s)$
$V_i(s)$
+
K
$E(s)$
$\Theta(s)$
−
Sensor
$V_o(s)$
(b)

Fig. P1.5-1

(b) The system block diagram is given in Fig. P1.5-1(b), with the angle signals shown in degrees and the voltages in volts. Add the required gains and the transfer functions to this block diagram.

(c) Make the changes necessary in the gains in part (b) such that the units of $\theta(t)$ are radians.

(d) A step input of $\theta_i(t) = 10°$ is applied at the system input at $t = 0$. Find the response $\theta(t)$.

(e) The response in part (d) reaches steady state in approximately how many seconds?

Solution:

(a) $$\Theta(s) = \frac{20}{s^2(s+6)} = \frac{3.33}{s^2} + \frac{k}{s} + \frac{5/9}{s+6}$$

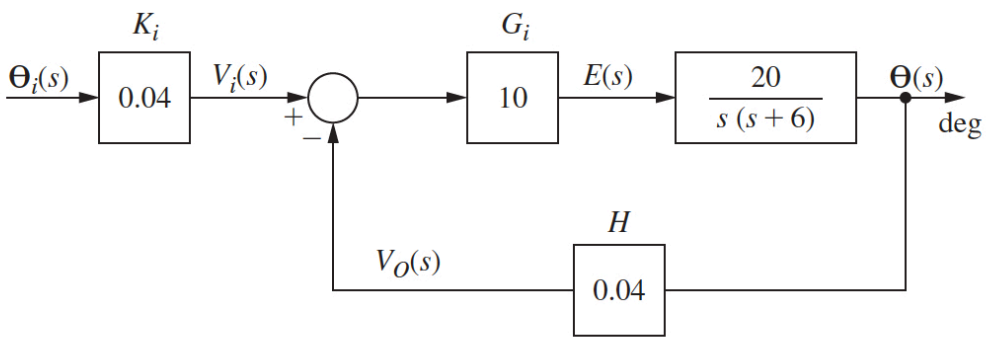
$$k = \frac{d}{ds}\left[\frac{20}{s+6}\right]_{s=0} = \left.\frac{-20}{(s+6)^2}\right|_{s=0} = -\,5\!\!\diagup\!\!9$$

$$\therefore\ \theta_{ss}(t) = \left(3.33t - \frac{5}{9}\right) \text{ in degrees}$$

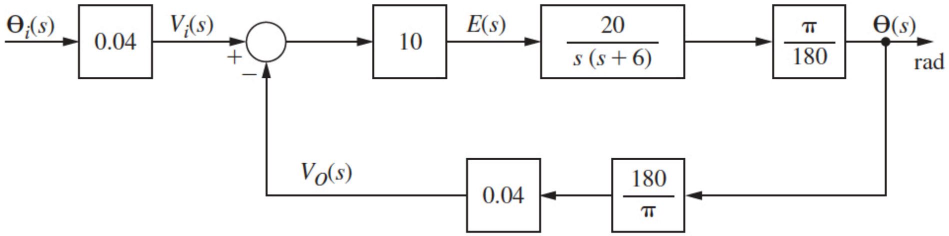
$$\dot{\theta}_{ss}(t) = 3.33 \ \text{deg/s}$$

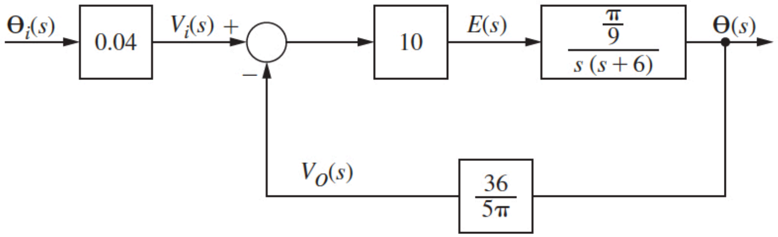
$$\therefore\ 3.33\frac{d}{s} \times \frac{60s}{1\text{ min}} \times \frac{\text{lrcv}}{360^\circ} = \frac{5}{9}\text{rpm}$$

(b)



(c) 1 degree $= \frac{\pi}{180}$ rad





(d) $$\frac{\Theta(s)}{\Theta_i(s)} = \frac{(0.04)(10)\frac{20}{s(s+6)}}{1+(0.04)(10)\frac{20}{s(s+6)}} = \frac{8}{s^2+6s+8}$$

$$\Theta(s) = \frac{8}{(s+2)(s+4)} \times \frac{10}{s} = \frac{10}{s} + \frac{-20}{s+2} + \frac{10}{s+4}$$

$$\therefore \theta(t) = 10 - 20\varepsilon^{-2t} + 10e^{-4t},\ t \geq 0$$

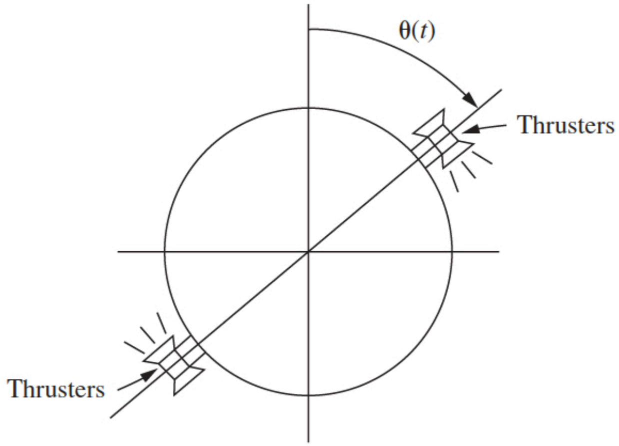
(e) $$\varepsilon^{-t/\tau} \Rightarrow \frac{1}{\tau_1} = 2,\ \tau_1 = 0.5,\ \tau_2 = 1/4 = 0.25,\ \therefore t_{ss} \approx 4(0.5) = 2s$$

1.4-1

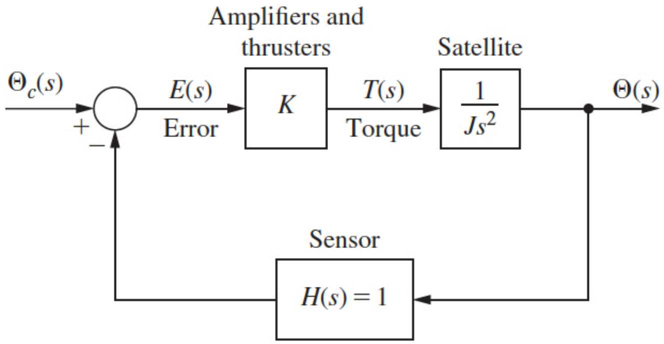
The satellite of Section 1.4 is connected in the closed-loop control system shown in Fig. P1.4-1. The torque is directly proportional to the error signal.

(a) Derive the transfer function $\Theta(s)/\Theta_c(s)$, where $\theta(t) = \mathscr{L}^{-1}[\Theta(s)]$ is the commanded attitude angle.

(b) The state equations for the satellite are derived in Section 1.4. Modify these equations to model the closed-loop system of Fig. P1.4-1.


θ(t)
Thrusters
Thrusters

(a)


Amplifiers and thrusters
Satellite
Θc(s)
+
−
E(s)
Error
K
T(s)
Torque
1/Js²
Θ(s)
Sensor
H(s) = 1

(b)

Fig. P1.4-1

Solution:

(a) $$\Theta(s) = \frac{1}{Js^2}T(s) = \frac{K}{Js^2}E(s) = \frac{K}{Js^2}[\Theta_c(s) - \Theta(s)]$$

$$\therefore \left[1 + \frac{K}{Js^2}\right]\Theta(s) = \frac{K}{Js^2}\Theta_c(s)$$

$$\therefore \frac{\Theta(s)}{\Theta_c(s)} = \frac{K/Js^2}{1 + \dfrac{K}{Js^2}} = \frac{K/J}{s^2 + K/J}$$

(b) $(1-4)$: $x_1 = 0$; $(1-5)$: $x_2 = \dot{\theta} = \dot{x}_1$

$$\therefore \dot{x}_2(t) = \ddot{\theta}(t) = \frac{1}{J}\tau(t) = \frac{K}{J}e(t) = \frac{K}{J}[\theta_c(t) - \theta(t)]$$

$$= \frac{K}{J}[\theta_c(t) - x_1(t)]$$

$$\therefore \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ K/J & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K/J \end{bmatrix} \theta_c(t)$$

$$y(t) = [1 \quad 0] x(t)$$