$f: 2^V \to \mathbb{R}$ sym submod

$f(B) \neq f(A)\ \forall\, B \subseteq V: B \neq A, \bar{A}$ ⟶ (I)

$(V_1, \dots, V_k) = \arg\min \left\{ \max_{i \in [k]} f(V_i) : (V_1, \dots, V_k) \text{ is a } k\text{-partition of } V \right\}$ (OPT_k)

Let $(U, \bar{U})$ be s.t. $f(U) \le OPT_k$ $= \arg\min\{f(A) : S \subseteq A \subseteq V - T\}$

⟶ Thm 1. $\exists\, S, T \subseteq V$: (i) $(U, \bar{U}) = \arg\min\, (S,T)$-cut
(ii) $|S|, |T| \le k-1$

⟶ Thm 2. $\exists\, S \subseteq U$: (i) $(U, \bar{U}) = \arg\min\, (S, \bar{U})$-cut
(ii) $|S| \le k-1$

Thm 2 $\Downarrow$

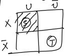
Coro 3. $\exists\, T \subseteq \bar{U}$: (i) $(U, \bar{U}) = \arg\min\, (U, T)$-cut
(ii) $|T| \le k-1$

Fix S as in Thm 2 & T as in Coro 3

Obs. $(U, \bar{U}) = \arg\min\, (S, T)$-cut

Pf $(X, \bar{X})$ be a min (S,T)-cut

Say $X \neq U \Rightarrow X \cap U \neq U$ or $X \cup U \neq U$



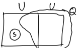
$2 f(U) \ge f(X) + f(U)$

$\ge f(X \cap U) + f(X \cup U)$ ($\ge f(U)$, $\ge f(U)$)

by (I) ⟶ $> 2 f(U) \Rightarrow \Leftarrow$ □

Pf of Thm 2

$\mathcal{C} := \{ Q \subseteq V : \bar{U} \subseteq Q \ \&\ f(Q) < f(U) \}$



Let S be a minimal subset in U s.t. $S \cap Q \neq \emptyset\ \forall\, Q \in \mathcal{C}$ (II)

Prop 1. $(U, \bar{U}) = \arg\min\, (S, \bar{U})$-cut

Pf. $(X, \bar{X})$ be a min $(S, \bar{U})$-cut ⟵

Say $X \neq U \Rightarrow \bar{U} \subseteq \bar{X}$ & $f(\bar{X}) < f(U)$

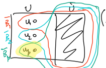
$\Rightarrow \bar{X} \in \mathcal{C}$

$\Rightarrow \bar{X} \cap S \neq \emptyset$

$\Rightarrow \Leftarrow S \subseteq X$ □

Lemma 2. $|S| \le k-1$

Pf. Let $S = \{u_1, \dots, u_p\}$ for $p \ge k$



$(B_i, \bar{B}_i) = \arg\min\, (S - u_i, \bar{U})$-cut $\forall\, i \in [p]$

Claim 1. $u_i \in \bar{B}_i\ \forall\, i \in [p]$

Pf (II) $\Rightarrow \exists\, Q \in \mathcal{C} : S \cap Q = \{u_i\}$

$f(\bar{B}_i) = f(B_i) \le f(Q) < f(U) \le OPT_k$

$\because (\bar{Q}, Q)$ is a $(S - u_i, \bar{U})$-cut

$\bar{U} \subseteq \bar{B}_i$ & $f(\bar{B}_i) < f(U) \Rightarrow \bar{B}_i \in \mathcal{C}$

$\Rightarrow S \cap \bar{B}_i \neq \emptyset$

$\Rightarrow u_i \in \bar{B}_i$ □

$(\bar{A}_i, A_i) := (B_i, \bar{B}_i)\ \forall\, i \in [p]$

wlog let $f(A_1) \le f(A_2) \le \dots \le f(A_k)$ ⟵

Claim 2. $\exists$ a $(k-1)$-partition $(P_1, \dots, P_{k-1})$ of $\bigcup_{i=1}^{k-1} A_i$: $f(P_i) \le f(A_i)\ \forall\, i \in [k-1]$

Claim 3. $f\left(\bigcup_{i=1}^{k-1} A_i\right) \le f(A_k)$.

$\Rightarrow \exists$ a k-partition $\left(P_1, \dots, P_{k-1}, P_k := \overline{\bigcup_{i=1}^{k-1} A_i}\right)$:

$\max_{i \in [k]} f(P_i) \le f(A_k)$

$= f(B_k)$ (sym)

$< f(U)$

$\le OPT_k$

$\Rightarrow \Leftarrow$ □

Pf of claim 3. (Uses submod.)

To show: $f\left(\bigcap_{i=1}^{k-1} B_i\right) \le f(B_k)$

Will show: $f\left(\bigcap_{i=1}^{r} B_i\right) \le f(B_k)\ \forall\, r = 1, \dots, k-1$ by indn on r

Base: $r = 1$

Indn: $f(B_k) + \cancel{f(B_{r+1})}$

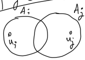
$\ge f\left(\bigcap_{i=1}^{r} B_i\right) + f(B_{r+1})$ (by indn hyp)

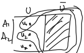
$\ge f\left(\bigcap_{i=1}^{r+1} B_i\right) + f\left(\left(\bigcap_{i=1}^{r} B_i\right) \cup B_{r+1}\right)$ (by submod)

$(S - u_{r+1}, \bar{U})$-cut; $\ge f(B_{r+1})$

$\ge f\left(\bigcap_{i=1}^{r+1} B_i\right) + \cancel{f(B_{r+1})}$ □

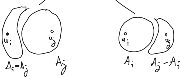
Pf of claim 2





posimod: $f(A_i) + f(A_j) \ge f(A_i - A_j) + f(A_j - A_i)$

$f(A_i) \ge f(A_i - A_j)$ $f(A_j) \ge f(A_j - A_i)$



Repeatedly uncross via posimod ineq. □