Problem 1 (**Linearization and State-space Representation of a Dynamical System**) Consider the following system of nonlinear differential equations:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} x_1^3 + x_2^2 + x_3\cos(x_1) + u\sin(x_1) \\ x_1 + x_1\cos(x_2) + x_3 + u\cos(x_2) \\ (1+x_1)x_1 + (1+x_2)x_3 + u\cos(x_3) \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 \\ x_3 + x_3^2 \end{bmatrix}$$

(a) Linearize the given system about the equilibrium point $\mathbf{x}^{eq} = \begin{bmatrix} x_1^{eq} \\ x_2^{eq} \\ x_3^{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u^{eq} = 0$ to find the Jacobian matrices $A = \left(\frac{\partial f}{\partial x}\right)\Big|_{(x^{eq},u^{eq})}$, $B = \left(\frac{\partial f}{\partial u}\right)\Big|_{(x^{eq},u^{eq})}$, $C = \left(\frac{\partial g}{\partial x}\right)\Big|_{(x^{eq},u^{eq})}$ and $D = \left(\frac{\partial g}{\partial u}\right)\Big|_{(x^{eq},u^{eq})}$.

(b) Represent the linearized system in the state-space form.

Solutions

(a) There are three nonlinear functions $f_i(\mathbf{x}, u)$, defined as:

$$\begin{aligned} f_1(\mathbf{x}, u) &= x_1^3 + x_2^2 + x_3\cos(x_1) + u\sin(x_1) \\ f_2(\mathbf{x}, u) &= x_1 + x_1\cos(x_2) + x_3 + u\cos(x_2) \\ f_3(\mathbf{x}, u) &= (1+x_1)x_1 + (1+x_2)x_3 + u\cos(x_3) \end{aligned}$$

We obtain Jacobian matrices A and B by taking the partial derivatives with respect to $\mathbf{x}$ and u as follows:

$$\begin{aligned}
\frac{\partial f_1(\mathbf{x}, u)}{\partial x_1}\bigg|_{(x^{eq},u^{eq})} &= 3(x_1^{eq})^2 - x_3^{eq}\sin(x_1^{eq}) + u^{eq}\cos(x_1^{eq}) = 0 \\
\frac{\partial f_1(\mathbf{x}, u)}{\partial x_2}\bigg|_{(x^{eq},u^{eq})} &= 2x_2^{eq} = 0 \\
\frac{\partial f_1(\mathbf{x}, u)}{\partial x_3}\bigg|_{(x^{eq},u^{eq})} &= \cos(x_1) = 1 \\
\frac{\partial f_2(\mathbf{x}, u)}{\partial x_1}\bigg|_{(x^{eq},u^{eq})} &= 1 + \cos(x_2^{eq}) - u^{eq}\sin(x_2^{eq}) = 2 \\
\frac{\partial f_2(\mathbf{x}, u)}{\partial x_2}\bigg|_{(x^{eq},u^{eq})} &= -x_1^{eq}\sin(x_2^{eq}) + u^{eq}\cos(x_2^{eq}) = 0 \\
\frac{\partial f_2(\mathbf{x}, u)}{\partial x_3}\bigg|_{(x^{eq},u^{eq})} &= 1 \\
\frac{\partial f_3(\mathbf{x}, u)}{\partial x_1}\bigg|_{(x^{eq},u^{eq})} &= 1 + 2x_1^{eq} = 1 \\
\frac{\partial f_3(\mathbf{x}, u)}{\partial x_2}\bigg|_{(x^{eq},u^{eq})} &= x_3^{eq} = 0 \\
\frac{\partial f_3(\mathbf{x}, u)}{\partial x_3}\bigg|_{(x^{eq},u^{eq})} &= 1 + x_2^{eq} - u^{eq}\sin(x_3^{eq}) = 1 \\
\frac{\partial f_1(\mathbf{x}, u)}{\partial u}\bigg|_{(x^{eq},u^{eq})} &= \sin(x_1^{eq}) = 0 \\
\frac{\partial f_2(\mathbf{x}, u)}{\partial u}\bigg|_{(x^{eq},u^{eq})} &= \cos(x_2^{eq}) = 1 \\
\frac{\partial f_3(\mathbf{x}, u)}{\partial u}\bigg|_{(x^{eq},u^{eq})} &= \cos(x_3^{eq}) = 1
\end{aligned}$$

Matrices A and B are given as:

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Similarly, there are two nonlinear functions $g_i(\mathbf{x}, u)$, defined as:

$$g_1(\mathbf{x}, u) = x_1 + x_2$$
$$g_2(\mathbf{x}, u) = x_3 + x_3^2$$

We obtain Jacobian matrices C and D by taking the partial derivatives with respect to $\mathbf{x}$ and u as follows:

$$\left.\frac{\partial g_1(\mathbf{x}, u)}{\partial x_1}\right|_{(x^{eq}, u^{eq})} = 1$$
$$\left.\frac{\partial g_1(\mathbf{x}, u)}{\partial x_2}\right|_{(x^{eq}, u^{eq})} = 1$$
$$\left.\frac{\partial g_1(\mathbf{x}, u)}{\partial x_3}\right|_{(x^{eq}, u^{eq})} = 0$$
$$\left.\frac{\partial g_2(\mathbf{x}, u)}{\partial x_1}\right|_{(x^{eq}, u^{eq})} = 0$$
$$\left.\frac{\partial g_2(\mathbf{x}, u)}{\partial x_2}\right|_{(x^{eq}, u^{eq})} = 0$$
$$\left.\frac{\partial g_2(\mathbf{x}, u)}{\partial x_3}\right|_{(x^{eq}, u^{eq})} = 1 + 2x_3^{eq} = 1$$
$$\left.\frac{\partial g_1(\mathbf{x}, u)}{\partial u}\right|_{(x^{eq}, u^{eq})} = 0$$
$$\left.\frac{\partial g_2(\mathbf{x}, u)}{\partial u}\right|_{(x^{eq}, u^{eq})} = 0$$

Matrices C and D are given as:

$$C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

(b) The state-space representation of the linearized system is given as:

$$\dot{x} = Ax + Bu = \begin{bmatrix} 0 & 0 & 1 \\ 2 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u$$
$$y = Cx + Du = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u$$

Problem 2 (**Transfer Function and Time Response of an LTI System**) Consider a linear time-invariant system:

$$\dot{x} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 1 & 1 \end{bmatrix} x$$

(a) Find the transfer function of the given system. Is the obtained transfer function a proper rational function?
(b) Assume $t_0 = 0$ and compute the state transition matrix, $\Phi(t, 0)$ of the given system.
(c) Given the initial state $x(0) = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \end{bmatrix}$, compute the zero-input response of the given system.

Solutions:
(a) The transfer function of a linear time-invariant system can be find as:

$$G(s) = C[sI - A]^{-1}B$$

We thus first compute the inverse of matrix $[sI - A]$ as:

$$(sI - A) = \begin{bmatrix} s & -3 \\ -3 & s \end{bmatrix}$$
$$(sI - A)^{-1} = \frac{1}{s^2 - 9} \begin{bmatrix} s & 3 \\ 3 & s \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2-9} & \frac{3}{s^2-9} \\ \frac{3}{s^2-9} & \frac{s}{s^2-9} \end{bmatrix}$$

We now compute:

$$G(s) = C[sI - A]^{-1}B = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \frac{s}{s^2-9} & \frac{3}{s^2-9} \\ \frac{3}{s^2-9} & \frac{s}{s^2-9} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s+3}{s^2-9} & \frac{s+3}{s^2-9} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+3}{(s+3)(s-3)} = \frac{1}{s-3}$$

The obtained transfer function is a proper rational function.
(b) The state transition matrix of the given system can be computed as:

$$\Phi(t, 0) = \exp\{At\} = \mathcal{L}^{-1} \left\{ \begin{bmatrix} \frac{s}{s^2-9} & \frac{3}{s^2-9} \\ \frac{3}{s^2-9} & \frac{s}{s^2-9} \end{bmatrix} \right\}$$

Using the following formulas from the Laplace transform table:

$$\frac{a}{s^2 - a^2} \Rightarrow \sinh(at)$$
$$\frac{s}{s^2 - a^2} \Rightarrow \cosh(at)$$

we obtain the state transition matrix as $\Phi(t, 0) = \begin{bmatrix} \cosh(3t) & \sinh(3t) \\ \sinh(3t) & \cosh(3t) \end{bmatrix}$.
(c) The zero-input response (ZIR) of the given system is defined as $y_{ZIR} = Ce^{At}x_0$. We thus compute:

$$\begin{aligned} y_{ZIR} = C\exp(At)x(0) &= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} \cosh(3t) & \sinh(3t) \\ \sinh(3t) & \cosh(3t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \begin{bmatrix} \sinh(3t) + \cosh(3t) & \sinh(3t) + \cosh(3t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &= \{\sinh(3t) + \cosh(3t)\}x_1(0) + \{\sinh(3t) + \cosh(3t)\}x_2(0) \\ &= \sinh(3t) + \cosh(3t) \end{aligned}$$

Problem 3 (**Functions of a Square Matrix**) Consider a matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 5 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

(a) Find the characteristic polynomial $\Delta(\lambda)$ of matrix A.
(b) Find the eigenvalues of matrix A.
(c) Find matrix power A^{25}.

Solutions
(a) The characteristic polynomial $\Delta(\lambda)$ is defined as $\Delta(\lambda) = \det(\lambda I - A)$. We therefore compute:

$$\begin{aligned} \Delta(\lambda) = \det(\lambda I - A) &= \begin{vmatrix} \lambda - 1 & 0 & -1 \\ 0 & \lambda - 5 & -2 \\ 0 & 0 & \lambda - 3 \end{vmatrix} \\ &= (\lambda - 1) \begin{vmatrix} \lambda - 5 & -2 \\ 0 \lambda - 3 & \end{vmatrix} = (\lambda - 1)(\lambda - 3)(\lambda - 5) \\ &= \lambda^3 - 9\lambda^2 + 23\lambda - 15 \end{aligned}$$

(b) The eigenvalues are defined as the roots of the characteristic polynomial. Thus, we can read off the eigenvalues as:

$$\lambda_1 = 1, \qquad \lambda_2 = 3, \qquad \lambda_3 = 5$$

Note: Given matrix is an upper triangular matrix, so the eigenvalues can simply be read off the diagonal.

(c) The matrix power A^{25} can be computed using the Cayley-Hamilton theorem. We define two functions, $f(\lambda) = \lambda^{25}$ and $h(\lambda) = \beta_0 + \beta_1\lambda + \beta_2\lambda^2$. By equating these two function of the spectrum of A:

$$f^l(\lambda) = h^l(\lambda)$$

we obtain the following system of equations:

f(1)=h(1): 1=b0+b1+b2
f(3)=h(3): 3^25 = b0+3b1+9b2
f(5)=h(5): 5^5 = b0+5b1+25b2

Solve for bo, b1, b2

Problem 4 (**Jordan Decomposition of Matrices**) Consider matrices:

$$A_1 = \begin{bmatrix} 2 & 6 & 2 \\ 2 & 0 & 0 \\ 2 & 6 & 2 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} -1 & 7 & 4 & 9 \\ 0 & 0 & -2 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

(a) Find the characteristic polynomial and eigenvalues of matrices A_1 and A_2.
(b) Find the Jordan form representation of matrices A_1 and A_2.

Solutions
(a) The characteristic polynomial $\Delta(\lambda)$ is defined as $\Delta(\lambda) = \det(\lambda I - A)$. We therefore compute:

$$\begin{aligned} \Delta(\lambda)_{A_1} &= \det(\lambda I - A_1) = \begin{vmatrix} \lambda-2 & -6 & -2 \\ -2 & \lambda & 0 \\ -2 & -6 & \lambda-2 \end{vmatrix} = -2\,(-1)^{1+2} \begin{vmatrix} -6 & -2 \\ -6 & \lambda-2 \end{vmatrix} + \lambda \begin{vmatrix} \lambda-2 & -2 \\ -2 & \lambda-2 \end{vmatrix} \\ &= \lambda(\lambda+2)(\lambda-6) = \lambda^3 - 4\lambda^2 - 12\lambda \end{aligned}$$

$$\begin{aligned} \Delta(\lambda)_{A_2} &= \det(\lambda I - A_2) = \begin{vmatrix} \lambda+1 & -7 & -4 & -9 \\ 0 & \lambda & 2 & 3 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda+4 \end{vmatrix} = (\lambda+1) \begin{vmatrix} \lambda & 2 & 3 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda+2 \end{vmatrix} \\ &= \lambda(\lambda+1) \begin{vmatrix} \lambda & 0 \\ 0 & \lambda+2 \end{vmatrix} \\ &= \lambda^2(\lambda+1)(\lambda+2) = \lambda^4 + 3\lambda^2 + 2\lambda^2 \end{aligned}$$

(b) The eigenvalues are defined as the roots of the characteristic polynomial. Thus, we can read off the eigenvalues as:

$$\begin{aligned} &\lambda(A_1)_1 = 0, \qquad \lambda(A_1)_2 = 2, \qquad \lambda(A_1)_3 = 6 \\ &\lambda(A_2)_1 = -1, \qquad \lambda(A_2)_2 = -2, \qquad \lambda(A_2)_3 = 0, \text{ with multiplicity 2} \end{aligned}$$

(c) Matrix A_1 has all three distinct eigenvalues. It can therefore be put in the diagonal form as follows:

$$J_{A_1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix}$$

Matrix A_2 has one eigenvalue with multiplicity 2 (larger than 1). Its diagonal form therefore may not exist and in order to find Jordan form representation of A_2, we first determine the nullity of matrix $A_2 - \lambda_3 I = A_2$. The second column of matrix A_2 is equal to the first column multiplied by (-7). Thus, the rank of A_2 is 3 and correspondingly, the nullity is 1. Therefore, there is one Jordan block associated with eigenvalue $\lambda_3 = 0$. The Jordan form representation of matrix A_2 is thus given as:

$$J_{A_2} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}$$

Problem 5 (**Stability of Linear Time-invariant Systems**)

(a) Consider a system in Problem 2. Is the given system BIBO stable? Explain your response.

(b) Consider matrices A_1 and A_2 given in Problem 4. Assume the matrices A_1 and A_2 define the following continuous-time homogeneous LTI systems:

$$\dot{x}_1 = A_1 x_1$$
$$\dot{x}_2 = A_2 x_2$$

Are continuous-time systems defined by matrices A_1 and A_2 asymptotically stable? Are they marginally stable? Explain your response.

(c) Consider the following discrete-time system:

$$x[k+1] = \begin{bmatrix} 0.3 & 0.5 & -2 \\ 0 & 0.75 & 0 \\ 0 & 1 & -0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u[k]$$

Is the given system asymptotically stable? Is it marginally stable? Explain your answer.

(d) Consider the following continuous system:

$$\dot{x} = \begin{bmatrix} -1 & -3 \\ 0 & -5 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Use the Lyapunov test for stability to check if the given system is asymptotically stable. Use positive definite matrix $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in your computations.

Solutions

(a) The transfer function of the system given in Problem 2 is $G(s) = \frac{1}{s-3}$. This function has one positive pole, $s = 3$. Hence the system is not BIBO stable.

(b) Matrix A_1 from Problem 4 has three distinct eigenvalues $\lambda_1 = 0, \lambda_2 = 2$ and $\lambda_3 = 6$. Since eigenvalues λ_2 and λ_3 are positive, the given system is neither asymptotically nor marginally stable.
Matrix A_2 has three distinct eigenvalues $\lambda_1 = -1, \lambda_2 = -2$ and $\lambda_3 = 0$, with multiplicity 2. Since there exist an eigenvalue 0, the system is not asymptotically stable. Also, since the size of Jordan block corresponding to the eigenvalue 0 is equal to 0, the system is not marginally stable.

(c) In order to determine if the given system is asymptotically and marginally stable, we find the eigenvalues of the given system. We start by finding the characteristic polynomial $\Delta(\lambda)$:

$$\Delta(\lambda) = \det[\lambda\, I - A] = \begin{vmatrix} \lambda - 0.3 & -0.5 & 2 \\ 0 & \lambda - 0.75 & 0 \\ 0 & -1 & \lambda + 0.5 \end{vmatrix} = (\lambda - 0.3) \begin{vmatrix} \lambda - 0.75 & 0 \\ -1 & \lambda + 0.5 \end{vmatrix}$$
$$= (\lambda - 0.3)(\lambda + 0.5)(\lambda - 0.75)$$

The eigenvalues of the given system are all distinct and they are equal to $\lambda_1 = 0.3, \lambda_2 = -0.5$ and $\lambda_3 = 0.75$. The given system is a discrete-time system and all eigenvalues of matrix A have real parts within a unit-circle. Thus, the given system is both asymptotically and marginally stable.

(d) Lyapunov test for stability says that the system is asymptotically stable if there exists a unique positive definite solution P to the following equation:

$$A^T P + PA = -Q$$

where Q is a positive definite symmetric matrix. For the given system, the Lyapunov equation is defined as:

$$\begin{bmatrix} -1 & 0 \\ -3 & -5 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & -3 \\ 0 & -5 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$\begin{bmatrix} -2p_{11} & -3p_{11} - 6p_{12} \\ -3p_{11} - 6p_{12} & -6p_{12} - 10p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

From the given equation, we compute $p_{11} = 0.5$, $p_{12} = -0.25$ and $p_{22} = 0.25$ and matrix P is given as:

$$P = \begin{bmatrix} 0.5 & -0.25 \\ -0.25 & 0.25 \end{bmatrix}$$

The characteristic polynomial of the matrix P is given as:

$$\Delta(\lambda) = \begin{vmatrix} \lambda - 0.5 & 0.25 \\ 0.25 & \lambda - 0.25 \end{vmatrix} = \lambda^2 - 0.75\lambda + 0.0625$$

and the eigenvalues of P are given as:

$$\lambda_1 = \frac{0.75 + \sqrt{0.5}}{2}, \qquad \lambda_2 = \frac{0.75 - \sqrt{0.5}}{2}$$

Both eigenvalues of are positive. Hence, the matrix P is positive definite, which implies that the system is asymptotically stable.