MATH237 Lecture 06

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1 Clicker!

Is

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

continuous at $(0,0)$?

Yes! We check if the limit exists at (0,0) and if it does, check if it's 0. We will try $y = mx$.

$$\begin{aligned} \lim_{x\to 0} f(x, mx) &= \lim_{x\to 0} \frac{m^4x^5}{x^2 + m^6x^6} \\ &= \lim_{x\to 0} \frac{m^4x^3}{1 + m^6x^4} \\ &= 0 \end{aligned}$$

we suspect the limit is 0. We prove it

Proof.

$$\begin{aligned} 0 \leq \left| \frac{xy^2}{x^2+y^6} - 0 \right| &\leq \frac{|x|y^4}{2|x||y|^3} \qquad \text{Young's Inequality} \\ &= \frac{1}{2}|y| \end{aligned}$$

Since $\lim_{\underline{x}\to\underline{0}} \frac{1}{2}|y| = 0, \lim_{\underline{x}\to\underline{0}} \frac{xy^4}{x^2+y^6} = 0 = f(0,0)$. Thus the function is continuous at $(0,0)$. □

2 Partial Derivatives

Definition 2.1 (Partial Derivative)**.** Let $f : B(r, \underline{a}) \subset \mathbb{R}^2 \to \mathbb{R}$. The *partial derivative of* $f(x,y)$ *with respect to* x at $\underline{a} = (a,b)$ is (if the limit exists)

$$\frac{\partial f}{\partial x}(a,b) = \lim_{t\to 0} \frac{f(a+t,b) - f(a,b)}{t}$$

and the *partial derivative of* $f(x,y)$ *with respect to* y at $\underline{a} = (a,b)$ is (if the limit exists)

$$\frac{\partial f}{\partial x}(a,b) = \lim_{t \to 0} \frac{f(a,b+t) - f(a,b)}{t}$$

2.1 Notation

- point in $\mathbb{R}^n$ indicated by underscore (class) or bold (text): $\underline{a}$ or **a**
- partial derivative of $f(x,y)$ with respect to x indicated by:

$$\frac{\partial f}{\partial x}, f_x, D_1 f, D_x$$

Definition 2.2. Suppose that $f : B(r, \underline{a}) \subset \mathbb{R}^n \to \mathbb{R}$ has partial dervatives $\underline{a}$. The *gradient* of f at $\underline{a}$ is

$$\nabla f(\underline{a}) = (f_{x_1}(\underline{a}), \ldots, f_{x_n}(\underline{a}))$$

2.2 Example 1

Let

$$f(x,y) = x^2 + xy^3$$

Then we have

$$\begin{aligned} \lim_{t \to 0} \frac{f(x+t), y - f(x,y)}{t} &= \lim_{t \to 0} \frac{(x+t)^2 + (x+t)y^3 - x^2 - xy^3}{t} \\ &= \lim_{t \to 0} \frac{2xt + t^2 + ty^3}{t} \\ &= 2x + y^3 \end{aligned}$$

Thus $\frac{\partial f}{\partial x}(x,y) = 2x + y^3$. Or,
We can also use regular rules if we treat y is constant.

2.3 Example 2

Let

$$f(x,y) = \sqrt{x^2+y^2} = (x^2+y^2)^{1/2}, (x,y)$$

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{x}{\sqrt{x^2+y^2}} \\ \frac{\partial f}{\partial y} &= \frac{y}{\sqrt{x^2+y^2}} \end{aligned}$$

at $(x, y) \neq (0,0)$. For $(x, y) = (0,0)$, we must use the definition and check if the limit even exists:

$$\begin{aligned}\lim_{t\to 0} \frac{f(0+t,0)-f(0,0)}{t} &= \lim_{t\to 0} \frac{\sqrt{t^2}-0}{t} \\ &= \lim_{t\to 0} \frac{|t|}{t}\end{aligned}$$

The limit does not exist so $\frac{\partial f}{\partial x}(0,0)$ doesn't exist. Similarly, $\frac{\partial f}{\partial y}(0,0)$ doesn't exist. f is continuous, so this example shows that continuity does not imply existence of partial derivative.

2.4 Example 3

Let

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Then for $(x, y) \neq (0,0)$:

$$\frac{\partial f}{\partial x} = \frac{y(x^2+y^2) - xy2x}{(x^2+y^2)^2}$$

At $(x, y) = (0,0)$:

$$\begin{aligned}\lim_{t\to 0} \frac{f(t+0,0)-f(0,0)}{t} &= \lim_{t\to 0} \frac{0-0}{t} \\ &= 0\end{aligned}$$

So $\frac{\partial f}{\partial x}(0,0) = 0$. Similarly, by symmetry, $\frac{\partial f}{\partial y}(0,0) = 0$. It can be shown that $\lim_{\underline{x}\to\underline{0}} f(\underline{x})$ doesn't exist, so the function is not continuous. This example shows that even if all the partial derivatives exist, the function may not be continuous.

2.5 Remarks

- Calculate a partial derivative just as you would for single variable calculus: all other variables are held constant
- Examples illustrate that:
  - A continuous function may fail to have partial derivatives
  - A discontinuous function may have partial derivatives