

Proving Techniques

By- Piyush Jain

CS01: Mathematics-I

Join the class

Today's Agenda

- Rules of Inference
- Logical Fallacies

Homework from Last Class

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

Some Basic Terminology

Theorem	A statement that can be true. Also known as facts or results.	Pythagorean Theorem: In a right triangle, $a^2+b^2=c^2$
Proof	A valid argument from known or established true statements. <i>Proof validates theorem.</i>	Pythagorean theorem proof using two different distribution of area
Axiom (Postulate)	A statement that is assumed to be true without any proof.	Euclid's Axiom : Through any two points, there is exactly one straight line.
Conjecture	A statement proposed to be true. Not yet proved. Becomes theorem after proved.	Goldbach's Conjecture: Every even number greater than 2 is the sum of two prime numbers

Some Basic Terminology

Lemma	Less important theorem used to prove other results. Proved separately.	Euclid's Lemma: If a prime p divides ab , then p divides a or b . Used in proving the Fundamental Theorem of Arithmetic .
Corollary	Another theorem directly proved from previously proved theorem.	Theorem: The sum of the angles in a triangle is 180° . Corollary: Each angle in an equilateral triangle is 60° .

How Theorems are Stated

Generally, theorems state a property that is valid over a domain given a precondition.

For example,

- “All positive real numbers less than 1 have its square less than itself.”

Precondition \Rightarrow Being a real number between 0 and 1.

Property \Rightarrow Its square being less than itself.

$$P(x) = "0 < x < 1, x \in \mathbb{R}"$$

$$Q(x) = "x^2 < x, x \in \mathbb{R}"$$

$$\boxed{\forall x(P(x) \rightarrow Q(x))}$$

Rules of Inference

Prerequisite Concepts

- Proposition - Definition
- Logical Operation and Truth Table
- Negation, Conjunction, Disjunction (\neg , \wedge , \vee)
- Logical Implication (\rightarrow) & it's contrapositive
- Logical Equivalence Laws
- Tautology, Contradiction, Contingency
- Predicate and Quantifiers
- Operations on Quantifiers

Argument vs Proposition

Statements in mathematics are called **propositions**.

In contrast,

an **argument** is **sequence of statements**
with **premises** and **a conclusion**.

The **last statement** in an argument is called **conclusion**.

Whereas, **all statements except last** are called **premises**.

Argument - Example

Here is an example of argument.

- 1) If you are logged in on twitter, you can access your twitter inbox.
- 2) You are logged in on twitter.

hence

- 3) You can access your twitter inbox.

Here (1) and (2) are **premises**. Whereas (3) is a **conclusion**.

Argument - Formal Notation

If we extract unique statements from previous examples, we get

- p = “You are logged in on twitter.”
- q = “You can access your twitter inbox.”

Argument - Formal Notation

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- p = “You are logged in on twitter.”
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Then the argument in formal notation is written as

Argument Form

$$p \rightarrow q$$

$$p$$

$$\therefore q$$

Statement Form

$$((p \rightarrow q) \wedge p) \rightarrow q$$

Validity of an Argument

- An *argument* is said to be *valid*, “**if all the premises are true, then the conclusion must be true.**”
- A valid argument “derives a conclusion from given premises”.
- A valid argument can be represented as a **logical implication that is a tautology**.
- Formally, if conclusion q is drawn from premises $p_1, p_2, p_3, \dots, p_n$. Then the argument is said to be valid if and only if $(p_1 \wedge p_2 \wedge p_3 \wedge \dots \wedge p_n) \rightarrow q$ is tautology.

Rules of Inference - Valid Argument

Logical Implication is a mathematical proposition that evaluates to either true or false. Whereas a valid argument is always a tautology in the form

$$(p_1 \wedge p_2 \wedge \dots \wedge p_n) \rightarrow q$$

Since, a valid argument “derives conclusion from premises”, we can interpret this Logical Implication as “premises implies conclusion”. This is kind of rule that conjures a new proposition(conclusion) from given proposition.

The ways to “derive conclusion from premises”, using valid arguments are known as **rules of inference**.

Rules of Inference - Modus Ponens

Rule:-

If $p \rightarrow q$ and p , then q .

Example:-

If it is raining, then the ground will be wet, and

It is raining.

Then: *The ground is wet.*

Rules of Inference - Modus Ponens

Example:-

Premise 1: *If you study hard, then you pass the exam.*

Premise 2: *You study hard.*

Conclusion: *You pass the exam.*

Rules of Inference - Modus Tollens

Rule:-

If $p \rightarrow q$ and not q , then not p .

Example:-

If the alarm goes off, then the house is not secure.

The house is secure.

Then: *The alarm did not go off.*

Rules of Inference - Modus Tollens

Example:-

Premise 1: *If it is raining, then the ground is wet.*

Premise 2: *The ground is not wet.*

Conclusion: *It is not raining.*

TABLE 1 Rules of Inference.

<i>Rule of Inference</i>	<i>Tautology</i>	<i>Name</i>	<i>Example</i>
$\begin{array}{l} p \\ p \rightarrow q \\ \hline \therefore q \end{array}$	$(p \wedge (p \rightarrow q)) \rightarrow q$	Modus ponens	<p>You have a car.</p> <p>If you have a car, you are below poverty line.</p> <hr/> <p>you are below poverty line.</p>
$\begin{array}{l} \neg q \\ p \rightarrow q \\ \hline \therefore \neg p \end{array}$	$(\neg q \wedge (p \rightarrow q)) \rightarrow \neg p$	Modus tollens	<p>Raman does not like science.</p> <p>If Raman is scientist, then Raman likes science.</p> <hr/> <p>Raman is not scientist.</p>
$\begin{array}{l} p \rightarrow q \\ q \rightarrow r \\ \hline \therefore p \rightarrow r \end{array}$	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	Hypothetical syllogism	<p>If it snows, then it is cold.</p> <p>If it is cold, then you might get sick.</p> <hr/> <p>If it snows, then you might get sick.</p>
$\begin{array}{l} p \vee q \\ \neg p \\ \hline \therefore q \end{array}$	$((p \vee q) \wedge \neg p) \rightarrow q$	Disjunctive syllogism	<p>Raman studies either Science or Maths everyday.</p> <p>Raman did not studied science today.</p> <hr/> <p>Raman studied Maths today.</p>

Logical Fallacies

Logical **fallacies** are common mistakes in reasoning that lead to **invalid arguments**. Being aware of these helps avoid errors in mathematical proofs.

Example:-

- “If you don’t study, you fail the exam.”
- “You failed the exam”
- Therefore, “You must not have studied.”

This feels like valid argument, but it is not.

Here, the given student’s pen might have stopped working while writing exam.

So, always try to make valid arguments.

Logical Fallacies

- **Affirming the Consequent** : Proving the hypothesis if conclusion is true.

$$((p \rightarrow q) \wedge q) \rightarrow p$$

- *Example :-*
 - If it rains, the ground becomes wet.
 - The ground is wet.
 - Incorrect conclusion: Therefore, it must have rained.
(perhaps someone watered the lawn!)

Logical Fallacies

- **Denying the Antecedent:** Proving the hypothesis if conclusion is true.

$$((p \rightarrow q) \wedge \neg p) \rightarrow \neg q$$

- *Example :-*
 - If you study, you will pass the exam
 - You didn't study.
 - Incorrect conclusion: Therefore, you won't pass the exam.
(perhaps you already know the material!)

Proof By Example and Counterexample

	Proof By Example	Proof By Counterexample
To Prove	$\exists xP(x)$	$\forall xP(x)$
Method	Find a “c” such that $P(c)$ is true. This will prove above statement.	Find a “c” such $P(c)$ is false. This will disprove above statement.
Example $P(x)$	“There is a integer who is bigger than its cube”	“For all prime number p, $2^p - 1$ is prime”
c	-2	11
$P(c)$	$(-2) > (-2)^3 \Rightarrow (-2) > (-8)$	$2^{11} - 1 = 2047 = 23 \times 89$

Quiz Quiz Quiz



**See You Guys
in Next
Session :)**