Basic Arithmetic Foundations (Part 3)

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CS01: Mathematics-I

Join the class

Today's Agenda

- LCM
- GCD
- Euclidean Algorithm on Numbers
- Co-prime numbers

Final Thought on Fundamental Theorem of Arithmetic

“What seems obvious to the child becomes awe-inspiring to the philosopher.”

Uniqueness of prime factorization is not trivial.

It is a miracle of structure — and we must pause to appreciate it.

Perfect Number

- A Perfect Number is a positive integer where the sum of its proper divisors (divisors other than itself) adds back to itself.

Example 1: Consider 6

- Divisors of 6 are 1, 2, 3, 6.
- Proper divisors are 1, 2, 3.
- Sum of proper divisors:
  1 + 2 + 3 = 6
- Therefore, 6 is a perfect number.

Example 2: Consider 28

- Divisors of 28 are 1, 2, 4, 7, 14, 28.
- Proper divisors are 1, 2, 4, 7, 14.
- Sum of proper divisors:
  1 + 2 + 4 + 7 + 14 = 28
- Therefore, 28 is a perfect number.

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Twin Prime Conjecture

- In the list of primes, it is sometimes true that consecutive odd numbers are prime (Twin Primes).

- **Examples:** (3,5), (5,7), (11,13), (17,19), (29,31), (41,43), (59,61), (71,73),...

- **Question:** Are there infinitely many twin primes?

Goldbach Conjecture

- Every even integer greater than 2 can be expressed as the sum of two prime numbers.
- **Examples:**
    - 4 = 2 + 2
    - 6 = 3 + 3
    - 8 = 3 + 5
    - 10 = 3 + 7 = 5 + 5
    - 12 = 5 + 7,...

Can you prove it?

Do you want to earn millions? Solve!

- While mathematicians have found very large twin primes, no proof exists that the number of twin primes is infinite.

- Goldbach Conjecture has been verified for extremely large numbers (up to 4×10^{18}), but no general proof exists according to Wikipedia.

- So, **Twin Prime Conjecture** and **Goldbach Conjecture** both are **unsolved** [with prize money in millions!].

- Terence Tao is working on these!

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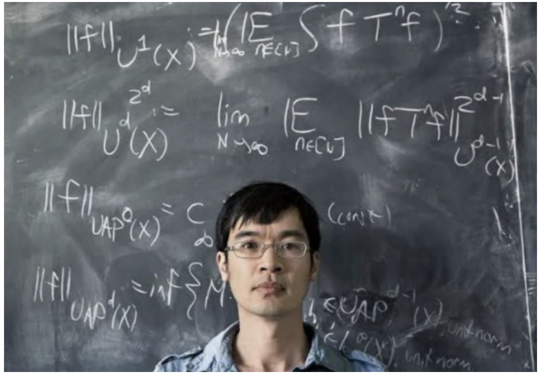
Terence Tao’s Strategy

- **STEP 1:** Collect data — write down numbers (by hand or on computer).
- **STEP 2:** Look for patterns and relationships.
- **STEP 3:** Make a guess, formulate a conjecture.
- **STEP 4:** Test your conjecture on more examples.
- **STEP 5:** Try to prove it!



Voila! 🎉

💡 It takes patience, practice, and perseverance to follow these 5 steps — but the reward is deep mathematical insight.

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Division Algorithm

Introduction to Division

- Among integers, we can perform addition, multiplication, subtraction, and division.
- Division gives birth to interesting pattern hunting tools!
- Let's learn the **Division Algorithm**.
- **Why call it an Algorithm?**
  - An Algorithm is a "finite" set of rules or steps to solve a problem or to achieve a certain task.
  - So, while dividing an integer 'a' with an integer 'b' (assuming $a > b$), we need a "finite" set of "rules" to achieve Division.

- Consider dividing 7 by 3.
- We can take two groups of 3 from 7:

$$7 = 2 \times 3 + 1$$

- Here, 1 is the remainder.
- Why algorithm? I could have done $7 = 3 \times \mathbf{3} - 2$ or $7 = 0 \times \mathbf{3} + 7$ nobody stops me!
- We need to fix the copies of b.How much it leaves behind.One rule we impose is;
- **The remainder 'r' should be greater than or equal to 0 and less than 'b' (the divisor)**.

The Division Algorithm

- Given integers a and b, with $b \neq 0$, there exist **unique** integers q (quotient) and r (remainder) such that:

$$a = q \cdot b + r$$

- where $0 \leq r < |b|$.
- This ensures a unique remainder.
- Examples:
  - Single digit: $7 = 2 \cdot 3 + 1$
  - Two digits: $15 = 4 \cdot 3 + 3$
  - Three digits: $125 = 10 \cdot 12 + 5$
  - Seven digits: $1234567 = 123456 \cdot 10 + 7$

Rapid fire

Can the remainder ever be negative?

Rapid fire

Can the remainder ever be negative?

Answer: No

- ➔ If we allow negative remainder, then we lose the uniqueness property of division algorithm.
- ➔ Remainder is defined to be non-negative in math because we want the remainder to represent a "leftover" quantity. So, it should always be a small, clean, and positive offset.

Divisibility and Notation

- Now when remainder is Zero i.e. $r = 0$, we record that situation with a notation.
- **English:** "d divides a"
- **Notation:** $d \mid a$
- **Examples:**
  - English: "2 divides 6" $\rightarrow$ Notation: $2 \mid 6$ (since $6 = 2 \times 3 + 0$)
  - English: "5 divides 15" $\rightarrow$ Notation: $5 \mid 15$ (since $15 = 5 \times 3 + 0$)
  - English: "2 does not divide 7" $\rightarrow$ Notation: $2 \nmid 7$ (since $7 = 2 \times 3 + 1$)
- A divisor d of an integer a means $d \mid a$.

Common Divisors and Properties

- If a, b are arbitrary integers, then d is said to be a **common divisor** if d divides both a and b.
  - i.e., $d \mid a$ and $d \mid b$.
- **Definition of Divisibility:** $d \mid a$ means there exists an integer k such that $a = k \cdot d$.
- **Properties of Divisibility:** If $d \mid a$ and $d \mid b$, then:
  - $d \mid (a + b)$ ✓
  - $d \mid (a - b)$ ✓
  - $d \mid (ab)$ ✓

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Properties of Divisibility

If $d \mid a$ and $d \mid b$, then:

1. $d \mid (a+b)$:

$$a = dk,\ b = dl \implies a + b = d(k+l).$$

2. $d \mid (a-b)$:

$$a - b = d(k-l).$$

3. $d \mid (ab)$:

$$ab = (dk)(dl) = d^2(kl).$$

👉 Hence, d divides $a+b$, $a-b$, and ab.

Problems on Divisibility

- **Problem 1:** Show that:
  - The product of any 3 consecutive integers is divisible by 6.
  - The product of any 4 consecutive integers is divisible by 24.
  - The product of any 5 consecutive integers is divisible by 120.

Do you see a pattern?

3 consecutive ⇒ divisible by 6

Among any 3 consecutive integers, one is a multiple of 3, and at least one is even.

So the product has a factor $3 \times 2 = 6$.

4 consecutive ⇒ divisible by 24

In any 4 consecutive integers:

- Exactly two are even, and among them one is a multiple of 4 ⇒ even part contributes $4 \times 2 = 8$.
- One is a multiple of 3.

  Thus the product has $8 \times 3 = 24$.

5 consecutive ⇒ divisible by 120

In any 5 consecutive integers:

- There are at least two evens, one of which is a multiple of 4 ⇒ even part gives $4 \times 2 = 8$.
- One is a multiple of 3.
- One is a multiple of 5.

  Hence the product has $8 \times 3 \times 5 = 120$.

Problems on Divisibility

Observed Pattern:

The product of any k consecutive integers is divisible by k!.

Problems on Divisibility

- **Problem 2:** Check if $2^{35} - 1$ is divisible by 3.

Problems on Divisibility

▶ **Problem 2:** Check if $2^{35} - 1$ is divisible by 3.

- If the exponent is **odd**, 2^n leaves remainder **2** when divided by 3.
- If the exponent is **even**, 2^n leaves remainder **1** when divided by 3.

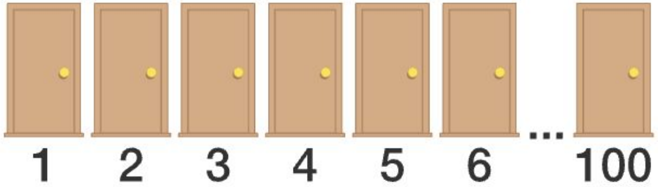
So the remainders alternate:

$$2, 1, 2, 1, 2, 1, \ldots$$

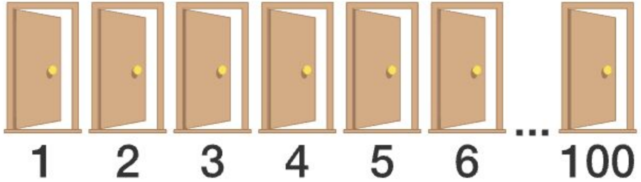
- 2^{35} leaves remainder 2 when divided by 3.
- So, 2^{35} - 1 leaves remainder 2 - 1 = **1** when divided by 3.

100 Doors Revisited

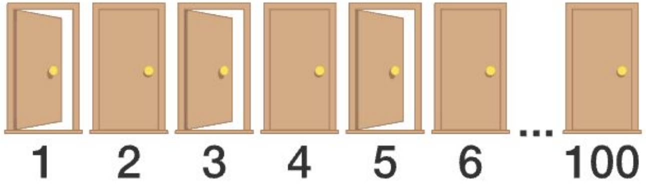
- Let's take a look at the hallway of 100 doors.
- In the hallway of 100 doors, 100 people numbered 1 to 100 are standing in a long hallway that has 100 closed doors also numbered 1 to 100:



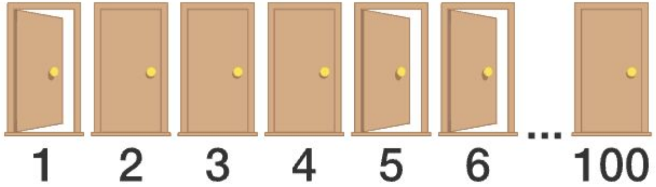
- Person 1 walks down the hallway and opens every door:



- Person 2 walks down the hallway and closes every door that is a multiple of 2:



- Person 3 walks down the hallway and changes every door that is a multiple of 3. That is, if the door is open, they close it, and if it is closed, they open it:



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- Person 4 changes every door that is a multiple of 4, Person 5 every door that is a multiple of 5, etc. This continues until all 100 people have walked down the hallway and changed their doors.

Question:

What is the number of the first door changed by both Person 6 and Person 8?

Lowest Common Multiple

- **Person 6** changes the doors that are **multiples of 6**:
  6, 12, 18, 24, 30, 36……….

- **Person 8** changes the doors that are **multiples of 8**:
  8, 16, 24, 32, 40………..

Question:
What is the number of the first door changed by both Person 6 and Person 8?
→ **24**

Lowest Common Multiple

Q. Find the LCM of 294 and 364.

Finding LCM using Prime Factorization

$$294$$

$$2^1 \times 3^1 \times 7^2 \times 13^0$$

$$364$$

$$2^2 \times 3^0 \times 7^1 \times 13^1$$

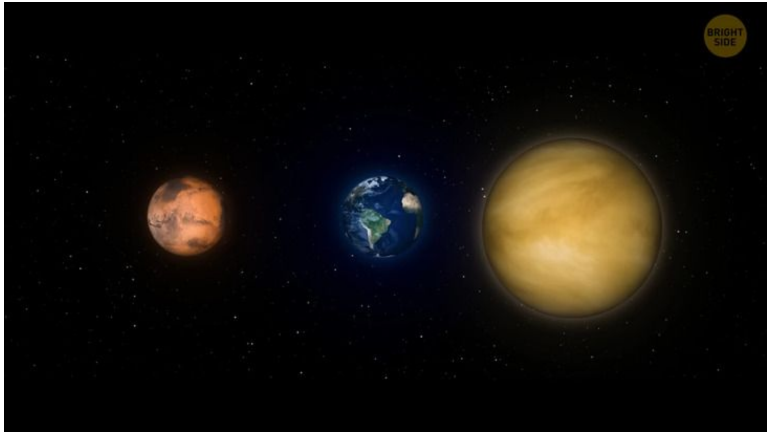
$$2^2 \times 3^1 \times 7^2 \times 13^1$$

$$\operatorname{lcm}(294, 364)$$

Ans. 7644

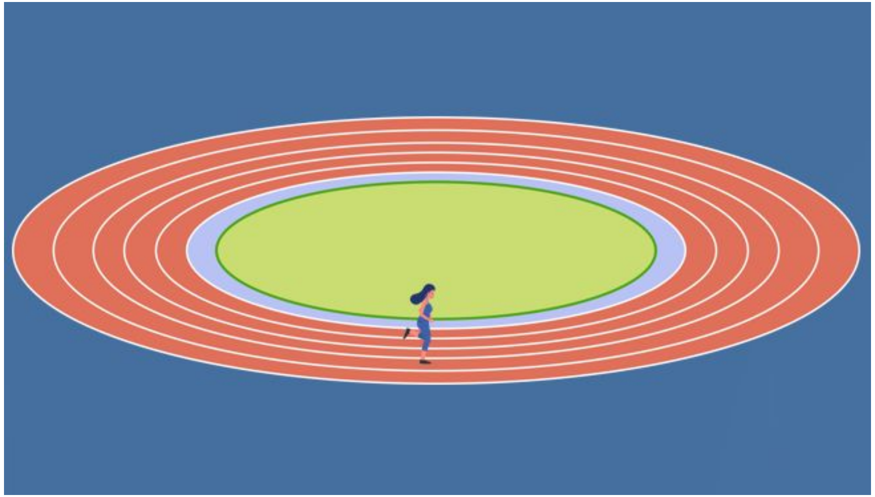
Example

Earth rotates around the sun in 365 days and Venus rotates around the sun in 730 days. Whenever Earth and Venus cross each other, a unique phenomenon occurs due to which the population of the Olive Ridley turtle reduces to half. Considering the initial population of Olive Ridley turtle is 10000, what will their population after 10 years?



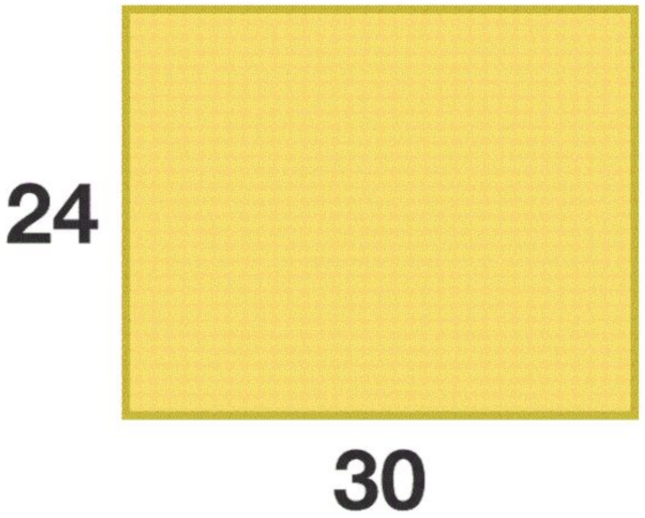
Example

Q. Three runners Manshu, Pintu, and Dally who are running around a circular track can complete one lap in 250, 400, and 600 seconds, respectively. After how long will they meet at the starting point next if they start together?



Example

What is the largest size of square tiles you can use to tile a 24×30 rectangle if all the tiles must be the same size?



Greatest Common Divisor

- If a and b are arbitrary integers, and if d is a common divisor, and when one of the integers is non-zero, then there will be a “finite” number of positive common divisors for a and b. The greatest among them is the **GCD**.

- **Mathematical Definition:** Let a and b be given integers, with at least one of them different from 0. An integer d is the Greatest Common Divisor of a and b, denoted as gcd(a, b), if:
    - 1. $d \mid a$ and $d \mid b$ (i.e., d is a common divisor)
    - 2. If $c \mid a$ and $c \mid b$ for any integer c, then $c \leq d$. (i.e., d is the greatest among common divisors)

- **GCD is also called as Highest Common Factor (HCF).**

Greatest Common Divisor

Q. Find the GCD of 294 and 364.

How to Find GCD? (School Days Method)

- **Method 1:** Listing Divisors
    1. Find all divisors of a.
    2. Find all divisors of b.
    3. List their common divisors.
    4. Find the greatest among the common divisors.
- **Limitation:** It is cumbersome for big a and b.

Finding GCD using Prime Factorization

$$294 \qquad\qquad\qquad 364$$

$$2^1 \times 3^1 \times 7^2 \times 13^0 \qquad 2^2 \times 3^0 \times 7^1 \times 13^1$$

$$2^1 \times 3^0 \times 7^1 \times 13^0$$

$$\gcd(294, 364)$$

Therefore, we have

$$\begin{aligned} \gcd(294, 364) &= 2^1 \times 3^0 \times 7^1 \times 13^0 \\ &= 2^1 \times 1 \times 7^1 \times 1 \\ &= 2^1 \times 7^1. \end{aligned}$$

Euclidean Algorithm

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Efficient Way to Find GCD: Euclidean Algorithm

- The idea is to use the repeated application of the Division Algorithm.
- Euclid wrote about this in his 7th book, ”Elements”.

Euclidean Algorithm

- Given a and b, with $b \neq 0$, assuming $a \geq b > 0$.
- First apply Division Algorithm (DA) on a and b, which gives unique q_1 and r_1:

$$a = q_1 b + r_1; \quad 0 \leq r_1 < b$$

- Apply DA repeatedly:

$$\begin{aligned} b &= q_2 r_1 + r_2; \quad & 0 \leq r_2 < r_1 \\ r_1 &= q_3 r_2 + r_3; \quad & 0 \leq r_3 < r_2 \\ &\vdots & \vdots \end{aligned}$$

Euclidean Algorithm

- The process continues until the remainder is zero:

$$r_{n-2} = q_n r_{n-1} + r_n; \quad 0 < r_n < r_{n-1}$$
$$r_{n-1} = q_{n+1} r_n + 0 \quad \rightarrow \text{STOPPING CONDITION}$$

- The **Greatest Common Divisor (GCD)** is the last non-zero remainder, r_n.

Finding GCD using Euclidean Algorithm

Compute **GCD(1701, 3768)**

- $3768 = 2 \times 1701 + 366$
- $1701 = 4 \times 366 + 237$
- $366 = 1 \times 237 + 129$
- $237 = 1 \times 129 + 108$
- $129 = 1 \times 108 + 21$
- $108 = 5 \times 21 + 3$
- $21 = 7 \times 3 + 0$ (STOP!, **GCD = 3**)

Imagine, how much time method 1 will take? (Computational Thinking)

Euclidean Algorithm for finding GCD of 2 numbers

Question: Find the GCD of 48 and 18.

Solution:

Apply Division Algorithm:

- **Step 1:** 48 = 2 x 18 + 12
- **Step 2:** 18 = 1 x 12 + **6**
- **Step 3:** 12 = 2 x 6 + 0

So, gcd(48, 18) = 6

Relation between LCM and GCD

The product of the LCM and GCD of two numbers equals the product of the numbers themselves: **gcd(a, b) × lcm(a, b) = a × b**

- LCM(12,15) × GCD(12,15) = 12×15
- 60×3 = 180, and 12×15 = 180

Example

You have two positive integers, A and B, with the following properties:

1. The Least Common Multiple (LCM) of A and B is 90.
2. The Greatest Common Divisor (GCD) of A and B is 3.
3. The sum of A and B is 33.

What is A and B?

Example

Q. Find sum of two positive integers x and y such that the LCM of x and y is 360 and their GCD is 15.

Example

Given the numbers 35 and 64, identify the common divisors of these two numbers.

Ans:

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Example

Given the numbers 35 and 64, identify the common divisors of these two numbers.

Ans: 1

Relatively Prime

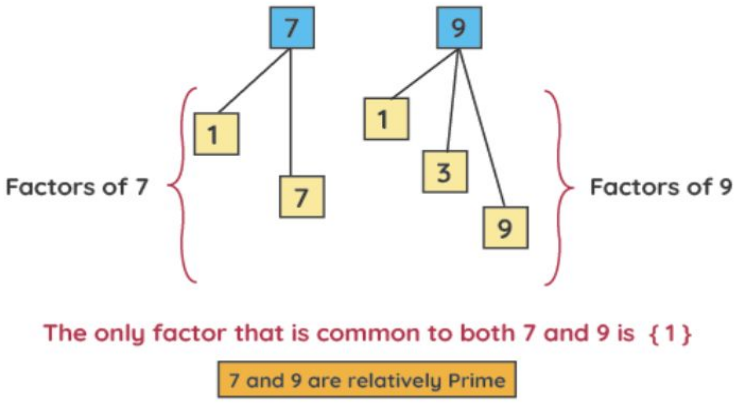
If gcd(a, b) = 1, we say that a and b are relatively prime.

Ex: gcd(*17*, *11*) = 1

Relatively Prime

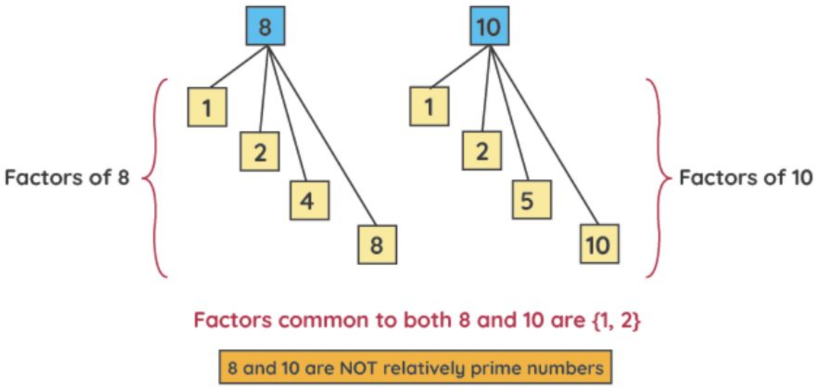
Example 1: gcd(7,9) = 1

So, 7 and 9 are relatively prime numbers.



Example 2: gcd(8,10) = 2

So, 8 and 10 are not relatively prime numbers.



Example

Q. How many of the numbers 1, 2, 3 ,…, 100 are relatively prime to 101?

Ans:

Example

Q. How many of the numbers 1, 2, 3 ,…, 100 are relatively prime to 101?

Ans: 100

Quiz Quiz Quiz

Key Takeaways

Today we learnt :

- LCM
- GCD
- Euclidean Algorithm on Numbers
- Co-prime numbers

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See You Guys in Next Session :)

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