eg. no. of accidents in a week
no. of printing mistake on page of a book

Poisson Distribution

Poisson Distribution - discovered by French Mathematician Poisson.

Poisson distribution is the limiting case of binomial distribution under the following conditions.

(i) n, the number of trials is very infinitely large. $(n \to \infty)$

(ii) probability of occurance (or success) in each trial is very small (i.e $p \to 0$)

(iii) The average of success $\lambda = np$ is finite.

* A discrete random variable X is said to follow Poisson Distribution if its probability mass function is

$$P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!} \quad ; \quad x = 0, 1, 2, \ldots$$

19) Derive the p.d.f of Poisson distribution as a limiting form of B.D and hence find the mean.

Proof:-

Consider $P(x) = {}^{n}C_{x}\, p^{x} q^{n-x}$

$$= {}^{n}C_{x} \left(\frac{p}{q}\right)^{x} \cdot q^{n} = {}^{n}C_{x}\left(\frac{p}{1-p}\right)^{x}(1-p)^{n}$$

Let $np = \lambda$

$\therefore$ put $p = \frac{\lambda}{n}$

$$= {}^{n}C_{x} \frac{(\lambda/n)^{x}}{(1-\lambda/n)^{x}} \left(1-\frac{\lambda}{n}\right)^{n}$$

$$= \frac{n!}{(n-x)!\,x!} \frac{(\lambda/n)^{x}}{(1-\lambda/n)^{x}} \left(1-\frac{\lambda}{n}\right)^{n}$$

$$P(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)\cancel{(n-x)!}}{\cancel{(n-x)!}\, x!} \frac{(\lambda/n)^x \left(1-\frac{\lambda}{n}\right)^n}{\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\cancel{n^x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\frac{\lambda^x}{\cancel{n^x}}\left(1-\frac{\lambda}{n}\right)^n}{x!\left(1-\frac{\lambda}{n}\right)^x}$$

$$= \frac{\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right)\lambda^x\left(1-\frac{\lambda}{n}\right)^n}{x!\left(1-\frac{\lambda}{n}\right)^x}$$

Since $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^n = e^{-\lambda}$

and $\lim_{n\to\infty}\left(1-\frac{\lambda}{n}\right)^x = 1$

Taking limit of b.s as $n\to\infty$.

$$\boxed{p(x) = \frac{\lambda^x e^{-\lambda}}{x!}}$$

Mean =

$$\mu_1' = E(x) = \sum p_i x_i$$

$$= \sum_{x=0} \frac{e^{-\lambda}\lambda^x}{x!} x$$

$$= \sum_{x=1} \frac{e^{-\lambda}\lambda^x}{(x-1)!}$$

$$= \lambda e^{-\lambda} \sum_{x=1} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\cdots\right)$$

$$= \lambda e^{\lambda}$$

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(2) In a certain factory turning out blades there is a small chance $\frac{1}{500}$ for any blade to be defective. The blades are supplied in packets of 10 each. Use poisson distribution to calculate the approximate number of packets containing no defective, one defective, two defective blades in a consignment of 10,000 packets.

Soln We have $p = \frac{1}{500}$ & $n = 10$ so that

$$\lambda = np = \frac{10}{500} = 0.02$$

Let X be the number of defectives. Then X has poisson distribution.

The number of packets out of N = 10,000 in which X = x are defective in

$$N\,P(X=x) = \frac{N e^{-\lambda} \lambda^x}{x!}$$

$$= 10000\,\frac{e^{-0.02}(0.02)^x}{x!}$$

$$= \frac{9802\,(0.02)^x}{x!}$$

∴ The required nos. are

$N\,P(X=0) = 9802$

$N\,P(X=1) = 196$

$N\,P(X=2) = 2$

~~$N\,P(X=3) = 0$~~.

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~~Using Poisson distribution~~

(22) Find the approximate value of
$^{300}C_2 (0.02)^2 (0.98)^{298} + {}^{300}C_3 (0.02)^3 (0.98)^{297}$

Sol^n :- Clearly the above probabilities are the probabilities of B.D.
comparing with $P(X=x) = {}^nC_x p^x q^{n-x}$.

$n = 300, \; p = 0.02, \; q = 0.98, \; x = 2$ and 3.

$\lambda = np = 300(0.02) = 6$

Using poisson distribution

$$P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-6}.6^x}{x!}$$

We can approximate the given quantity as.
$P(X=2) + P(X=3)$.

$$= \frac{e^{-6}6^2}{2!} + \frac{e^{-6}6^3}{3!}$$

$$= e^{-6}\left[\frac{36}{2} + \frac{216}{6}\right]$$

$$= e^{-6}[18 + 36]$$
$$= e^{-6}(54)$$
$$= 0.1338$$

20) In sampling a large number of parts manufactured by a machine the ~~nth~~ mean number of defectives in a sample of 20 is 2. Out of 100 such samples, how many would you expect to contain 3 defectives ~~digits~~ using
(i) Binomial Distribution
(ii) Poisson Distribution.

Sol^n Let X be the no. of defectives.

Mean = np.

$2 = 20p$.

$\therefore p = \frac{2}{20} = \frac{1}{10} = 0.1$

By Binomial Dist

$P(X=x) = {}^{n}C_{x} p^{x} q^{n-x}$.

$N P(X=3) = 100 \; {}^{20}C_{3} (0.1)^{3} (0.9)^{17}$

$= 100 (1140) (0.1)^{3} (0.9)^{17}$.

$= 19.011$

$\approx \underline{19}$

By Poission Dist,

$P(X=x) = \frac{e^{-\lambda} \lambda^{x}}{x!}$

$\lambda = np \Rightarrow 2.$

$\therefore N P(X=3) = \left[\frac{e^{-2} (2)^{3}}{3!}\right] (100)$

$= 18.044$

$\approx \underline{\underline{18}}$

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(21) An insurance company found that only 0.01% of the population is involved in a certain type of accident each year. If its 1000 policy holders were randomly selected from the population, what is the probability that no more than two of its clients are involved in such accident next year?

Sol^n We have $p = \frac{0.01}{100} = 0.0001$, $n = 1000$

$\lambda = np = 1000 \times 0.0001 = 0.1$

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} = \frac{e^{-0.1} (0.1)^x}{x!}$$

$$P(X \leq 2) = P(X=0) + P(X=1) + P(X=2)$$

$$\therefore P(X \leq 2) = e^{-0.1}\left[\frac{(0.1)^0}{0!} + \frac{(0.1)^1}{1!} + \frac{(0.1)^2}{2!}\right]$$

$$= \underline{\underline{0.9998}}$$

* Properties of Poisson Distribution

Mean $= \lambda$
Variance $= \lambda$
Mgf $= e^{\lambda(e^t - 1)}$
$P(X = x+1) = \frac{\lambda}{x+1} P(X = x)$

(28) If the variance of the poisson distribution is 3, find the probability that $P(X=2)$ and $P(X \geq 4)$.

Sol^n $P(X=x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$ $x = 0, 1, 2, \dots$

$\therefore P(X=2) = \frac{e^{-3}(3)^2}{2!} = 0.224$

$P(X \geq 4) = 1 - [P(X=0) + P(X=1) + P(X=2) + P(X=3)]$

$= 1 - \left[\frac{e^{-3}(3)^0}{0!} + \frac{e^{-3}(3)^1}{1!} + \frac{e^{-3}(3)^2}{2!} + \frac{e^{-3}(3)^3}{3!}\right]$

$= 1 - [0.04978 + 0.1494 + 0.224 + \cancel{0.0747}\ 0.224]$

$= 1 - 0.6472$

$= 0.35378$

29) If a r.v. X follows Poisson Distribution such that $P(X=1) = 2P(X=2)$ then find the mean & variance of dist. Also find $P(X=3)$.

Sol^n $\because P(X=1) = 2P(X=2)$ where $X \sim P.D.$

$\frac{\cancel{e^{-\lambda}}\cancel{\lambda^1}}{\cancel{1!}} = \cancel{2}\frac{\cancel{e^{-\lambda}}\lambda^2}{\cancel{2!}}$

$\lambda = 1$

Mean and variance $= \lambda = 1$

$\therefore P(X=3) = \frac{e^{-\lambda}\lambda^3}{3!} = \frac{e^{-1}(1)^3}{3!} = 0.0613$

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30) Fit a Poisson distribution to the following data.

X	0	1	2	3	4	Total
F	123	59	14	3	1	200.

Sol^n Fitting a Poisson distribution means finding the expected frequencies of $X = 0, 1, 2, 3, 4$.

$$\text{Mean} = \lambda = \frac{\sum x_i f_i}{N} = \frac{100}{200} = 0.5.$$

$$\therefore \quad P(X=x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-0.5}(0.5)^x}{x!} \quad \cdots (1).$$

$$\text{Expected freq.} = N\,P(X=x).$$

$$= 200\left[\frac{e^{-0.5}(0.5)^x}{x!}\right]$$

Putting $x = 0, 1, 2, 3, 4$.

$x = 0$. , $N\,P(x=0) = 200 \times 0.6065 = 121.30$
≈ 121

$121, 61, 15, 2, 1$.

Or:

$\cancel{f(x+1) = \frac{\lambda}{x+1} f(x)}$ $P(X=x+1) = \frac{\lambda}{x+1} P(X=x)$

$$P(X=1) = \frac{0.5}{0+1}(0.6065) = 0.151625.$$

$$N\,P(X=1) = 60.65 \approx 61$$

$$P(X=2) = P(X=1+1) = \frac{0.5}{1+1}(61) = 15$$