Standard Distributions

Binomial Distribution

Discovered by James Bernoulli
B.D expresses probabilities of events which results only two ways, success or failure.

Consider an experiment which results in either success or failure. Let it be repeated n times; the probability p of success remaining constant everytime and let $q = 1-p$ the probability of failure.

$\therefore$ The probability of x successes and hence $(n-x)$ failures in a trail in a particular order, SSS ... (x times) and FFF ... $(n-x)$ times is given by

$$= pp \ldots (x \text{ times}) \times qq \ldots (n-x) \text{ times}$$
$$= p^x q^{n-x}$$

But x successes can occur in ^nC_x ways

$$\therefore \boxed{p(x) = {^nC_x}\, p^x q^{n-x}}$$

Def^n The number of successes X in n independent trails, in each of which the probability of success p (and probability of failure is $q = 1-p$) is constant, is called binomial random variable or binomial variate and its probability mass function is given by

$$P(X=x) = {^nC_x}\, p^x q^{n-x} \; ; \; x = 0, 1, 2, \ldots n$$

This is called binomial distribution because for $x = 0, 1, 2, \ldots, n$ this gives gives the successive terms of binomial expansion $(q+p)^n = \sum_{x=0}^{n} {}^nC_x p^x q^{n-x}$

If there are N sets of n trials each then the number of x successes occur in $N \cdot P(X=x) = N \, {}^nC_x p^n q^{n-x}$ sets

i.e. $x = 0, 1, 2, \ldots, n$ successes are given by the successive terms of $N(q+p)^n$.

* Mean of Binomial Distribution $= np$

* Variance $= npq$

* Problems: (Determine the B.D with mean 2 and Var $\frac{4}{3}$)

① If X is binomially distributed with mean 2 and variance $\frac{4}{3}$. Find the probability distribution of X.

Sol^n $E(x) = np = 2$.

$Var(x) = npq = \frac{4}{3}$

$\therefore \cancel{\frac{np}{npq} = \frac{1}{q}}$

$\frac{np}{npq} = \frac{2}{\frac{4}{3}}$

$\therefore \frac{1}{q} = \frac{3}{2}$

$\therefore \boxed{q = \frac{2}{3}}$. $\therefore p = 1 - q = 1 - \frac{2}{3} = \frac{1}{3}$

$\therefore \boxed{p = \frac{1}{3}}$.

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Also $np = 2$

$\therefore n\left(\frac{1}{3}\right) = 2$

$\therefore \boxed{n = 6}$

$P(X=x) = {}^{n}C_x\, p^x q^{n-x}$

$P(X=x) = {}^{6}C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

putting $x = 0, 1, 2, 3, 4, 5, 6$ we get

x	0	1	2	3	4	5	6
$P(X=x)$	$\frac{64}{729}$	$\frac{192}{729}$	$\frac{240}{729}$	$\frac{160}{729}$	$\frac{60}{729}$	$\frac{12}{729}$	$\frac{1}{729}$

Q. What is the expectation of heads if an unbaised coin is tossed 12 times?

Sol^n Let X be the no. of head obtained

X follows B.D with parameters

$n = 12,\ p = \frac{1}{2}$

$\therefore E(X) = np = 12\left(\frac{1}{2}\right) = 6$

$\boxed{E(X) = 6}$

① Find the probability that in tossing a fair coin 3 times, there will appear
(i) 3 heads (ii) 2 tails and 1 head
(iii) Atleast one head
(iv) Not more than 1 tail.

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Sol^4 Let X be the d.r.v showing no. of heads.
Then X has binomial distribution with $P(X=x) = {}^nC_x\, p^x q^{n-x}$

$$P(x=x) = {}^3C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{3-x}$$

$$= {}^3C_x \left(\frac{1}{2}\right)^3$$

$$P(X=x) = \frac{1}{8}\,{}^3C_x$$

(i) $P(X=3) = \frac{1}{8}\,{}^3C_3 = \frac{1}{8}$

(ii) $P(X=1) = \frac{1}{8}\,{}^3C_1 = \frac{3}{8}$

(iii) P(atleast one head)

$= P(X=1) + P(X=2) + P(X=3)$

$= \frac{1}{8}\,{}^3C_1 + \frac{1}{8}\,{}^3C_2 + \frac{1}{8}\,{}^3C_3$

$= \frac{3}{8} + \frac{3}{8} + \frac{1}{8} = \frac{7}{8}$

(iv) P(Not more the one tail)
ie either 0 tail (3 Heads) or 1 tail (2 Heads)

$= P(X=2) + P(X=3)$

$= \frac{1}{8}\,{}^3C_2 + \frac{1}{8}\,{}^3C_3$

$= \frac{3}{8} + \frac{1}{8} = \frac{4}{8} = \frac{1}{2}$

2) Out of 2000 families with 4 children each how many would you expect to have (i) Atleast one boy
(ii) 2 Boys
(iii) 1 or 2 girls.
(iv) At the most 2 girls.
(V) No girls
Assume equal probabilities for boys and girls.

Sol^n Let X be the no. of boys
then X has Binomial distribution with

$P(X=x) = {}^nC_x p^x q^{n-x}$

$= {}^4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

$= \left(\frac{1}{2}\right)^4 {}^4C_x$

$P(X=x) = \frac{1}{16} {}^4C_x$

(i) $NP(X=0) = 2000 \cdot \frac{1}{16} {}^4C_0 = 125$ is the number of families with no boys.
$2000 - 125 = \underline{1875}$ families have atleast one boy.

(iii) 1 girl (3 boys) and or 2 girls (2 boys)

$N[P(X=3) + P(X=2)] = 2000\left[\frac{1}{16} {}^4C_3 + \frac{1}{16} {}^4C_2\right]$

$= 2000\left[\frac{4}{16} + \frac{6}{16}\right]$

$= \underline{1250}$

(ii) $P(X=2) = 2000 \frac{1}{16} {}^4C_2 = \frac{2000}{16} 6 = \underline{750}$

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(iv) N P(At most 2 girls)

$= 2000[P(X=2) + P(X=3) + P(X=4)]$

$= \frac{2000}{16}[{}^4C_2 + {}^4C_3 + {}^4C_4]$

$= \frac{2000}{16}(6+4+1)$

$= \frac{2000 \times 11}{16} = 125 \times 11 = \underline{\underline{1375}}$

(v) P(No girls) $= P(X=4)$

$= 2000 \frac{1}{16} {}^4C_4$

$= \underline{\underline{125 \text{ families}}}$

Q3) An irregular six-sided die is thrown. The expectation that in 10 throws it will give 5 even numbers is twice the expectation it will give 4 even numbers. How many times in 10,000 sets of 10 throws would you expect to get no even numbers?

Solⁿ Let X be the number of even numbers turning up.

$\therefore$ X has binomial distribution with

$P(X=x) = {}^nC_x p^x q^{n-x}$

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04) In a precision bombing attack there is a 50% chance that any one bomb will strike the target. Two direct hits are required to destroy the target completely. How many bombs must be dropped to give a 99% chance or better of completely destroying the target.

Sol^n Let X be a d.r.v representing the number of bombs striking the target (ie. success)

$\therefore$ X has B.D with
$$P(X=x) = {}^nC_x \, p^x q^{n-x}.$$
where n is the number of bombs dropped and p is the success of each bomb hitting the target given as $\frac{1}{2}$

Given $P(X \geq 2) \geq 0.99$

$$1 - P(X<2) \geq 0.99$$

$$1 - [P(X=0) + P(X=1)] \geq 0.99$$

$$1 - \left[{}^nC_0 \left(\tfrac{1}{2}\right)^0 \left(\tfrac{1}{2}\right)^n + {}^nC_1 \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right)^{n-1}\right] \geq 0.99.$$

$$1 - \left[\frac{1}{2^n} + \frac{n}{2^n}\right] \geq 0.99.$$

$$1 - 0.99 \geq \frac{1+n}{2^n}$$

$$0.01 \geq \frac{n+1}{2^n}$$

$$\therefore \frac{2^n}{100} \geq n+1$$

This is found to be true for n = 11 by trial & error.

$\therefore$ 11 bombs are required.

05) A communication system consists of n components, each of which will independently function with probability p. The total system will be able to operate effectively if atleast one half of its components function. For what values of p is a 5-component system more likely to operate effectively than a 3-component system.

Sol^n We have a B.D with parameters n & p.

$P(X=x) = {}^nC_x\, p^x q^{n-x}$; $x = 0,1,2,\cdots,n$.

P(5 components system will work effectively) = P(x = 3 or 4 or 5)

$= P(X=3) + P(X=4) + P(X=5)$

$= \sum_{x=3}^{5} {}^5C_x\, p^x q^{5-x}$ $\quad \cdots (\because n=5)$

P(3-components system will work effectively) = P(x=2 or 3)

$= P(X=2) + P(X=3)$

$= \sum_{x=2}^{3} {}^3C_x\, p^x q^{3-x}$ $\quad (\because n=3)$

5-components system will work effectively than 3-components system if

$$\sum_{x=3}^{5} {}^5C_x\, p^x q^{5-x} \geq \sum_{x=2}^{3} {}^3C_x\, p^x q^{3-x}$$

$\therefore {}^5C_3 p^3 q^2 + {}^5C_4 p^4 q + {}^5C_5 p^5 q^0 \geq {}^3C_2 p^2 q + {}^3C_3 p^3 q^0$

$\therefore \left(10p^3(1-p)^2 + 5p^4(1-p) + p^5\right) \geq \left[3p^2(1-p) + p^3\right]$

$10p^3 - 20p^4 + 10p^5 + 5p^4 - 5p^5 + p^5 - 3p^2 + 3p^3 - p^3 \geq 0$

$6p^5 - 15p^4 + 12p^3 - 3p^2 \geq 0$

$3p^2(2p^3 - 5p^2 + 4p - 1) = 0$

$\therefore 3p^2(p-1)^2(2p-1) \geq 0$

$\therefore 2p - 1 \geq 0 \quad (\because p^2 \geq 0, (p-1)^2 \geq 0)$

$\therefore p \geq \frac{1}{2}$ is the required probability.

* Properties of Binomial Distribution

Mean $= np$

Variance $= npq$

M.G.F about origin $M(t) = (q + pe^t)^n$

$P(X = x+1) = \frac{n-x}{x+1} \cdot \frac{p}{q} \cdot P(X = x)$

(14) Comment on the statement that the mean of B.D is 3 and variance is 4.

Sol^n Mean $= np = 3$ --- (1)

Var$(X) = npq = 4$ --- (2)

$\frac{(2)}{(1)}$ gives

$\frac{\cancel{np}q}{\cancel{np}} = \frac{4}{3} > 1$

This probability being > 1 we have contradiction

∴ The given Binomial distribution is not correct

(15) If the probability of a defective bolt is 0.1, find the mean and variance of defective bolts in total of 400 bolts.

Sol^n: $p = 0.1$, $n = 400$.
Let X be the r.v representing number of defective bolts which is a binomial variate.

$\text{mean} = np = 400(0.1) = 40$.
$\text{Variance} = npq = 400(0.1)(0.9) = 36$.

(16) If m.g.f of d.r.v is $\left(\frac{1}{3} + \frac{2}{3}e^t\right)^5$
Find $P(X = 2 \text{ or } 3)$.

Sol^n Here $q = \frac{1}{3}$, $p = \frac{2}{3}$, $n = 5$.

∴ for a B.D ; $P(X = x) = {}^nC_x\, p^x q^{n-x}$.
$P(X = 2 \text{ or } 3)$
$= P(X=2) + P(X=3) = {}^5C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right)^3 + {}^5C_3\left(\frac{2}{3}\right)^3\left(\frac{1}{3}\right)^2$

$= 10 \cdot \frac{4}{9}\,\frac{1}{27} + 10\left(\frac{8}{27}\right)\left(\frac{1}{9}\right)$

$= \frac{40}{243} + \frac{80}{243}$

$= \frac{120}{243}$

(17) Seven coins are tossed and the number of heads obtained is noted. The experiment is repeated 128 times and the following distribution is obtained

No. of heads	0	1	2	3	4	5	6	7	Total
f	7	6	19	35	30	23	7	1	128

Fit a binomial distribution and find the mean and variance if
(i) coins are unbaised
(ii) The nature of coins is not known.

sol^n To fit a distribution to a given data means to find the constants of distribution ~~w.r.t~~ which will adequately describe the given ~~function~~ situation.

Let X be r.v which denotes no. of heads (success)

(i) when the coins are unbaised :-

$$p=\frac{1}{2},\ q=\frac{1}{2} \text{ and } n=7$$

$$P(X=x) = {}^7C_x\left(\frac{1}{2}\right)^x\left(\frac{1}{2}\right)^{7-x}$$

At $x=0$, $P(X=0) = {}^7C_0\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^7 = \frac{1}{2^7} = \frac{1}{128}$

$x=1$, $P(x=1) = {}^7C_1\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^6 = \frac{7}{2^7} = \frac{7}{128}$

$x=2$ $\qquad \frac{21}{2^7}$

$x=3$ $\qquad \frac{35}{2^7}$

$x=4$ $\qquad \frac{35}{2^7}$

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$x=5$ $\qquad P(X=5) = \frac{\cancel{21}}{\cancel{2^7}} \quad \frac{21}{2^7} = \frac{21}{128}$

$x=6$ $\qquad P(X=6) = \frac{\cancel{7}}{\cancel{2^7}} \quad \frac{7}{2^7} = \frac{7}{128}$

$x=7$ $\qquad P(X=7) = \frac{\cancel{1}}{\cancel{2^7}} = \frac{1}{2^7} = \frac{1}{128}$

Expected qt freq. = Np where N = 128.

Mult the above probabilities by 128 i.e. by 2^7, we get expected q frequencies as

1, 7, 21, 35, 35, 21, 7, 1

(ii) When the nature of coins is not known

we have $\bar{X} = \frac{\sum x_i f_i}{N} = \frac{433}{128} = 3.38$

But $\bar{X} = np$

$\therefore 3.38 = 7 \cdot p$

$\therefore p = \frac{3.38}{7} \quad \therefore \boxed{p = 0.48}$

$\boxed{\therefore q = 1-p = 0.52}$

$P(X=x) = {}^7C_x (0.48)^x (0.52)^{n-x}$

putting $x = 0, 1, 2, 3, \ldots, 7$

$P(0) = 0.01, \quad P(1) = 0.066, \quad P(2) = 0.184$

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Mult these probabilities by 128 we get frequencies as

1, 8, 23, 36, 33, 18, 6, 3.