MATH237 Lecture 07

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1 Last Class

Clicker!: What are the partial derivatives of $f(x, y) = e^{x^2 y}\cos(x) + y$?

1.1 Second Partial Derivatives

The partial derivatives of the (first) partial derivatives of f are called the <u>second partial derivatives</u> of f. Notation:

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = f_{xx} = D_x^2 f$$

1.2 Example 1

$$f(x, y) = x^2 + xy^3$$

$$\frac{\partial f}{\partial x}(x, y) = 2x + y^3, \frac{\partial f}{\partial y}(x, y) = 3xy^2$$

$$\frac{\partial^2 f}{\partial x^2}(x, y) = 2, \frac{\partial f^2}{\partial y \partial x}(x, y) = 3y^2$$

$$\frac{\partial^2 f}{\partial x \partial y}(x, y) = 3y^2$$

Consider $f : B(r, \underline{a}) \subset \mathbb{R}^2 \to \mathbb{R}$. If f_{xy} and f_{yx} are defined on $B(r, \underline{a})$ and continuous at $\underline{a}$, then $f_{ya}(\underline{a}) = f_{xy}(\underline{a})$

2 Linear Approximation

For functions $f : \mathbb{R} \to \mathbb{R}$, the tangent line

$$L_a(x) = f(a) + f'(a)(x - a)$$

can be used to approximate f when $f'(a)$ is defined.

For $f : \mathbb{R}^2 \to \mathbb{R}$, a linear approximation at (a, b) would have the form

$$L_{(a,b)}(x, y) + A + B(x - a) + C(y - b)$$

What should our constants be?

$$L_{(a,b)}(a, b) = f(a, b) \implies A = f(a, b)$$

$$\begin{aligned}\lim_{t \to \underline{a}} \frac{L_{(a,b)}(a + t, b) - L_{(a,b)}(a, b)}{t} &= \lim_{t \to \underline{a}} \frac{A + Bt + 0 - A}{t} \\ &= B \implies B = f_x(a, b)\end{aligned}$$

Similarly, $C = f_y(a, b)$.
Thus,

Definition 2.1. Consider $f : B(r, \underline{a}) \subset \mathbb{R}^2 \to \mathbb{R}$ and assume all partial derivatives of f exist at $\underline{a}$. The **linearization** or **lienar approximation** of f at $\underline{a}$ implies

$$L_{\underline{a}}(x) = f(\underline{a}) + \nabla f(a) \cdot (\underline{x} - \underline{a})$$

2.1 Increment form of linear approximation

$$f(x, y) \approx L_{(a,b)}(x, y)$$

$$\begin{aligned}L_{(a,b)}(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ L_{(a,b)}(x, y) - f(a, b) &= f_x(a, b)(x - a) + f_y(a, b)(y - b)\end{aligned}$$

2.2 Example 1

Consider

$$f(x, y) = x^2 + y^2$$

Calculate $L(x, y)$ at $(1, 2) : L_{(1,2)}(x, y)$.

$$f(1, 2) = 1 + 4 = 5$$

$$\begin{aligned}f_x(x, y) &= 2x, f_x(1, 2) = 2 \\ f_y(x, y) &= 2y, f_y(1, 2) = 4\end{aligned}$$

$$\begin{aligned}L(x, y) &= f(1, 2) + f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2) \\ &= 5 + 2(x - 1) + 4(y - 2)\end{aligned}$$

Comparision:

point	L	f
(1.1, 2.1)	5.6	5.62
(.9, 2)	4.8	4.81
(1, 1.9)		

2.3 Example 2

$$f(x,y) = \sqrt{|xy|}$$

Use linear approximation to approximate f near $(0,0)$.

$$f(0,0) = 0$$

$$f_x : \frac{\partial}{\partial x}(|xy|^{1/2}) = \frac{1}{2}|xy|^{-1/2}\frac{\partial}{\partial x}|xy| \text{ Not valid!}$$

$$\lim_{t\to 0}\frac{f(t+0,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0$$

$$\text{Similarly, } \frac{\partial f}{\partial y}(0,0) = 0$$

So, $L(x,y) = 0$.

2.4 Example 3

$$f(x,y) = |xy|$$

Calculate $L(x,y)$ at $(0,0)$.

$$f(0,0) = 0$$

$$f_x(0,0) = \lim_{t\to 0}\frac{f(t+0,0)-f(0,0)}{t} = \lim_{t\to 0}\frac{0-0}{t} = 0$$

$$f_y(0,0) = 0 \text{ by similar analysis}$$

$$L_{(0,0)}(x,y) = f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 0$$

Comparision:

point	L	f
(.1, -.1)	0	.01
(.01, .1)	0	.001

2.5 Example 4

Same f as before but at $(1,2)$.

$$
\begin{aligned}
f(0,0) &= 2 \\
f_x : f(x,y) &= xy \qquad \text{near } (1,2) \implies f_x(1,2) = 2 \\
f_y : f(x,y) &= x \implies f_y(1,2) = 1 \\
L_{(1,2)}(x,y) &= 2 + 2(x-1) + (y-2)
\end{aligned}
$$