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Proving Techniques

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CS01: Mathematics-I

Join the class

Today's Agenda

- Proof Introduction : Why ? What ?
  - Know proofs
  - Invent Proofs
  - Enjoy proofs
- Types of Proofs
  - Direct proofs
  - Indirect proofs

Hilbert's Noble Dream

Mathematician David Hilbert once proposed a mathematical model that will

- Formulate all of the mathematics
- Complete - Can prove all true statement
- Consistent - Lack of Contradictions
- Decidable - Can prove or disprove any statement

David Hilbert
(1862 - 1943)

Hilbert’s Failed Dream

Spoiler Alert: No such system exists. Any mathematical system-

- Cannot formulate all true statements. True statement exists outside all of formalism.
- Incomplete - some statements cannot be proven or disproven
- Undecidable - No single algorithm can decide truth value of all statements.

Why Study Proof Techniques

- **Ensures Rigor:** Removes ambiguity with precise justification
- **Develops Reasoning:** Trains logical and critical thinking
- **Validates Truth:** Proves statements beyond examples
- **Builds Formal Frameworks:** Helps categorize, connect, and verify mathematical ideas
- **Avoids Common Mistakes:** e.g. logical fallacies.
- **Enables Program Verification:** Powers computer-assisted proofs and correctness checks

Proofs We have seen

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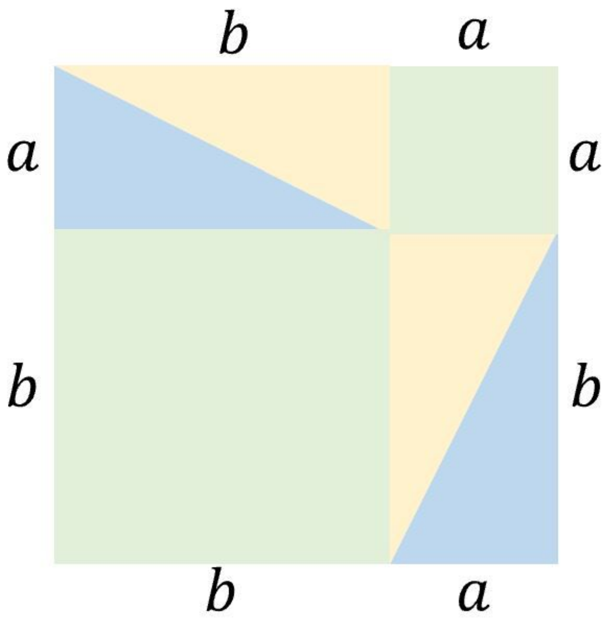
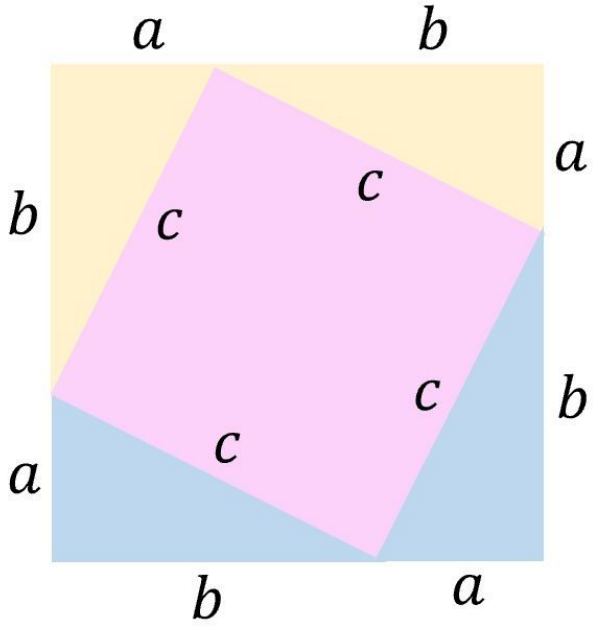
LHS = RHS

- **LHS (Left-Hand Side)** $\rightarrow$ **RHS (Right-Hand Side)** or vice versa

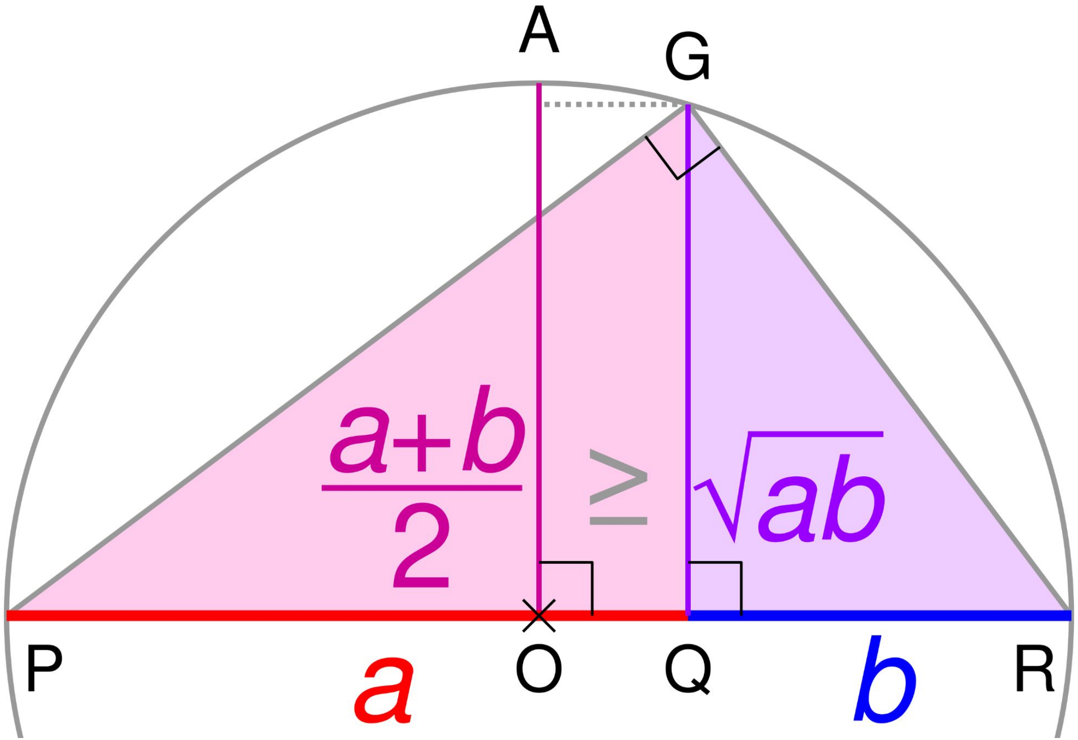
- **Example:** Prove, $a^2 - b^2 = (a - b)(a + b)$
  - RHS $= (a - b)(a + b)$
    $= a(a + b) - b(a + b)$
    $= a^2 + ab - ba - b^2$
    $= a^2 - b^2$
    $=$ LHS
  - Hence Proved

Visual Proofs - Pythagorean Theorem





AM ≥ GM - Visual Proof



$$\frac{GQ}{QR} = \frac{PQ}{GQ}$$

$$GQ \cdot GQ = PQ \cdot QR$$

$$GQ^2 = a \cdot b$$

$$GQ = \sqrt{ab}$$

AM ≥ GM - Algebraic Proof

To prove ,

$$(x + y) / 2 > \sqrt{xy}$$

when x and y are distinct real positive numbers, we can use **backward reasoning.**

$$
\begin{aligned}
(x + y)/2 &> \sqrt{xy}, \\
(x + y)^2/4 &> xy, \\
(x + y)^2 &> 4xy, \\
x^2 + 2xy + y^2 &> 4xy, \\
x^2 - 2xy + y^2 &> 0, \\
(x - y)^2 &> 0.
\end{aligned}
$$

False Proof - The Infamous 1=2 Proof

"Proof": We use these steps, where a and b are two equal positive integers.

Step	Reason
1. $a = b$	Given
2. $a^2 = ab$	Multiply both sides of (1) by a
3. $a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides of (2)
4. $(a - b)(a + b) = b(a - b)$	Factor both sides of (3)
5. $a + b = b$	Divide both sides of (4) by $a - b$
6. $2b = b$	Replace a by b in (5) because $a = b$ and simplify
7. $2 = 1$	Divide both sides of (6) by b

Algebraic Proof

- Prove that "The sum of two odd numbers is always even".
  - Odd numbers are represented as $(2k + 1)$ for some integer k.
  - Let those two odd numbers be $(2a + 1)$ and $(2b + 1)$, for some integers a, b.
  - Sum:

$$
\begin{aligned}
(2a+1) + (2b+1) &= 2a + 2b + 2 \\
&= 2(a + b + 1) \\
&= 2m
\end{aligned}
$$

  - Since $(a + b + 1)$ is an integer, the sum is divisible by 2.
  - Hence, it is even.

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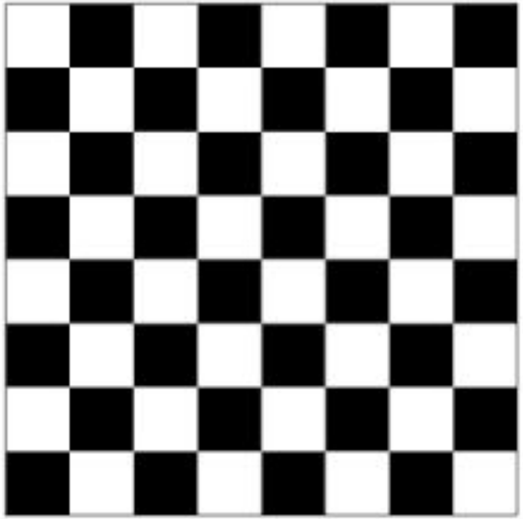
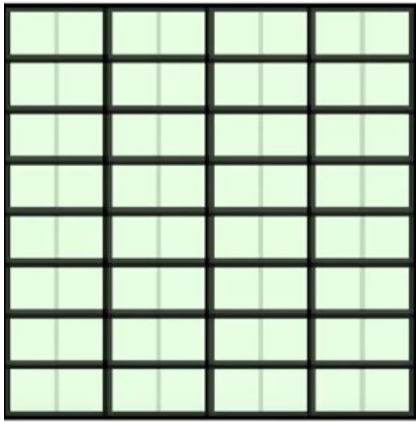
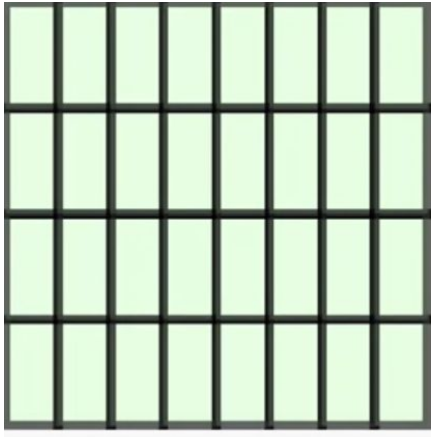
Example

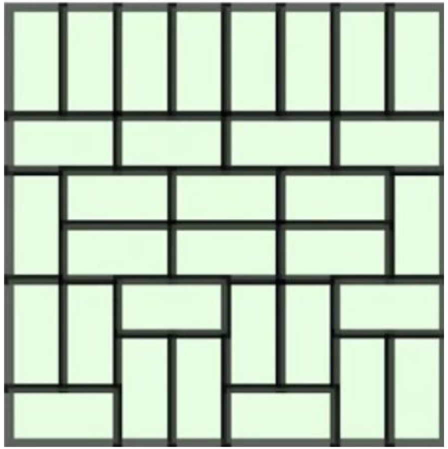
Can you tile the chessboard with 1x2 tiles?



Proof by Example

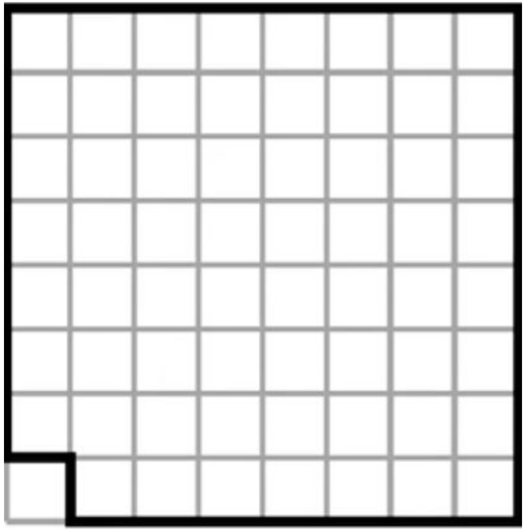






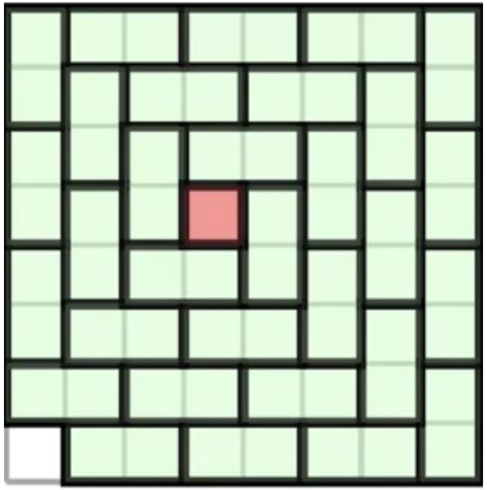
Example

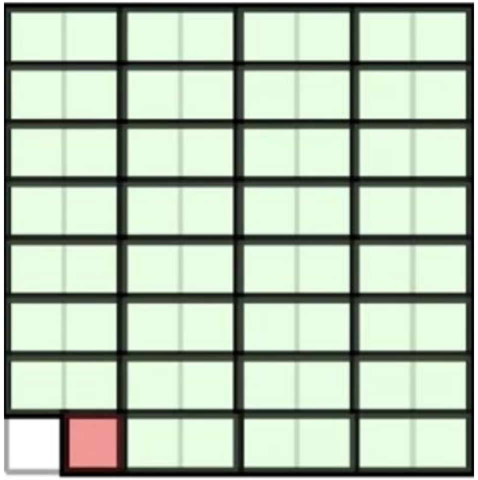
Can you tile the chessboard with 1x2 tiles.



Proof of Impossibility

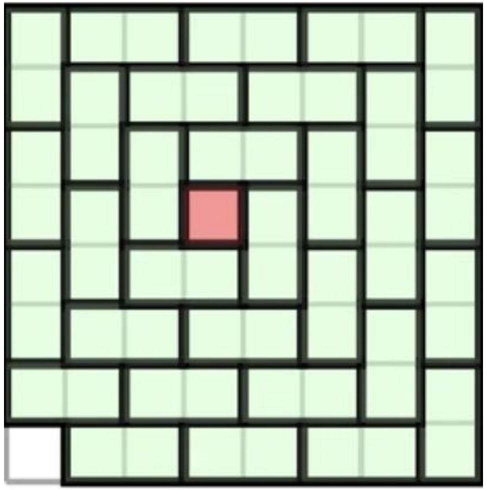
- Mission probably impossible

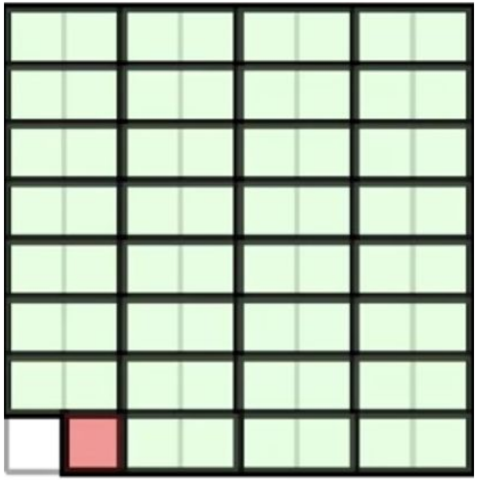




Proof of Impossibility

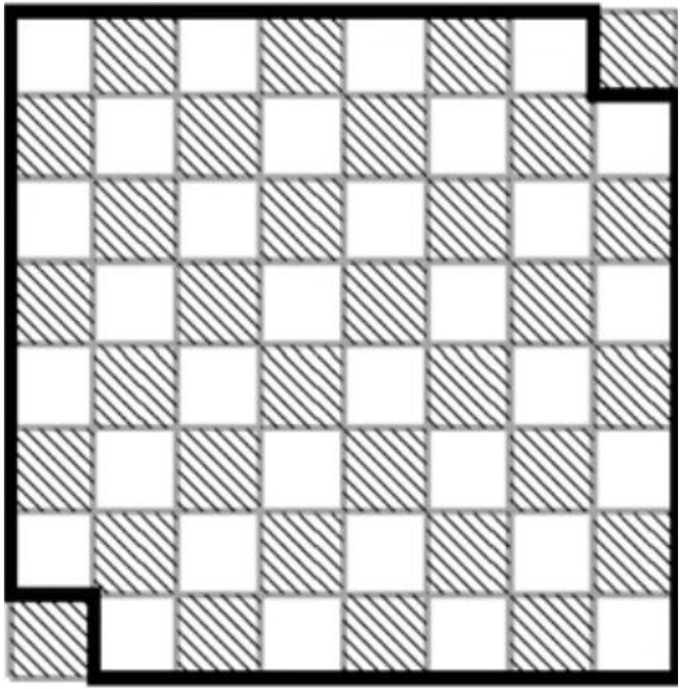
- One cell will always remain, because…
- There are 63 = 8 × 8 − 1 cells, an odd number
- 31 tiles cover 62 cells, one remains
- Mission probably impossible





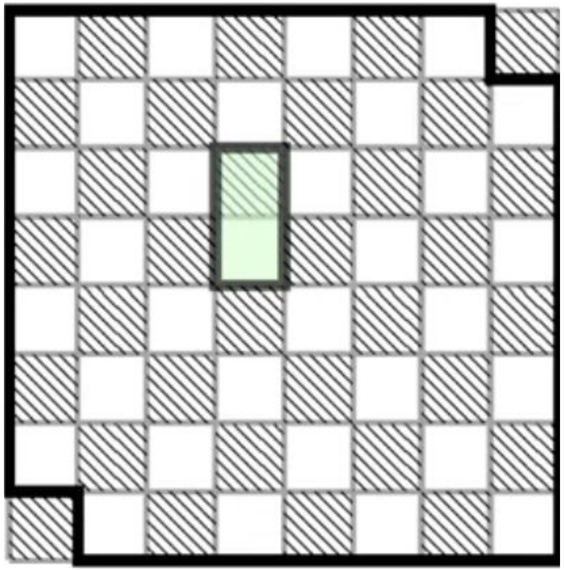
Example

Can you tile the chessboard with 1x2 tiles.



Proof of Impossibility

- Opposite corners are (say) black
- 30 black and 32 white
- A tile: two different colors



Theorems in Proving

A theorem is a statement that has been proven, or can be proven.

Example

Theorem: A chess board 8 x 8 without two opposite corners cannot be tiled by 1x2 tiles.

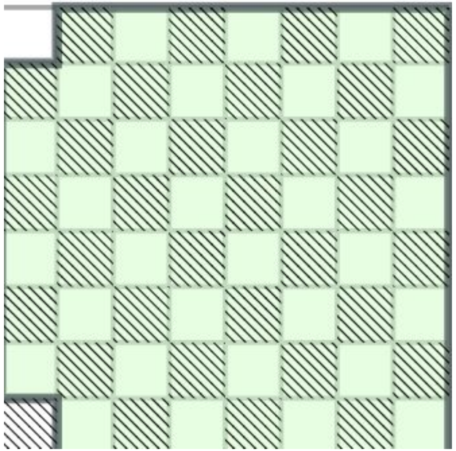
Proof:

- opposite corners are (say) black
- 30 black and 32 white
- A tile: two different colors

Example

Can we tile the 8×8 board without two non-opposite corners (say, the left bottom and left top ones)?

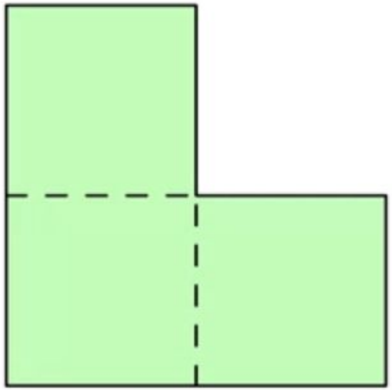
Yes, because these corners are of different colors: this board has 31 white and 31 black cells and therefore can be tiled.



Existential Proofs

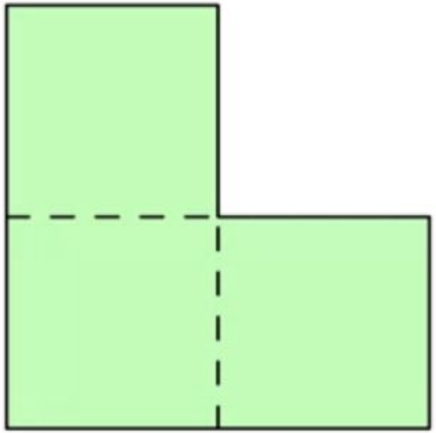
Existential Proofs

- One example is enough
- Ex: Prove that this figure can be cut into 2 identical pieces.

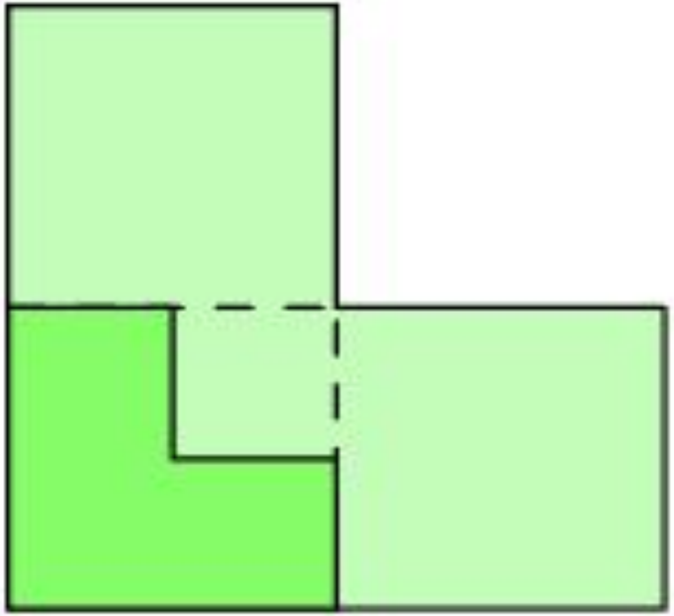


Example

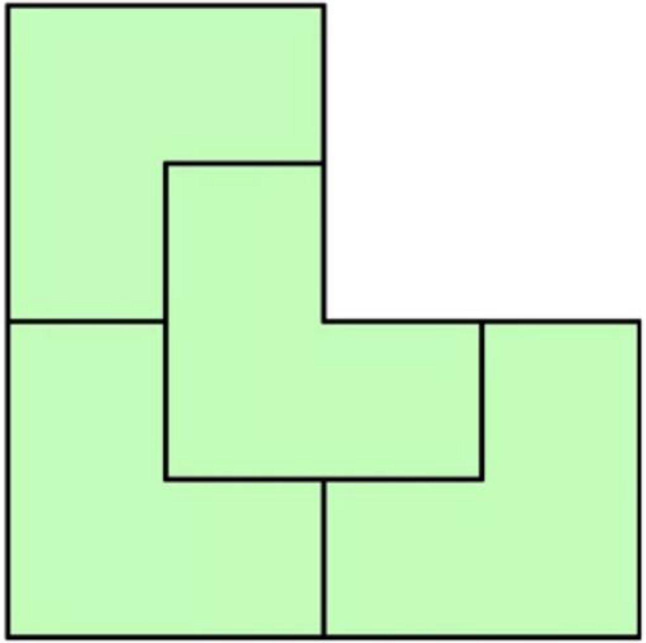
Prove that this figure can be cut in 4 identical pieces?



Hint



Solution



Direct Proofs

Direct Proofs

A direct proof shows that a conditional statement $p \rightarrow q$ is true by showing that if p is true, then q must also be true.

Example

If n is an odd integer, then $\mathbf{n^2}$ is odd.

Solution

To begin a direct proof of this theorem, we assume that the hypothesis of this conditional statement is true, namely, we assume that n is odd.

By the definition of an odd integer, it follows that **$n = 2k + 1$,** where k is some integer.

We want to show that **n^2** is also odd.

We can square both sides of the equation $n = 2k + 1$ to obtain a new equation that expresses **n^2**.

When we do this, we find that **$n^2 = (2k + 1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$.**

Example

If **m** and **n** are both perfect squares, then **n.m** is also a perfect square.

(An integer a is a perfect square if there is an integer b such that a = b^2)

Solution

We assume that **m and n** are both perfect squares.

By the definition of a perfect square, it follows that there are integers s and t such that $\mathbf{m = s^2}$ and $\mathbf{n} = \mathbf{t^2}$.

The goal of the proof is to show that mn must also be a perfect square when m and n are;

looking ahead we see how we can show this by substituting s^2 for m and t^2 for n into mn.

This tells us that $\mathbf{mn = s^2 * t^2}$. Hence, mn = $\mathbf{s^2} * \mathbf{t^2}$ = (ss)(tt) = (st)(st) = $(\mathbf{st})^2$, using commutativity and associativity of multiplication.

By the definition of perfect square, it follows that mn is also a perfect square, because it is the square of st, which is an integer. We have proved that if m and n are both perfect squares, then mn is also a perfect square.

Example

Prove that if n is an integer and n^2 is odd, then n is odd.

Indirect Proofs

Proof by Contraposition

Proof by Contraposition

Proofs by contraposition make use of the fact that the conditional statement $p \rightarrow q$ is equivalent to its contrapositive, $\neg q \rightarrow \neg p$.

To perform an indirect proof, do a direct proof on the contrapositive

Example

Prove that if n is an integer and n^2 is odd, then n is odd.

Solution

The contrapositive of the statement “If n is an integer and n^2 is odd, then n is odd” will be **“If n is even then n^2 is even”**.

If n is even then, n=2k

$$n^2 = (2k)^2 = 4.k^2 = 2(2k^2)$$

Our proof by contraposition succeeded; we have proved the theorem “If n^2 is odd, then n is odd.

Which method to use?

When do you use a direct proof versus an indirect proof?

If it’s not clear from the problem, try direct first, then indirect second.

If indirect fails, try the other proofs.

Example

Prove that if n is an integer and $3n + 2$ is odd, then n is odd.

Homework

Prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

Proof by Contradiction

Proof by Contradiction

Establishes the truth of a statement by showing that assuming the statement is false leads to a contradiction.

Example

Prove that $\sqrt{2}$ is irrational by giving a proof by contradiction.