PHY 375 Final Project

Final Project Presentation: March 30th
Final Project Report: April 6th

1 Motivation

The goal of the project is to combine the various aspects of stellar structure discussed in the course to produce a sequence of stellar models, and with these grossly reproduce the Main Sequence, the properties of stars on it, and how key elements in the underlying physics manifest themselves. Some of the key features necessary to do this are collected here, along with some wisdom regarding execution.

Because of the intrinsic difficulty of constructing the numerical machinery, everyone is assigned to work in a group which should collectively produce a code (in whatever language the group prefers) capable of constructing the desired stars. The project is therefore separated into three parts:

1. *Collectively* (within the group) creating the ability to numerically construct a variety of stellar models using the equations of stellar structure.

2. *Collectively* (within the group) assessing the impact of a group-specific set of variations about the standard model, and making a 15 minute presentation on the results on March 30th.

3. *Individually* constructing and interpreting key features in a set of stellar models generated with the group code, collecting this into a report to be submitted by April 6th.

All group members must be able to describe what the code is doing and running it independently.

Section 2 present the standard set of stellar structure equations that should be solved and provides some suggestions on how to do this in practice. Section 3 gives some example results that are useful to compare against. Section 4 describes the expected results from the independent projects. Finally, section 5 lists the specific group projects, who is assigned to each, and suggestions for how to implement them.

2 Solving for the Standard Main Sequence

2.1 Default Equations of Stellar Structure

The equations of stellar structure we seek to solve are hydrostatic equilibrium, the energy transport equation, the definition of the enclosed mass, and the energy generation equation. As these are normally written, they are not amenable to direct integration. The natural variables are ρ, T, M, and L, and thus the first must be recast as a differential equation for one of these instead of P. However, this can be done simply if the partial derivatives of P can be readily computed, giving

$$\frac{dP}{dr} = -\frac{GM\rho}{r^2} \quad \rightarrow \quad \frac{d\rho}{dr} = -\left[\frac{GM\rho}{r^2} + \frac{\partial P}{\partial T}\frac{dT}{dr}\right] \Big/ \frac{\partial P}{\partial \rho} \,. \tag{1}$$

In addition, to define the "surface", corresponding to the photosphere, it will be necessary to supplement the four equations mentioned above with one for the optical depth, as described in lecture. Thus, the *five* equations we must solve are:

$$\begin{aligned}
\frac{d\rho}{dr} &= -\left[\frac{GM\rho}{r^2} + \frac{\partial P}{\partial T}\frac{dT}{dr}\right] \Big/ \frac{\partial P}{\partial \rho} \,,\\
\frac{dT}{dr} &= -\min\left[\frac{3\kappa\rho L}{16\pi a c T^3 r^2}, \left(1 - \frac{1}{\gamma}\right)\frac{T}{P}\frac{GM\rho}{r^2}\right] \,,\\
\frac{dM}{dr} &= 4\pi r^2 \rho \,,\\
\frac{dL}{dr} &= 4\pi r^2 \rho \epsilon \,,\\
\frac{d\tau}{dr} &= \kappa\rho \,.
\end{aligned} \tag{2}$$

As described in lecture, boundary conditions must be set both at the center and at the surface. The center boundary conditions are:

$$M(0) = 0 \qquad \text{and} \qquad L(0) = 0 \,. \tag{3}$$

The surface boundary conditions are

$$\tau(\infty) - \tau(R_*) = \frac{2}{3} \qquad \text{and} \qquad L(R_*) = 4\pi\sigma R_*^2 T_*^4 \,. \tag{4}$$

Note that the first defines R_*. These are sufficient to specify all but one quantity, which parameterizes the Main Sequence. A variety of choices may be made for the final boundary condition, parameterizing it in terms of central density, central temperature, total mass, etc., though we will say more about this below.

To be well defined, the above must be supplemented with definitions for the equation of state, opacity, and specific energy generation rate. For the former, three equations of state must be considered: non-relativistic degenerate, ideal gas, and photon gas. It is sufficient to approximate the total pressure as the sum of the three since one will typically dominate in the appropriate regime. That is, consider

$$P(\rho, T) = \frac{(3\pi^2)^{2/3}}{5}\frac{\hbar^2}{m_e}\left(\frac{\rho}{m_p}\right)^{5/3} + \rho\frac{kT}{\mu m_p} + \frac{1}{3}aT^4 \,, \tag{5}$$

where

$$\mu = \left[2X + 0.75Y + 0.5Z\right]^{-1} \tag{6}$$

is the mean molecular weight for fully ionized gas (you may neglect the variation in μ associated with the lack of ionization in the stellar atmosphere). The partial derivatives, needed for the modified hydrostatic equilibrium equation, are

$$\frac{\partial P}{\partial \rho} = \frac{(3\pi^2)^{2/3}}{3}\frac{\hbar^2}{m_e m_p}\left(\frac{\rho}{m_p}\right)^{2/3} + \frac{kT}{\mu m_p} \qquad \text{and} \qquad \frac{\partial P}{\partial T} = \rho\frac{k}{\mu m_p} + \frac{4}{3}aT^3\,. \tag{7}$$

The energy generation is dominated by the PP-chain and CNO-cycle. While more sophisticated estimates of the specific energy generation rates are available (see, e.g., Clayton or Kippenhahn & Weigert), these are sufficiently well approximated by the power laws given in lecture:

$$\epsilon_{PP} = 1.07\times10^{-7}\rho_5 X^2 T_6^4 \text{ W/kg} \qquad \text{and} \qquad \epsilon_{CNO} = 8.24\times10^{-26}\rho_5 X X_{CNO} T_6^{19.9} \text{ W/kg}\,, \tag{8}$$

where $\rho_5 \equiv \rho/10^5$ kg/m^3 and $T_6 \equiv T/10^6$ K. You may adopt the solar CNO abundance: $X_{CNO} = 0.03X$. The total specific energy generation rate is then simply

$$\epsilon(\rho, T) = \epsilon_{PP} + \epsilon_{CNO}\,. \tag{9}$$

The computation of the opacities can be extremely complicated, and thus these are tabulated for practical use by various groups. Among these are the OPAL tables, `http://opalopacity.llnl.gov/`, which are commonly employed. Since the structure and surface temperature of the star are exceedingly sensitive to the particular values of κ, these are crucial to getting the internal structure correct. In addition, since the photosphere is defined by the optical depth, the opacities define the surface location. There importance can be directly inferred from enormous temperature gradient present in most stars, e.g., the Sun's temperature varies from more than 10^7 K at the center to roughly 5800 K at the surface! Despite this, an adequate approximation can be obtained via three processes: electron scattering, a Kramers-like approximation to free-free, and that due to H^- at the sufficiently low temperatures near the surface, where the hydrogen is no longer ionized. The associated Rosseland mean opacities for these processes are approximated by

$$\kappa_{es} = 0.02(1+X) \text{ m}^2/\text{kg}\,, \tag{10}$$
$$\kappa_{ff} = 1.0\times10^{24}(Z+0.0001)\rho_3^{0.7}T^{-3.5} \text{ m}^2/\text{kg}\,, \tag{11}$$
$$\kappa_{H^-} = 2.5\times10^{-32}(Z/0.02)\rho_3^{0.5}T^9 \text{ m}^2/\text{kg}\,, \tag{12}$$
$$\tag{13}$$

where $\rho_3 \equiv \rho/10^3$ kg/m^3 and the deviation in the density dependence from the normal Kramers opacity arises to fit the OPAL tables for solar composition more closely. The corresponding radiative opacity is then obtained by an appropriate combination off these:

$$\kappa(\rho, T) = \left[\frac{1}{\kappa_{H^-}} + \frac{1}{\max(\kappa_{es}, \kappa_{ff})}\right]^{-1}\,. \tag{14}$$

At high temperatures both free-free and scattering occur due to the large numbers of free electrons, and thus the opacity is set by the maximum of the two. However, below 10^4 K, hydrogen recombines and the opacity is dominated by that due the H^- ion. Of course, at higher temperatures there are no H^- ions. Hence near and below 10^4 K, the net opacity is the *minimum* of the that due to free-free/scattering and that due to H^-, which has been smoothed via the expression given above. *While these processes were mentioned in lecture, explicit expressions were not given for them.* Conduction is important only at the centers of evolved stars, and thus is completely neglected.

2.2 Numerically Integrating the Stellar Structure Equations

Because boundary conditions are defined at both the center and surface of the star, some care must be taken when integrating the coupled set of ODEs. A common solution is the "shooting method". The idea is simply to choose *all four* of the conditions at the center of the star (i.e., choose $M(0) = L(0) = 0$ and $\rho(0) = \rho_c$ and $T(0) = T_c$), integrate until the surface is reached, and vary the choices at the center until the desired boundary condition at the surface is satisfied. We can vary either ρ_c or T_c, keeping the other as our Main Sequence parameterization. *In practice it will be numerically much more convenient to choose T_c to be the Main Sequence parameter and vary ρ_c to satisfy the surface boundary condition on luminosity and surface temperature.* Thus, the procedure is:

1. Choose ρ_c and T_c.
2. Integrate equations (2) to obtain R_*, $L(R_*)$ and $T(R_*)$.
3. Compare $L(R_*)$ and $4\pi\sigma R_*^2 T_*^4$ and modify ρ_c accordingly.
4. Repeat!

Most of the complication appears in steps 2 and 3, which we now discuss.

2.2.1 Integrating to Obtain a Trial Solution (Step 2)

There are two main complications in integrating equations (2), the behavior near $r = 0$ and the practical determination of R_*.

Near $r = 0$ some of the source terms appear to diverge. As discussed in class this is avoided physically by choosing the appropriate conditions on L and M at the center. However, numerically, the computer will complain nonetheless. The solution is to begin at some small r_0 which is chosen to be much, much smaller than any characteristic in the problem, at which we have

$$\rho = \rho_c\,, \quad T = T_c\,, \quad M = \frac{4\pi}{3} r_0^3 \rho_c\,, \quad L = \frac{4\pi}{3} r_0^3 \rho_c \epsilon(\rho_c, T_c)\,. \tag{15}$$

The second difficulty is to define the surface, which required $\tau(\infty)$. Again the solution is to integrate to finite radii. However, some care must be taken to ensure that this is far enough. This may be achieved via an opacity proxy:

$$\tau(\infty) - \tau \approx \delta\tau \equiv \frac{\kappa\rho^2}{|d\rho/dr|}\,, \tag{16}$$

which provides an order-of-magnitude estimate of the remaining optical depth. Integration proceeds until $\delta\tau \ll 1$. In practice it is often convenient to introduce a mass limit as well, e.g., $M < 10^3 M_\odot$, since some fully radiative trial solutions can erroneously extend to very large radii. The resulting value of τ at the largest radii considered provides an approximate value of $\tau(\infty)$, thus setting the arbitrary offset in τ. From this, the surface is obtained by simple interpolation from $\tau(\infty) - \tau(R_*) = 2/3$. *In practice, the actual optical depth is somewhat smaller than that implied by $\delta\tau$ since it neglects the temperature structure of the photosphere.*

Finally, since a variety of scales are present, I've used the 4th-5th order adaptive step-sized Runge-Kutta from Numerical Recipes (`http://www.nr.com/`*) to resolve the various features. I have not attempted to do this with a straight Eulerian integrator. I suspect that this would produce significant problems since it would fail to accurately resolve the surface layers of the star.*

2.2.2 Evaluating a Trial Solution (Step 3)

The procedure in the previous section effectively provides a way to construct the function $L(R_*; \rho_c, T_c)$. That is, given the location of the surface, the remaining quantities can be obtained, again by interpolation, most importantly giving $L(R_*)$ and $T(R_*)$. For a given trial, this will typically fail to satisfy the desired surface boundary condition. However, we may imagine encapsulating (and you should do exactly that!) this procedure into a function of ρ_c, returning a value that is dependent upon the error in the surface luminosity boundary condition:

$$f(\rho_c) = \frac{L_* - 4\pi\sigma R_*^2 T_*^4}{\sqrt{4\pi\sigma R_*^2 T_*^4 L_*}}\,, \tag{17}$$

where we have set $L_* = L(R_*)$ and $T_* = T(R_*)$, and $f(\rho_c)$ is conveniently normalized so that near the solution it represents a fractional error while far from it it diverges. The desired stellar solution is then obtained by numerically finding the root of f, giving the specific ρ_c for which the surface boundary condition is satisfied.

Typical solutions will either be radiatively dominated near the surface, and thus vastly over-shoot T_* (typical surface temperatures for these are 10^6 K!), or convectively dominated near the surface, and thus under-shoot T_*. The true solution is a finely tuned value of ρ_c corresponding to near where these meet. However, near this location the value of f appears to diverge. Thus, gradient-based root-finders are ill-equipped for this task. However, bisection proved robust, obtaining solutions rapidly. For bisection to work, the desired root must be bounded; in practice $\rho_c = 0.3$–500 covered the entire relevant range.

Frequently it will not be possible to obtain a solution that converges f before it converges ρ_c numerically (i.e., variations in ρ_c are not measurable at `long double` precision in C++). Nevertheless, in this case the stellar structure has completely converged, and thus the surface temperature is simply identified via setting the boundary condition to zero by hand, the true solution being guaranteed to exist.

3 Example Results

Solutions to individual stars vary tremendously as a function of T_c, as anticipated. At low T_c, and thus low masses and late spectral types, the stars are fully convective, powered by the PP-chain, and supported by degeneracy pressure. *Note that the neglect of molecular line opacity makes these models rather suspect, potentially substantially modifying T_* and thus increasing the mass.* Near $T_* \simeq 2000$ K the star develops a radiative core and is dominated by ideal gas pressure; nevertheless, it is still powered by the PP-chain. As the temperature rises the radiative region expands. However, near $1M_\odot$ the CNO-cycle begins to become important and rapidly dominates. The strong temperature dependence of ϵ_{CNO} results in a convective core, surrounding by a radiative envelope. At the highest masses the photon pressure begins to become important, and the opacity is dominated by electron scattering. An example solution is shown in Figure 1.

The resulting HR diagram is shown in the left panel of Figure 2. A summary of the stellar structure is shown in the right panel of Figure 2, similar to Fig. 22.7 of Kippenhahn & Weigert. Again, the expected general trends in the structure are observed.

4 Independent Project

The individual projects consist of the following parts:

1. Describe the specific algorithmic choices made by your group to solve the stellar structure equations. Your descriptions must be in your own words, identical descriptions are not permitted!

2. Produce a Main Sequence from your code and plot::

   (a) The associated HR diagram.

   (b) $L/L_\odot$ as a function of $M/M_\odot$ and compare to the expression in the text.

   (c) $R/R_\odot$ as a function of $M/M_\odot$ and compare to the expression in the text.

3. Choose two stars, one with a mass below $0.75M_\odot$ and one with a mass above $2M_\odot$:

   (a) plot ρ, T, M, L, P, dL/dr, and κ as functions of r/R_*

   (b) Identify the convection zones and describe why they are where they are (compare and contrast the two stars).

   (c) Determine the dominant energy generation source from your calculation (compare and contrast the two stars).

   (d) Determine the dominant opacities in each region from your calculation (compare and contrast the two stars).

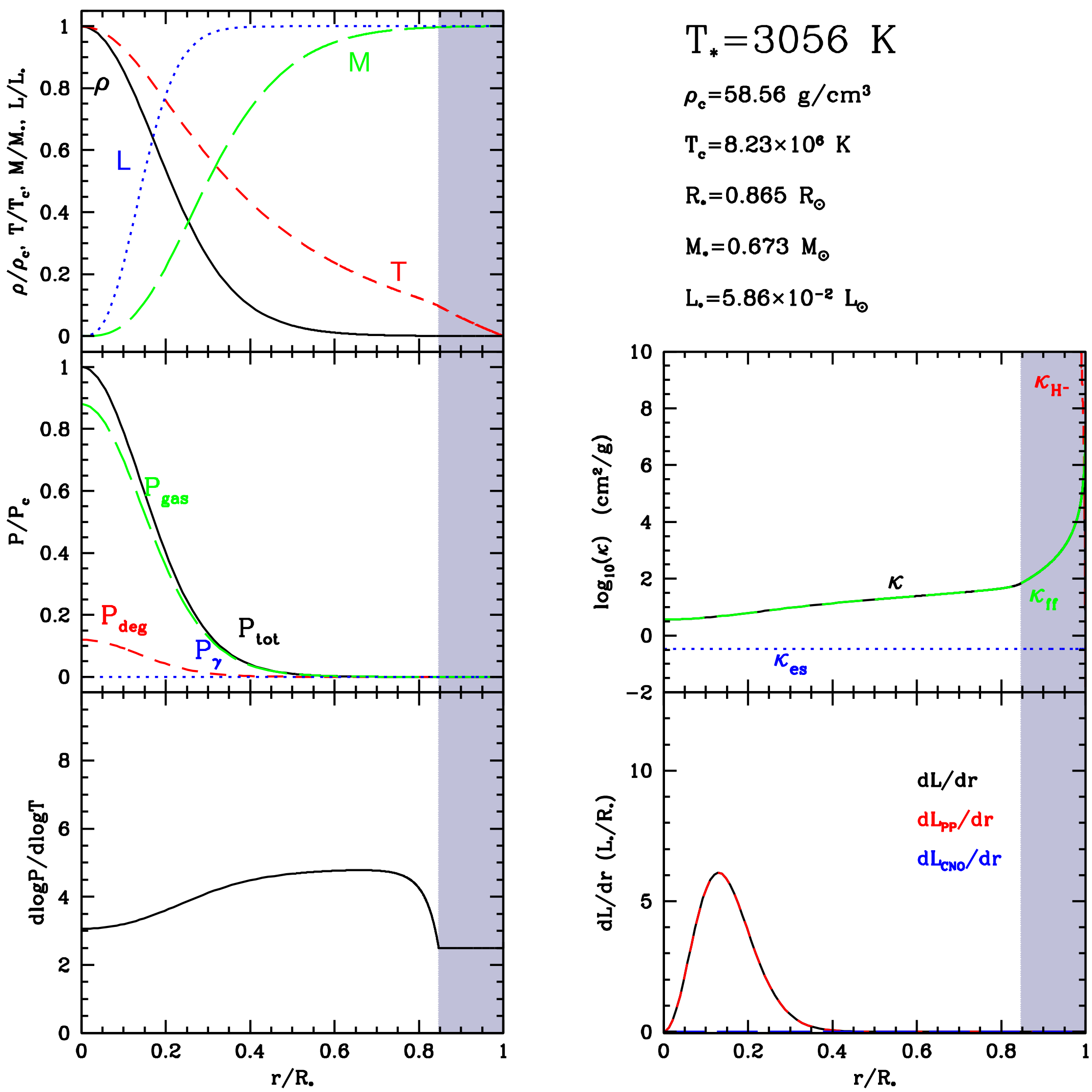

Figure 1: Stellar structure solution for the parameters shown in the upper-right. Convective regions are shown in gray. Upper left: Normalized density, temperature, mass, and luminosity. Middle left: Pressure and pressure by components. Bottom left: $d\log P/d\log T$, which must lie above $1 - 1/\gamma = 2.5$ for the star to be locally convectively stable. Middle right: Opacity and opacity by components; near the surface the opacity is usually dominated by H^-. Bottom right: Local contribution to the stellar luminosity, and by components.

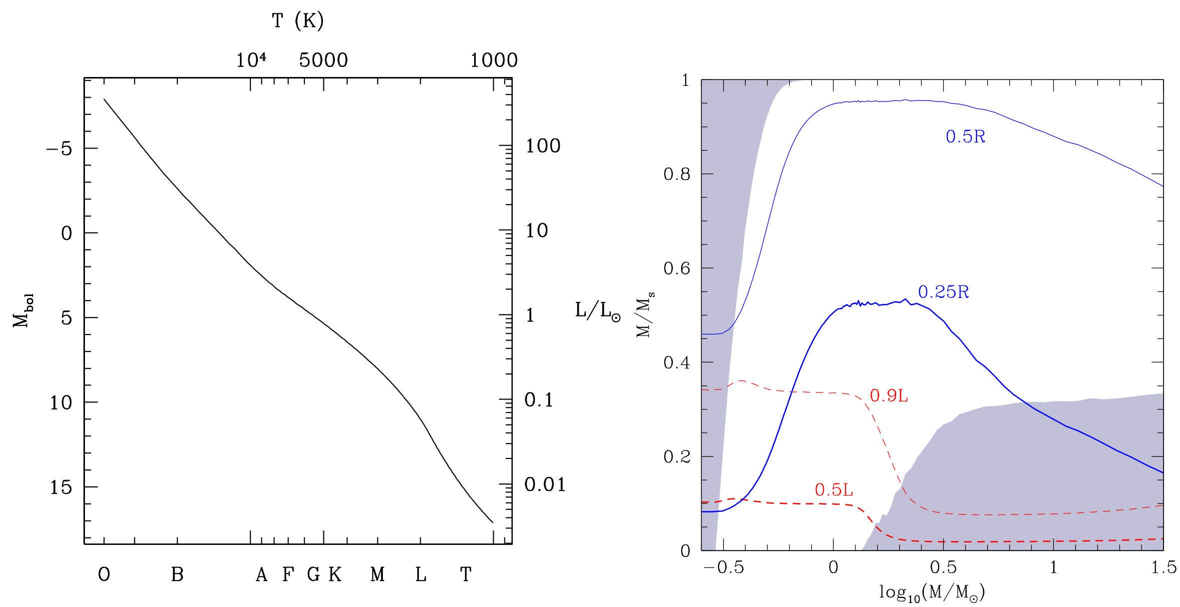

Figure 2: Left: HR diagram. Right: Summary of stellar structure for the family of solutions as a function of mass. The vertical axis shows the fraction of the total mass enclosed. Gray regions are convective. Contours showing the regions enclosed by $0.5L$ and $0.9L$ are shown in red, contours showing the regions inside $0.25R$ and $0.5R$ are shown in blue. This is a reproduction of Fig. 22.7 of Kippenhahn & Weigert.

5 Group Projects

Behind each group project is some modification of the equations of stellar structure, with the intention of exploring how stars respond to various changes in the underlying physics.

There are 5 group projects, described in each of the subsections below. The names of the people assigned to each group are listed within. Because it will be helpful if each group meets with me as they are getting started, please discuss your availability on Mondays and Wednesdays over the next week.

5.1 Gravity Group

Jamie Breault, Brandon Hiles, Jacob Koenig, Phillip Lam, Phillip Lemoine

This group will modify gravity in a specific way and infer the limits that may be placed on the modification length scale given that the Main Sequence is known to better than 10%. While many potential modifications of gravity may be considered, two particularly simply cases are:

Small Scale Below some scale λ we imagine that the gravitational force has a component that scales as r^{-3}, i.e.,

$$g = \frac{GM}{r^2}\left(1 + \frac{\lambda}{r}\right) . \tag{18}$$

The goal is then to constrain λ. Note that λ can be both positive and negative, and in this case will produce a linear cusp of one sort or another at the center of the star.

Large Scale Above some scale Λ we imagine that the gravitational force scales more weakly, as r^{-1}, i.e.,

$$g = \frac{GM}{r^2}\left(1 + \frac{r}{\Lambda}\right) . \tag{19}$$

Again the goal is to constrain Λ. As before, Λ can be positive or negative. Sufficiently large, negative Λ will produce unbound stellar exteriors.

In all cases the group should generate a variety of main sequences as functions of λ and Λ and explore the implications for the HR diagram, and the L-M and M-R relations.

5.2 Metallicity Group

Kalle Inberg, Mohamad Issawi, Aditya Iyer, Bailey Robison, Anam Yunus

A key systematic variable among stellar populations is the metallicity during formation. This is group will explore the impact that metallicity has by creating a number of main sequences with metallicities smaller and larger than the solar value. In addition, they will generate a main sequence with zero-metallicity. In all cases the group should quantitatively estimate the impact of metallicity on the HR diagram and stellar relations (e.g., L-M and M-R relations).

5.3 Nuclear Group

Emma Ellingwood, Neeti Patel, Federico Rivera Keszti, Prabhakar Uthayakumar, Christopher Zaworski

This group will explore the way in which different nuclear processes and models impact the stellar structure and correspondingly the Main Sequence. The two key things to study are the impact of varying the coefficients in the nuclear generation rates (by large factors!) and the temperature power-law indexes (by small factors!). Given that the Main Sequence is known to better than 10%, the group should estimate the permitted range over which each of these can vary. In all cases the group should generate a variety of main sequences as functions of λ and Λ and explore the implications for the HR diagram, and the L-M and M-R relations.

5.4 Pressure Group

Timothy Brett, Michael Chapman, Adrien Delazzer Meunier, Andrew McDonald, Jean-Christophe Robertson

This group will explore the way in which different sources of pressure imprint themselves on the stellar structure and correspondingly the Main Sequence. Key things turn on and off are the contributions from:

- ideal gas pressure,
- radiation pressure,
- degeneracy pressure.

For each generate the Main Sequence and illustrative stellar examples and indicate where each is important.

5.5 Other Main Sequences

Caleb Cote, Nazban Darukhanawalla, Rishita Gudapati, Hiral Patel, Cheok Vong

This group will explore the consequence of stellar evolution via the helium and carbon main sequences. To do this, the group should modify the composition in the "core" of the star, defined as the region in which 99% of the luminosity is produced, to be composed of entirely helium or carbon. The appropriate specific energy generation rates for He- and C-burning are, respectively,:

$$\varepsilon_{3\alpha} = 3.85 \times 10^{-8} Y^3 \rho_5^2 T_8^{44} \text{ W/kg} \qquad \text{and} \qquad \varepsilon_{CC} = 5.0 \times 10^4 X_C^2 \rho_5 T_9^{30} \text{ W/kg} \tag{20}$$

where X_C is the carbon mass fraction and $T_8 = T/10^8$ K and $T_9 = T/10^9$ K. The group should explore the associated main sequences for He cores and C cores, the M-R and L-M relationships and estimate the lifetimes of stars on each. Based on this, make an estimate of which regions of the associated sequences are sufficiently long-lived to justify the equilibrium assumptions underlying their construction.