MATH237 Lecture 04

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January 13 2020

1 Last Class

Recall the definition of a limit.

Definition 1.1 (Neighbourhood)**.** The neighbourhood of a point a is a ball $B(r, a), r > 0$.

Definition 1.2 (Limit)**.**

1.1 Showing a limit doesn't exist

Calculate the limit along the lines $y = mx$ - If different values are obtained along different lines, the limit does not exist.
If a single value of L is obtained, the limit might exist. If it does, then it's L.

1.2 Showing a limit exists

- Limit theorems
- Squeeze theorem

1.3 Exercise

Show that

$$\lim_{\underline{x} \to \underline{0}} \frac{xy}{\sqrt{x^2 + y^2}}$$

exists. What is the limit?
Question: What technique should be used? It should be the Squeeze theorem.
What is the limit? - 0
Proof that the limit is 0:

$$\begin{aligned} 0 \leq \left|\frac{xy}{\sqrt{x^2 + y^2}} - 0\right| &\leq \frac{|x||y|}{\sqrt{x^2}} \\ &= |y| \end{aligned}$$

Since $\lim_{\underline{x} \to \underline{0}} |y| = 0$, by the Squeeze theorem $\lim_{\underline{x} \to \underline{0}} \frac{xy}{\sqrt{x^2+y^2}} = 0$

2 Continuous Functions

Definition 2.1. A function $f : \mathbb{R}^n \to \mathbb{R}$ is continuous at $\underline{a}$ if

$$\lim_{\underline{x} \to \underline{a}} f(\underline{x}) = f(\underline{a}).$$

There are 3 parts to the definition:

- f is defined at $\underline{a}$
- $\lim_{\underline{x} \to \underline{a}} f(x)$ exists
- $\lim_{\underline{x} \to \underline{a}} f(x) = f(\underline{a})$

Let $f : D(f) \subset \mathbb{R}^n \to \mathbb{R}, g : D(g) \subset \mathbb{R}^n \to \mathbb{R}$, then define for $\underline{x} \in D(f) \cap D(g)$,

- sum - $(f + g)(\underline{x}) = f(\underline{x}) + g(\underline{x})$
- product - $(fg)(\underline{x}) = f(\underline{x})g(\underline{x})$
- quotient - $(f/g)(\underline{x}) = f(\underline{x})/g(\underline{x})$
- composition - $(g \circ f)(\underline{x}) = g(f(\underline{x}))$

Continuity theorems: Consider $f : B(r, \underline{a}) \subset \mathbb{R}^n \to \mathbb{R}$ and $g : B(r, \underline{a}) \subset \mathbb{R}^n \to \mathbb{R}$ for some $r > 0$. If f and g are continuous at $\underline{a}$,

- $f + g$ is continuous at $\underline{a}$
- fg is continuous at $\underline{a}$.
- if $g(\underline{a}) \neq 0$, f/g is continuous at $\underline{a}$.
- If $f : \mathbb{R}^n \to \mathbb{R}$ at $\underline{a}$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous at $f(\underline{a})$ then $(g \circ f)$ is continuous at $\underline{a}$.