Instance-dependent Sample Complexity Bounds for Zero-sum Matrix Games

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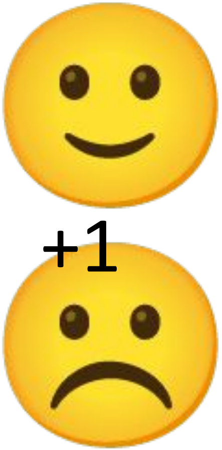
Two player zero-sum games



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We have two players X and Y.

X player has m different strategies

Y player has n different strategies

If X choses a strategy i and Y choses a strategy j, then X receives a reward A(i,j) and Y receives a reward -A(i,j)

Two player zero-sum matrix games

Zero-sum games can be represented by a matrix $A \in \mathbb{R}^{n \times m}$.

Let Δ_m be m-dimensional probability simplex.

$(x^*, y^*) \in \Delta_n \times \Delta_m$ is a Nash Equilibrium if the following holds:

$$\langle x, Ay^* \rangle \leq \langle x^*, Ay^* \rangle \leq \langle x^*, Ay \rangle$$

for all $(x, y) \in \Delta_n \times \Delta_m$

Example

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} \quad x^* = (0.75, 0.25) \quad y^* = (0.5, 0.5)$$

$$\langle x^*, Ay^* \rangle = 2.5$$

$$A^\top x^* = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix} \qquad Ay^* = \begin{bmatrix} 2.5 \\ 2.5 \end{bmatrix}$$

Two player zero-sum matrix games

Let $V_A^* := \langle x^*, Ay^* \rangle$ denote the value of the game.

Due to Von Neumann's minimax theorem we have:

$$V_A^* = \max_{x \in \Delta_n} \min_{y \in \Delta_m} \langle x, Ay \rangle = \min_{y \in \Delta_m} \max_{x \in \Delta_n} \langle x, Ay \rangle$$

One can formulate any zero-sum game as an LP and find the Nash Equilibrium.

Noisy setting

In this setting, the matrix A is not directly available.

Instead, one can query an element (i,j) of the matrix A and receive a value $X_{i,j} = A_{i,j} + \eta$ where η is zero-mean 1-sub-gaussian noise.

The aim is now to compute some kind of approximate solution concept.

Solution concepts

1. ε-good solution: We aim to find $(x, y) \in \Delta_n \times \Delta_m$ such that:

$$|V_A^* - \langle x, Ay \rangle| \leq \varepsilon$$

2. ε-Nash Equilibrium: We aim to find $(x, y) \in \Delta_n \times \Delta_m$ such that:

$$\langle x, Ay \rangle \geq \langle x', Ay \rangle - \varepsilon$$
$$\langle x, Ay' \rangle \geq \langle x, Ay \rangle - \varepsilon$$

for all $(x', y') \in \Delta_n \times \Delta_m$

Goal

Minimize the number of samples required to find the approximate solution concepts

Simple Algorithm

Sample each element of the matrix $A \in \mathbb{R}^{n\times m}$ for $\frac{8\log(2mn/\delta)}{\varepsilon^2}$ times.

Return the Nash Equilibrium of the empirical matrix $\bar{A}$

Minimax Upper Bound

Using sub-gaussian tail bounds, we get the following:

$$|\bar{A}_{ij} - A_{ij}| \leq \varepsilon/2 \text{ for all } (i, j) \in [m] \times [n]$$

Minimax Upper Bound

For any x’ in the m-dimensional simplex, we have the following:

$$\begin{aligned}
\langle x, Ay\rangle &= \langle x, \bar{A}y\rangle + \sum_{i,j}(A_{ij} - \bar{A}_{i,j})x_i y_j \\
&\geq \langle x, \bar{A}y\rangle - \frac{\varepsilon}{2} \\
&\geq \langle x', \bar{A}y\rangle - \frac{\varepsilon}{2} \qquad \text{(as } (x, y) \text{ is a NE of } \bar{A}\text{)} \\
&\geq \langle x', Ay\rangle - \varepsilon
\end{aligned}$$

Minimax Upper Bound

Similarly, for any y’ in the n-dimensional simplex, we have the following:

$$\langle x, Ay' \rangle \geq \langle x, Ay \rangle - \varepsilon$$

Hence, (x,y) is an ε-Nash Equilibrium of A.

Instance-dependent bounds

Stochastic Multi-armed bandits is a special case of our noisy setting. Instance dependent sample complexity bounds to find the best arm is well studied.

We aim to establish similar instance dependent bounds to find either ε-good solution or ε-Nash Equilibrium in zero-sum games.

Multi-Armed Bandits

We have n arms with means $\mu_1 > \mu_2 \geq \ldots \geq \mu_n$

The problem can be considered an $n \times 1$ game as follows.

$$\text{Optimal strategy} \rightarrow \begin{bmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_n \end{bmatrix} \Delta_g > 0$$

n×2 matrix games (n>2)

Consider a matrix game $A \in \mathbb{R}^{n\times 2}$ with unique Nash Equilibrium (x^*, y^*) which is not a PSNE.

It can be shown that $|Supp(x^*)| = |Supp(y^*)| = 2$

$$Supp(x^*) \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix} \right.$$

n×2 matrix games (n>2)

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Examples

	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.05 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0.9 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.15 & 0.95 \end{bmatrix}$
x^*	$(0.5, 0.5, 0)$	$(0.5, 0.5, 0)$ $(4/9, 0, 5/9)$	$(4/9, 0, 5/9)$
y^*	$(0.5, 0.5)$	$(0.5, 0.5)$	$(19/36, 17/36)$

Examples

y^*	$(0.5, 0.5)$	$(0.5, 0.5)$	$(19/36, 17/36)$
	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.05 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0.9 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.15 & 0.95 \end{bmatrix}$
$\langle y^*, (A_{i1}, A_{i2}) \rangle$	$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.45 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 19/36 \\ 17/36 \\ 19/36 \end{bmatrix}$

Examples

y^*	$(0.5, 0.5)$	$(0.5, 0.5)$	$(19/36, 17/36)$
	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.05 & 0.85 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.1 & 0.9 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0.15 & 0.95 \end{bmatrix}$
$\langle y^*, (A_{i1}, A_{i2}) \rangle$	$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.45 \end{bmatrix}$	$\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$	$\begin{bmatrix} 19/36 \\ 17/36 \\ 19/36 \end{bmatrix}$

Instance dependent parameters for n$\times$2 games

Suppose that A has a unique Nash equilibrium (x^*, y^*) such that Supp(x^*) = {i_1, i_2}. Let $A_{i_1 1} > A_{i_1 2}$ and $A_{i_2 1} > A_{i_2 2}$ and define

$$\Delta_g := \min_{i \in [n] \setminus \{i_1, i_2\}} r_i \cdot (V_A^* - \langle y^*, (A_{i1}, A_{i2}) \rangle)$$

where

$$r_i = \frac{|A_{i_1 1} - A_{i_1 2}| + |A_{i_2 1} - A_{i_2 2}|}{|A_{i_1 1} - A_{i_1 2}| + |A_{i_2 1} - A_{i_2 2}| + |A_{i1} - A_{i2}|}$$

Instance dependent parameters for n×2 games

$$\Delta_{\min} = \min\{\min_i\{|A_{i1} - A_{i2}|\}, \min_{j,k:j\neq k}\{|A_{j1} - A_{k1}|\}, \min_{j,k:j\neq k}\{|A_{j2} - A_{k2}|\}\}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{bmatrix}$$

Results for n$\times$2 games

Consider a game defined by a fixed 3 × 2 matrix A which has a unique Nash equilibrium which is not a pure-strategy Nash equilibrium.

We show under some mild assumptions that to find an ε-good solution for the matrix game A with probability at least 1 − δ, we require at least $\Omega\left(\frac{1}{\Delta_g^2}\log(1/\delta)\right)$ samples from A

Results for n×2 games

We complement our lower bound result by designing an algorithm that, with probability at least $1 - \delta$, samples each element of an $n \times 2$ matrix A for

$$O\left(\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{1}{\Delta_g^2}\right\}\right\}\log(1/\delta)\right)$$

times (ignoring some logarithmic factors) and either returns $\mathrm{Supp}(x^*)$ and $\mathrm{Supp}(y^*)$ or concludes that Δ_g is not sufficiently large compared to ε.

Key ideas behind the Algorithm to find Support

$$\Delta_g \approx V_A^* - \min_{i \notin Supp(x^*)} \langle y^*, (A_{i1}, A_{i2}) \rangle$$

$$\bar{\Delta}_g \approx V_{\bar{A}}^* - \min_{i \notin Supp(x^*)} \langle \bar{y}^*, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

$$\underbrace{\begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} \\ \begin{matrix} a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{matrix} \end{bmatrix}}_{A} \Big\} \Delta_g \qquad \underbrace{\begin{bmatrix} \boxed{\begin{matrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{matrix}} \\ \begin{matrix} \bar{a}_{31} & \bar{a}_{32} \\ \vdots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} \end{matrix} \end{bmatrix}}_{\bar{A}} \Big\} \bar{\Delta}_g$$

$(\bar{x}^*, \bar{y}^*)$ is the NE in $\bar{A}$

If every element of A is sampled for $\frac{1}{\Delta_g^2}$ then $\Delta_g \approx \bar{\Delta}_g$

Key ideas behind the Algorithm to find Support

$$\Delta_g \approx V_A^* - \min_{i \notin Supp(x^*)} \langle y^*, (A_{i1}, A_{i2}) \rangle$$

$$\bar{\Delta}_g \approx V_{\bar{A}}^* - \min_{i \notin Supp(x^*)} \langle \bar{y}^*, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

If we sample each element of A for $\frac{1}{\Delta^2}$ times then $|\Delta_g - \bar{\Delta}_g| \leq \Delta$

So in $\frac{1}{\Delta_g^2}$ steps we can estimate Δ_g and we are done.

What's the problem here?

Key ideas behind the Algorithm to find Support

$$\Delta_g \approx V_A^* - \min_{i \notin Supp(x^*)} \langle y^*, (A_{i1}, A_{i2}) \rangle$$

$$\bar{\Delta}_g \approx V_{\bar{A}}^* - \min_{i \notin Supp(x^*)} \langle \bar{y}^*, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

If we sample each element of A for $\frac{1}{\Delta^2}$ times then $|\Delta_g - \bar{\Delta}_g| \leq \Delta$

So in $\frac{1}{\Delta_g^2}$ steps we can estimate Δ_g and we are done.

What's the problem here?

We don't know $Supp(x^*)$ hence can't measure $\bar{\Delta}_g$

Key ideas behind the Algorithm to find Support

What we can measure is instead the following:

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

where $(\bar{x}, \bar{y})$ is the NE of the $\bar{A}$

$$\begin{bmatrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \\ \bar{a}_{31} & \bar{a}_{32} \\ \vdots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} \end{bmatrix} \Big\} \tilde{\Delta}_g$$

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

Saving grace:

$$\Delta'_g = V^*_A - \min_{i \in Supp(x)} \langle y, (A_{i1}, A_{i2}) \rangle < 0$$

where (x, y) is the NE of the submatrix of A with rows $Supp(\bar{x})$

$$\begin{bmatrix} \boxed{\begin{matrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{matrix}} \\ \begin{matrix} a_{31} & a_{32} \\ \vdots & \vdots \\ a_{n1} & a_{n2} \end{matrix} \end{bmatrix} \} \Delta'_g < 0 \qquad \begin{bmatrix} \boxed{\begin{matrix} \bar{a}_{11} & \bar{a}_{12} \\ \bar{a}_{21} & \bar{a}_{22} \end{matrix}} \\ \begin{matrix} \bar{a}_{31} & \bar{a}_{32} \\ \vdots & \vdots \\ \bar{a}_{n1} & \bar{a}_{n2} \end{matrix} \end{bmatrix} \} \; \tilde{\Delta}_g < \Delta \text{ if we sample for } \frac{1}{\Delta^2} \text{ times}$$

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V_{\bar{A}}^* - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

Hence, if we have sampled each element for $\frac{1}{\Delta^2}$ times, we stop only when the following holds:

$$\tilde{\Delta}_g > \Delta$$

And then we return the support of the empirical matrix.

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

Hence, if we have sampled each element for $\frac{1}{\Delta^2}$ times, we stop only when the following holds:

$$\tilde{\Delta}_g > \Delta$$

Are we done? NO

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

Hence, if we have sampled each element for $\frac{1}{\Delta^2}$ times, we stop only when the following holds:

$$\tilde{\Delta}_g > \Delta$$

Are we done? NO

Why? $\tilde{\Delta}_g > \Delta$ may-not hold at any time step

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

$$\Delta_g \approx V^*_{A} - \min_{i \notin Supp(x^*)} \langle y^*, (A_{i1}, A_{i2}) \rangle$$

Hence, if we have sampled each element for $\frac{1}{\Delta^2}$ times, we stop only when the following holds:

$$\tilde{\Delta}_g > \Delta$$

$$\bar{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(x^*)} \langle \bar{y}^*, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

It can be shown that $\tilde{\Delta}_g \geq \bar{\Delta}_g$ and eventually $\bar{\Delta}_g$ becomes greater than Δ as $\Delta_g > 0$

Key ideas behind the Algorithm to find Support

$$\tilde{\Delta}_g \approx V^*_{\bar{A}} - \min_{i \notin Supp(\bar{x})} \langle \bar{y}, (\bar{A}_{i1}, \bar{A}_{i2}) \rangle$$

Hence, if we have sampled each element for $\frac{1}{\Delta^2}$ times, we stop only when the following holds:

$$\tilde{\Delta}_g > \Delta$$

In this case the support of the NE of the empirical matrix will be equal to $Supp(x^*)$

2×2 matrix games

Instance dependent parameters for 2✕2 games

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Let a>c, a>b, d>b, d>c. Then we have the following closed form solution for the Nash Equilibrium.

$$x^* = (\tfrac{d-c}{D}, \tfrac{a-b}{D}) \qquad y^* = (\tfrac{d-b}{D}, \tfrac{a-c}{D})$$

Here D=a-b-c+d

Instance dependent parameters for 2✕2 games

$$D = A_{11} - A_{12} - A_{21} + A_{22}$$

$$\Delta_{\min} = \min\{|A_{11} - A_{12}|, |A_{11} - A_{21}|, |A_{22} - A_{21}|, |A_{22} - A_{12}|\}$$

$$\Delta_{m_2} = \max\{\min\{|A_{11} - A_{12}|, |A_{22} - A_{21}|\}, \min\{|A_{11} - A_{21}|, |A_{22} - A_{12}|\}\}$$

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

Results for 2✕2 games: ε-good solution

Consider a game defined by a fixed 2×2 matrix A which has a unique Nash equilibrium which is not a pure-strategy Nash equilibrium.

We show under some mild assumptions that to find an ε-good solution for the matrix game A with probability at least $1 - \delta$, we require at least

$$\Omega\left(\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{1}{\varepsilon|D|}\right\}\right\}\log(1/\delta)\right)$$

samples from A

Proof Idea

$$\begin{bmatrix} a-\Delta & b+\Delta \\ c-\Delta & d+\Delta \end{bmatrix} \quad \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad \begin{bmatrix} a+\Delta & b-\Delta \\ c+\Delta & d-\Delta \end{bmatrix}$$

There is no pair (x,y) which is an ε-good solution for all the three matrices

Results for 2×2 games: ε-good solution

Complementing the lower bound result, we design an algorithm that, with probability $1 - \delta$, identifies an ε-good solution using a number of samples matching the lower bound up to logarithmic factors.

Easy Instance for 2✕2 games: ε-good solution

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{1}{\varepsilon|D|}\right\}\right\}$$

$$\approx \frac{1}{\varepsilon}$$

Hard Instance for 2×2 games: ε-good solution

$$\begin{bmatrix} 0.5+\varepsilon & 0.5 \\ 0.5 & 0.5+\varepsilon \end{bmatrix}$$

$$\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{1}{\varepsilon|D|}\right\}\right\}$$

$$\approx \frac{1}{\varepsilon^2}$$

Results for 2✕2 games: ε-Nash Equilibrium

Consider a game defined by a fixed 2 × 2 matrix A which has a unique Nash equilibrium which is not a pure-strategy Nash equilibrium.

We show under some mild assumptions that to find an ε-Nash Equilibrium for the matrix game A with probability at least 1 − δ, we require at least

$$\Omega\left(\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{\Delta_{m_2}^2}{\varepsilon^2 D^2}\right\}\right\}\log(1/\delta)\right)$$

samples from A

Results for 2$\times$2 games: ε-Nash Equilibrium

Complementing the lower bound result, we design an algorithm that, with probability $1 - \delta$, identifies an ε-Nash Equilibrium using a number of samples matching the lower bound up to logarithmic factors.

Hard Instance for 2✕2 games: ε-Nash Equilibrium

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{\Delta_{m_2}^2}{\varepsilon^2 D^2}\right\}\right\}$$

$$\approx \frac{1}{\varepsilon^2}$$

Easy Instance for 2✕2 games: ε-Nash Equilibrium

$$\begin{bmatrix} 1 & 1-\sqrt{\varepsilon} \\ 0 & 1 \end{bmatrix}$$

$$\min\left\{\frac{1}{\varepsilon^2}, \max\left\{\frac{1}{\Delta_{\min}^2}, \frac{\Delta_{m_2}^2}{\varepsilon^2 D^2}\right\}\right\}$$

$$\approx \frac{1}{\varepsilon}$$

Results for 2×2 games with Non-unique Nash Equilibrium

We show under some mild assumptions that to find an ε-good solution for the matrix game A with probability at least $1 - \delta$, we require at least $\Omega(\frac{1}{\varepsilon^2})$ samples from A.

$n \times m$ games with unique Nash Equilibrium

Our techniques can be extended to the case $n \times m$ games with unique Nash Equilibrium. However, they are not optimal.

Open Question: Find tight upper and lower bounds for $n \times m$ games.

Thank You