IBL-Infinity Model of String Topology from Perturbative Chern-Simons Theory

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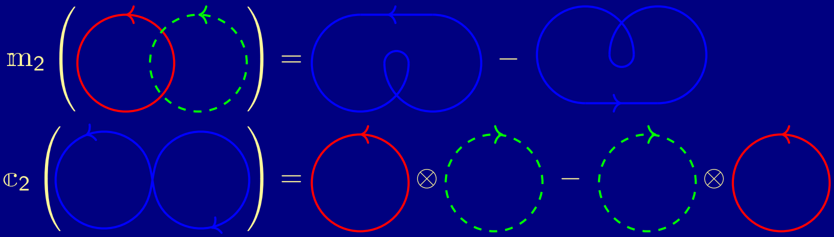
$\arg\min(|DDMM - 2019|)$

Bracket and cobracket in string topology

- Σ oriented surface (Goldman 86', Turaev 91'):



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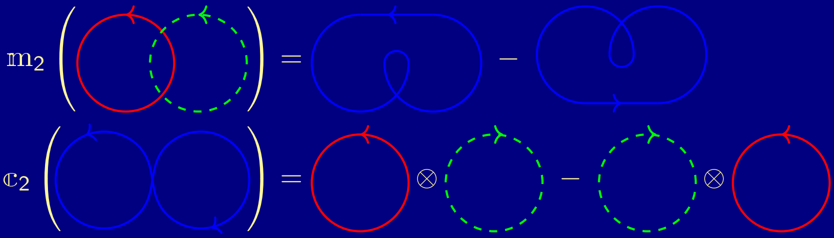

- Chas-Sullivan 99': M^n oriented, $\mathrm{L}M$ loop space

$$
\begin{aligned}
\mathbb{m}_2 :&\quad \mathrm{H}^{\mathbb{S}^1}_i(\mathrm{L}M) \otimes \mathrm{H}^{\mathbb{S}^1}_j(\mathrm{L}M) \longrightarrow \mathrm{H}^{\mathbb{S}^1}_{i+j+2-n}(\mathrm{L}M) \\
\mathbb{c}_2 :&\quad \bar{\mathrm{H}}^{\mathbb{S}^1}_k(\mathrm{L}M) \longrightarrow \bigoplus_{i+j=k+2-n} \bar{\mathrm{H}}^{\mathbb{S}^1}_i(\mathrm{L}M) \otimes \bar{\mathrm{H}}^{\mathbb{S}^1}_j(\mathrm{L}M)
\end{aligned}
$$

IBL-algebra

Theorem (Chas-Sullivan 04')

$(\bar{\mathrm{H}}^{\mathbb{S}^1}(\mathrm{L}M), \mathrm{m}_2, \mathrm{c}_2)$ *is an involutive bi-Lie algebra of degree* $2-n$.

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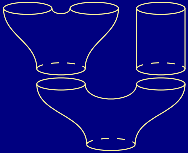
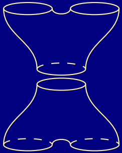
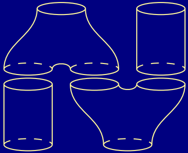
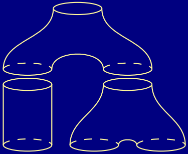
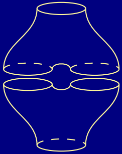
$0 =$ 

$0 =$  $+$ 

$0 =$ 

$0 =$ 

Chain model of string topology

Definition (IBL-infinity chain model)

Strong htpy bi-Lie alg. $(C, \mathfrak{q}_{110}, \mathfrak{q}_{210}, \mathfrak{q}_{120}, \dots)$, whe $(C, \mathfrak{q}_{110}) \sim (C(\mathrm{L}_{\mathbb{S}^1} M), \partial)$ s.t. $(\mathrm{H}(C), \mathfrak{q}_{210}, \mathfrak{q}_{120}) \simeq (\mathrm{H}^{\mathbb{S}^1}(\mathrm{L}M), \mathbb{m}_2, \mathbb{c}_2)$.

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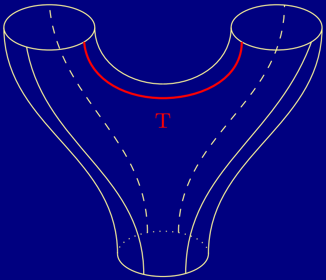
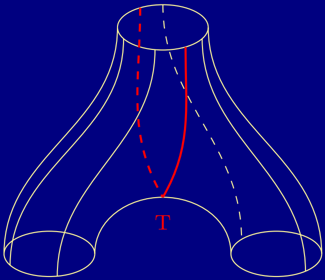
Natural candidate: IBL_∞-chain model on *cyclic Hochschild cochains* $\hat{\mathrm{B}}^*_{\mathrm{cyc}}\Omega$ (Cielebak, Fukaya, Latschev, Volkov 15'–):

- $\mathrm{B}^{\mathrm{cyc}}\Omega \ni \omega_1 \dots \omega_{k-1}\omega_k = \pm \omega_k \omega_1 \dots \omega_{k-1}$
- $\mathrm{b}(\omega_1 \cdots \omega_k) = \sum_c \pm \mathrm{d}(\omega_c) \dots \omega_{c+k} \pm (\omega_c \wedge \omega_{c+1}) \dots \omega_{c+k}$
- $I : \mathrm{B}^{\mathrm{cyc}}\Omega \longrightarrow C^*(\mathrm{L}_{\mathbb{S}^1} M)$ *Chen's iterated integral*

$$\omega_1 \cdots \omega_k \longmapsto \Big(\sigma \mapsto \pm \int_{K_\sigma \times \Delta^k} \omega_1(\sigma(x, t_1)) \cdots \omega_k(\sigma(x, t_k))\Big)$$

Operations $\mathfrak{q}_{210}$ and $\mathfrak{q}_{120}$





$\mathfrak{q}_{\mathbf{210}} : (\mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega})^{\otimes \mathbf{2}} \to \mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega}$ $\qquad$ $\mathfrak{q}_{\mathbf{120}} : \mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega} \to (\mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega})^{\otimes \mathbf{2}}$

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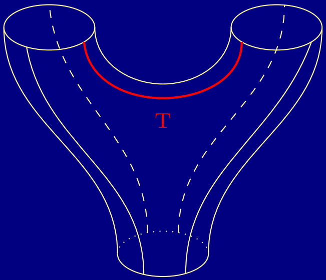

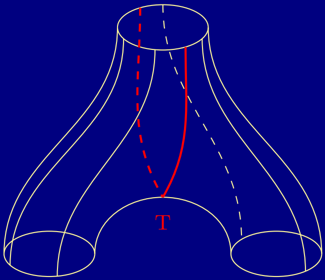

$\mathfrak{q}_{210} : (\mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega})^{\otimes 2} \to \mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega}$ $\qquad$ $\mathfrak{q}_{120} : \mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega} \to (\mathrm{B}^*_{\mathrm{cyc}}\boldsymbol{\Omega})^{\otimes 2}$

$\Longrightarrow$ **Not well-defined** since Schwartz kernel $T = \sum e^i \otimes e_i$ (*identity propagator*) of $\mathbb{1} : \Omega \to \Omega$ w.r.t. $\langle \omega_1, \omega_2 \rangle = \int_M \omega_1 \wedge \omega_2$ is bad. Really bad. Not good at all.

Fields and strings of fields in BV-formalism

Field theory: dg-Frobenius algebra $(\Omega, \mathrm{d}, \wedge, \langle\cdot,\cdot\rangle)$

- $S_{CS}(\omega) = \frac{1}{2}\int_M \omega \wedge \mathrm{d}\omega + \frac{1}{3}\int_M \omega \wedge \omega \wedge \omega$, $S \in \mathcal{F}(\Omega[1])$
  Chern-Simons BV*-action*
- *QME:* $\hbar\, \Delta\, S + \frac{1}{2}\{S, S\} = 0$ for Schwartz's BV-operator
  $\Delta : \mathcal{F}(\Omega[1]) \to \mathcal{F}(\Omega[1])$ for odd symplectic v.s. $\Omega[1]$.

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$\Longrightarrow \langle f\rangle = \int_L f e^S\,\mathrm{d}\omega$ (quantum expectation value) for $f \in \mathcal{F}(\Omega[1])$ closed under $\Delta^S = e^{-S}\,\Delta\,e^S$ (observable) well-defined and independent of adding Δ^S-exact term (regularization) and deforming Lagrangian subspace L (gauge fix).

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$\xrightarrow{\text{cyclization}}$

Theory of strings of fields:

- $S(\omega_1 \dots \omega_k) = \delta_{k,2}\int_M \omega_1 \wedge \mathrm{d}\omega_2 + \delta_{k,3}\int_M \omega_1 \wedge \omega_2 \wedge \omega_3$, $S \in \mathcal{F}(\mathrm{B}^{\mathrm{cyc}}\Omega[1])$ *Hochschild* BV*-action*
- satisfies QME for *string* BV*-op.* $\Delta = \hat{\mathfrak{q}}_{120} + \hbar\hat{\mathfrak{q}}_{210}$.

Path integral and effective theory

Deformation retract: $\mathcal{P} \circlearrowright (X, \mathrm{d}) \underset{\iota}{\overset{\pi}{\rightleftarrows}} (X_1, \mathrm{d}_1)$

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- *(partial) path integral* $Z : \mathcal{F}(X) \to \mathcal{F}(X_1)$ s.t. $Z\,\Delta = \Delta_1\,Z$
  $\sim$ compute Gaussian $Z(f)(x_1) = \int_{X_2} f(x_1, x_2) e^{S_{\text{free}}(x_1,x_2)}\,\mathrm{d}x_2$
  ($\Longrightarrow$ sum of integrals indexed by graphs via Wick's theorem)
- *effective action*

$$W = \log(Z(e^{S_{\text{int}}})) \in \mathcal{F}(X_1)$$

- *(twisted) path integral*

$$Z^S = e^{-W} Z(e^{S_{\text{int}}} \cdot) : \mathcal{F}(X) \to \mathcal{F}(X_1)$$

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$\implies \int_{X_1} Z_S(f) e^W\, \mathrm{d}x_1 = \int_X f e^S\, \mathrm{d}x$ and $Z^S\, \Delta^S = \Delta_1^W\, Z^S$.

Effective action for Chern-Simons theory

- **3D-Chern-Simons** (Cattaneo, Mnev 09'):
  $X = \Omega$, $X_1 = \mathcal{H} \simeq \mathrm{H}_{\mathrm{dR}}$, $\pi = \pi_{\mathcal{H}}$, $\mathcal{P}\omega(x) = \int_M \mathrm{P}(x,y)\omega(y)$
  Hodge homotopy and *propagator*

$$W = \sum I \left(\text{connected non-oriented loop-free multi-graphs with } m \text{ circles, trivalent interior vertices, univalent exterior vertices decorated with elements of } \mathcal{H} \text{ and interior edges decorated with } \mathrm{P} \right) \hbar^m$$

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$$\bar{W} = \sum_{l,g} I \left(+ \text{ cyclic order of half edges at every interior vertex, with } l \text{ boundary components and genus } g\text{, at least one exterior vertex} \right) \hbar^{l+g}$$

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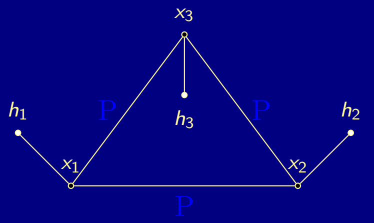
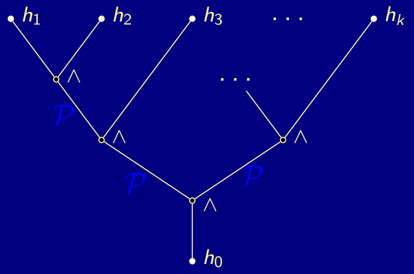
$$f \in \widehat{\mathrm{Sym}}(X_1^*) \simeq \prod \mathrm{Sym}(X_1^{\otimes i *}) \Longleftrightarrow (f_i : X_1^{\otimes i} \to \mathbb{R})$$

$$\bar{W} \simeq (\mathfrak{n}_{lg} : \mathrm{B}^{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}^{\otimes l} \to \mathbb{R})_{l \geq 1, g \geq 0},\ (W_{lg}^{\text{no input}} \in \mathbb{R})\ \textbf{CS-inv.}$$

Feynman integrals



$$\int_{x_1,x_2,x_3} \mathrm{P}(x_1,x_2)\mathrm{P}(x_2,x_3)\mathrm{P}(x_3,x_1)$$
$$h_1(x_1)h_2(x_2)h_3(x_3)$$



$$\langle h_0, \wedge\circ(\mathcal{P}\otimes\mathcal{P})\circ(\wedge\otimes\wedge)\circ(\mathcal{P}\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1})$$
$$\circ(\wedge\otimes\mathbb{1}\otimes\cdots\otimes\mathbb{1})(h_1,h_2,h_3,\ldots,h_k)\rangle$$

Effective Hochschild IBL-infinity chain model

$\Longrightarrow \Delta^{\mathfrak{n}} = \hat{\mathfrak{q}}^{\mathfrak{n}}_{110} + \hbar\hat{\mathfrak{q}}^{\mathfrak{n}}_{210} + \sum_{l\geq 2, g\geq 0} \hat{\mathfrak{q}}^{\mathfrak{n}}_{1lg}\hbar^{l+g-1}$ on $\mathcal{F}(\mathrm{B}^{\mathrm{cyc}}\mathcal{H})$

- $\Delta^{\mathfrak{n}}\,\Delta^{\mathfrak{n}} = 0 \Longleftrightarrow (\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathcal{H}, \mathfrak{q}^{\mathfrak{n}}_{klg})$ *quantum co-*L_∞*-algebra with Drinfeld compatible Lie bracket* $\mathfrak{q}^{\mathfrak{n}}_{210} = \mathfrak{q}_{210}$.

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Recall: *Kontsevich-Soibelman* $(\Omega, \mathrm{d}, \wedge) \xrightarrow{\mathrm{A}_\infty - h.t.} (\mathrm{H}_{\mathrm{dR}}, (m_k))$

- $[(\mathrm{H}_{\mathrm{dR}}, m_k)]_{\mathrm{A}_\infty}$ topological invariant containing rational homotopy theory for 1-connected M.
- $m_k = \sum I(\text{trees})$ depends on $(\Omega, \mathrm{d}, \wedge)$ and $\mathcal{P}$, not P, nor $\langle\cdot,\cdot\rangle$
- $\mathfrak{q}^{\mathfrak{n}}_{110} = \mathrm{b}^*_{\mathrm{A}_\infty}$, $\mathrm{b}_{\mathrm{A}_\infty}(\omega_1\ldots\omega_k) = \sum_{c,i} \pm m_i(\omega_c\ldots\omega_{c+i-1})\ldots\omega_{c+k}$

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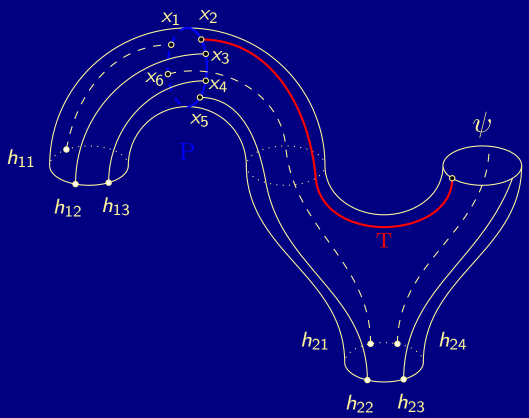
$[(\hat{\mathrm{B}}^*_{\mathrm{cyc}}\mathrm{H}_{\mathrm{dR}}, \mathfrak{q}^{\mathfrak{n}}_{klg})]$ depends on $(\Omega, \mathrm{d}, \wedge, \langle \cdot, \cdot \rangle)$ as *Poincaré DGA* because P is needed to evaluate graphs with circles.

Contribution to $\mathfrak{q}^{\mathfrak{n}}_{120}(\psi) : \mathrm{B}^{\mathrm{cyc}}\mathcal{H} \otimes \mathrm{B}^{\mathrm{cyc}}\mathcal{H} \to \mathbb{R}$



$$\sum_{a,b} \sum_{c=1}^{4} \pm \mathrm{T}^{ab}\psi(e_a h_{2,c+2} h_{2,c+3}) \Big(\int_{x_1 x_2 x_3 x_4 x_5 x_6} \mathrm{P}(x_1,x_2)\mathrm{P}(x_2,x_3)\mathrm{P}(x_3,x_4)$$

$$\mathrm{P}(x_4,x_5)\mathrm{P}(x_5,x_6)\mathrm{P}(x_6,x_1)\big(h_{11}(x_1)h_{12}(x_3)h_{13}(x_4)\big)\big(e_b(x_2)h_{2,c}(x_6)h_{2,c+1}(x_5)\big)\Big)$$

Vanishing result

Theorem (Vanishing for 1-connected manifolds $\neq \mathbb{S}^2$)

$\exists \mathrm{P}$: $\mathrm{H}^1_{\mathrm{dR}} = 0 \Longrightarrow \mathfrak{q}^{\mathfrak{n}}_{110} = \mathrm{b}^*_{\mathrm{A}_\infty}$, $\mathfrak{q}^{\mathfrak{n}}_{210} = \mathfrak{q}_{210}$, $\mathfrak{q}^{\mathfrak{n}}_{120} = \mathfrak{q}_{120}$ *and* $\mathfrak{q}^{\mathfrak{n}}_{klg} = 0$ *otherwise.*

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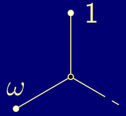
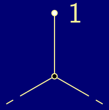
Proof.



$\int_x \mathrm{P}(x,y)\omega(x) = 0$



$\int_x \mathrm{P}(x,y)\mathrm{P}(x,z) = 0$

$\implies$ strong degree restriction $\deg(\omega_i) \geq 2$. $\square$

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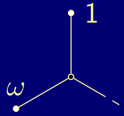

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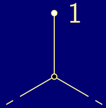

$\int_x \mathrm{P}(x,y)\mathrm{P}(x,z) = 0$

$\Longrightarrow$ strong degree restriction $\deg(\omega_i) \geq 2$. $\square$

- M geometrically formal $\Longrightarrow \mathrm{b}_{\mathrm{A}_\infty} = \mathrm{b}$.

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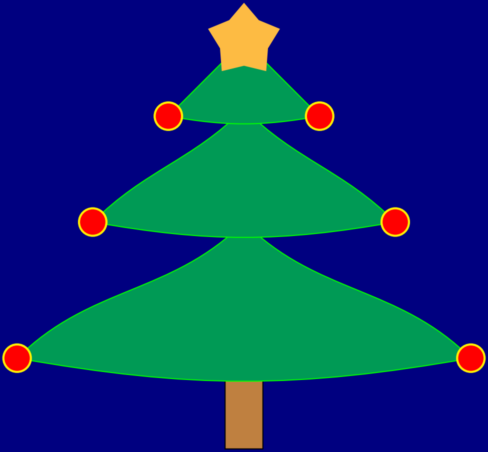
- Observation: $(\Omega, \mathrm{d}, \wedge, \langle\cdot,\cdot\rangle) \overset{\text{whe}}{\sim} (\mathrm{H}_{\mathrm{dR}}, \wedge, \langle\cdot,\cdot\rangle)$ as Poincaré DGA's iff as DGA's $\Longrightarrow$ study Poincaré duality models of Poincaré DGA's and functoriality of dIBL-construction.

Thank you!



Merry Christmas!