Fundamentals of Fuzzy Logic Systems

Origins

- Fuzzy logic was first developed by L.A. Zadeh in 1960's to extend conventional (binary) crisp logic to make it suitable to incorporate knowledge and mimic human-like approximate reasoning.
- Two-state (bivalent) crisp logic uses two quantities: true (T) and false (F) as truth values.
- Real-life situations are usually characterized by ambiguity and partial truth.
  $\Rightarrow$ cannot always assume two crisp states T and F with a crisp line dividing them.

Illustration 1

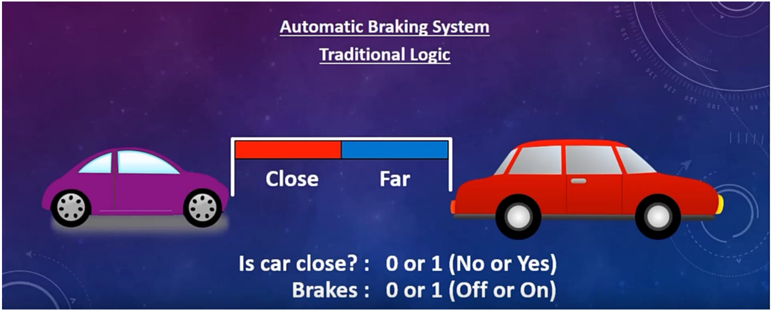


Illustration 1 (cont.)

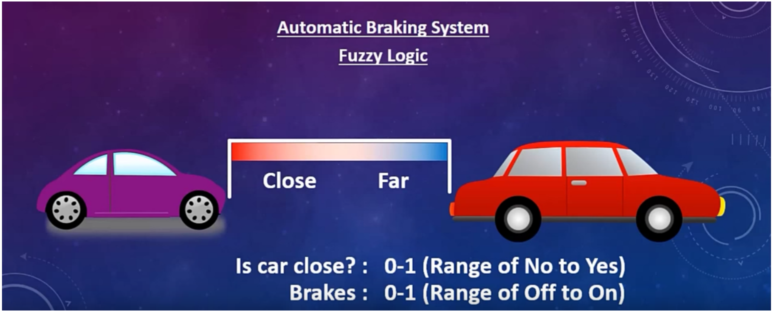


Illustration 2

- The linguistic descriptors "fast", "warm", and "large" are not crisp quantities and tend to be quite subjective, approximate, and qualitative.
- The statement "the water is warm" may have varying degrees of truth values (not always T or F).

Illustration 3: Fish Cutting (System Architecture)

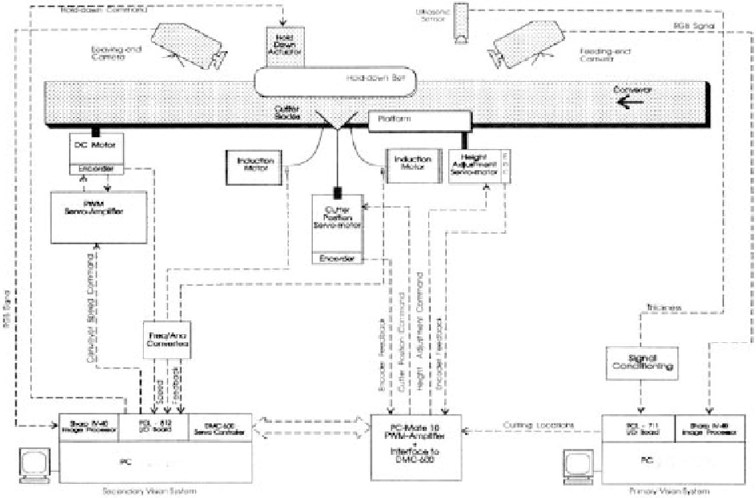


Illustration 3: Fish Cutting (cont.)

- Machine adjustments (control system's outputs) may include descriptive changes (e.g., small, medium, large) in the cutter blade speed, conveyor speed, and holding force.

- Human knowledge is represented in the form of a set of if-then rules:

  If the "quality" of the processed fish is "not acceptable", **and**
  if the cutting load appears to be "high",
  then "moderately" decrease the conveyor belt speed.

Illustration 3: Fish Cutting (cont.)

- These rules contain qualitative, descriptive, and linguistic terms such as "quality", "high", and "slight", which reflects human knowledge/expertise in operating such systems.
- It is clear that these fuzzy descriptors cannot be directly incorporated by conventional bivalent logic theories.

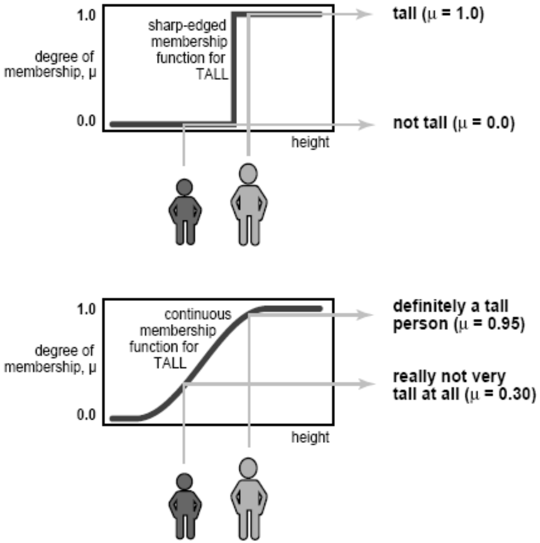
Applications of Fuzzy Logic

Consumer	Automotive	Industrial	Decision support
VCRs	Air conditioning	Extruding	Building HVAC
Microwave ovens	Emission control	Printing	Transportation
Camcorders	Engine control	Painting	Medical diagnostics
Washing machines	Fuel control	Food processing	Stock market analysis
Vacuum cleaners	Suspension control	Injection molding	Database processing
Clothes dryers	Cruise control	Packaging	
		Conveyor control	
		Temperature control	
		AGV Control	
		Mixing	
		Furnace control	
		Plating	

Fuzzy Sets

- Fuzzy sets represent the corner stone of fuzzy logic theory.
- Unlike a crisp set, where an element either belongs to it or not, partial membership in a fuzzy set is possible.
- An example of a fuzzy set is “the set of narrow streets in a city.”
- How to quantify the term “narrow”? $\Rightarrow$ **Gray** zone.

Definition of Tall: Crisp and Fuzzy Representation



Universe of Discourse and a Fuzzy Set

Definition

Let X be a set that contains every set of interest in the context of a given class of problems (e.g., set of all students). Then, X is called the **universe of discourse** (or simply the **universe**), whose elements are denoted by x.

- A fuzzy set A in the universe of discourse X may be represented by a Venn diagram.

- Generally, the elements of a fuzzy set are not numerical quantities.
- However, for analytical convenience, they are assigned real values.

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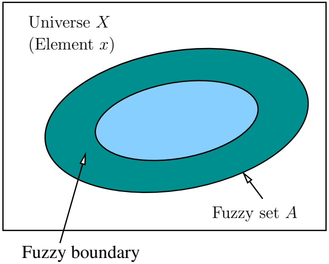

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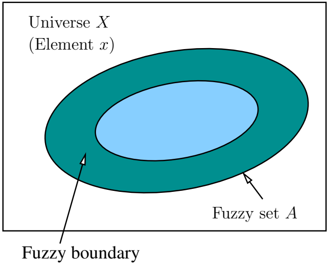

- Generally, the elements of a fuzzy set are not numerical quantities.
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Membership Function

- A fuzzy set may be represented by a membership function, which gives the grade (or degree) of membership within the set, of every element of the universe of discourse.
- The membership function maps the elements of the universe of discourse onto numerical values in the interval $[0, 1]$.

Membership function

A fuzzy set A may be represented as a set of ordered pairs:

$$A = \{(x, \mu_A(x)); x \in X, \mu_A(x) \in [0, 1]\},$$

where $\mu_A(x)$ is the membership function of the fuzzy set A.

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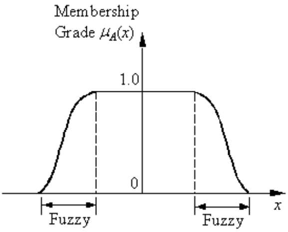

Membership Function

- The membership function $\mu_A(x)$ represents the grade of possibility (not probability) that an element x belongs to the fuzzy set A.

$$\begin{aligned} \mu_A(x) = 1 \quad &\Rightarrow x \text{ is definitely an element of } A. \\ \mu_A(x) = 0 \quad &\Rightarrow x \text{ is definitely not an element of } A. \\ \mu_A(x) = 0.2 \quad &\Rightarrow \text{the possibility that } x \text{ is an element of } A \text{ is } 0.2. \end{aligned}$$

Remark

A crisp set is a special case of a fuzzy set where the membership function can take only two values, 0 and 1.

Symbolic Representation

- If the universe of discourse is discrete with elements x_i, then a fuzzy set A may be symbolically represented with a "formal series"

$$A = \mu_A(x_1)/x_1 + \mu_A(x_2)/x_2 + \cdots + \mu_A(x_n)/x_n = \sum_{x_i \in X} \mu_A(x_i)/x_i$$

- If the universe of discourse is continuous, an equivalent form notation is given in terms of a "symbolic" integration

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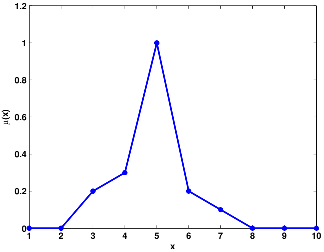
Fuzzy Representation of 5

Discrete Universe

Suppose that the universe of discourse X is the set of positive integers (or natural numbers). Consider the fuzzy set A in this discrete universe, given by the Zadeh notation:

$$A = 0.2/3 + 0.3/4 + 1.0/5 + 0.2/6 + 0.1/7$$

This set may be interpreted as a fuzzy representation of the integer 5.



Fuzzy Representation of “$x = a$”

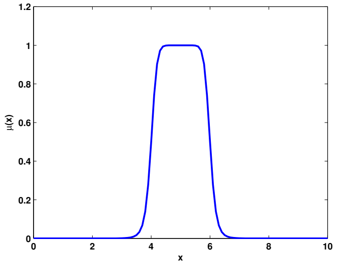
Continuous Universe

Consider the continuous universe of discourse X representing the set of real number. The membership function

$$\mu_A(x) = \frac{1}{1 + (x - a)^{10}}$$

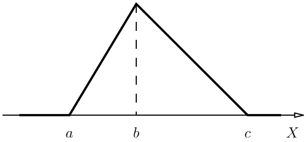
Defines a fuzzy set A whose elements x vaguely represent those satisfying the crisp relation $x = a$. This fuzzy set corresponds to a *fuzzy relation*.

$a = 5$



Triangular Membership Function

A triangular membership function is characterized by 3 parameters $\{a, b, c\}$, $a < b < c$.

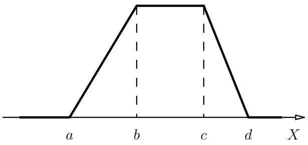


$$triangle(x|a, b, c) = \begin{cases} 0 & , x \leq a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ \frac{c-x}{c-b} & , b \leq x \leq c \\ 0 & , x \geq c \end{cases}$$

$$= \max\left[\min\left(\frac{x-a}{b-a}, \frac{c-x}{c-b}\right), 0\right]$$

Trapezoidal Membership Function

A trapezoidal membership function is characterized by 4 parameters $\{a, b, c, d\}$, $a < b < c < d$.

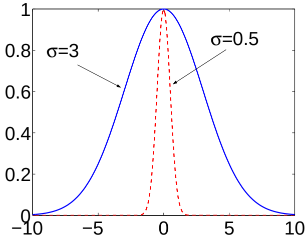


$$trapezoid(x|a,b,c,d) = \begin{cases} 0 & , x \leq a \\ \frac{x-a}{b-a} & , a \leq x \leq b \\ 1 & , b \leq x \leq c \\ \frac{d-x}{d-c} & , c \leq x \leq d \\ 0 & , x \geq d \end{cases}$$

$$= \max\left[\min\left(\frac{x-a}{b-a}, 1, \frac{d-x}{d-c}\right), 0\right]$$

Gaussian Membership Function

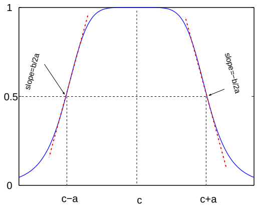
A Gaussian membership function is characterized by 2 parameters $\{\sigma, c\}$, with σ indicating the width, and c being the center of the membership function, respectively.



$$gauss(x|\sigma, c) = e^{-0.5\left(\frac{x-c}{\sigma}\right)^2}$$

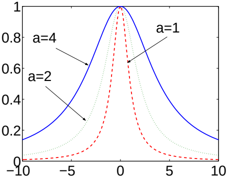
Generalized Bell Membership Function

A Generalized Bell (or simply Bell) membership function is characterized by 3 parameters $\{a, b, c\}$, with a and b defining the width and the slope, and c being the center of the membership function, respectively.

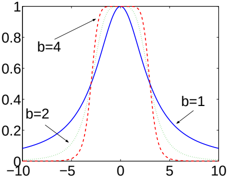


$$gbell(x|a, b, c) = \frac{1}{1 + \left|\frac{x-c}{a}\right|^{2b}}$$

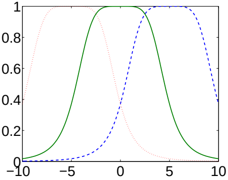
Bell MF: Parameters Significance



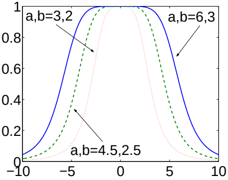
Varying a



Varying b



Varying c



Varying a and b

Remark

- Triangular and trapezoidal membership functions:
  - $+$ Linear $\Rightarrow$ Computationally inexpensive.
  - $-$ Non-differentiable $\Rightarrow$ May not be suitable to be used with gradient-descent optimization algorithms.
- Gaussian and Bell membership functions:
  - $+$ Differentiable $\Rightarrow$ Suitable to be used with gradient-descent optimization algorithms.
  - $-$ Nonlinear $\Rightarrow$ Computationally expensive.

Fuzzy Logic Operations

- Like in crisp sets, several operations can be performed on fuzzy sets.
- A fuzzy logic operation is an operation applied to "one" or more membership functions (or membership grades) at a time, resulting in one membership function (or grade).
- The associated mapping is indicated by:

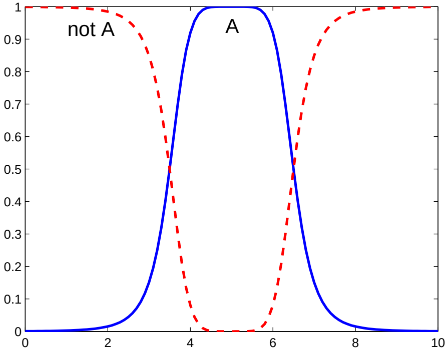
$$f : [0,1] \times [0,1] \times \ldots \times [0,1] \to [0,1]$$

- Among the most useful operations are:
  - the complement
  - the union
  - the intersection

Fuzzy Complement (Negation, Not)

- Consider a fuzzy set A in a universe of discourse X. It's complement A' is a fuzzy set with membership function:

$$\mu_{A'}(x) = 1 - \mu_A(x)\ ,\ \ x \in X$$



Fuzzy Union and Intersection

Assume two fuzzy sets A and B are given in universe X.

- The fuzzy union of A and B:

$$\mu_{A\cup B}(x) = max(\mu_A(x), \mu_B(x)), \forall x \in X\cdot$$

- The fuzzy intersection of A and B:

$$\mu_{A\cap B}(x) = min(\mu_A(x), \mu_B(x)), \forall x \in X\cdot$$

Example: Fuzzy Complement

Discrete Universe

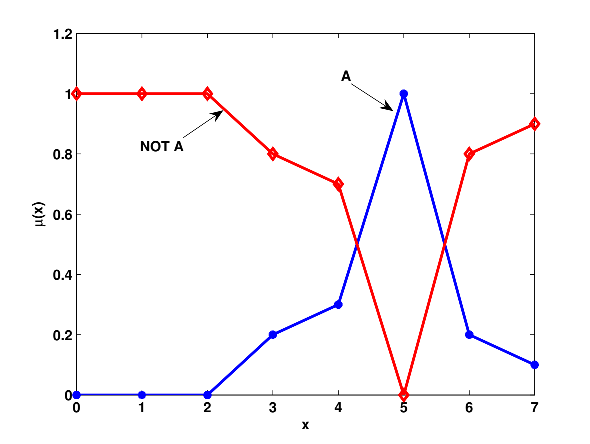
- Suppose that the universe of discourse X is defined as $X = \{0, 7\}$.
- Consider the fuzzy set A in this discrete universe given by:

$$A = 0.2/3 + 0.3/4 + 1.0/5 + 0.2/6 + 0.1/7$$

- The complement of fuzzy set A is as follows:
  $A' = 1.0/0 + 1.0/1 + 1.0/2 + 0.8/3$
  $+0.7/4 + 0.0/5 + 0.8/6 + 0.9/7$

Example: Fuzzy Complement (cont.)

Graphical Representation



Example: Fuzzy Union and Intersection

Discrete Universe

Suppose that the universe of discourse X denotes the driving speeds in a highway. Consider the fuzzy sets A and B are given by:

- Fuzzy state **Slow** (A)
  $A = 1.0/40 + 1.0/50 + 0.8/60 + 0.4/80 + 0.1/100$
- Fuzzy state **Moderate** (B)
  $B = 0.6/50 + 0.8/60 + 1.0/70 + 1.0/80 + 1.0/90 + 1.0/100$

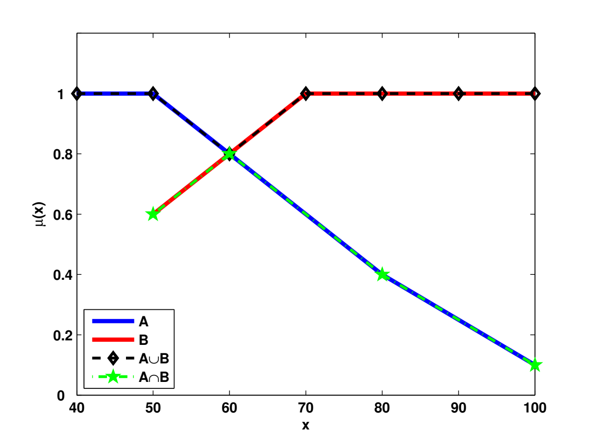
The fuzzy union and intersection of A and B are as follows:

$$A \cup B = 1.0/40 + 1.0/50 + 0.8/60 + 1.0/70 + 1.0/80 + 1.0/90 + 1.0/100$$

$$A \cap B = 0.6/50 + 0.8/60 + 0.4/80 + 0.1/100$$

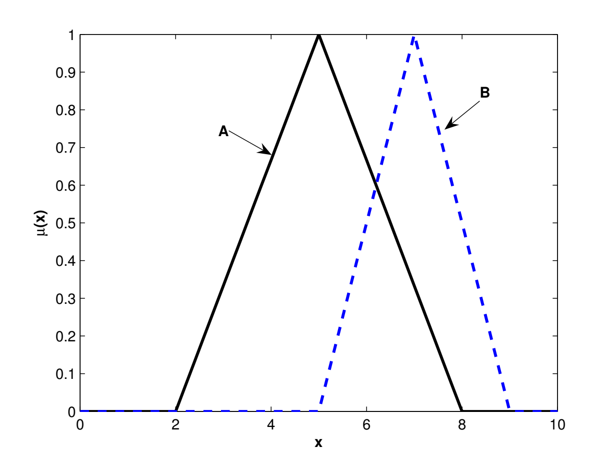
Example: Fuzzy Union and Intersection (cont.)

Graphical Representation



Example: Continuous Universe

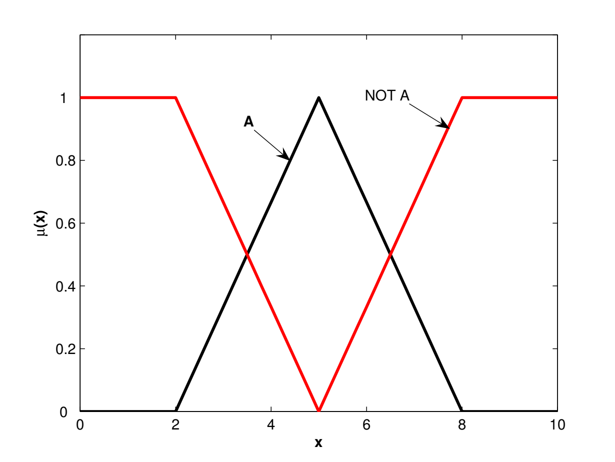
Consider Two Fuzzy Sets A and B



We are interested in finding the fuzzy complement of A as well as fuzzy union and intersection of A and B.

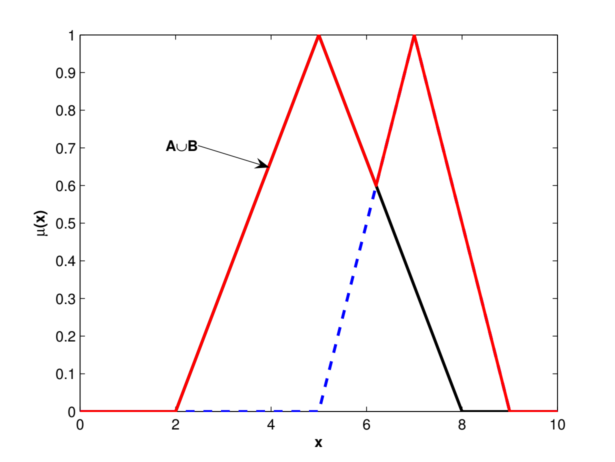
Example (cont.)

Fuzzy Complement of A



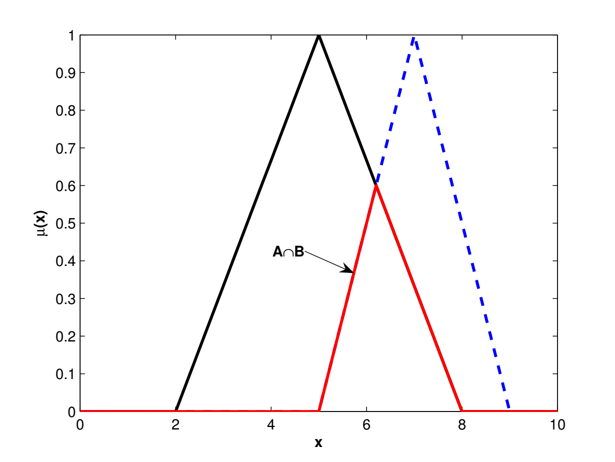
Example (cont.)

Fuzzy Union of A and B



Example (cont.)

Fuzzy Intersection of A and B



Generalized Fuzzy Complement

A generalized complement operation, denoted by $C : [0,1] \to [0,1]$, should satisfy the following axioms:

- Boundary conditions: $C(\emptyset) = X$ and $C(X) = \emptyset$, where X is the universe of discourse and $\emptyset$ is the null set.
- Non-increasing: if $\mu_A(x) < \mu_B(x)$ then $C(\mu_A(x)) \geq C(\mu_B(x))$, and vice versa.
- Involutive: $C(C(A)) = A$.

Two of the Well Known Fuzzy Complement Operators

1. Sugeno's complement: $C(a) = \dfrac{1-a}{1+pa}$, $p \in (-1, \infty)$
2. Yager's complement: $C(a) = (1-a^p)^{1/p}$, $p \in (0, \infty)$

Fuzzy Intersection and T-norm

- The intersection of two fuzzy sets A and B is given by an operation T which maps two membership functions to:

$$\mu_{A \cap B}(x) = T(\mu_A(x), \mu_B(x)).$$

- $T(\cdot)$ is known as the T-norm operator.

T-norm (Generalized Intersection)

- Consider two membership functions that are given by $a = \mu_A(x)$ and $b = \mu_B(x)$.
- The t-norm operation or generalized intersection may be represented by $T(a, b)$ or more commonly aTb.

T-norm Properties

1. It is non-decreasing in each argument. i.e., if $a \leq b$ and $c \leq d$ then $aTc \leq bTd$.
2. It satisfies commutativity. i.e., $aTb = bTa$.
3. It satisfies associativity. i.e., $(aTb)Tc = aT(bTc)$.
4. It satisfies the boundary conditions, i.e., $aT1 = a$ and $aT0 = 0$.

Well Known T-norm Operators

- Two general forms of T-norms are:

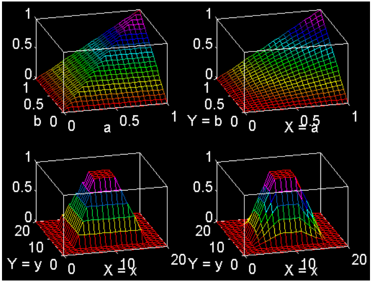
$$1 - \min[1, ((1-a)^p + (1-b)^p)^{1/p}] \ , \ \ p \geq 1$$
$$\max[0, (\lambda+1)(a+b-1) - \lambda ab] \ , \ \ \lambda \geq -1$$

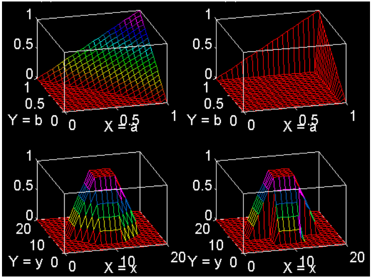
- Four of the well known T-norm operators are:

Min:	$T(a,b) = \min(a,b) = a \wedge b$
Algebraic product:	$T(a,b) = a \times b$
Bounded product:	$T(a,b) = 0 \vee (a+b-1)$
Basic product:	$T(a,b) = \begin{cases} a & \text{if } b = 1 \\ b & \text{if } a = 1 \\ 0 & \text{if } a, b < 1 \end{cases}$

Well Known T-norm Operators: Graphical Example

Minimum | Algebraic product | Bounded product | Basic product





Fuzzy Union and S-norm

- The union of two fuzzy sets A and B is given by an operation S which maps two membership functions to:

$$\mu_{A\cup B}(x) = S(\mu_A(x), \mu_B(x)).$$

- $S(\cdot)$ is known as the S-norm operator, or T-conorm operator.

Theorem: DeMorgan's Laws

According to DeMorgan's Law, there exists a complementary S-norm associated to every T-norm, and vice versa.

$$aSb = 1 - (1-a)T(1-b)$$
$$aTb = 1 - (1-a)S(1-b)$$

Example

Prove that the S-norm associated to the T-norm *min* is *max*.

Proof

- Use DeMorgan's Law: $aSb = 1 - (1-a)T(1-b)$
- Direct substitution of *min* for T gives:

$$\begin{aligned} \text{aSb} \quad &= 1 - min[(1-a),(1-b)] = 1-(1-a) = a \quad \text{if } a \geq b \\ &= 1-(1-b) = b \quad \text{if } b < a \end{aligned}$$

- Hence: $aSb = max(a, b)$

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S-norm (Generalized Union)

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- The s-norm operation or generalized union may be represented by $S(a, b)$ or more commonly aSb.

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Well Known S-norm Operators

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$$\min[1, a + b + \lambda ab] \ , \ \ \lambda \geq -1$$

- Four of the well known S-norm operators are:

Max: $S(a,b) = \max(a,b) = a \vee b$

Algebraic sum: $T(a,b) = a + b - ab$

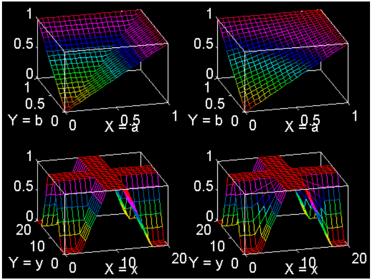
Bounded sum: $T(a,b) = 1 \wedge (a+b)$

Basic sum: $S(a,b) = \begin{cases} a & \text{if } b = 0 \\ b & \text{if } a = 0 \\ 1 & \text{if } a, b < 1 \end{cases}$

Well Known S-norm Operators: Graphical Example

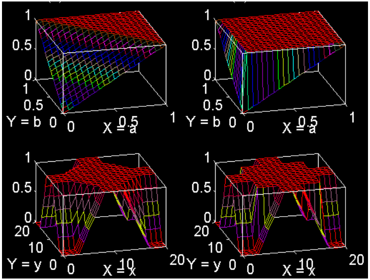
Maximum

Algebraic sum



Bounded sum

Basic sum



Set Inclusion ($A \subseteq B$)

- The concept of subset (or, a set "included" in another set) in crisp sets may be conveniently extended to the case of fuzzy sets.
- Specifically, a fuzzy set A is considered a subset of another fuzzy set B, in universe X, if and only if $\mu_A(x) \leq \mu_B(x)$, for all $x \in X$.
  - This is denoted by $A \subseteq B$.

Set Inclusion

$$A \subset B \Leftrightarrow \mu_A(x) < \mu_B(x), \ \forall x \in X$$
$$A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \ \forall x \in X$$

Definition: Proper Subset

In the case where $A \subset B$, A is called a proper subset of B.

Grade of Inclusion

In the context of fuzzy logic, a set may be partially included in another set.

- Then, it is convenient to define a grade of inclusion of a fuzzy set A in another fuzzy set B.
- This may be interpreted as the membership function of the *fuzzy relation* $A \subset B$ (or $A \subseteq B$).

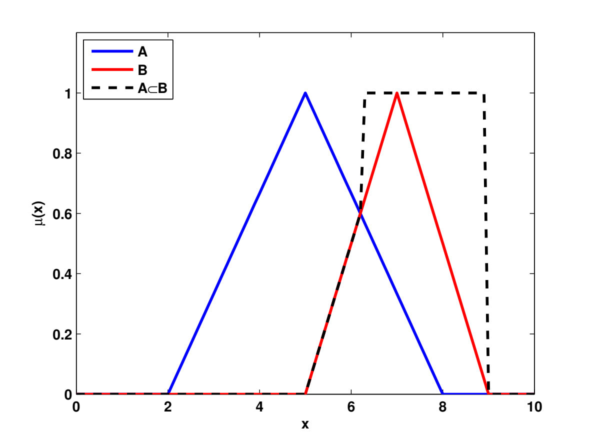
Grade of Inclusion

$$\mu_{A\subset B}(x) = \begin{cases} 1 & , \text{ if } \mu_A(x) < \mu_B(x) \\ \mu_A(x)\, T\, \mu_B(x) & , \text{ otherwise} \end{cases}$$

$$\mu_{A\subseteq B}(x) = \begin{cases} 1 & , \text{ if } \mu_A(x) \leq \mu_B(x) \\ \mu_A(x)\, T\, \mu_B(x) & , \text{ otherwise} \end{cases}$$

Example

Graphical Representation



Set Equality ($A = B$)

- The equality of two fuzzy sets is a special case of set inclusion.
- A fuzzy set A is equal to another fuzzy set B, in universe X, if and only if $\mu_A(x) = \mu_B(x)$, for all $x \in X$.
    - This is denoted by $A = B$.

Set Equality

$$A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \;\forall x \in X$$

Grade of Equality

- A grade of equality for two fuzzy sets may be defined similar to the grade of inclusion.
- This may be interpreted as the membership function of the *fuzzy relation* $A = B$.

Grade of Equality

$$\mu_{A=B} = \begin{cases} 1 & \text{, if } \mu_A(x) = \mu_B(x) \\ \mu_A(x) T \mu_B(x) & \text{, otherwise} \end{cases}$$

Dilation and Contraction

- Let A be a fuzzy set in the universe X with membership function μ_A.
  - Its Kth dilation is a fuzzy set A' in the universe X with membership function $\mu_{A'}(x) = \mu_A^{1/K}(x)$.
  - Its Kth contraction is a fuzzy set A'' in the universe X with membership function $\mu_{A''}(x) = \mu_A^K(x)$.

Dilation

$$\mathrm{dil}(A) = A^{1/2} = \int \frac{\sqrt{\mu_A(x)}}{x} \equiv \{\text{more or less}\} \equiv \{\text{somehow}\}$$

Contraction

$$\mathrm{con}(A) = A^2 = \int \frac{(\mu_A(x))^2}{x} \equiv \{\text{very}\} \equiv \{\text{too}\}$$

Example 1

Discrete Case

- Consider the following fuzzy set:

$$A = 0.0/1 + 0.5/2 + 1.0/3 + 0.5/4 + 0.0/5$$

- The second dilation of fuzzy set A is given as follows:

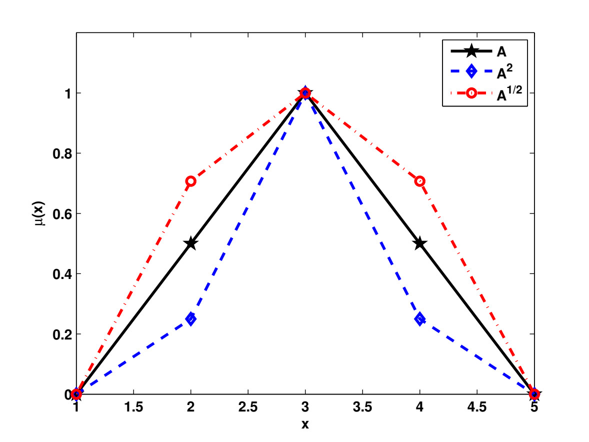
$$\begin{aligned} A^{1/2} &= 0.0^{1/2}/1 + 0.5^{1/2}/2 + 1.0^{1/2}/3 + 0.5^{1/2}/4 + 0.0^{1/2}/5 \\ &= 0.0/1 + 0.7071/2 + 1.0/3 + 0.7071/4 + 0.0/5 \end{aligned}$$

- Contraction of fuzzy set A is given as follows:

$$\begin{aligned} A^{2} &= 0.0^{2}/1 + 0.5^{2}/2 + 1.0^{2}/3 + 0.5^{2}/4 + 0.0^{2}/5 \\ &= 0.0/1 + 0.25/2 + 1.0/3 + 0.25/4 + 0.0/5 \end{aligned}$$

Example 1 (cont.)

Graphical Illustration

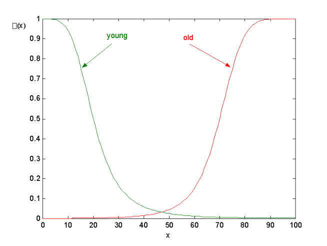


Example 2

Continuous Case

$$\mu_{\text{young}}(x) = gbell(x|20,2,0) = \frac{1}{1+\left|\frac{x}{20}\right|^4}$$

$$\mu_{\text{old}}(x) = gbell(x|30,3,100) = \frac{1}{1+\left|\frac{x-100}{30}\right|^6}$$



Example 2 (cont.)

Construction of linguistic Representation

$$\text{More or less old} \equiv \int_X \left[\sqrt{\frac{1}{1+\left|\frac{x-100}{30}\right|^6}}\right] \Big/ x$$

$$\text{Not young and not old} \equiv \overline{\text{young}} \cap \overline{\text{old}}$$

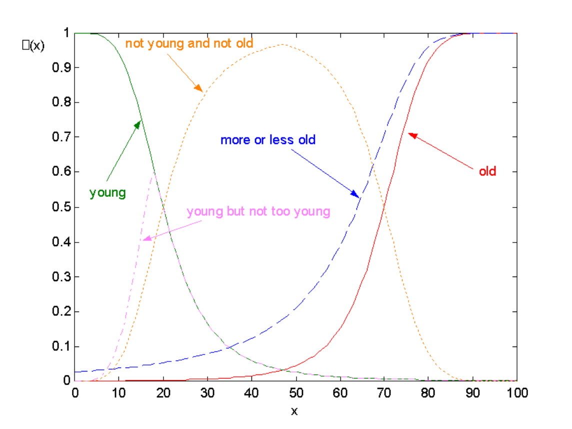
$$\equiv \int_X \left[1-\frac{1}{1+\left|\frac{x}{20}\right|^4}\right] \wedge \left[1-\frac{1}{1+\left|\frac{x-100}{30}\right|^6}\right] \Big/ x$$

$$\text{Young but not too old} \equiv \text{young} \cap \overline{\text{too old}}$$

$$\equiv \int_X \left[\frac{1}{1+\left|\frac{x}{20}\right|^4}\right] \wedge \left[1-\left\{\frac{1}{1+\left|\frac{x-100}{30}\right|^6}\right\}^2\right] \Big/ x$$

Example 2 (cont.)

Graphical Representation



Implication (if-then)

- Consider two fuzzy sets: A in X and B in Y.
- The fuzzy implication $A \to B$ is a fuzzy relation in the Cartesian product $X \times Y$ defined by:

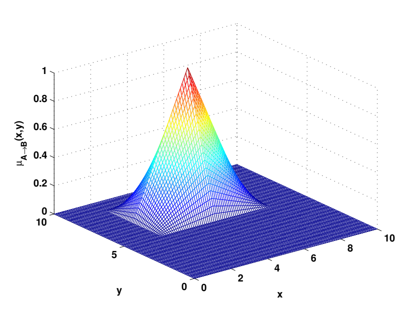
Implication

- Larsen implication: $\underset{\forall (x,y) \in (X \times Y)}{\mu_{A \to B}(x, y)} = \mu_A(x)\mu_B(y)$
- Mamdai implication: $\underset{\forall (x,y) \in (X \times Y)}{\mu_{A \to B}(x, y)} = \min\left[\mu_A(x), \mu_B(y)\right]$
- Zadeh implication:
  $\underset{\forall (x,y) \in (X \times Y)}{\mu_{A \to B}(x, y)} = \max\left[\min\{\mu_A(x), \mu_B(y)\}, 1 - \mu_A(x)\right]$
- Dienes-Rascher implication: $\underset{\forall (x,y) \in (X \times Y)}{\mu_{A \to B}(x, y)} = \max\left[1 - \mu_A(x), \mu_B(y)\right]$
- Lukasiewicz implication: $\underset{\forall (x,y) \in (X \times Y)}{\mu_{A \to B}(x, y)} = \min\left[1, 1 - \mu_A(x) + \mu_B(y)\right]$

Example

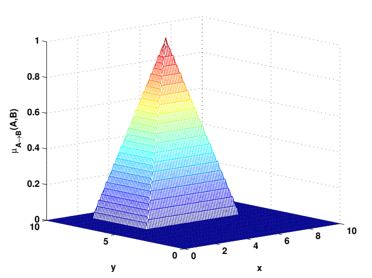
- Assume two fuzzy sets A and B are defined on universes X, Y, respectively: triangular membership functions.
- Both sets A and B have triangular membership functions given by $A = triangle(x|2,5,8)$, $B = triangle(y|5,7,9)$.

Larsen Implication

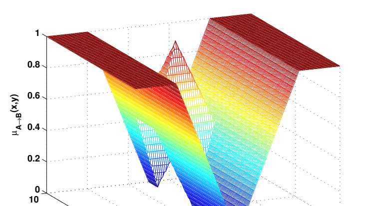


Example (cont.)

Mamdani Implication

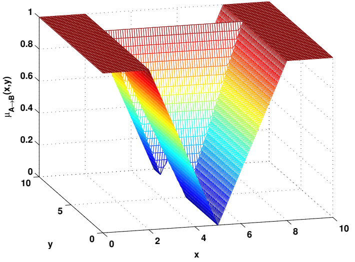


Zadeh Implication

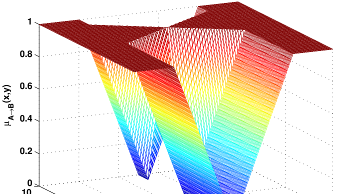


Example (cont.)

Deines-Rescher Implication



Lukasiewicz Implication



Height of a Fuzzy Set

- The **height** (also called **modal grade**) of a fuzzy set is the maximum value of its membership function.

Height of a Fuzzy Set

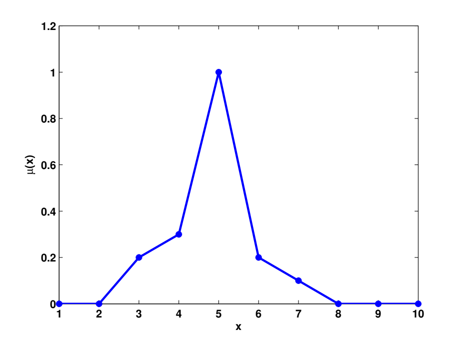
For a fuzzy set A in the universe X, with membership function μ_A, the height of A is defined as

$$\mathrm{hgt}(A) = \sup_{x \in X} \mu_A(x)$$

- The element $x^* \in X$ corresponding to the modal grade of the fuzzy set (i.e., $\mu_A(x^*) = \mathrm{hgt}(A)$) is called the **modal element value**, or simply **modal point**.

Example

Fuzzy Set A



- Height of fuzzy set A: $hgt(A) = 1$
- Modal point: $x = 5$

- The **support set** of a fuzzy set is a crisp set containing all the elements (in the universe) whose membership grades are greater than zero.

Support Set

The support set S of a fuzzy set A in the universe X, with membership function μ_A, is defined by

$$S = \{x \in X \mid \mu_A(x) > 0\}$$

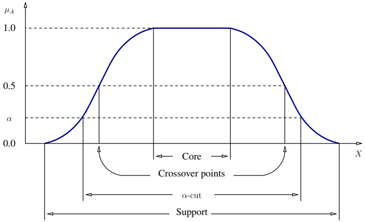
α-cut of a Fuzzy Set

The α-**cut** of a fuzzy set A is the crisp set denoted by A_α formed by the elements of A whose membership function grades are greater than or equal to a specified threshold value $\alpha \in [0, 1]$.

α-cut

Mathematically, A_α is defined by

$$A_\alpha = \{x \in X \mid \mu_A(x) \geq \alpha\}\ ,\ \ \alpha \in [0, 1]$$

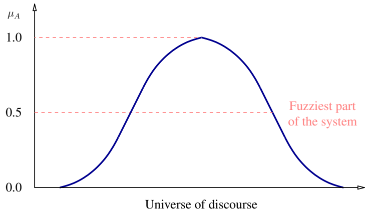


Measure of Fuzziness

A measure of fuzziness of a fuzzy set A in the universe X defines the closeness of its membership function μ_A to the most fuzzy grade (0.5).

Different Measures of Fuzziness

1. Closeness to grade 0.5 (A_1),
2. Distance from $1/2-$cut (A_2),
3. Inverse of distance from the complement (A_3).



Measures of Fuzziness

Closeness to grade 0.5

$$A_1 = \int_{x \in S} f(x)dx, \quad S \text{ denotes the support set.}$$

$$f(x) = \begin{cases} \mu_A(x) & \textit{for } \mu_A(x) \leq 0.5 \\ 1 - \mu_A(x) & \textit{otherwise} \end{cases}$$

Distance from 1/2−cut

$$A_2 = \int_{x \in X} \left| \mu_A(x) - \mu_{A_{1/2}}(x) \right| dx$$

Measures of Fuzziness

Inverse of distance from the complement

$$A_3 = 2 \int\limits_{x \in X} |\mu_A(x) - 0.5| \, dx = \int\limits_{x \in X} |2\mu_A(x) - 1| \, dx$$

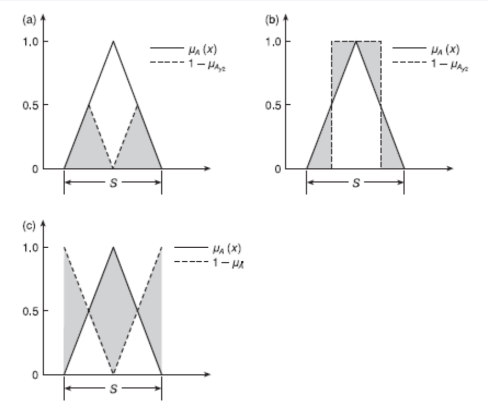
$$= \int\limits_{x \in X} |\mu_A(x) - (1 - \mu_A(x))| \, dx = \int\limits_{x \in X} |\mu_A(x) - \mu_{\bar{A}}(x)| \, dx$$

Relationship Between Different Measures of Fuzziness

$$A_1 = A_2 = \frac{1}{2}(S * 1 - A_3)$$

Illustration of Three Measures of Fuzziness

(a) A_1, (b) A_2, (c) A_3



- Consider the following boolean relation as an example in crisp sets and binary logic.

$$a = x \cdot \bar{y} + z$$

- Then, the relation a is equivalent to the logical expression (proposition)

$$\text{“}x \text{ AND NOT } y \text{ OR } z\text{”}$$

  and is denoted by a.
- An equivalent expression may be written using three sets and the operations: Complement, Intersection, and Union, but we will not do so here.

Truth Table of the Relation a

x	y	z	$\bar{y}$	$x \cdot \bar{y}$	a
0	0	0	1	0	0
0	0	1	1	0	1
0	1	0	0	0	0
0	1	1	0	0	1
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	0	0	0
1	1	1	0	0	1

Example of Fuzzy Relation

Consider a fuzzy logic proposition A with membership function μ_A.

- Then, μ_A may be taken to represent the degree of validity of the relation "A is True".
- Similarly, the complement membership function $\mu_{\bar{A}} = 1 - \mu_A$ represents the degree of validity of the fuzzy relation "$\bar{A}$ is True", or equivalently "A is NOT True".

Example of Fuzzy Relation (cont.)

Consider a fuzzy logic proposition "A AND B".

- If μ_A is the membership function of A and μ_B is the membership function of B, then using the min T-norm to represent AND, the membership function of "A AND B" is given by

$$R(x, y) = \min\left[\mu_A(x), \mu_B(y)\right]$$

- This membership function represents the degree of validity of the fuzzy relation "A AND B is True"

Analytical Representation of a Fuzzy Relation

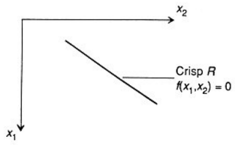
- Any membership function represents a fuzzy relation in the universe (domain, or space) of definition of the particular membership function.
- The space can be one dimensional; say the real line $\mathbb{R}$ as in the case of $\mu_R(x)$, and this gives a 1-D fuzzy relation.
- This idea can be extended to higher dimensions.

Analytical Representation of a Fuzzy Relation (cont.)

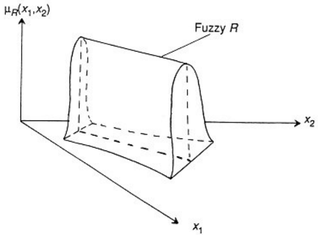
- Consider two universes $X_1 = \{x_1\}$ and $X_2 = \{x_2\}$.
- A crisp set consisting of a subset of ordered pairs (x_1, x_2) is a crisp relation R in the two-dimensional Cartesian product space $X_1 \times X_2$.
- We may imagine that a truth value of 1 is associated to each of these ordered pairs, giving the characteristic function (special case of membership function) of the crisp relation.

Relation Representation in Crisp and Fuzzy Cases

(a)



(b)



Cartesian Products

Recall for the Crisp Case

- Consider a crisp set A_1 defined on the universe X_1 and a second crisp set A_2 defined on a different (independent, orthogonal) universe X_2.
- The Cartesian product $A_1 \times A_2$ is the rectangular area which is a subset of the Cartesian product space $X_1 \times X_2$ defined on the usual manner, which is the entire 2-D space (plane) containing the two axes x_1 and x_2.

Cartesian Product of Fuzzy Sets

- Consider a fuzzy set A_1 defined on the universe X_1 and a second fuzzy set A_2 defined on a different (independent, orthogonal) universe X_2.

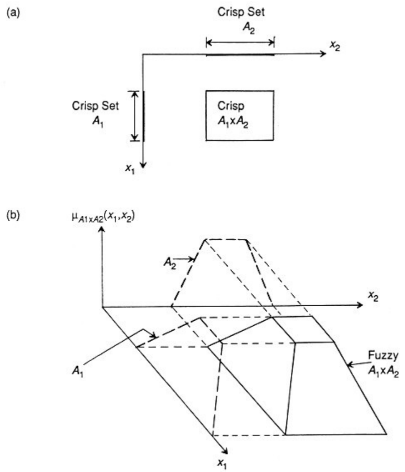
- The Cartesian product $A_1 \times A_2$ is the fuzzy subset of the Cartesian space $X_1 \times X_2$. Its membership function is given by

$$\mu_{(A_1 \times A_2)}(x_1, x_2) = \min\left[\mu_{A_1}(x_1), \mu_{A_2}(x_2)\right] \ , \ \forall (x_1, x_2) \in (X_1 \times X_2)$$

Cartesian Product of Fuzzy Sets (cont.)

- The min combination applies here because, each element (x_1, x_2), in the Cartesian product is formed by taking both elements x_1 "and" x_2 together (an "AND" operation), not just one or the other.
- The Cartesian product $A \times B$ provides the region of definition of the two fuzzy sets A and B.
- It is a subset of the Cartesian product of their support sets, which in turn is a subset of the Cartesian product of their universes $(X \times Y)$.

Cartesian Product: Graphical Representation



Extension Principle

- Consider a mapping function $f : X \to Y$
- Let A be a fuzzy set on X

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots + \frac{\mu_A(x_n)}{x_n}$$

- The Extension Principle states that the image of A under mapping $f(\cdot)$ is expressed on Y

$$B = f(A) = \frac{\mu_A(x_1)}{y_1} + \frac{\mu_A(x_2)}{y_2} + \cdots + \frac{\mu_A(x_n)}{y_n} \text{ , where}$$
$$y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$$

presuming that the function f is a one-to-one mapping.

- If f is a many-to-one mapping, i.e., $\exists\ x_1 \neq x_2$ such that $y_1 = f(x_1) = f(x_2) = y_2$, Then $\mu_B(y) = \max[\mu_A(x_1), \mu_A(x_2)]$.

Extension Principle

- Consider a mapping function $f : X \to Y$
- Let A be a fuzzy set on X

$$A = \frac{\mu_A(x_1)}{x_1} + \frac{\mu_A(x_2)}{x_2} + \cdots + \frac{\mu_A(x_n)}{x_n}$$

- The Extension Principle states that the image of A under mapping $f(\cdot)$ is expressed on Y

$$B = f(A) = \frac{\mu_A(x_1)}{y_1} + \frac{\mu_A(x_2)}{y_2} + \cdots + \frac{\mu_A(x_n)}{y_n}\ , \text{ where}$$
$$y_1 = f(x_1), y_2 = f(x_2), \ldots, y_n = f(x_n)$$

  presuming that the function f is a one-to-one mapping.
- If f is a many-to-one mapping, i.e., $\exists\ x_1 \neq x_2$ such that $y_1 = f(x_1) = f(x_2) = y_2$, Then $\mu_B(y) = \max[\mu_A(x_1), \mu_A(x_2)]$.

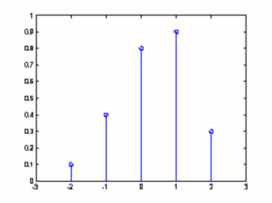
Example 1

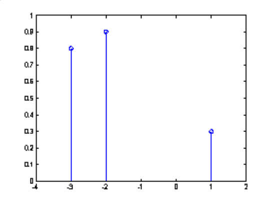
Discrete Case

$$A = \frac{0.1}{-2} + \frac{0.4}{-1} + \frac{0.8}{0} + \frac{0.9}{1} + \frac{0.3}{2}$$

$$f : x \mapsto x^2 - 3$$

$$\begin{aligned} B &= \frac{0.1}{1} + \frac{0.4}{-2} + \frac{0.8}{-3} + \frac{0.9}{-2} + \frac{0.3}{1} \\ &= \frac{(0.1 \vee 0.3)}{1} + \frac{(0.4 \vee 0.9)}{-2} + \frac{(0.8)}{-3} \\ &= \frac{0.3}{1} + \frac{0.9}{-2} + \frac{0.8}{-3} \end{aligned}$$





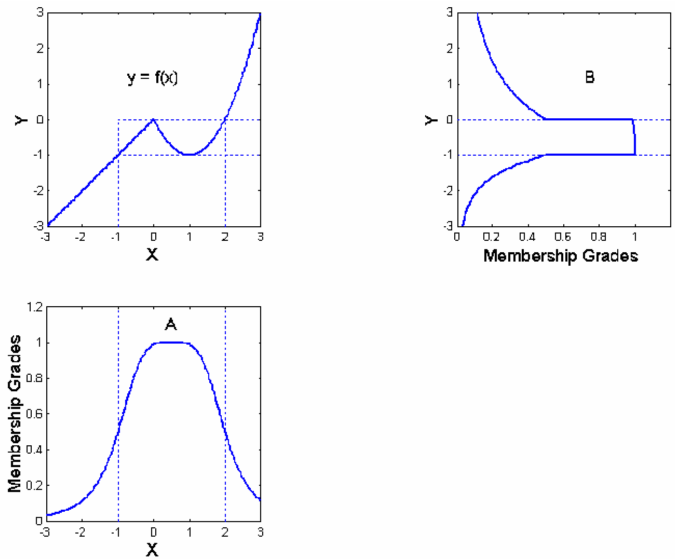
Example 2

Continuous Case

$$\mu_A(x) = \text{gbell}(x|1.5, 2, 0.5)$$
$$f(x) = \begin{cases} (x-1)^2 - 1 & \text{, if } x \geq 0 \\ x & \text{, if } x < 0 \end{cases}$$

Define B and $\mu_B(y)$.

Example 2: Graphical Solution

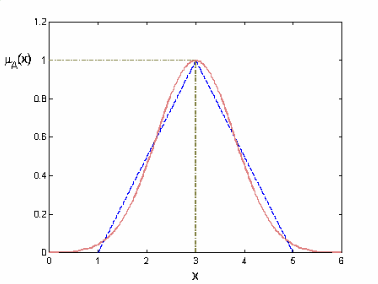


Example 3

Continuous Case

Let f be a crisp mapping $f : x \mapsto y = \sqrt{x}$, with x being a fuzzy number around 3.

$$\mu_A(x) = \begin{cases} (x-1)/2 & \text{, if } x \leq 3 \text{ monotonic } (f(x)) \\ (5-x)/2 & \text{, if } 3 \leq x \leq 5 \text{ monotonic } (f(x)) \end{cases}$$



Find out $\mu_B(y)$.

Example 3: Solution

Solution

$$B = f(A) = \int_y \frac{\mu_A(x)}{f(x)} = \int_y \frac{\mu_A(f^{-1}(y))}{f(x)} = \int_y \frac{\mu_A(f^{-1}(y))}{y}$$

Case 1: $1 \leq x \leq 3$ $\quad x = f^{-1}(y) = y^2$

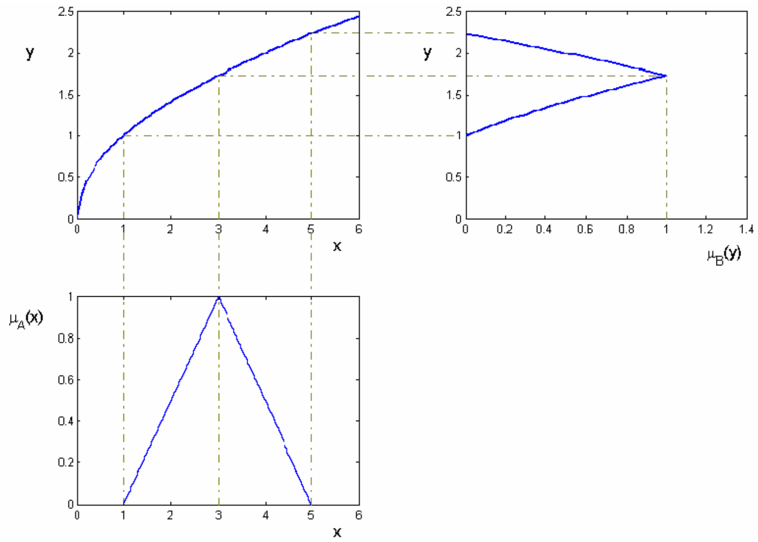
$\mu_B(y) = (y^2 - 1)/2 \quad 1 \leq y \leq \sqrt{3}$

Case 2: $3 \leq x \leq 5$

$\mu_B(y) = (5 - y^2)/2 \quad \sqrt{3} \leq y \leq \sqrt{5}$

$$\mu_B(y) = \bigvee_y \mu_A\left[f^{-1}(y)\right]$$

Example 3: Graphical Solution



Projection

Given the relation R defined by $R = \int_{X \times Y} \frac{\mu_R(x,y)}{(x,y)}$

The first projection is a fuzzy set that results by eliminating the second set Y of $X \times Y$ by projecting the relation on X.

$$R_1 = \int_X \frac{\mu_{R_1}(x)}{x}, \qquad \mu_{R_1}(x) = \bigvee_y \max\left[\mu_R(x,y)\right]$$

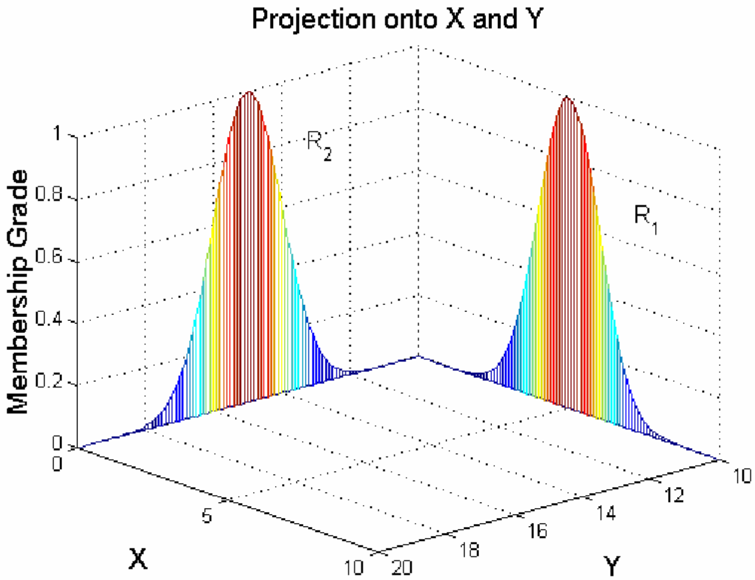
The second projection is a fuzzy set that results by eliminating the first set X of $X \times Y$ by projecting the relation on Y.

$$R_2 = \int_Y \frac{\mu_{R_2}(y)}{y}, \qquad \mu_{R_2}(y) = \bigvee_x \max\left[\mu_R(x,y)\right]$$

The **total projection** is a combined projection over the spaces X and Y and is defined by

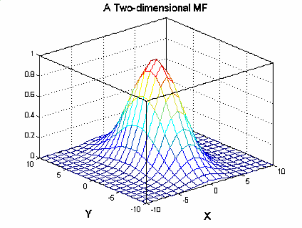
$$\mu_{R_T}(x,y) = \bigvee_x \bigvee_y \max\left[\mu_R(x,y)\right] = \bigvee_y \bigvee_x \max\left[\mu_R(x,y)\right]$$

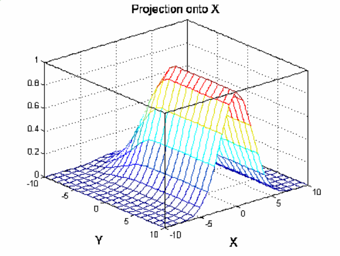
First and Second Projections of Relation R

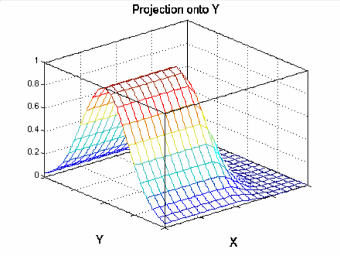


Example 1

Continuous Case







Example 2

Discrete Case

$$R = \begin{array}{cc} \begin{bmatrix} 0.1 & 0.2 & 0.4 & 0.8 & 1.0 & 0.6 \\ 0.2 & 0.4 & 0.8 & 0.9 & 0.8 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.8 & 0.4 & 0.2 \end{bmatrix} & x \\ y & \end{array}$$

$$R^1 = \sum_i \frac{\mu_{R^1}(x_i)}{x_i} = \frac{1.0}{x_1} + \frac{0.9}{x_2} + \frac{1.0}{x_3} = \begin{bmatrix} 1.0 \\ 0.9 \\ 1.0 \end{bmatrix}$$

$$R^2 = \sum_j \frac{\mu_{R^2}(y_j)}{y_j} = \frac{0.5}{y_1} + \frac{0.9}{y_2} + \frac{1.0}{y_3} + \frac{0.9}{y_4} + \frac{1.0}{y_5} + \frac{0.6}{y_6}$$

$$= \begin{bmatrix} 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \end{bmatrix}^T$$

$$R^T = 1$$

Cylindrical Extension

$$C(R) = \int_{X_1 \times X_2 \times \ldots X_n} \frac{\mu_R(x_1, x_2, \ldots, x_n)}{(x_1, x_2, \ldots, x_n)}$$

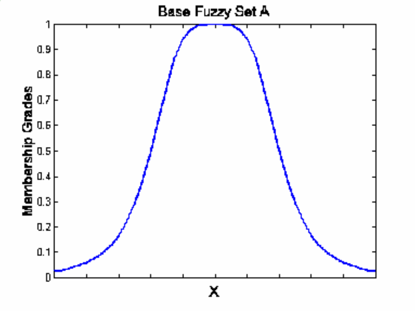
Example: Discrete Case

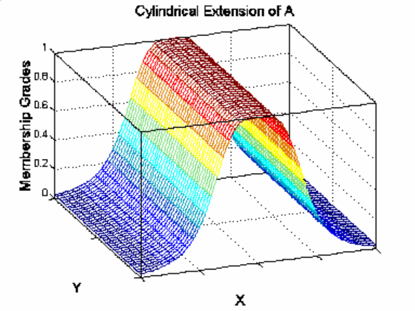
$$C(R^1) = \begin{bmatrix} 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \end{bmatrix}$$

$$C(R^2) = \begin{bmatrix} 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \\ 0.5 & 0.9 & 1.0 & 0.9 & 1.0 & 0.6 \end{bmatrix}$$

Cylindrical Extension (cont.)

Example: Continuous Case





Compositional Rule of Inference

- A typical fuzzy logic system's knowledge base is composed of a series of if-then rules.
- Each rule is a fuzzy relation employing the (if-then) fuzzy implication.
- When all individual rules (relations) are aggregated (executed), they result in one fuzzy relation represented by a fuzzy set, say K, and a multi-variable membership function.

Compositional Rule of Inference (cont.)

Fuzzy Inference

1. In a fuzzy decision making process, the rule base K is first collectively matched with the available data.
2. Next, an inference is made on another fuzzy variable (consequent part) that is represented in the knowledge base.

- The above process is conducted based on the **compositional rule of inference** (CRI).

Compositional Rule of Inference (cont.)

Suppose that the available data (context) is denoted by the fuzzy set (or relation) D and the inference (action) is denoted by a fuzzy set (or relation) I.

- Then, the compositional rule of inference states that

$$I = D \circ K \ , \qquad\qquad \circ : \text{“composed with”}$$

- The membership function of the inference I (also called decision, or action) is determined as

$$\mu_I = \sup_y \left\{ \min[\mu_D, \mu_K] \right\}$$

  where the inference I is the output of the knowledge base decision making system.
- This is known as the **sup-min** (or max-min) composition.
- Another method to compute the inference I is through the **sup-product** composition.

$$\mu_I = \sup_y \left\{ \mu_D \cdot \mu_K \right\}$$

Example

Rule Base

- Fuzzy set A of universe Y represents the output of a process.
- Fuzzy set C of universe Z represents the control input to the process.

$$A = 0.0/y_1 + 0.2/y_2 + 1.0/y_3 + 0.8/y_4 + 0.1/y_5$$
$$C = 0.1/z_1 + 0.7/z_2 + 1.0/z_3 + 0.4/z_4$$

- A fuzzy relation $R : A \rightarrow C$ is defined by applying the min operation to A and C.

$$\mu_R(y_i, z_i) = \begin{array}{cc} z_i & \\ \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} & y_i \end{array}$$

Example (cont.)

Context

A process measurement y_0 is made and found to be closest to y_4. This crisp set may be represented by a fuzzy singleton A_0 defined by

$$A_0 = 0.0/y_1 + 0.0/y_2 + 0.0/y_3 + 0.8/y_4 + 0.0/y_5$$

Inference

The membership function of the corresponding fuzzy control inference C' is obtained using the compositional rule of inference.

$$C' = A_0 \circ R$$

Define C'.

Remark

Note how the matrix R represents the rule base in the fuzzy decision making.

Example (cont.)

Solution: max-min Composition Rule

$$\mu_{C'}(z_i) = \max_{\text{rows}} \left(\min_{\text{columns}} \left(\begin{bmatrix} 0.0 \\ 0.0 \\ 0.0 \\ 0.8 \\ 0.0 \end{bmatrix}, \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \right) \right)$$

$$= \max_{\text{rows}} \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$= [0.1, 0.7, 0.8, 0.4]$$

Hence, $C' = 0.1/z_1 + 0.7/z_2 + 0.8/z_3 + 0.4/z_4$.

Example (cont.)

Solution: max-product Composition Rule

$$\mu_{C'}(z_i) = \max_{\text{row}} \left(\begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.8 & 0.0 \end{bmatrix} \cdot \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.1 & 0.2 & 0.2 & 0.2 \\ 0.1 & 0.7 & 1.0 & 0.4 \\ 0.1 & 0.7 & 0.8 & 0.4 \\ 0.1 & 0.1 & 0.1 & 0.1 \end{bmatrix} \right)$$

$$= \max_{\text{rows}} \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \\ 0.08 & 0.56 & 0.64 & 0.32 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

$$= \begin{bmatrix} 0.08 & 0.56 & 0.64 & 0.32 \end{bmatrix}$$

Hence, $C' = 0.08/z_1 + 0.56/z_2 + 0.64/z_3 + 0.32/z_4$.

Properties of Composition

- In general, any **sup-T** operator can be used to perform the compositional rule of inference, where T is a T-norm operator.
  - The sup-min and sup-product compositions are special cases of the sup-T composition where the min and product operations are used as T-norms, respectively.
- Let P and R be two fuzzy sets represented by two membership functions $\mu_P(x,z)$ and $\mu_R(z,y)$.

Definition: Sup-T and Inf-S Compositions

The Sup-T composition is characterized by

$$\mu_{P \circ R}(x,y) = \sup_{z \in Z}[\mu_P(x,z) \; T \; \mu_R(z,y)] \qquad T : \text{T-norm}$$

The Inf-S composition is characterized by

$$\mu_{P \otimes R}(x,y) = \inf_{z \in Z}[\mu_P(x,z) \; S \; \mu_R(z,y)] \qquad S : \text{S-norm}$$

Properties of Composition

Commutativity:
$$P \circ R = R \circ P$$
$$P \otimes R = R \otimes P$$

Associativity:
$$P \circ (Q \circ R) = (P \circ Q) \circ R$$
$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$$

Distributivity: $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$

DeMorgan's Laws:
$$\overline{P \circ R} = \overline{P} \otimes \overline{R}$$
$$\overline{P \otimes R} = \overline{P} \circ \overline{R}$$

Inclusion: if $R_1 \subset R_2$, then $P \circ R_1 \subset P \circ R_2$

Properties of Composition

Commutativity:
$$P \circ R = R \circ P$$
$$P \otimes R = R \otimes P$$

Associativity:
$$P \circ (Q \circ R) = (P \circ Q) \circ R$$
$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$$

Distributivity: $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$

DeMorgan's Laws:
$$\overline{P \circ R} = \overline{P} \otimes \overline{R}$$
$$\overline{P \otimes R} = \overline{P} \circ \overline{R}$$

Inclusion: if $R_1 \subset R_2$, then $P \circ R_1 \subset P \circ R_2$

Properties of Composition

Commutativity:
$$P \circ R = R \circ P$$
$$P \otimes R = R \otimes P$$

Associativity:
$$P \circ (Q \circ R) = (P \circ Q) \circ R$$
$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$$

Distributivity: $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$

DeMorgan's Laws:
$$\overline{P \circ R} = \overline{P} \otimes \overline{R}$$
$$\overline{P \otimes R} = \overline{P} \circ \overline{R}$$

Inclusion: if $R_1 \subset R_2$, then $P \circ R_1 \subset P \circ R_2$

Properties of Composition

Commutativity:
$$P \circ R = R \circ P$$
$$P \otimes R = R \otimes P$$

Associativity:
$$P \circ (Q \circ R) = (P \circ Q) \circ R$$
$$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$$

Distributivity: $(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$

DeMorgan's Laws:
$$\overline{P \circ R} = \overline{P} \otimes \overline{R}$$
$$\overline{P \otimes R} = \overline{P} \circ \overline{R}$$

Inclusion: if $R_1 \subset R_2$, then $P \circ R_1 \subset P \circ R_2$

Properties of Composition

Commutativity: $$P \circ R = R \circ P$$ $$P \otimes R = R \otimes P$$

Associativity: $$P \circ (Q \circ R) = (P \circ Q) \circ R$$ $$P \otimes (Q \otimes R) = (P \otimes Q) \otimes R$$

Distributivity: $$(P \cup Q) \circ R = (P \circ R) \cup (Q \circ R)$$

DeMorgan's Laws: $$\overline{P \circ R} = \overline{P} \otimes \overline{R}$$ $$\overline{P \otimes R} = \overline{P} \circ \overline{R}$$

Inclusion: if $R_1 \subset R_2$, then $P \circ R_1 \subset P \circ R_2$

Case Study

Problem Statement

- A discrete fuzzy relation $R(x_i, y_j)$ is given by the following membership function matrix, defined in $X \times Y$ where $X = \{0, 1, 2, 3, 4\}$ and $Y = \{0, 1, 2, 3, 4\}$.

Fuzzy Relation R

	$y_0 = 0$	$y_1 = 1$	$y_2 = 2$	$y_3 = 3$	$y_4 = 4$
$x_0 = 0$	0.0	0.4	0.7	0.3	0.0
$x_1 = 1$	0.1	0.5	0.8	0.4	0.1
$x_2 = 2$	0.6	0.7	1.0	0.5	0.2
$x_3 = 3$	0.3	0.4	0.9	0.7	0.4
$x_4 = 4$	0.0	0.1	0.5	0.3	0.1

Problem Statement

- The following discrete fuzzy sets are derived from $R(x_i, y_j)$:
  - $A(x_i) = Projection_{x_i} R(x_i, y_j)$
  - $B(y_j) = Projection_{y_j} R(x_i, y_j)$
  - $A_1(x_i) = R(x_i, 1)$
  - $A_2(x_i) = R(x_i, 2)$

Problem Statement

Part (a)

Determine and sketch the membership functions of the following fuzzy sets and fuzzy relations:

Required Membership Functions

1. $A(x_j)$
2. $A_1(x_i)$
3. $A_2(x_i)$
4. $B(y_j)$
5. $A_1 \cup A_2$
6. $A_1 \cap A_2$
7. α-cut of A_1 for $\alpha = 0.2, 0.42,$ and 0.5
8. $A \longrightarrow B$
9. $A \times B$
10. Cylindrical extension of A in $X \times Y$
11. Cylindrical extension of B in $X \times Y$

Problem Statement

Part (b)

Crisp Mapping of $R(x_i, y_j)$ in $X \times Y$ to $C(z_z)$ in Z

- The fuzzy relation $R(x_i, y_j)$ in $X \times Y$ is to be mapped to a fuzzy set $C(z_k)$ in Z, using the crisp function $z = x + y$.
- Determine and sketch C.

Solution (a).1

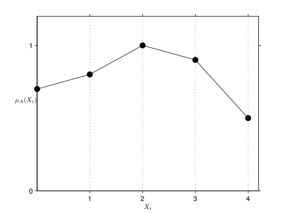
Membership Function

$$A(x_i) = Projection_{x_i} R(x_i, y_j)$$

$$\mu_A(x_i) = sup_{y_j}[\mu_R(x_i, y_j)]$$

$$A(x_i) = [\frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4}]$$

Graphical Representation



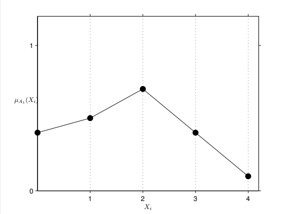
Solution (a).2

Membership Function

$$A_1(x_i) = R(x_i, 1)$$

$$A_1(x_i) = [\frac{0.4}{0}, \frac{0.5}{1}, \frac{0.7}{2}, \frac{0.4}{3}, \frac{0.1}{4}]$$

Graphical Representation



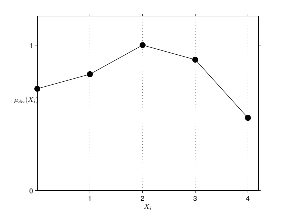
Solution (a).3

Membership Function

$$A_2(x_i) = R(x_i, 2)$$

$$A_2(x_i) = [\frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4}]$$

Graphical Representation

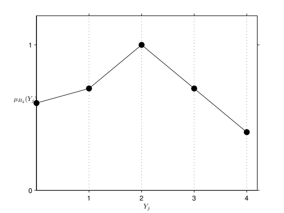


Solution (a).4

Membership Function

$$B(y_j) = Projection_{y_j} R(x_i, y_j)$$
$$\mu_B(y_j) = sup_{x_i}[\mu_R(x_i, y_j)]$$
$$B(y_j) = [\frac{0.6}{0}, \frac{0.7}{1}, \frac{1.0}{2}, \frac{0.7}{3}, \frac{0.4}{4}]$$

Graphical Representation



Solution (a).5

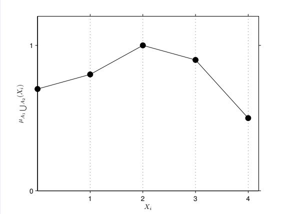
Membership Function

$$A_1 \cup A_2$$

$$\mu_{A_1 \cup A_2}(x_i) = max\{\mu_{A_1}(x_i), \mu_{A_2}(x_i)\}$$

$$A_1 \cup A_2 = [\frac{0.7}{0}, \frac{0.8}{1}, \frac{1.0}{2}, \frac{0.9}{3}, \frac{0.5}{4}$$

Graphical Representation



Solution (a).6

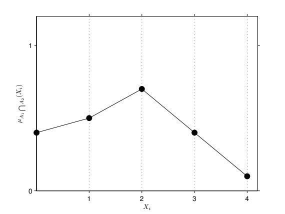
Membership Function

$A_1 \cap A_2$

$\mu_{A_1 \cap A_2}(x_i) = min\{\mu_{A_1}(x_i), \mu_{A_2}(x_i)\}$

$A_1 \cap A_2 = [\frac{0.4}{0}, \frac{0.5}{1}, \frac{0.7}{2}, \frac{0.4}{3}, \frac{0.}{4}$

Graphical Representation

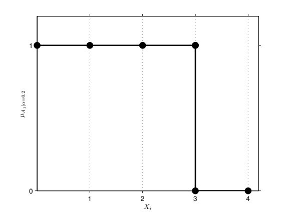


Solution (a).7 ($\alpha = 0.2$)

Membership Function

$$\mu_{F\alpha}(x) = 1 \quad \text{for } \mu_F(x) \geq \alpha$$
$$= 0 \quad \text{for } \mu_F(x) < \alpha$$
$$A_{1\alpha|\alpha=0.2} = [\frac{1.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{1.0}{3}, \frac{0.0}{4}$$

Graphical Representation

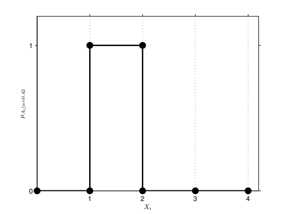


Solution (a).7 ($\alpha = 0.42$)

Membership Function

$$A_{1\alpha|\alpha=0.42} = [\frac{0.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{0.0}{3}, \frac{0.0}{4}$$

Graphical Representation

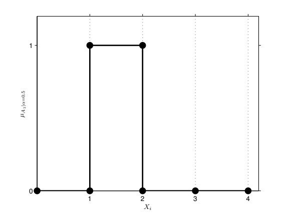


Solution (a).7 ($\alpha = 0.5$)

Membership Function

$$A_{1\alpha|\alpha=0.5} = [\frac{0.0}{0}, \frac{1.0}{1}, \frac{1.0}{2}, \frac{0.0}{3}, \frac{0.0}{4}$$

Graphical Representation

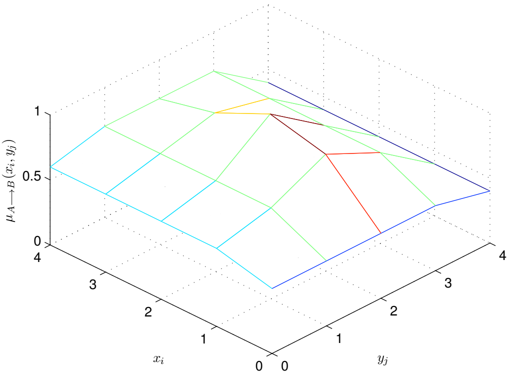


Solution (a).8 : Membership Function

$$\mu_{A \longrightarrow B} = min\{\mu_A(x_i), \mu_B(y_j)\}$$

$$A \longrightarrow B = \begin{array}{cc} & y_j \\ x_i & \begin{bmatrix} 0.6 & 0.7 & 0.7 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.8 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.9 & 0.7 & 0.4 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Solution (a).8 : Graphical Representation

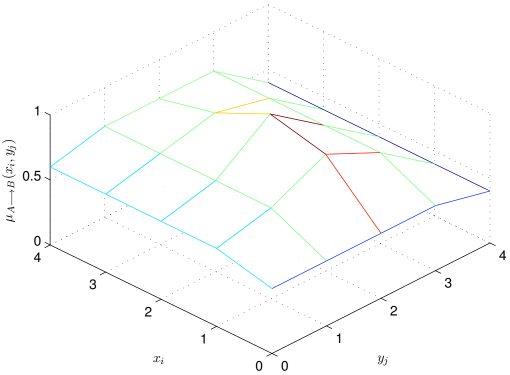


Solution (a).9 : Membership Function

$$\mu_{A\times B} = min\{\mu_A(x_i), \mu_B(y_j)\}$$

$$A \times B = \begin{array}{cc} & y_j \\ x_i & \begin{bmatrix} 0.6 & 0.7 & 0.7 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.8 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 0.9 & 0.7 & 0.4 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.4 \end{bmatrix} \end{array}$$

Solution (a).9 : Graphical Representation

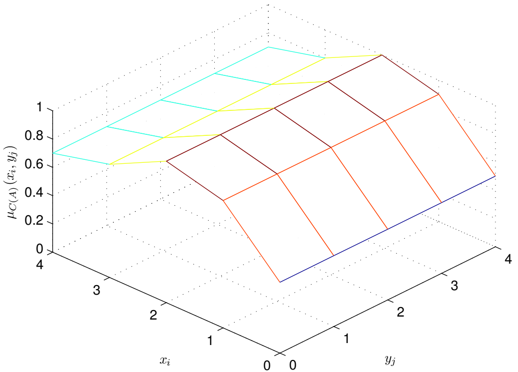


Solution (a).10 : Membership Function

$$C_{X \times Y}(A) = \sum_{x_i, y_j} \frac{\mu_A(x_i)}{x_i, y_j}$$

$$C_{X \times Y}(A) = \begin{array}{cc} & y_j \\ x_i & \begin{bmatrix} 0.7 & 0.7 & 0.7 & 0.7 & 0.7 \\ 0.8 & 0.8 & 0.8 & 0.8 & 0.8 \\ 1.0 & 1.0 & 1.0 & 1.0 & 1.0 \\ 0.9 & 0.9 & 0.9 & 0.9 & 0.9 \\ 0.5 & 0.5 & 0.5 & 0.5 & 0.5 \end{bmatrix} \end{array}$$

Solution (a).10 : Graphical Representation

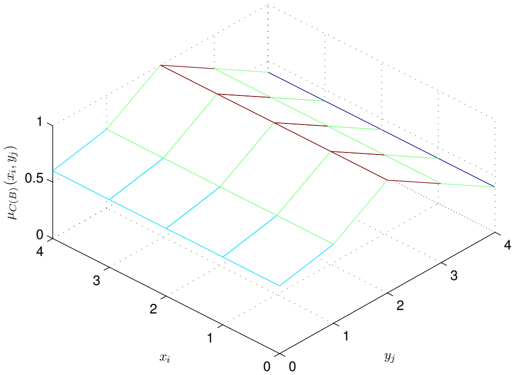


Solution (a).11 : Membership Function

$$C_{X\times Y}(B) = \sum_{x_i, y_j} \frac{\mu_B(y_j)}{x_i, y_j}$$

$$C_{X\times Y}(B) = \begin{array}{cc} & \begin{array}{ccccc} & & & y_j & \end{array} \\ x_i & \begin{bmatrix} 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \\ 0.6 & 0.7 & 1.0 & 0.7 & 0.4 \end{bmatrix} \end{array}$$

Solution (a).11 : Graphical Representation



Solution (b)

- The extension principle is applied, where the crisp relation $z = x + y$ is used.

$$\mu_C(z_k) = sup_{z=x+y}(\mu_R(x_i, y_j))$$

- The fuzzy relation $R(xi, yj)$ in the $X \times Y$ domain is mapped to $C(z_k)$ in the Z domain.
- Note that the diagonal broken lines are the $x + y = z$ Constant lines, for different feasible values of z.

$$C(z_k) = \{\frac{0.0}{0}, \frac{0.4}{1}, \frac{0.7}{2}, \frac{0.8}{3}, \frac{1.0}{4}, \frac{0.9}{5}, \frac{0.7}{6}, \frac{0.4}{7}, \frac{0.1}{8}\}$$

Solution (b)

Graphical Representation