MATH237 Lecture 05

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Recall that a function is continuous at $\underline{a}$ if $\lim_{\underline{x}\to\underline{a}} f(\underline{x}) = f(\underline{a})$. We also had the sum, quotient, and product continuity theorems.

Using the continuity theorems to show a function is continuous

Example 1

Show $e^{xy}\ln(x^2+y^2)$ is continuous for all $(x,y) \neq (0,0)$.

Proof. The functions x, y are continuous, as is x^2, y^2, so xy is continuous by the Product Continuity Theorem and x^2+y^2 is continuous by the Sum Continuity Theorem. Composition continuity theorem implies e^{xy} and $\ln(x^2+y^2)$ are continuous as well. By the Product Continuity Theorem, the final function is continuous. $\square$

Usually, the proof does not need to be in this detail.

Example 2

Prove that

$$f(x,y) = \begin{cases} \frac{\sin(x^2+y^2)}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$$

is continuous on $\mathbb{R}^2$. Hint:

$$\operatorname{sinc}(v) = \begin{cases} \frac{\sin(v)}{v} & v \neq 0 \\ 1 & v = 0 \end{cases}$$

is continuous for all $v \in \mathbb{R}$.

Proof. By the sum, quotient, and composition continuity theorems, $\frac{\sin(x^2+y^2)}{x^2+y^2}$ is continuous for $(x,y) \neq (0,0)$. Since $f(x,y) = \operatorname{sinc}(x^2+y^2)$, by the composition continuity theorem, f is continuous on $\mathbb{R}^2$. $\square$

Example 3

$$f(x,y) = \frac{\sin(xy)}{x^2+y^2}, (x,y) \neq (0,0)$$

f is continuous at all $(x,y) \in (0,0$ by the Continuity Theorems (exercise). Can $f(0,0)$ be defined so f is continuous at 0? No, since the limit does not exist.

The limit does not exist. $\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{0}{y^2} = 0$ but $\lim_{x\to 0} f(x,x) = \lim_{x\to 0} \frac{\sin(x^2)}{2x^2} = \frac{1}{2}$ □

Example 4

$$f(x,y) = \begin{cases} \frac{x^2y}{x^2+y^2} & (x,y) \neq (0,0) \end{cases}$$

Again, Continuity Theorems imply f is continuous for all $(x,y) \neq (0,0)$. Can we define $f(0,0)$ so its continuous at $(0,0)$? Since the limit along $y = mx$ is 0, we suspect the limit exists and is 0. We prove that with the Squeeze Theorem.

Proof.

$$\begin{aligned} 0 \leq |\frac{x^2y}{x^2+y^2} - 0| &\leq |\frac{x^2y}{x^2}| \\ &= |y| \end{aligned}$$

Since $\lim_{\underline{x}\to\underline{0}} |y| = 0$, the limit is indeed 0. □

Example 5 (Slide 54)

Can $f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ be defined at $\underline{0}$ so that it is continuous on all $\mathbb{R}^2$? Yes, define it to be 0. Is it currently continuous at (0,0)? No, nothing is defined there atm (automatic teller machine).
What if $f(0,0)$ is defined to be 5? Still not continuous lol.

Example 6

$$f(x,y) = \begin{cases} \frac{xy^4}{x^2+y^6} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

We check if the limit exists at (0,0) and if it does, check if it's 0. We will try $y = mx$.

$$\begin{aligned} \lim_{x\to 0} f(x,mx) &= \lim_{x\to 0} \frac{m^4x^5}{x^2+m^6x^6} \\ &= \lim_{x\to 0} \frac{m^4x^3}{1+m^6x^4} \\ &= 0 \end{aligned}$$

we suspect the limit is 0. We prove it

Proof.

$$0 \leq \left| \right|$$

$\square$