Basic Arithmetic Foundations (Part 2)

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CS01: Mathematics-I

Newton School of Technology

Today’s Agenda

- Exponentiation
- Logarithms
- Multiples
- Factors
- Prime Numbers
- Prime Factorization
- Number of Positive Divisors
- Division algorithm

Exponentiation

Exponentiation (a^b)

- Exponentiation is a mathematical operation used to express repeated multiplication of the same number.

- If a is a number and n is a positive integer, then $$a^n = \underbrace{a \cdot a \cdot a \cdots a}_{n \text{ times}}$$

  ➢ a is the base
  ➢ n is the exponent (or power)

- Multiplication is repeated addition.
  - 2 * 4 = 2 + 2 + 2 + 2

- Exponentiation is repeated multiplication.
  - 2^4 = 2 * 2 * 2 * 2

Exponentiation (a^b)

Type of Exponent	Example	Meaning
Positive Integer	2^3	$2 \cdot 2 \cdot 2 = 8$
Zero Exponent	a^0	Always 1 (if $a \neq 0$)
Negative Exponent	a^{-n}	$\frac{1}{a^n}$
Fractional Exponent	$a^{1/n}$	$\sqrt[n]{a}$ (the *n*th root of *a*)
	$a^{m/n}$	$\sqrt[n]{a^m}$

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Exponentiation (a^b)

Rule	Expression	Example
Product of Powers	$x^m \cdot x^n = x^{m+n}$	$2^3 \cdot 2^4 = 2^7$
Quotient of Powers	$\frac{x^m}{x^n} = x^{m-n}$	$\frac{5^6}{5^2} = 5^4$
Power of a Power	$(x^m)^n = x^{m \cdot n}$	$(3^2)^4 = 3^8$
Power of a Product	$(xy)^n = x^n \cdot y^n$	$(2 \cdot 5)^3 = 2^3 \cdot 5^3$
Power of a Quotient	$\left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$	$\left(\frac{4}{3}\right)^2 = \frac{4^2}{3^2}$
Zero Exponent	$x^0 = 1$, for $x \neq 0$	$7^0 = 1$
Negative Exponent	$x^{-n} = \frac{1}{x^n}$	$2^{-3} = \frac{1}{2^3} = \frac{1}{8}$

Logarithms

Why Logarithms?

- Exponentiation answers:

  👉 “What happens when you repeatedly multiply a number?”

- While logarithms reverse this process:

  👉 “How many times must I multiply a number to get a result?”

- **Logarithms are the inverse of exponentiation**, just like subtraction is the inverse of addition.

Logarithms

$$a^c = b \quad \rightleftarrows \quad \log_a b = c$$

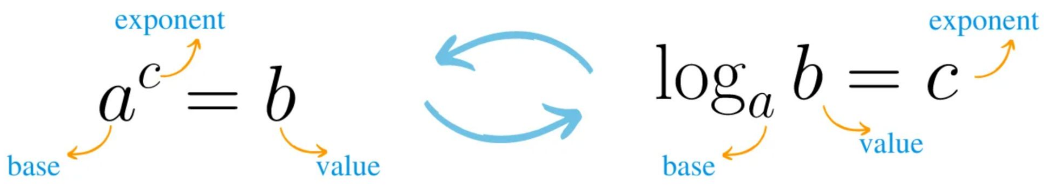
a = base, b = value, c = exponent

where

- a and b are positive real numbers
- a ≠ 1 (**Why???**)

Exponential Form	Logarithmic Form
$5^4 = 625$	$\log_5(625) = 4$
$15^2 = 225$	$\log_{15}(225) = 2$
$8^3 = 512$	$\log_8(512) = 3$

Logarithms



a = base, b = value, c = exponent

Exponential Form	Logarithmic Form
$5^4 = 625$	$\log_5(625) = 4$
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$8^3 = 512$	$\log_8(512) = 3$

where
- a and b are positive real numbers
- $a \neq 1$ (**Why???**)

Reason:

➔ $1^c = 1$

➔ We will not be able to fix c.

Logarithms

Type	Notation	Meaning
Common Log	$\log x$	Log base 10: $\log_{10} x$
Natural Log	$\ln x$	Log base e: $\log_e x$
Binary Log	$\log_2 x$	Log base 2: used in CS/IT

- ‘e’ refers to Euler's number, which is approximately 2.71828.
- ‘e’ is a mathematical constant that is the base of the natural logarithm and the exponential function.

Logarithms

Rule	Formula	Example
Product Rule	$\log_a(xy) = \log_a x + \log_a y$	$\log_{10}(2 \cdot 5) = \log 2 + \log 5$
Quotient Rule	$\log_a \left(\frac{x}{y}\right) = \log_a x - \log_a y$	$\log_{10} \left(\frac{8}{2}\right) = \log 8 - \log 2$
Power Rule	$\log_a(x^r) = r \log_a x$	$\log_2(8^2) = 2 \log_2 8$
Change of Base	$\log_a b = \frac{\log_c b}{\log_c a}$	$\log_2 8 = \frac{\log 8}{\log 2}$
Inverse Rule	$\log_a a = 1, \log_a 1 = 0$	$\log_5 5 = 1, \log_5 1 = 0$

Product Rule of Logarithm: Proof

Let:

$$\log_a m = p \Rightarrow a^p = m \quad (1)$$

$$\log_a n = q \Rightarrow a^q = n \quad (2)$$

Multiplying equations (1) and (2):

$$mn = a^p \cdot a^q = a^{p+q}$$

Now, taking the logarithm base a of both sides:

$$\log_a(mn) = \log_a(a^{p+q}) = p + q$$

Since $p = \log_a m$ and $q = \log_a n$, we substitute:

$$\log_a(mn) = \log_a m + \log_a n$$

Example: $\log_2 15 = \log_2(5 \times 3) = \log_2 5 + \log_2 3$

Quotient Rule of Logarithm: Proof

$$\log_a m = p \Rightarrow a^p = m \tag{1}$$

$$\log_a n = q \Rightarrow a^q = n \tag{2}$$

Now let

$$\log_a \left(\frac{m}{n}\right) = r \Rightarrow a^r = \frac{m}{n} \tag{3}$$

From equations (1) and (2), we substitute into (3):

$$\frac{m}{n} = \frac{a^p}{a^q} = a^{p-q}$$

So,

$$a^r = a^{p-q} \Rightarrow r = p - q$$

Now substituting back:

$$\log_a \left(\frac{m}{n}\right) = p - q = \log_a m - \log_a n$$

Power Rule of Logarithm 1: Proof

Let

$$\log_a(m^n) = p \Rightarrow m^n = a^p \Rightarrow m = a^{\frac{p}{n}}$$

Now, take logarithm base a of both sides:

$$\log_a m = \frac{p}{n} \Rightarrow n \log_a m = p \Rightarrow \log_a(m^n) = n \log_a m$$

Power Rule of Logarithm 2: Proof

$$\log_{a^n} b^m = \frac{m}{n} \log_a b$$

$$\log_{a^n} b^m = p \Rightarrow b^m = (a^n)^p = a^{np} \Rightarrow b = (a^{np})^{\frac{1}{m}} = a^{\frac{np}{m}}$$

Now take logarithm base a on both sides:

$$\log_a b = \frac{np}{m} \Rightarrow \frac{m}{n} \log_a b = p \Rightarrow \log_{a^n} b^m = \frac{m}{n} \log_a b$$

Change of Base Rule of Logarithm: Proof

If $a \neq 1, b \neq 1$ and a, b, m are positive real numbers, then:

$$\log_a m = \frac{\log_b m}{\log_b a}$$

$$\log_a m = p \Rightarrow a^p = m$$

Also,

$$\log_b a = q \Rightarrow a = b^q \Rightarrow a^p = (b^q)^p = b^{pq}$$

So,

$$m = b^{pq} \Rightarrow \log_b m = pq \Rightarrow p = \frac{\log_b m}{\log_b a}$$

Therefore,

$$\log_a m = \frac{\log_b m}{\log_b a}$$

Rapid Fire

- $\log_b a = 1 / \log_a b$

True or False ?

Rapid Fire

- $\log_b a = 1 / \log_a b$

→ True

→ Use change of base property and inverse rule of logarithms to prove it. $\log_b a = \log_a a / \log_a b = 1 / \log_a b$

Multiples, Factors, Prime Numbers, Composite Numbers

Multiples

- A **multiple** of a number is the result of multiplying that number by any integer.
- It can be said that **b is a multiple of a** if **b = n.a** for some integer **n**, which is called the **multiplier**.
- Think of it as the numbers you get in the table of that number.
- Every number has infinite multiples but 0 has only one unique multiple.
- Every number is a multiple of itself. For example, 5 is a multiple of 5 [5x1=5]
- Zero is a multiple of every number (since $n \times 0 = 0$)

Multiples

- **Multiples of 2:** …, -6, -4, -2, 0, 2, 4, 6, 8, 10, …
- **Multiples of 3:** …, -9, -6, -3, 0, 3, 6, 9, 12, 15, …
- **Multiples of 5:** …, -15, -10, -5, 0, 5, 10, 15, 20, 25, …
- **Multiples of 10:** …, -1000, -100, -10, 0, 10, 100, 1000, 10000, …

Notation

- 4/2 is an integer $\rightarrow$ 2 divides 4

$$\frac{4}{2} = 2 \in \mathbb{Z} \quad \Rightarrow \quad 2 \mid 4$$

- 7/2 is not an integer $\rightarrow$ 2 doesn't divide 7

$$\frac{7}{2} = 3.5 \notin \mathbb{Z} \quad \Rightarrow \quad 2 \nmid 7$$

Factors

- A **factor** of a number divides the number exactly, leaving no remainder.
- Factors are always finite.
- Factors are always less than or equal to the number.
- Every number has 1 and itself as factors.

Factors

- **Factors of 1:** 1 → **[Reason:** 1 | 1**]**
- **Factors of 2:** 1, 2 → **[Reason:** 1 | 2, 2 | 2**]**
- **Factors of 3:** 1, 3 → **[Reason:** 1 | 3, 3 | 3**]**
- **Factors of 4:** 1, 2, 4 → **[Reason:** 1 | 4, 2 | 4, 4 | 4**]**
- **Factors of 10:** 1, 2, 5, 10 → **[Reason:** 1 | 10, 2 | 10, 5 | 10, 10 | 10**]**
- **Factors of 16:** 1, 2, 4, 8, 16 → **[Reason:** 1 | 16, 2 | 16, 4 | 16, 8 | 16, 16 | 16**]**

Multiples v/s Factors

Feature	Multiples	Factors
Definition	Result of multiplying	Numbers that divide
Count	Infinite	Finite
Example	Multiples of 3: 3, 6, 9, 12...	Factors of 12: 1, 2, 3...
Size	Can be larger than the number	Always ≤ the number

Prime Number

- A prime number is a natural number greater than 1 that has exactly two distinct factors:

  👉 1 and itself

- **Examples of Prime numbers:** 2, 3, 5, 7, 11, 13, 17, 19, 23, …

- 2 is the only even prime number.

- 1 is NOT a prime number (It has only one factor — itself).

Composite Number

- A composite number is a natural number greater than 1 that has more than two factors.
- **Examples of Composite Numbers:** 4, 6, 8, 9, 10, …
- $4 = 2 \times 2 \rightarrow$ Factors: 1,2,4
- $6 = 2 \times 3 \rightarrow$ Factors: 1,2,3,6
- 1 is neither prime nor composite.

Prime Factorization

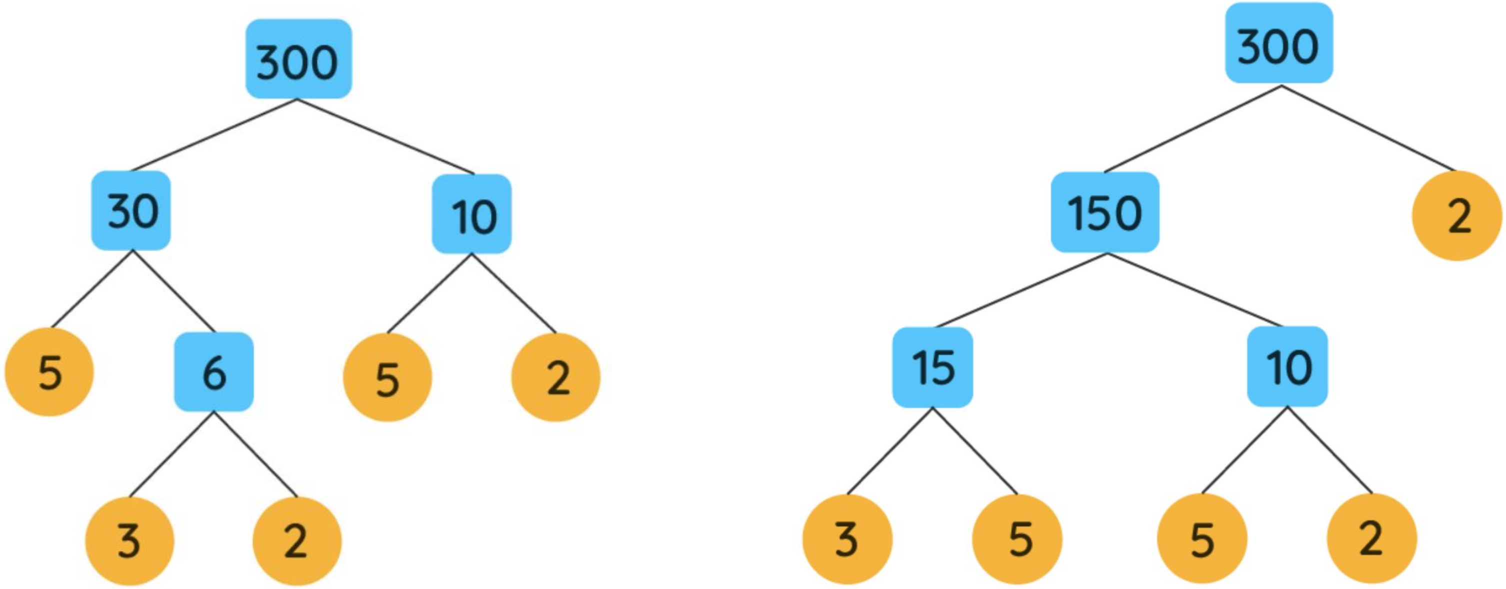
Prime Factorisation

- Prime factorisation is the process of breaking down a number into a product of its prime numbers.

- **Example:**
  - Prime factorisation of 36 = 6 x 6 = (2 x 3) x (2 x 3) = $2^2 \times 3^2$
    - Number of prime factors of 36 = 4 (with repetition) or 2 (distinct)
  - Prime factorisation of 60 = 10 x 6 = (2 x 5) x (2 x 3) = $2^2 \times 3 \times 5$
    - Number of prime factors of 60 = 4 (with repetition) or 3 (distinct)

- **Methods to find the Prime Factorisation of any number:**
  - Factor Tree Method
  - Repeated Division Method

Factor Tree Method



- Prime factorization of 300 = 2 × 2 × 3 × 5 × 5 = $2^2 \times 3 \times 5^2$

Repeated Division Method

2	40
2	20
2	10
5	5
	1

Prime factorization of 40 $= 2 \times 2 \times 2 \times 5$

$= 2^3 \times 5$

Prime Number Challenge

Q. What will be the parity of the final number when we multiply the first 100 prime numbers?

Even or Odd ?

Note: In mathematics, parity is the property of an integer whether it is even or odd.

Prime Number Challenge

Q. What will be the parity of the final number when we multiply the first 100 prime numbers?

Note: In mathematics, parity is the property of an integer whether it is even or odd.

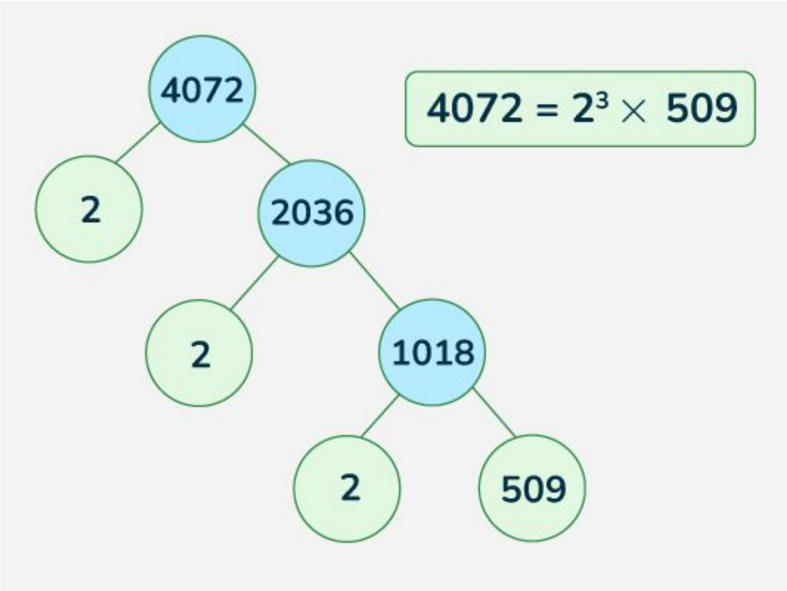
→ Even

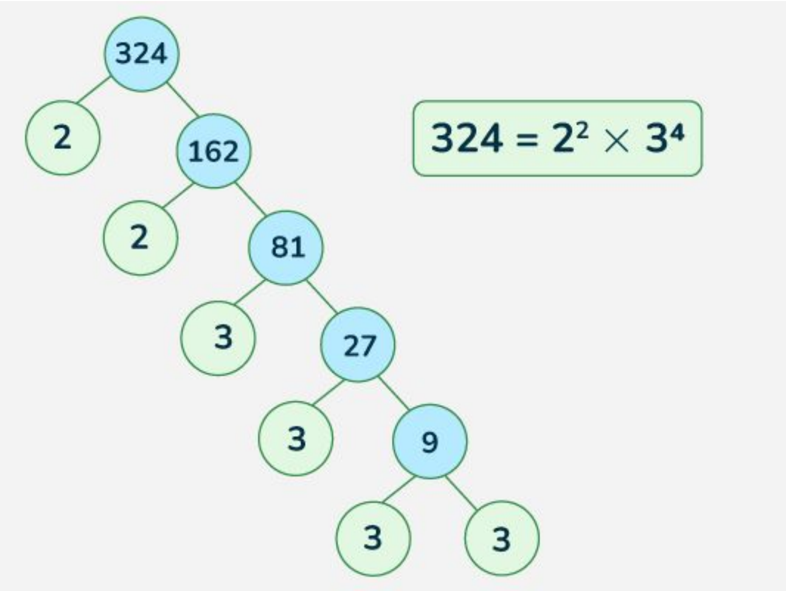
Reason:

- All prime numbers except 2 are odd.
- Odd x Odd = Odd
- 2 x Odd = Even

Fundamental Theorem of Arithmetic

- By the fundamental theorem of arithmetic, every integer greater than 1 can be uniquely represented as a product of prime numbers.





Finding number of divisors from Prime Factorization

n	Prime Factorization	# of Divisors
20	$2^2 \times 5^1$	$3 \times 2 = 6$
30	$2^1 \times 3^1 \times 5^1$	$2 \times 2 \times 2 = 8$
50	$2^1 \times 5^2$	$2 \times 3 = 6$
60	$2^2 \times 3^1 \times 5^1$	$3 \times 2 \times 2 = 12$
90	$2^1 \times 3^2 \times 5^1$	$2 \times 3 \times 2 = 12$
150	$2^1 \times 3^1 \times 5^2$	$2 \times 2 \times 3 = 12$
360	$2^3 \times 3^2 \times 5^1$	?

- $2^0 \times 5^0 = 1$
- $2^1 \times 5^0 = 2$
- $2^2 \times 5^0 = 4$
- $2^0 \times 5^1 = 5$
- $2^1 \times 5^1 = 10$
- $2^2 \times 5^1 = 20$

Finding number of divisors from Prime Factorization

In general, if n has the prime factorization

$$n = p_1^{k_1} \times p_2^{k_2} \times \cdots \times p_m^{k_m},$$

then n has

$$(k_1 + 1) \times (k_2 + 1) \times \cdots \times (k_m + 1)$$

divisors.

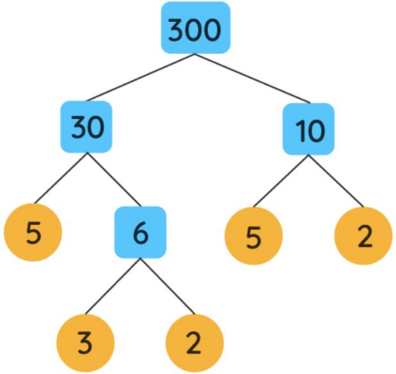
Example:

Scenario:

You have a pizza that is cut into 300 slices. You need to determine the number of all the possible group sizes that can share the pizza evenly, meaning no slices are left over.

Example: Dividing the pizza

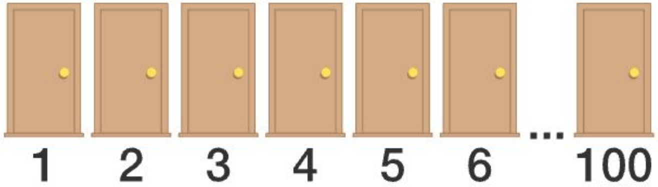
Ans.



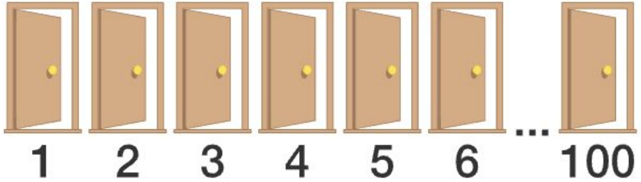
- Prime factorization of 300 = 2 × 2 × 3 × 5 × 5
  $= 2^2 \times 3 \times 5^2$
- Number of positive divisors of 300
  = (2 + 1) x (1 + 1) x (2 + 1)
  = 3 x 2 x 3
  = **18**

100 Doors Problem

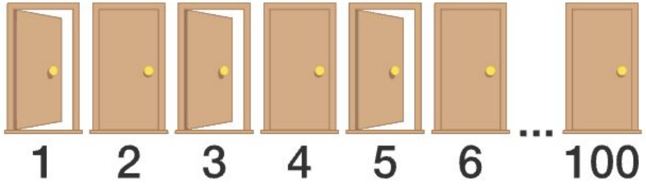
- There is a hallway of 100 doors.
- 100 people numbered 1 to 100 are standing in this long hallway that has 100 closed doors also numbered 1 to 100:



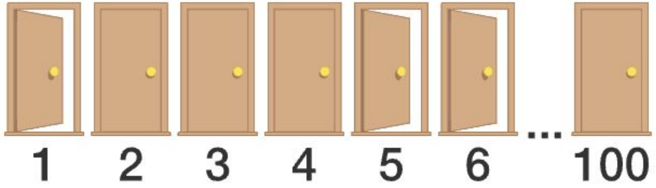
- Person 1 walks down the hallway and opens every door:



- Person 2 walks down the hallway and closes every door that is a multiple of 2:



- Person 3 walks down the hallway and changes every door that is a multiple of 3. That is, if the door is open, they close it, and if it is closed, they open it:

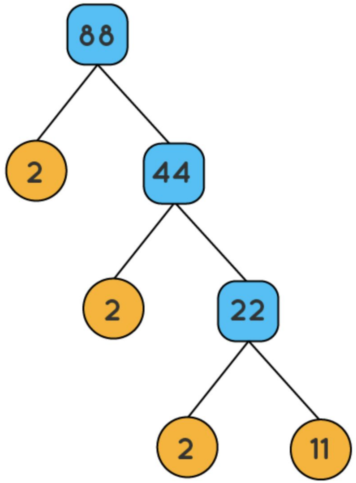


- Person 4 changes every door that is a multiple of 4, Person 5 every door that is a multiple of 5, etc. This continues until all 100 people have walked down the hallway and changed their doors.

Q. How many times door no. 88 is changed?

- Person 4 changes every door that is a multiple of 4, Person 5 every door that is a multiple of 5, etc. This continues until all 100 people have walked down the hallway and changed their doors.

Q. How many times door no. 88 is changed?



Approach: We need to find the number of positive divisors of 88.

➔ Prime factorisation of 88: $88 = 2^3 \times 11$

➔ Number of positive divisors of 88 = (3+1) x (1+1) = 4 x 2 = **8**

➔ So, **door no. 88 is changed 8 times.**

100 Doors Challenge

- There are 100 closed doors in a row, numbered from 1 to 100.
- You make 100 passes by the doors. On the i^{th} pass, you toggle every i^{th} door.
- Each "toggle" means:
  - If the door is closed, you open it.
  - If the door is open, you close it.

Question:

After 100 passes, which doors remain open?

Understanding Divisor Pairs

- When you find all the divisors (factors) of a number, they usually come in pairs.

- **Example:**
  n = 12 → Divisors of 12 are 1, 2, 3, 4, 6, 12.
  These form pairs that multiply to 12:
  ❖ 1×12
  ❖ 2×6
  ❖ 3×4
  → Total of 6 divisors = 3 pairs × 2

Understanding Divisor Pairs

- **Example:**
  n = 36 → Divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, 36.
  These form pairs that multiply to 36:
  - 1×36
  - 2×18
  - 3×12
  - 4×9
  - 6×6 ← notice this!

Here, 6×6=36, so both factors are the same. This is called a **repeated pair**, and it is only counted once.

→ Total of 9 divisors, which is odd.

Understanding Divisor Pairs

n is not a perfect square

↓

all divisor pairs are distinct

↓

even number of divisors

↓

door toggles even number of times, i.e., door is closed

n is a perfect square

↓

one pair contains repetition

↓

odd number of divisors

↓

door toggles odd number of times, i.e., door is open

100 Doors Challenge

Question:

After 100 passes, which doors remain open?

Solution:

- Only doors with perfect square numbers will remain open, i.e., the doors numbered **1, 4, 9, 16, 25, 36, 49, 64, 81, and 100**.
- So, overall, **10 doors** out of 100 doors will remain open.

Division Algorithm

Division Algorithm

- A division algorithm is an algorithm which, given two integers **a** and **b** (respectively the numerator and the denominator), computes their quotient and/or remainder, the result of division.

- For any integers **a** and **b** (with **b ≠ 0**), there exist **'unique' integers q** (quotient) and **r** (remainder) such that **a = bq + r**, where **0 ≤ r < |b|**

Dividend (a)	Divisor (b)	Quotient (q)	Remainder (r)	Expression (a = bq + r)
20	6	3	2	$20 = 6 \times 3 + 2$
17	5	3	2	$17 = 5 \times 3 + 2$
−17	5	−4	3	$-17 = 5 \times (-4) + 3$

Rapid fire

Can the remainder ever be negative?

Rapid fire

Can the remainder ever be negative?

Answer: No

- ➔ If we allow negative remainder, then we lose the uniqueness property of division algorithm.
- ➔ Remainder is defined to be non-negative in math because we want the remainder to represent a "leftover" quantity. So, it should always be a small, clean, and positive offset.

Key Takeaways

Today we learnt :

- Exponentiation
- Logarithms
- Multiples
- Factors
- Prime Numbers
- Prime Factorization
- Number of Positive Divisors
- Division algorithm

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See You Guys in Next Session :)