Trajectories with prescribed itineraries in CR3BP

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Outline:

1) Motivation: Orbits with prescribed itineraries

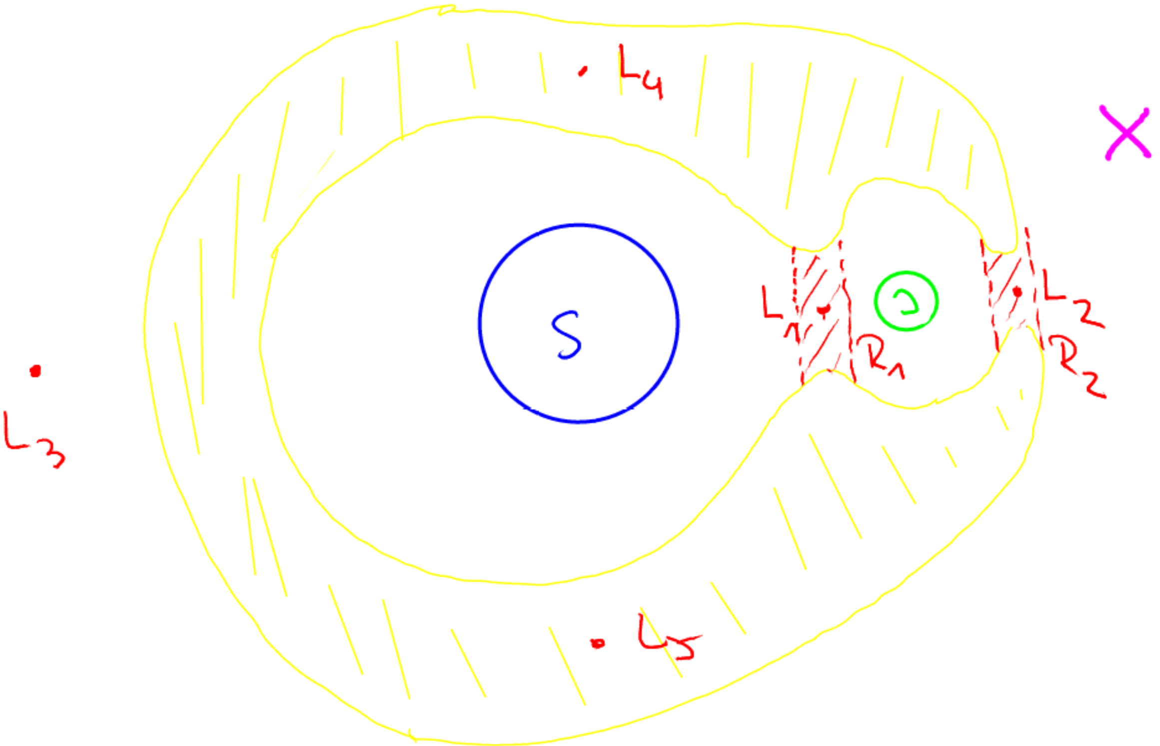
THEORY
2) Find'em using tube dynamics of Lagrange points
3) When is their existence granted?

PRAXIS
4) How to find Lyapunov orbits numerically?
5) How to find stable and unstable manifolds and Poincaré sections numerically?

PLAY
6) Program

0) Motivation: Orbits with prescribed itinerary

CR3BP, rotating frame, fixed E



Hill's Region → two necks → realms

X, ☽, S

Goal: Find IC of an orbit which follows

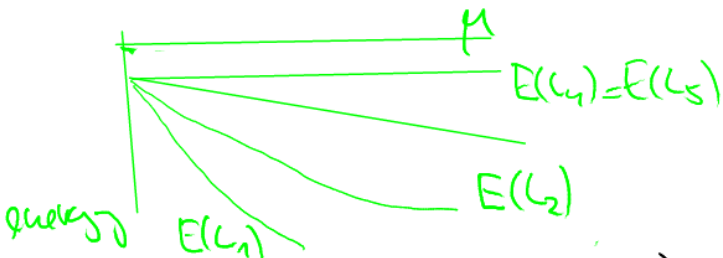
$$(\ldots, X, ☽, S, ☽, X, \ldots)$$

1) Find 'em using tube dynamics of Lagr. pts.

a) $\exists$ hyperbolic equilibrium points L_1 and L_2

(and L_3, L_4, L_5)

b) Linearize Ham. system around L for E slightly above $E(L)$

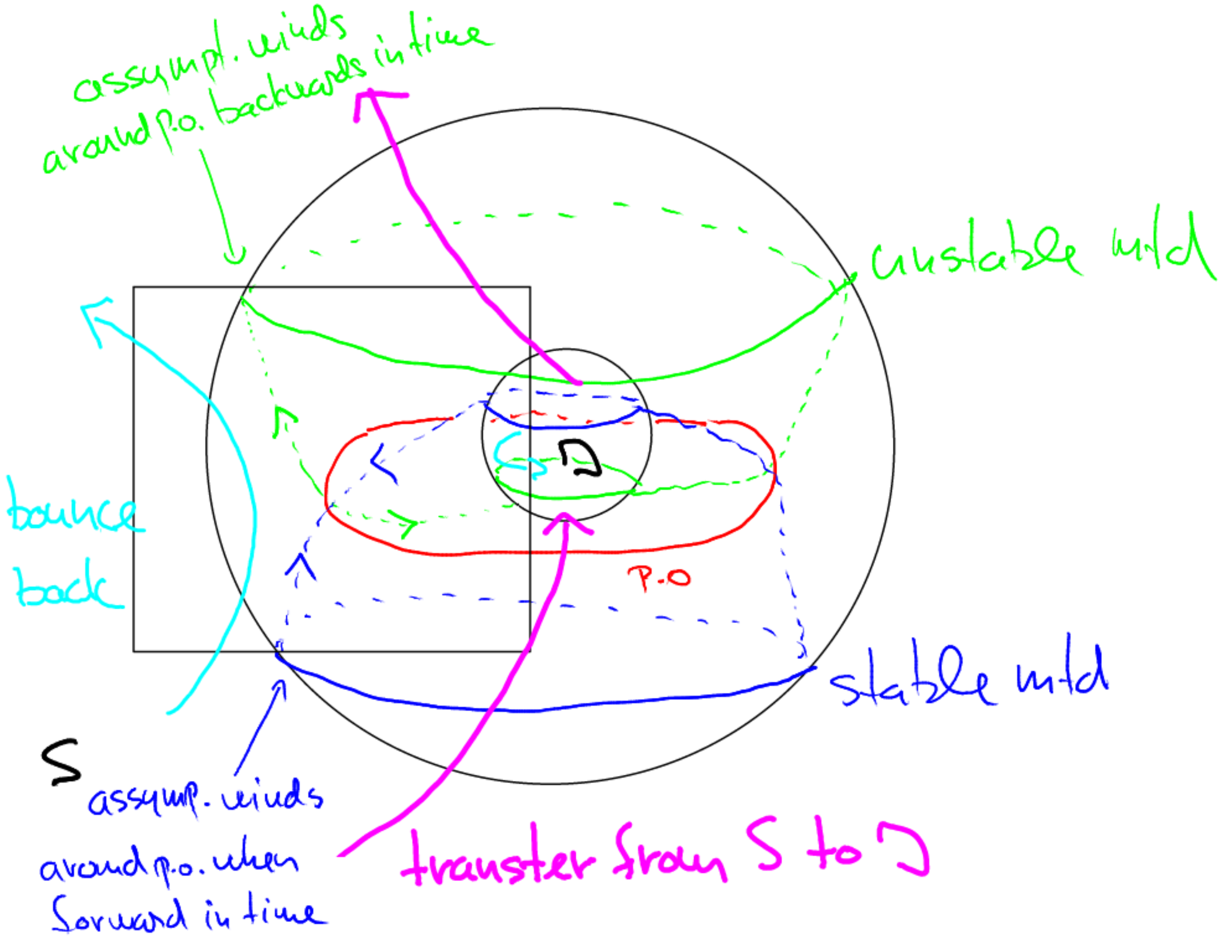


($\mu \ll 1 \longrightarrow E(L_2)$ only slightly above $E(L_1)$

so E can be chosen slightly above both)

S-J 9.537×10^{-4} (lin. $L \sim 0$, $E \sim E - E(L)$)

$\Longrightarrow \forall\, E > E(L)\ \exists!$ p.o. of the linearized system and its neighborhood in the energy hypersurface in $\mathcal{M}(E) \subset T^*\mathbb{R}^2$ (energy hypersurface) looks like $S^2 \times I$, where the dynamics is in the McGehee representation picturized as follows:



Note: 1) For $E \to E(L)$, the circles degenerate to points in the plane, the p.o. to L, in particular.

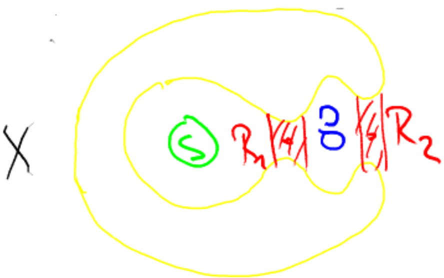
c) Moser's theorem about existence of a real analytic change of coordinates $(x, y, v_x, v_y) \rightsquigarrow (\xi, \eta, \zeta_1, \zeta_2)$ around L in $T^*\mathbb{R}^2$, where the solutions of the original system have manifestly the same form as the solutions of the linearized system $\longrightarrow$

$\Rightarrow \forall E > E(L)$ slightly above $\exists!$ periodic orbit
$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \parallel$
$\qquad\qquad\qquad\qquad\qquad\qquad$ Lyapunov orbit γ_L

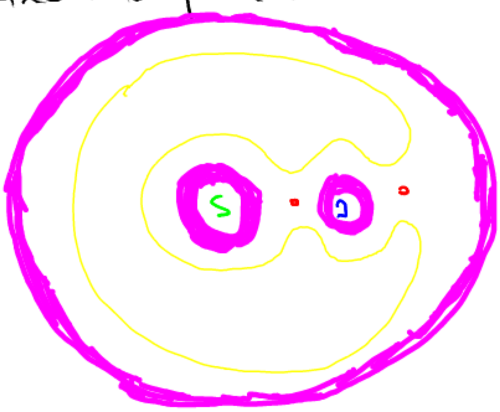
and its nbhd $\mathcal{R}$ in $\mathcal{M}(E)$ such that the dynamics in $\mathcal{R}$ is described qualitatively by the McGehee spheres.

d) Pair of McGehee theorems $\Rightarrow$

$\exists\ \sigma$ open nbhd of the graph $E_L(\mu)$ for $\mu \ll 1$, so that for $(\mu, e) \in \sigma$ the region $\mathcal{R}$ as above can be chosen, s.t. it projects to $R \subset \mathbb{R}^2$ in the form

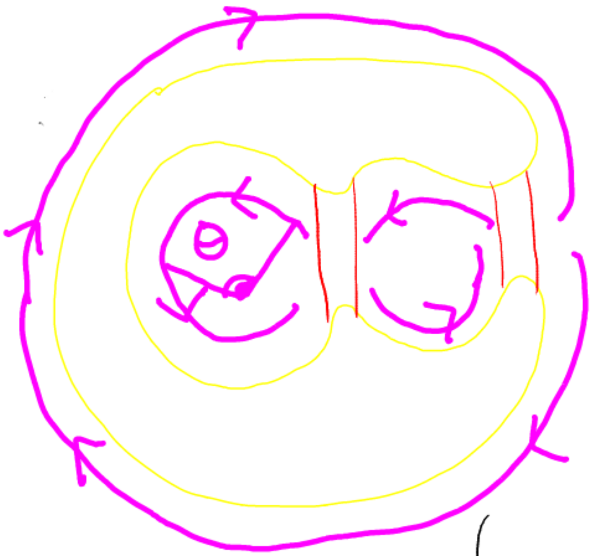


. Moreover (for $\mu \ll 1$) the S, J, X are separated by KAM-tori from the rest of the dynamics



(The smaller μ, the

closer are the KAM-tori to the bdd.). Moreover,



$\exists$ meridional angular coordinate Θ on complements of R in $M(\mu, e)$ which strictly increases along orbits
($\rightarrow$ orbits go around X, D, S in one direction)

2) To the hyperbolic equilibrium point L (normal definition), equivalently to γ_L (ω-limit set definition) are associated stable and unstable manifolds $W_s(L)$ and $W_u(L)$.

Properties: 1) Global 2-D submanifolds of $M(e)$,
(energy hypersurf)

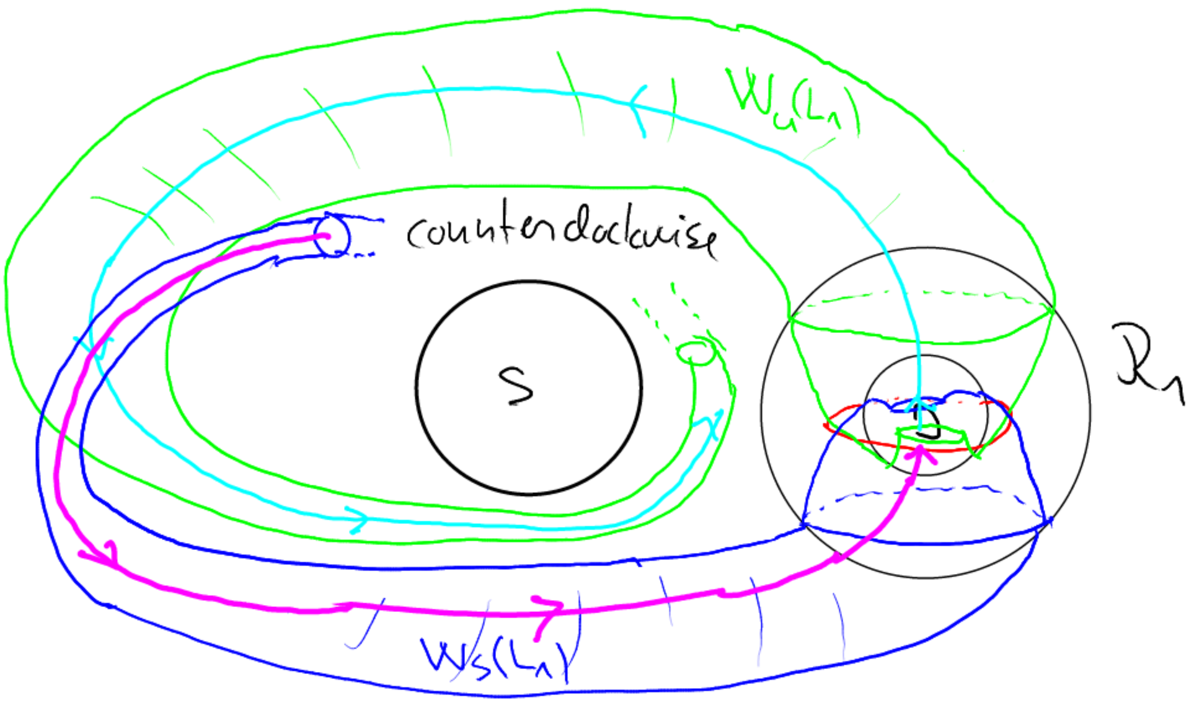
they are annuli (cylinders)

2) $W_s(L)$ is the set of orbits which assymptotically wind around γ_L in the forward time.

$W_u(L)$ is the set of orbits which assymptotically wind around γ_L in the backward time.

3) They wind around $S, ?, X$ in the direction of increasing Θ.

4) They start in $\mathcal{R}$ from γ_L. Eg (picture in $\Pi(e)$):

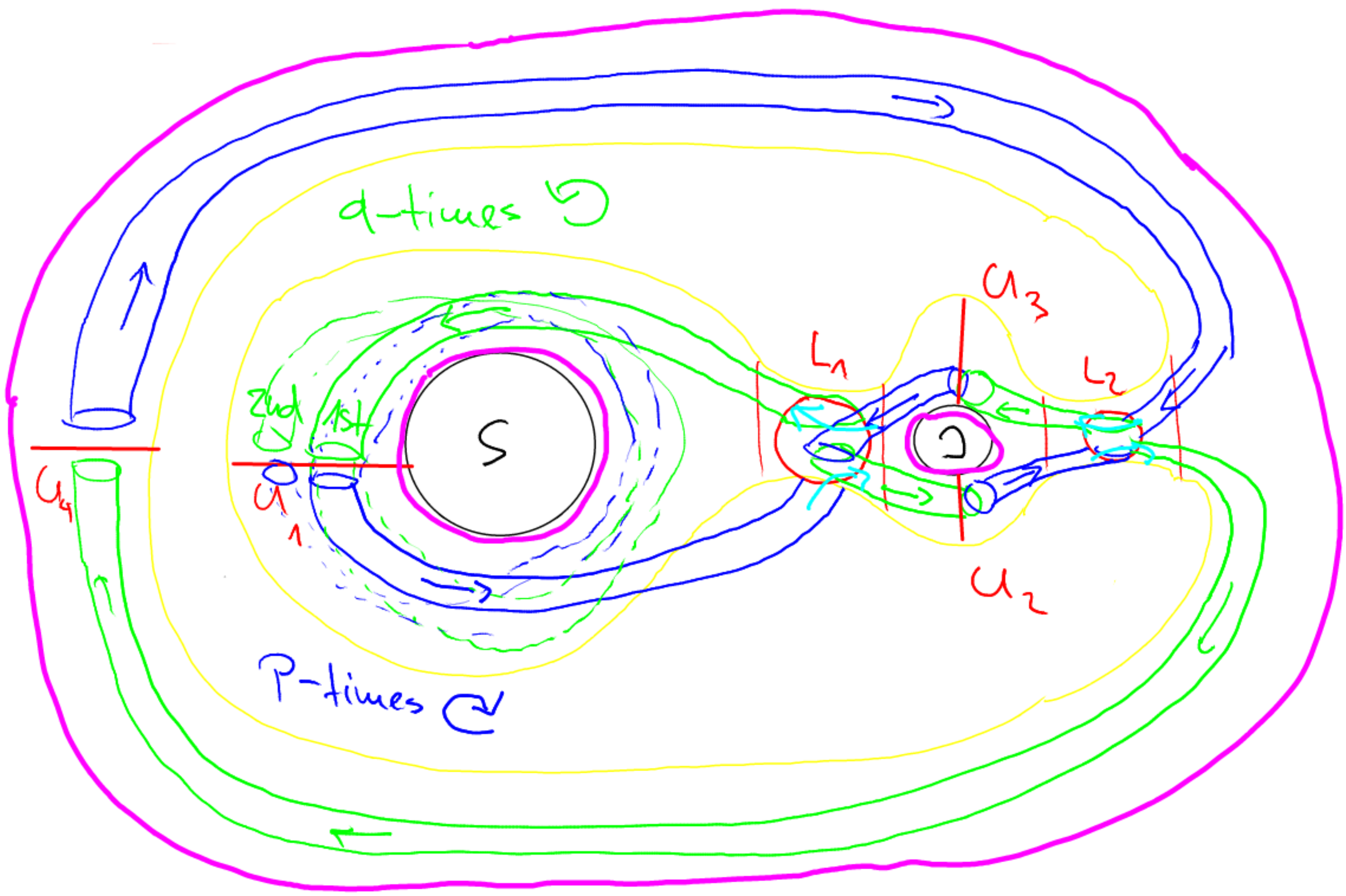


5) All trajectories inside of the tubes bounded by $W_s(L)$, resp. $W_u(L)$ are transfer trajectories $(\dots S ? \dots)$, resp. $(\dots ? S \dots)$. The closer to ∂W, the more time they spend winding around γ_L. In particular

(Note: $W_s(L) \cap W_u(L)$ homoclinic pts, $W_s(L_1) \cap W_u(L_2)$ heteroclinic pts) if $IC \in \mathring{W}_u(L_1) \cap \mathring{W}_s(L_1)$ $\Rightarrow$ the corresponding orbit has itinerary $(\ldots J S J \ldots)$.

f) The concept of the $\boxed{\text{tube dynamics}}$ is to search for trajectories with a given itinerary by searching for a "common intersections" of $\mathring{W}_s(L)$ and $\mathring{W}_u(L)$.

The whole picture:



This is feasible by looking at intersections with suitable sections U_1, U_2, U_3, U_4 in $\Pi(e)$:

$$U_1 = \{(x,\dot x) \mid y=0,\ x<0,\ \dot y(x,\dot x,e)<0\}$$

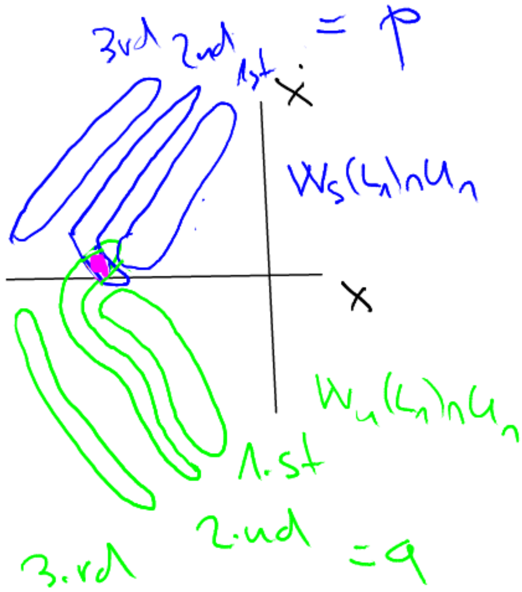
(i.e. projection of $\Pi(e)\cap\{y=0, x<0, \dot y<0\}$ to the $(x,\dot x)$-plane)

$$U_2 = \{(y,\dot y) \mid x \overset{!!}{=} \overset{1-\mu}{0},\ y<0,\ \dot x(y,\dot y,e)>0\}$$

$$U_3 = \{(y,\dot y) \mid x = 0,\ y>0,\ \dot x(y,\dot y,e)<0\}$$

$$U_4 = \{(x,\dot x) \mid y=0,\ x<-1,\ \dot y(x,\dot x,e)>0\}$$

$\Rightarrow$ Get pictures like this in U_1:

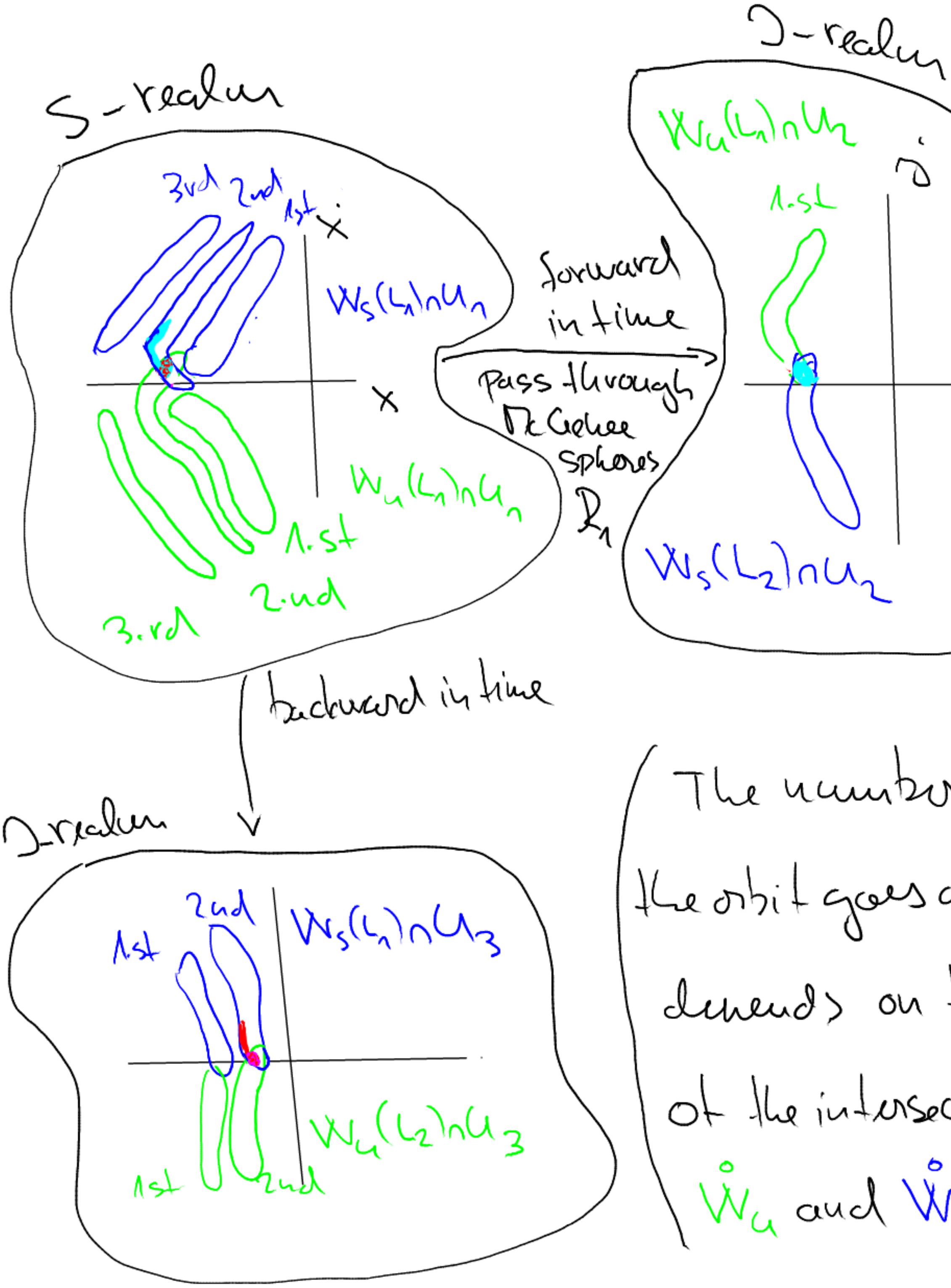


Note: the symmetry of the Eqn

$s: (x, y, v_x, v_y, t) \mapsto (x, -y, -v_x, v_y, -t)$

is manifested in U_1

Process of finding a trajectory by looking at $U \cap W$:



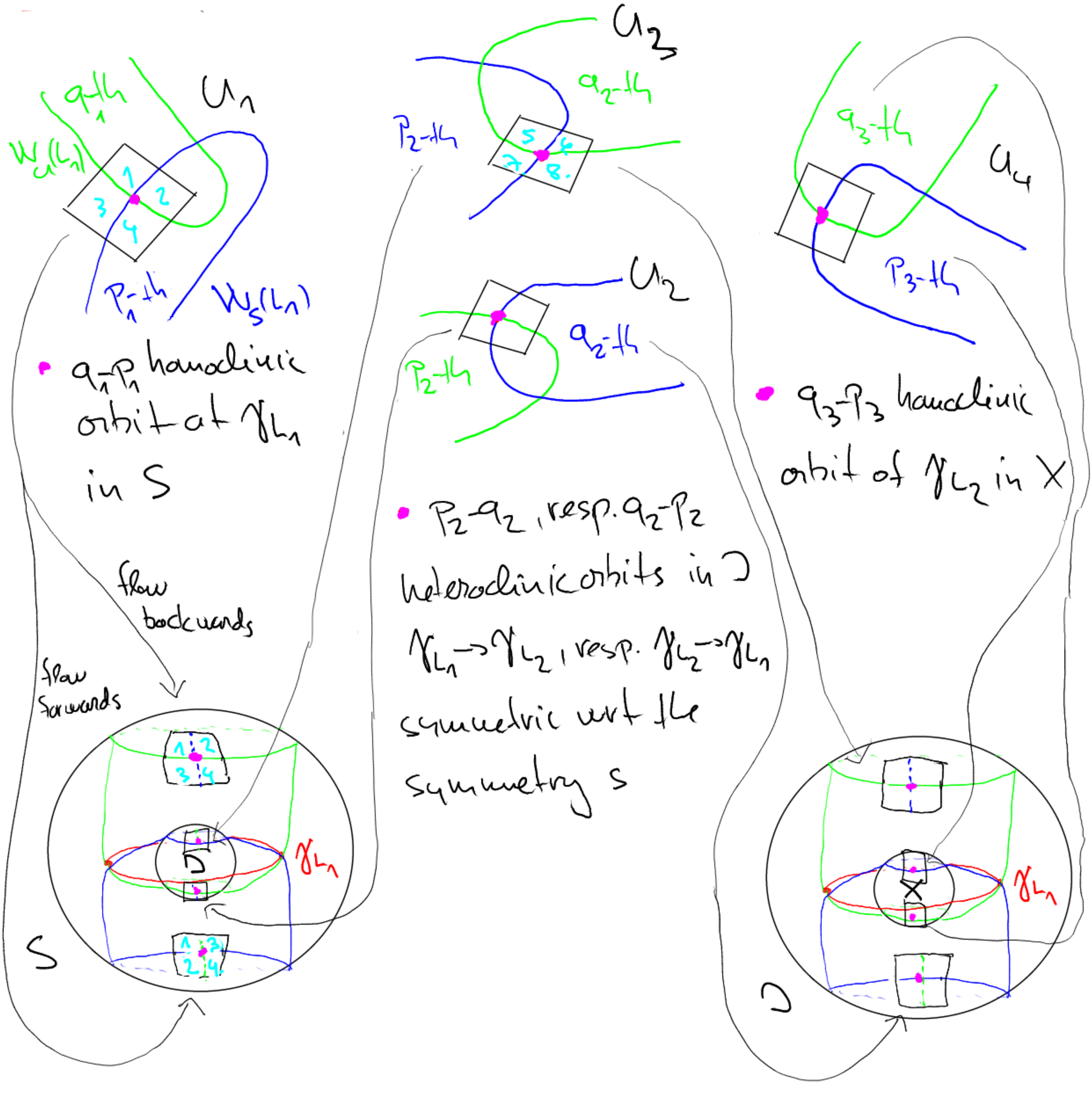
The number of times the orbit goes around S, J, X depends on the order q-p of the intersection of $\mathring{W}_u$ and $\mathring{W}_s$

The trajectories in • have itinerary

(... X J S J X ...)

2) When is the existence of the trajectories granted?

Assume $\exists$ transverse homoclinic-heteroclinic chain, i.e. for given E $\exists$ transverse intersections

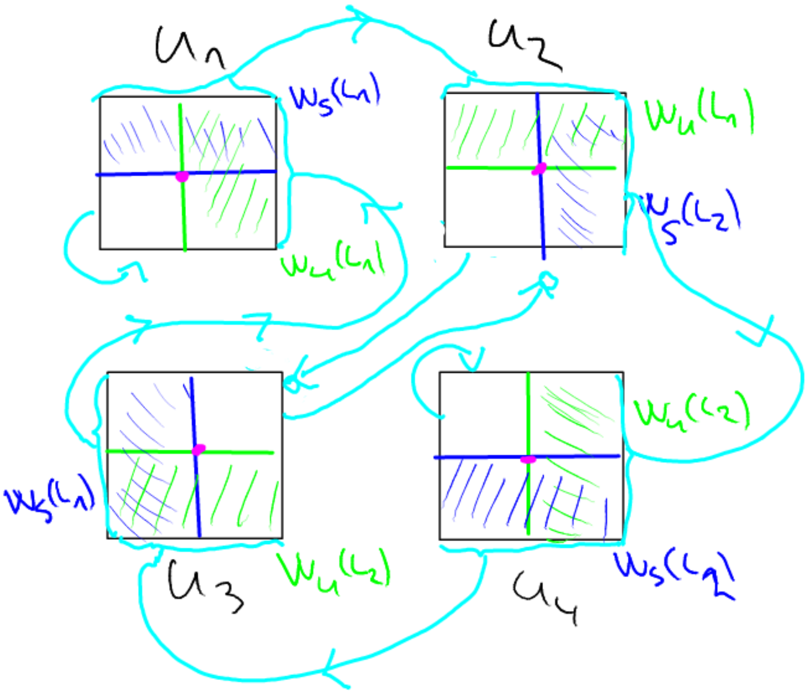


- q_1-p_1 homoclinic orbit at γ_{L_1} in S
- p_2-q_2, resp. q_2-p_2 heteroclinic orbits in D $\gamma_{L_1} \to \gamma_{L_2}$, resp. $\gamma_{L_2} \to \gamma_{L_1}$ symmetric wrt the symmetry s
- q_3-p_3 homoclinic orbit of γ_{L_2} in X

→ construct Poincaré map

$P: U := U_1 \cup U_2 \cup U_3 \cup U_4 \longrightarrow U$

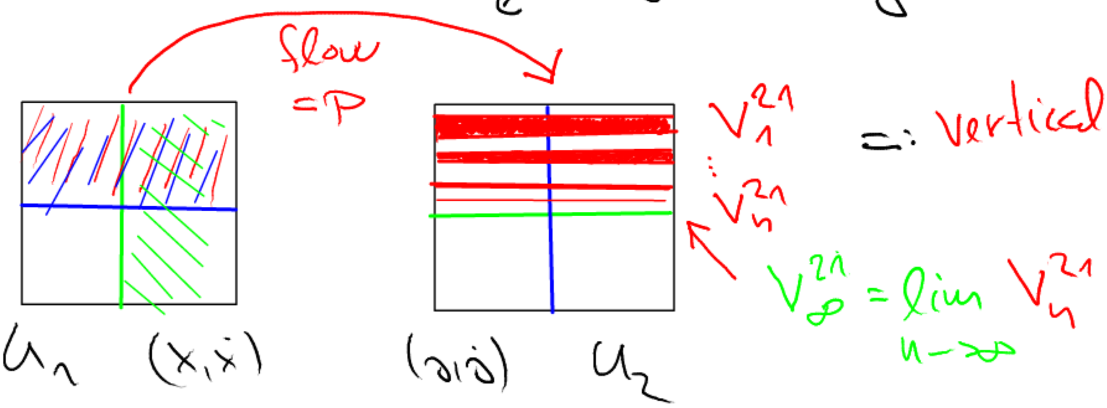
$q \longmapsto$ the 1.st intersection in the forward time of $\phi_t(q)$ with U if exists



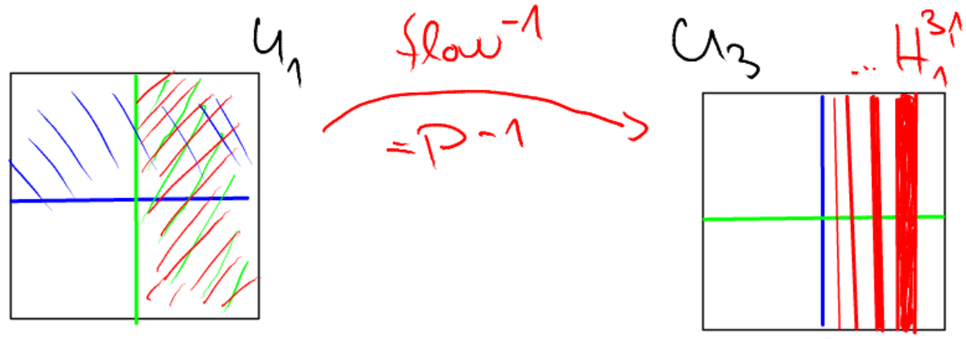
(P not defined on [blue/green cross] since the orbits there remain winding around γ_L for ∞ time)

Close up :

One gets this spiraling behaviour ↙ by studying the lin. system in R_1



V_n^{ji} n-th strip in U_j which came from U_i

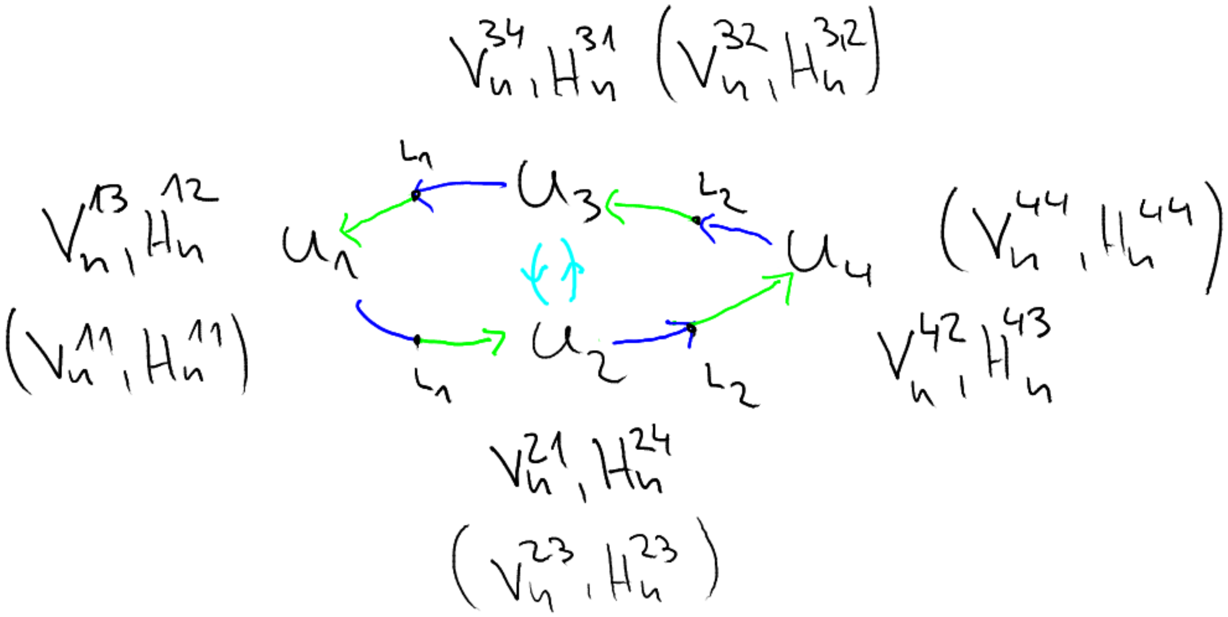


$=:$ horizontal strips

$H_\infty^{3,1} := \lim_{n \to \infty} H_n^{3,1}$

H_n^{ji} n-th strip in U_j which came from U_i by the negative flow, in other words, will go to U_i in forward time

$\longrightarrow$ 8 families of pairwise disjoint vertical and horizontal strips:



Can be proven, the strips satisfy generalized

Conley-Moser conditions :

1) P maps $(H,\partial H) \rightsquigarrow (V,\partial V)$ (strips condition)

2) P uniform contraction | expansion in the horizontal | vertical direction

$\Rightarrow$ "horseshoe-like" chaotic dynamics

In particular, the invariant set

$$\Lambda := \bigcap_{n=1}^{\infty} P^{-n}(U) \cap U \cap P^{n}(U) \quad (P(\Lambda) \subset \Lambda \text{ by def.})$$

of orbits remaining in U forever can be identified with

biinfinite sequences

address of a point in Λ
; (its history | future under P^{-1} | P)

$$(\ldots (u_{i_{-1}}, n_{-1}), (u_{i_0}, n_0), (u_{i_1}, n_1), \ldots)$$

- $u_i \in \{u_1, u_2, u_3, u_4\}$ ... where the point is
- n_i ... corresponds to index n of H_n^{ij}, V_n^{ij}
  = number of revolutions around γ_c on the way from j to i

By construction, every address is traced by an orbit γ going through u_i, winding around γ_L n_i-times in forward time as $i \mapsto i+1$.

$\rightarrow$ In particular, $\exists$ of an orbit with any given itinerary $(\ldots s_{-1} s_0 s_1 \ldots)$, $s_i \in \{X, J, S\}$

<u>Note</u>: The number of revolutions around S, J, X is the same for all orbits and is given by the type $q_i - p_i$ of the transversal homoclinic | heteroclinic points. E.g. q_1+p_1-1, q_3+p_3-1 for S and X, and $\frac{q_2+p_2-1}{2}$ for J.

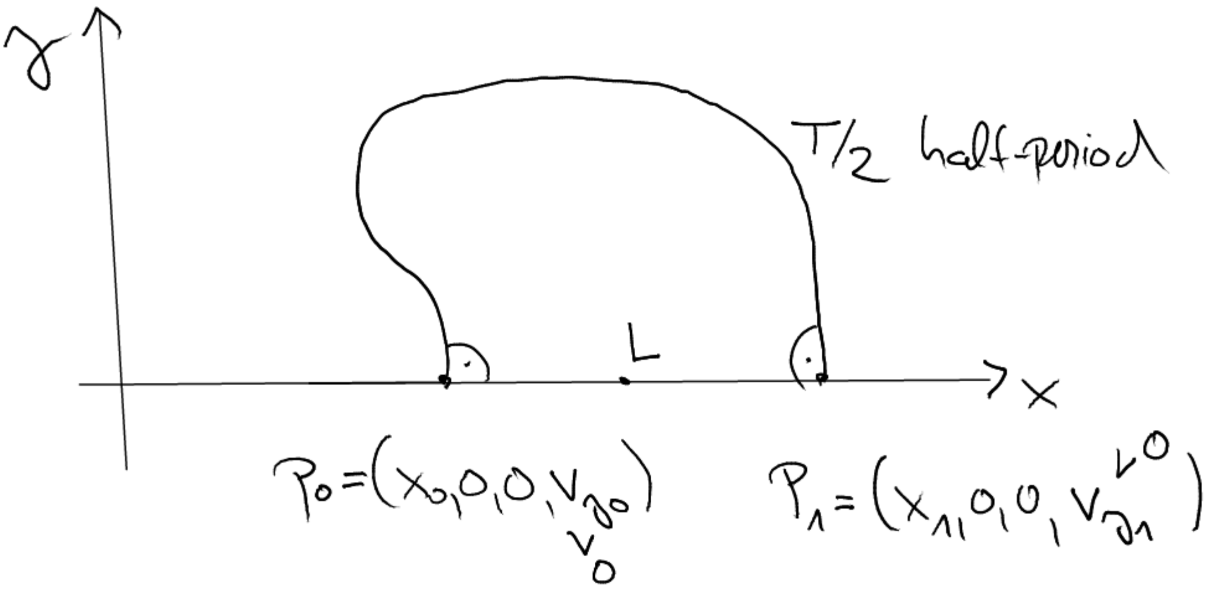
<u>Autors prove numerically existence of transverse</u>
<u>homo- and hetero-clinic chains for energies slightly</u>
<u>above $E(L_2)$ and $\mu = 9.537 \times 10^{-4}$ (Sun-Jupiter)</u>

4) How to find Lyapunov orbits numerically?

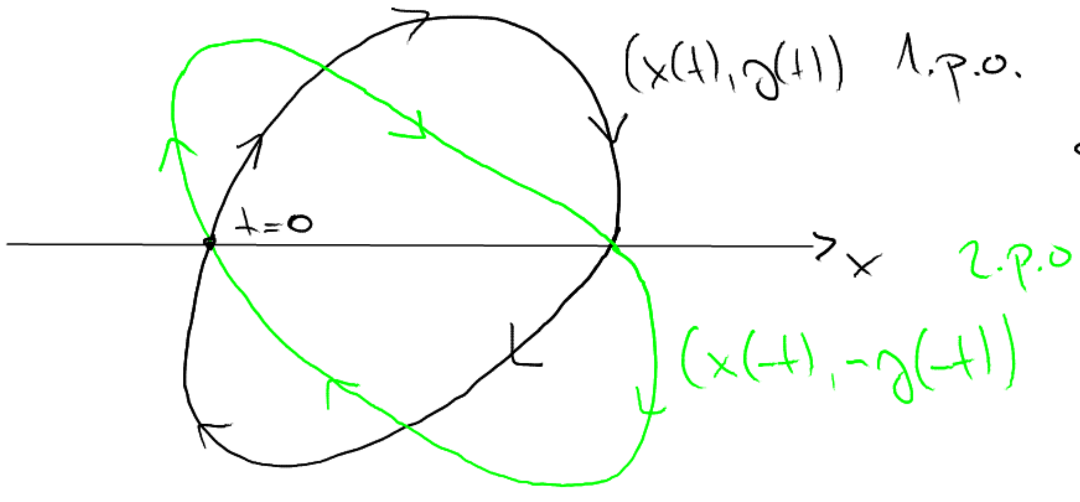
We know from the linearization:

$$\forall E > E(L) \text{ slightly } \exists! \text{ p.o. } \gamma_L \text{ around } L$$

Claim: Has to be symmetric on reflexion about the x-axis. It looks like:



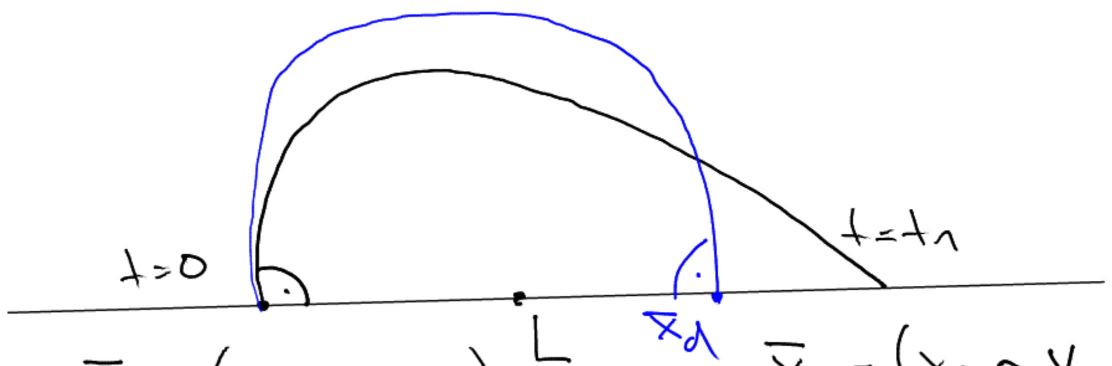
Proof: By contradiction with uniqueness:



since EoM has s-symmetry ↯

□

→ "Use shooting":



$\bar{x}_0 = (x_0, 0, 0, v_{y_0})$ $\qquad$ $\bar{x}_1 = (x_1, 0, v_{x_1}, v_{y_1})$

$$\bar{x}_d = (x_1, 0, 0, v_{y_1}) \;\to\; \delta\bar{x}_1 = \bar{x}_1 - \bar{x}_d = (0, 0, v_{x_1}, 0)$$

Naively adjust $\delta\bar{x}_0 = \Phi(\bar{x}_0, t_1)^{-1}(\bar{x}_1 - \bar{x}_d)$

↑ state transition matrix along γ

Φ is a solution of $\dot{\Phi}(t) = Df(\gamma(t))\,\Phi(t)$, $\Phi(0) = \mathbb{1}$

i.e. $\Phi(t) = T\exp\left(\int_0^t Df(\gamma(u))\,du\right) =$

$$= \mathbb{1} + \int_0^t Df(\gamma(u))\,du + \frac{1}{2!}\int_0^t\int_0^t T\, Df(\gamma(u_1))\,Df(\gamma(u_2))\,du_1\,du_2 + \dots$$

$$= \mathbb{1} + \int_0^t Df(\gamma(u))\,du + \int_0^t\int_0^{t_1} Df(\gamma(u_1))\,Df(\gamma(u_2))\,dt_2\,dt_1 + \dots$$

But we do the following:

1) Fix x_0 and adjust v_{y_0} only

$$\rightarrow \delta \bar{x}_0 = (0, 0, 0, \delta v_{y_0})$$

2) Take into account sensitivity on the final time t_1

flow

$$\delta \bar{x}(t+\delta t) = \varphi(t+\delta t, \bar{x}_0 + \delta \bar{x}_0) - \varphi(t, \bar{x}_0)$$

$$= \underbrace{\frac{\partial \varphi}{\partial t}(t, \bar{x}_0)}_{\dot{x}(t)} \delta t + \underbrace{\frac{\partial \varphi}{\partial x}(t, \bar{x}_0)}_{= \Phi(t, \bar{x}_0)} \delta \bar{x}_0 + \text{h.o.t}$$

Taylor

$$\rightarrow \delta \bar{x}_1 = \Phi(t, \bar{x}_0) \delta \bar{x}_0 + \dot{\bar{x}}(t) \delta t_1 \quad \text{to the 1st. order}$$

$$\begin{pmatrix} 0 \\ 0 \\ v_{x_1} \\ 0 \end{pmatrix} = \Phi \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta v_{y_0} \end{pmatrix} + \begin{pmatrix} v_{x_1} \\ v_{y_1} \\ \dot{v}_{x_1} \\ \dot{v}_{y_1} \end{pmatrix} \delta t_1$$

$$\left.\begin{aligned} v_{x_1} &= \Phi_{34} \delta v_{y_0} + \dot{v}_{x_1} \delta t_1 \\ 0 &= \Phi_{24} \delta v_{y_0} + v_{y_1} \delta t_1 \end{aligned}\right\} \text{solve for } \delta v_{y_0}$$

$$v_{x_1} v_{y_1} = \Phi_{34}\, \delta v_{y_0} v_{y_1} + \dot{v}_{x_1} \delta t_1\, v_{y_1}$$

substract

$$0 = \Phi_{24}\, \delta v_{y_0} \dot{v}_{x_1} + v_{y_1} \delta t_1\, \dot{v}_{x_1}$$

$$\Rightarrow \quad v_x v_{y_1} = \left(\Phi_{34} v_{y_1} - \Phi_{24} \dot{v}_{x_1}\right) \delta v_{y_0}$$

$$\Rightarrow \quad \delta v_{y_0} = \left(\Phi_{34} - \frac{1}{v_{y_1}} \Phi_{24} \cancel{\dot{v}_{x_1}}\right)^{-1} v_{x_1}$$

Why do they forget this?

Practical problem: How to pick the initial $(x_0, 0, 0, v_{y_0})$ so that the method converges to the Lyapunov orbit?

Solution: As $\mathrm{LinToRot}((0,0,-a_x,0)) =$

$$= (\underbrace{x_L - a_x}_{x_0}, 0, 0, \underbrace{-a_x \nu J}_{v_{y_0}}) = \bar{x}_0 \quad \text{for an } a_x$$

so that $E(\bar{x}_0) = e$. Here Lin is the coordinate system $(\xi, \eta, \zeta_1, \zeta_2)$ of $T^*\mathbb{R}^2$ with respect to the basis $u_1, u_2, \operatorname{Re} w_1, \operatorname{Im} w_1$, where $u_1, u_2 \in \mathbb{R}^4$, $w_1 = \overline{w_2} \in \mathbb{C}^4$ are eigenvectors of the matrix A of the system linearized at L $\dot{x} = Ax$, so that the linearized EOM looks like

$$\begin{cases} \dot{\xi} = \lambda \xi \\ \dot{\eta} = -\lambda \eta \\ \dot{\zeta}_1 = \nu \zeta_2 \\ \dot{\zeta}_2 = -\nu \zeta_1 \end{cases}$$

where $\pm\lambda$ are eigenvalues for u_1, u_2 and $\pm i\nu$ eigenvalues for w_1, w_2

so that the solution looks like

$$\xi(t) = \xi^0 e^{\lambda t}$$
$$\eta(t) = \eta_0 e^{-\lambda t}$$
$$\zeta(t) = \zeta_1(t) + i\zeta_2(t) = \underbrace{= \zeta^0 e^{-i\nu t}}_{\in \mathbb{C}}$$

$\rightarrow$ the periodic orbit $(\eta_0 = 0 = \xi_0)$ goes through $(0, 0, -a_x, 0)$ for some a_x

Problem IS: This initial seed works only for energies slightly above E(L) because of the validity of the linear approximation. When looking for γ_{L_1}, the neck at L_2 is often closed as $E(L_1) < E < E(L_2)$.

Sol1: But for very small μ, $E(L_2)$ is only slightly above $E(L_1)$ and so E can be chosen to be slightly above both! E.g. $\mu = 9.537 \times 10^{-4}$

Sol2: a) Find the Lyapunov orbit $\gamma_L^{E_0}$ for an energy E_0 slightly above E(L) using differential correction with the seed from the linearization

b) Calculate $\Phi_T^{E_0}$ (monodromy matrix $= \Phi(T,0)$ after one period) along $\gamma_L^{E_0}$.

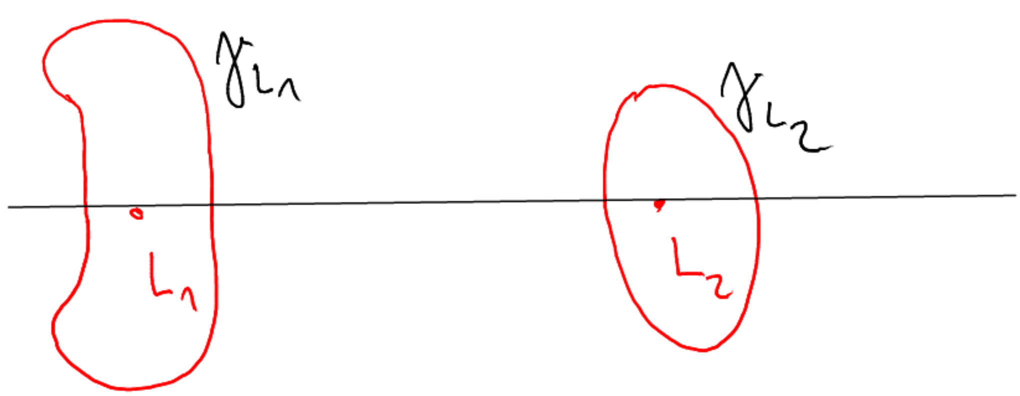
c) Φ_η has eigenvalues $1, 1, \lambda, \frac{1}{\lambda}$ where $\lambda > 1$ (without multiplicities) using the fact it is symplectic.

Pick e.v. w s.t. $\Phi_\eta w = w$ and $w \neq \dot{\gamma}_L^{E_0}(0)$

d) Shift initial condition by $\bar{x}_0' = \bar{x}_0 + \varepsilon w$, where $\varepsilon < 0$ or $\varepsilon > 0$ small, so that $E(\bar{x}_0 + \varepsilon w) > E_0$

e) Use differential correction with the seed $\bar{x}_0'$ to find the Lyapunov orbit with $E(\bar{x}_0') > E_0$

$\Rightarrow$ By this process with convenient E get Lyapunov orbits for all energies, so that both necks are open. This is what we need.



5) How to find stable and unstable manifolds and Poincaré sections numerically?

a) Calculate the monodromy matrix Φ_Π along γ_L,

$$\phi_\Pi = \phi(\bar{x}_0, T) \quad (\text{at } \bar{x}_0)$$

b) Φ_Π has e.v. $1, 1, \lambda, \frac{1}{\lambda}$ where $\lambda > 1$

$\rightarrow$ eigenvectors ψ_0^u of $\lambda > 1$

ψ_0^s of $1/\lambda < 1$

($\dot{\gamma}(0)$ of 1, ψ_0^p of 1 in the direction of the family of periodic orbits)

$\rightarrow$ $\psi^s(t) := \phi(t,0)\psi_0^s$ eigenvectors of $\Phi_\Pi(t)$

$\psi^u(t) := \phi(t,0)\psi_0^u$ (at $\gamma_L(t)$)

along γ

(This holds because $\Phi_\Pi(t) = \phi(t,0)\,\phi_\Pi\,\phi(t,0)^{-1}$)

c) Integrate the initial condition $\gamma_L(t_i) + \varepsilon\, u_s(t_i)$, resp.

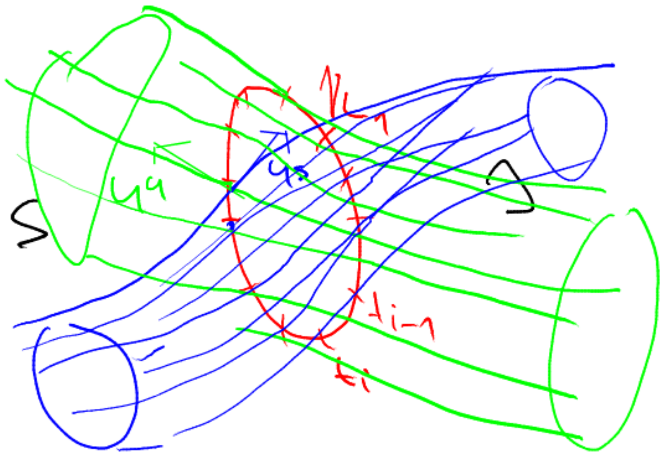
$\gamma_L(t_i) + \varepsilon\, u_u(t_i)$

backwards, resp. forwards in time with $\varepsilon > 0$, $\varepsilon < 0$ small

(but not too small so that it doesn't wind γ_L for too long

time) depending on to which realm we want to go.

Here $0 \le t_1 < \dots < t_n \le T$ are chosen points on γ_L.



6) Program

Mathematica $\boxed{\text{library CR3BP.wl}}$ and

a) The "explorer" – Demonstrate basic concepts for $\mu = 0.3$

b) The "new version" – Can be used for more sophisticated calculations

Ex.: μ = SunJupiter ($= 9.537 \times 10^{-4}$), L_2, $\Delta E = 0.0015$

$\Rightarrow$ transverse 1:1 homoclinic intersection point in U_4

Problem for EarthMoon : $\mu = 1.215 \times 10^{-2}$

Get Lyapunov orbit at L_1 for energy $E > E(L_2)$

$\Rightarrow$ TODO: Implement Lyapunov orbits for higher energies