Set Theory (Part – 2)

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CS01: Mathematics-I

Quick Revision:

In the last class you read:

- Sets
- Set Representation: Set Roaster, Set Builder
- Cardinality
- Types of Sets: Empty Set, Singleton Set, Finite Set, Infinite Set, Equivalent Sets, Equal Sets, Disjoint Sets
- Subset, Proper subset, Improper subset, Superset, Powerset
- Set Operations: Set Intersection, Set Union, Set Complement, Set Difference, Symmetric Difference

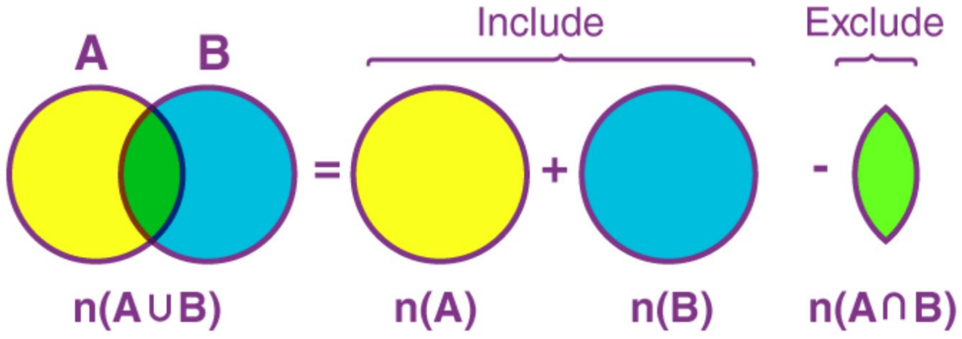
Today’s Agenda

- Principle of Inclusion Exclusion
- Demorgan’s Law
- Cartesian Product
- Relation on Sets
- Empty Relation
- Reflexive Relation
- Symmetric Relation
- Transitive Relation

Principle of Inclusion and Exclusion

Principle of Inclusion-Exclusion

- The Principle of Inclusion-Exclusion helps in calculating the size of the union of overlapping sets by adjusting for double-counted elements.

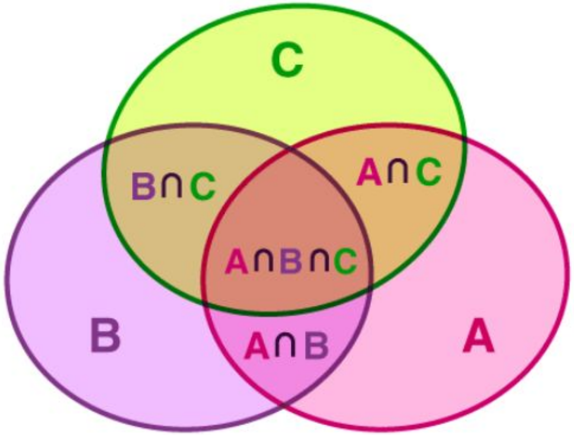


Formula: For two sets A and B

$$|A \cup B| = |A| + |B| - |A \cap B|$$

n(A∪B) = n(A) + n(B) – n(A∩B)

Principle of Inclusion-Exclusion



Formula: For three sets A, B and C

$$n(A \cup B \cup C) = n(A) + n(B) + n(C)$$

$$- n(A \cap B) - n(A \cap C) - n(B \cap C)$$

$$+ n(A \cap B \cap C)$$

Example

In your college, students have the option to participate in three different clubs: ICPC, SDG , and AI. The participation data for these clubs is as follows:

- 30 students are in the ICPC club.
- 25 students are in the SDG club.
- 20 students are in the AI club.
- 12 students are in both ICPC and SDG.
- 10 students are in both SDG and AI.
- 8 students are in both ICPC and AI.
- 5 students are in all three clubs: ICPC, SDG, and AI.

Example

Question:

1. How many students are participating in at least one of these clubs?
2. How many students are only in the ICPC club?
3. How many students are only in the SDG club?
4. How many students are only in the AI club?
5. How many students are in exactly two of these clubs?

Homework

There are 50 students in a class among which 20 students study Mathematics, 25 study PSP, and 10 study both Mathematics and PSP.

Find the number of students who:

1. study
2. study only Mathematics
3. study only PSP
4. study only one subject
5. neither study Mathematics nor PSP

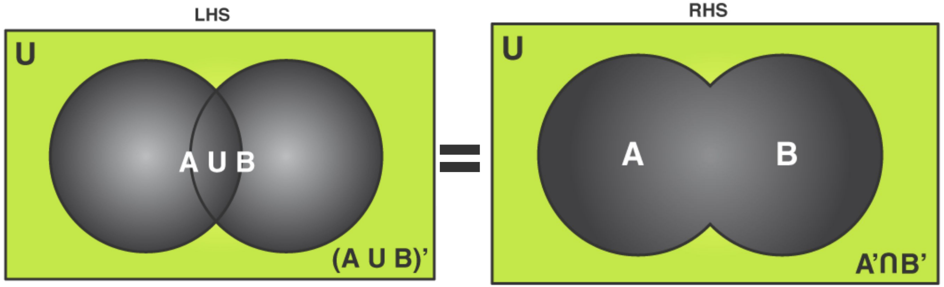
Homework

What is the formula for the union of 4 sets, i.e., n(A U B U C U D)?

De Morgan’s Laws on Sets

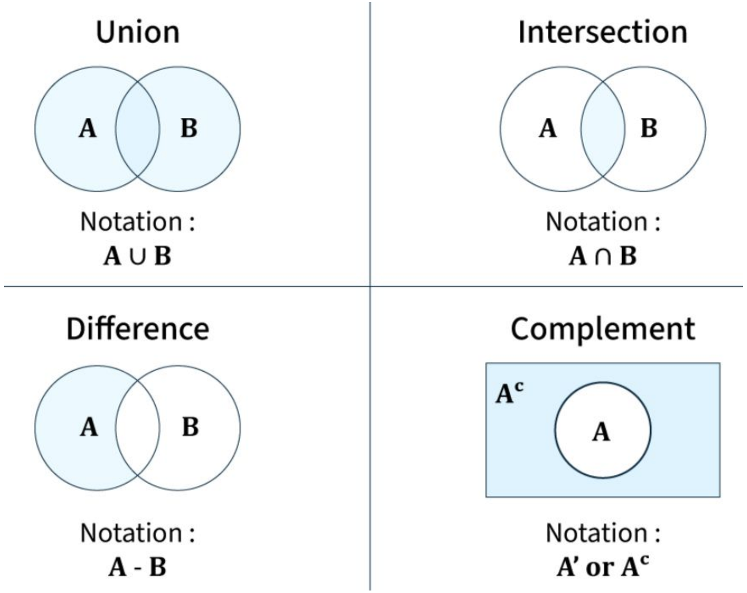
De Morgan’s Law

- **De Morgan’s 1st Law on Sets:**
  - **(A ∪ B)’ = A’ ∩ B’**
  - The complement of the union of two sets is equal to the intersection of their individual complements



- **De Morgan’s 2nd Law on Sets:**
  - **(A ∩ B)’ = A’ ∪ B’**
  - The complement of the intersection of two sets is equal to the union of their individual complements.

What all operations have we learnt in Sets so far??

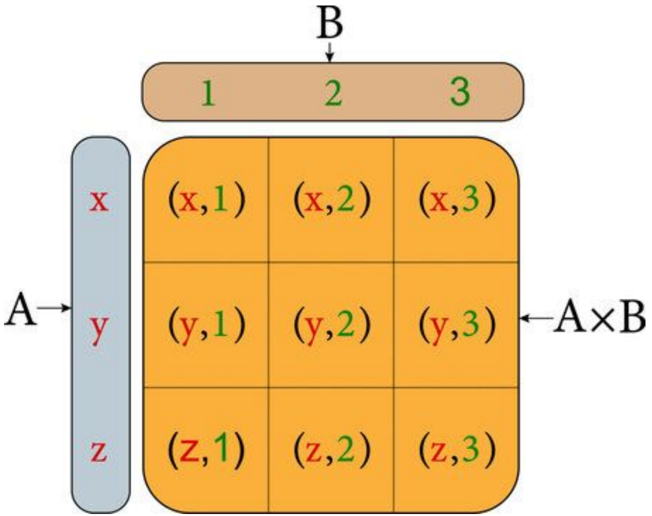


Cartesian Product

Definition:

The **Cartesian product** is a mathematical operation that returns a set of all ordered pairs from two or more sets. If you have two sets, A and B, the Cartesian product A×B is the set of all possible ordered pairs where the first element is from A and the second element is from B

$$A \times B = \{(a, b) \mid a \in A \land b \in B\}$$



Example

A = {red, blue}, B = {bag, coat,shirt}

A (Colors)	B (Items)	A × B (Pair)
🟥 red	👜 bag	(red, bag)
🟥 red	🧥 coat	(red, coat)
🟥 red	👕 shirt	(red, shirt)
🔵 blue	👜 bag	(blue, bag)
🔵 blue	🧥 coat	(blue, coat)
🔵 blue	👕 shirt	(blue, shirt)

There are 6 pairs in total.

$$|A \times B| = |A| \cdot |B| = 2 \cdot 3 = 6$$

Rapid fire

Are the Cartesian Products **A x B** and **B x A** same or different?

No, order matters.

What does that mean?

In a tuple like **(a, b)**:

- **(a, b)** is **not the same** as **(b, a)**

A × B	B × A
(🔴 Red, 👕 Shirt)	(👕 Shirt, 🔴 Red)
(🔴 Red, 🎒 Bag)	(👕 Shirt, 🔵 Blue)
(🔵 Blue, 👕 Shirt)	(🎒 Bag, 🔴 Red)
(🔵 Blue, 🎒 Bag)	(🎒 Bag, 🔵 Blue)

Rapid fire

$N \times Z = \{(a,b) | a \in N,\ b \in Z \land a=b\}$ So, does order matter?

It does, as the first element is a natural number and the second number is a integer.

How can we say that (a,b) is an ordered pair?

How can we say that (a,b) is an ordered pair?

If (a,b) = (c,d) → Rule

(a = c & b = d)

But how can you explain it???

How can we say that (a,b) is an ordered pair?

If $(a,b) = (c,d)$ $\longrightarrow$ Rule

$(a = c \,\&\, b = d)$

$(a,b) := \{\{a\}, \{a,b\}\}$

$(c,d) := \{\{c\}, \{c,d\}\}$

💡 Hint: Recall Equality of Sets

How can we say that (a,b) is an ordered pair?

If $(a,b) = (c,d)$ → Rule

$(a = c \;\&\; b = d)$

$(a,b) := \{\{a\}, \{a,b\}\}$

$(c,d) := \{\{c\}, \{c,d\}\}$

$\{a\} = \{c\} \rightarrow a = c$

$\{a,b\} = \{c,d\}$

$a = c\ ,\ b = d$

$A1 \times A2 \times A3$

$A = \{red, blue\}$

	A1	A2	A3
3	red	blue	red
4	red	blue	blue
5	blue	red	red
6	blue	red	blue
7	blue	blue	red
8	blue	blue	blue

Rapid fire

If **A** is a finite set with **n** elements, what is the cardinality of the Cartesian product A×A×A (three times) in terms of **n**?

Real Life Example

A = {DL, MP, KA}, B = {01, 02, 03}

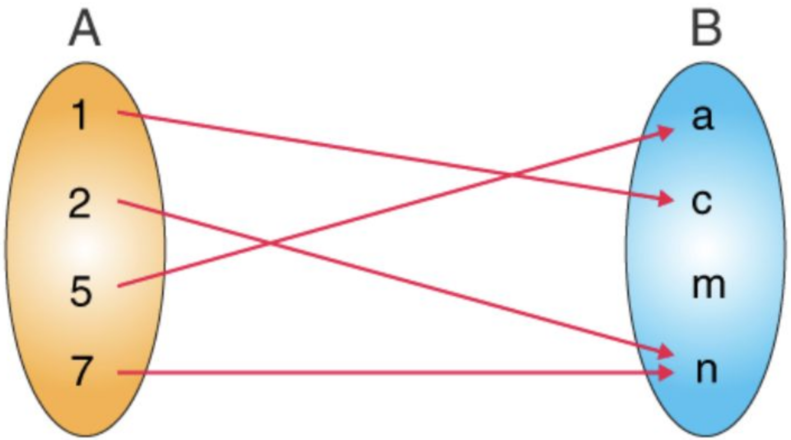
A × B = all possible state-code pairs.

Relations

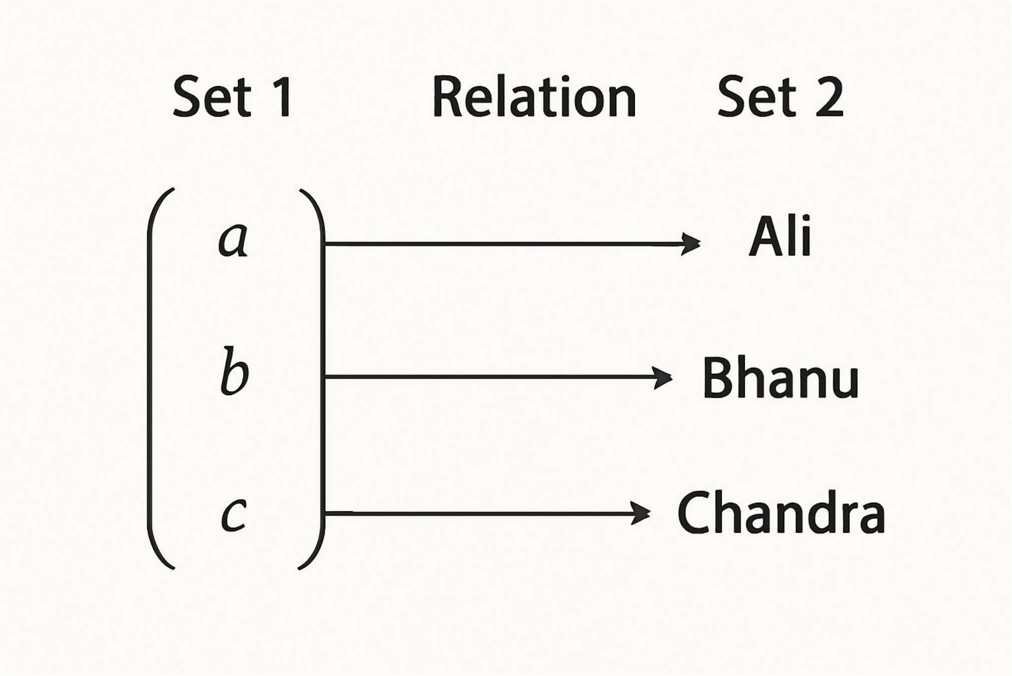
Definition:

If A and B are two sets, a relation **R** from **A** to **B** is a subset of the Cartesian Product A×B . This means that:

$$R \subseteq A \times B$$



Example



Arrow Diagrams and Set Representation

$f(x) = x^2$

Domain = $\{1,2,3,4,5\}$

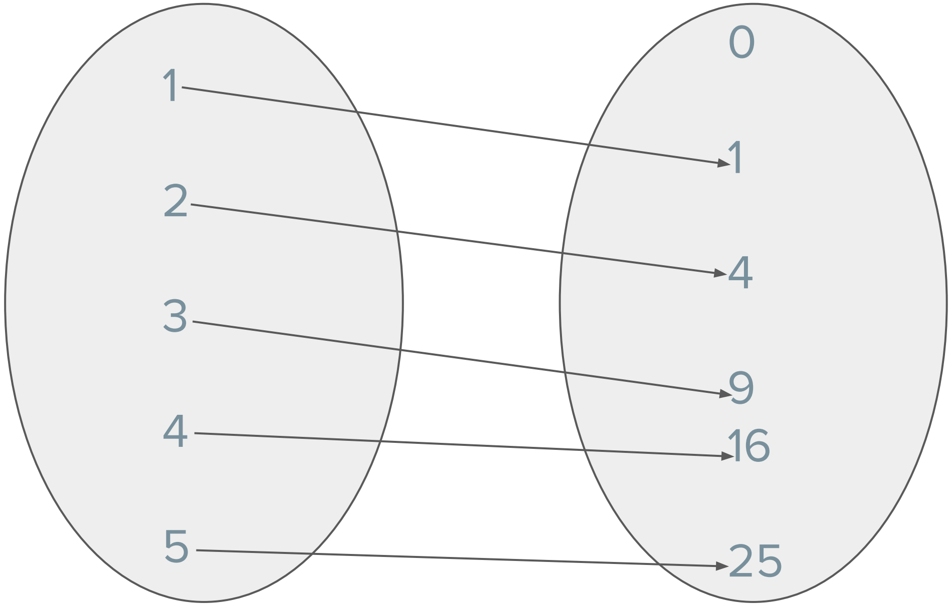
Domain = $\{x \mid x \in Z, 1 \le x \le 5\}$

Codomain = $\{0,1,4,9,16,25\}$

Codomain = $\{y \mid y \in Z, y \ge 0\}$

Range = $\{1,4,9,16,25\}$

Range = $\{f(x)|f(x)\, x^2 \;,\,,\; x \in \{1,2,3,4,5\}\}$



Reflexive Relation

A **reflexive relation** on a set A is a relation where every element of A is related to itself. In mathematical terms, a relation R on a set A is **reflexive** if for every element $a \in A$, the pair (a,a) is in the relation R.

Example:

Let $A=\{1,2,3\}$ and let the relation R on A be defined as:

$R=\{(1,1),(2,2),(3,3),(1,2)\}$
Here, since $(1,1),(2,2)$ and $(3,3)$ are present, this is a reflexive relation on A, because every element is related to itself.

Real Life Examples

Employee A is managed by employee A.

User A follows user A.

A person is the owner of their own car.

Transitive Relations

A **transitive relation** is one where, if an element a is related to b, and b is related to c, then a must also be related to c.

Mathematically, if aRb and bRc , then aRc.

- If a is an ancestor of b, and b is an ancestor of c, then a must be an ancestor of c.
- Example: If your grandfather is an ancestor of your father, and your father is an ancestor of you, then your grandfather is an ancestor of you.

Symmetric Relations

A **symmetric relation** is one where, if an element a is related to an element b, then b must also be related to a. Mathematically, for any a and b, if aRb , then bRa.

Example

If a is friends with b, then b must also be friends with a.

Reflexive and Transitive but not Symmetric

"Is less than or equal to" (≤)

Consider the relation (less than or equal to) on the set of real numbers.

Reflexive: For any real number x, $x \leq x$.

Transitive: If $a \leq b$, and $b \leq c$, then $a \leq c$.

Not Symmetric: If $a \leq b$, it does not mean that $b \leq a$ unless $a=b$. For example, $3 \leq 5$ but $5 \leq 3$ is not true.

Symmetric and Transitive but not Reflexive.

- **Symmetric**: If a is a sibling of b, then b is a sibling of a.
- **Transitive**: If a is a sibling of b, and b is a sibling of c, then a is a sibling of c (i.e., they share the same parents).
- **Not Reflexive**: A person cannot be their own sibling (i.e., a is not a sibling of a).

Reflexive and Symmetric but not Transitive

"Is a friend of"

Reflexive: A person can be their

own friend.

Symmetric: If person a is a friend of person b,

then person b is also a friend of person a.

Not Transitive: Just because person a is friends with person b, and person b is friends with person c, it doesn't imply that person a is friends with person c.

Equivalence Relations

- A relation **R** on **A** is an equivalence relation if:

– Reflexive: for all element a in **A**, (a , a) belongs to **R**

– Symmetric: If a and b in **A** with (a , b) belongs to **R** ⇒ (b , a) belongs to **R**

– Transitive: If a, b and c in **A** with (a, b) and (b,c) belongs to **R** ⇒ (a,c) belongs to **R**

Alternatively; using a notation R or ~

– Reflexive: for all element a in A, a R a

– Symmetric: If a and b in A with a R b ⇒ b R a

– Transitive: If a, b and c in A with a R b and b R c ⇒ a R c

Equivalence Relation: Gender Relation

Let A be the set of all people.

We define the relation ~ by:

a ~ b if and only if a and b have the same gender.

Is it a equivalence relation?

Verification of Equivalence Relation:

Reflexive: Each person has the same gender as themselves. So, a ~ a for any

a ∈ A.

Symmetric: If a ~ b,meaning a and b have the same gender, then b ~ a also holds because gender is mutual.

Transitive: If a ~ b and b ~ c, meaning a and b have the same gender.

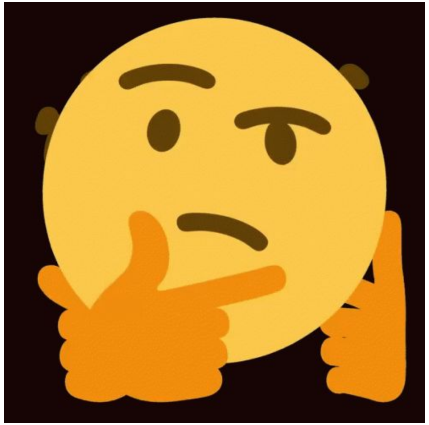
Example: Modulo 2 on Integers

We define the relation ~ on Z (the set of integers) as follows:

$$a \sim b \text{ if and only if } a \equiv b \pmod{2}$$

This means that a and b are related if their difference is divisible by 2, i.e., they both have the same parity (either both are even or both odd)

Is it a equivalence relation?

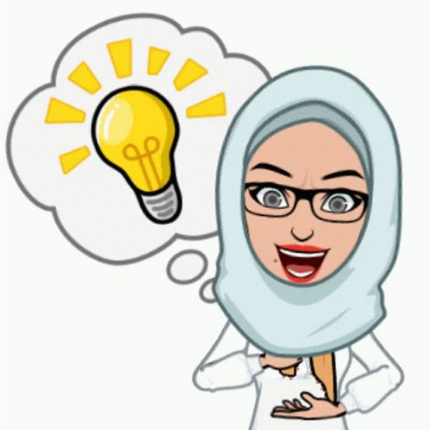


Reflexivity

For reflexivity, we need to show that a~a for all integers a.

- This means that $a \equiv a \pmod{2}$, which is true because the difference between any integer and itself is 0, and 0 is divisible by 2.

- So, for all $a \in Z$ a~a.



Symmetry

For symmetry, we need to show that if a~b then b~a.

- If a~b, then by definition, $a \equiv b \pmod 2$, meaning that $a-b$ is divisible by 2.

- This means that the difference $b-a$ is also divisible by 2 (since $b-a=-(a-b)$ a).

- Hence, $b \equiv a \pmod 2$, or b~a .

Thus, the relation is **symmetric**

Transitivity

For transitivity, we need to show that if a~b and b~c, then a~c.

- If a~b, then $a \equiv b \pmod{2}$, which means $a-b$ is divisible by 2.

- If b~c then $b \equiv c \pmod{2}$, meaning $b-c$ is divisible by 2.

- Now, consider the difference $a-c$. We can write it as:
  $a-c = (a-b)+(b-c)$
  Since both $a-b$ and $b-c$ are divisible by 2, their sum $a-c$ is also divisible by 2.

- Therefore, $a \equiv c \pmod{2}$, and hence a~c.

Thus, the relation is **transitive**.

Counter Example: Love Relation

Define a❤️b if person a loves person b.

- Not reflexive: not everyone loves themselves.
- Not symmetric: love is not always returned.
- Not transitive: love triangles!

Therefore: not an equivalence relation

Key Takeaways:

In today’s class you read :

- Cartesian Product
- Relation on Sets
- Reflexive Relation
- Symmetric Relation
- Anti-symmetric Relation
- Transitive Relation
- Equivalence Relations

Thank You!