MATH237 Lecture 10

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1 Last Class

Recall the defintion of a function being differentiable.

Definition 1.1. A function $f : B(r, \underline{a}) \subset \mathbb{R}^2 \to \mathbb{R}, r > 0$ is differentiable at $\underline{a} = (a, b)$ if the partial derivatives exist at $\underline{a}$ and the linear approximation

$$L_{\underline{a}}(\underline{x}) = f(\underline{a}) + f_x(\underline{a})(x - a) + f_y(\underline{a})(y - b)$$

satisfies

$$\lim_{\underline{x} \to \underline{a}} \frac{|f(\underline{x}) - L_{\underline{a}}(\underline{x})|}{||\underline{x} - \underline{a}||}$$

Intuitively, being differentiable exactly means having a good linear approximation at the point.

2 More on Differentiability

Proposition 1. *Consider $f : B(r, \underline{a}) \subset \mathbb{R}^n \to \mathbb{R}, r > 0$. If all the partial derivatives exist within B and are continuous at $\underline{a}$, then f is differentiable at $\underline{a}$.*

Theorem 2 (Mean Value Theorem on $\mathbb{R}$)**.** *Consider $g[c, d] \to \mathbb{R}$. If f is continuous on $[c, d]$ and differentiable on (c, d) then there is a point x_0 such that*

$$g(c) - g(d) = g'(x_0)(c - d)$$

Proof. Proof of the claim on slides. □

2.1 Example 1

Consider

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin(\frac{1}{\sqrt{x^2+y^2}}) & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

Where is f differentiable?

At $(x,y) \neq (0,0)$: Is f differentiable? (Clicker: Yes or No$\rightarrow$ Yes)
Compute the partial derivatives, and conclude that they are continuous for non 0 points.
At $(x,y) = (0,0)$:

$$\begin{aligned} f_x? : \lim_{h\to 0} \frac{f(h+0,0)-f(0,0)}{h} &= \lim_{h\to 0} \frac{h^2 \sin(\frac{1}{|h|})}{h} \\ &= \lim_{h\to 0} h \sin(\frac{1}{|h|}) \\ &= 0 \end{aligned}$$

Thus $f_x(0,0)$ exists and $f_x(0,0) = 0$. Similarly, $f_y(0,0) = 0$. Thus,

$$L(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0$$

Check differentiability using the definition:

$$\begin{aligned} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} &= \frac{|(x^2+y^2)\sin(\frac{1}{\sqrt{x^2+y^2}}) - 0|}{\sqrt{x^2+y^2}} \\ &= \sqrt{x^2+y^2}|\sin(\frac{1}{\sqrt{x^2+y^2}})| \\ &\leq \sqrt{x^2+y^2} \end{aligned}$$

So

$$\lim_{||\underline{x}-\underline{0}} \frac{|f(x,y)-L(x,y)|}{\sqrt{x^2+y^2}} = 0$$

by the Squeeze Theorem. Therefore, f is differentiable on $\mathbb{R}^2$.

2.1.1 Is f_x continuous at $(0,0)$?

$f_x(x,y) = 2x\sin(\frac{1}{\sqrt{x^2+y^2}}) - \frac{x}{\sqrt{x^2+y^2}}\cos(\frac{1}{\sqrt{x^2+y^2}})$. The limit of the first term is 0 as $\underline{x} \to \underline{0}$. As for the second term, the limit does not exist. So use the definition for niche/edge case points.

2.2 Relationships between stuff

- Differentiability implies existence of partial derivatives AND continuity
- Continuous partial derivatives implies differentiability