Lecture 10

Controllability

Lecture 11

Observability

Lecture 12

Decomposition

Usage - Necessary concepts to study
State feedback controllers
State observers

Motivation

Controllability: Satellite Attitude Control

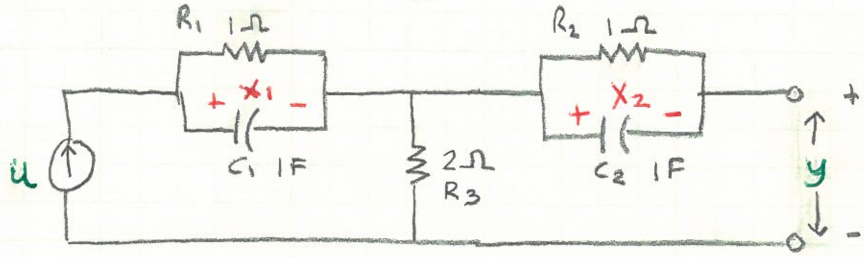
Satellite must change its orientation from one configuration (initial state) to another state in a certain amount of time.

Observability: Internal Combustion Engine

It is difficult/impossible to measure internal pressures & temperatures (states of the system). We can determine the states by observing the inputs & outputs over time.

Motivation

Consider the following network



current source input   voltage output

States: x_1 - voltage accross 1F capacitor C_1

x_2 - voltage across 1F capacitor C_2

We notice:

1) The input current u has no effect on x_2 because of the open circuit across y
2) The current passing through the 2Ω resistor R_3 always $= u \Rightarrow$ the response excited by the initial state of x_1 will not appear in y.

We say:

1) The input u cannot control x_2
2) The initial state of x_1 cannot be observed from the output y

Concepts for the internal structure of linear systems:

1) Controllability - whether or not the state of a state-space equation can be controlled from the input.
2) Observability - whether or not the initial state can be observed from the output.

Outline

Controllability of a Dynamical System

Controllability of a Linear System

Controllability of LTI Systems

Controllability Indices

Goal steer the state from the input

Controllability of a Dynamical System

Consider the dynamical system $\mathcal{D} = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$

p.5.4 Recall $\mathcal{U}$ = input set $= \{u(t),\ t \in T \subset \mathbb{R}\}$

Σ = state set $= \{x(t),\ t \in T\}$

$\mathcal{Y}$ = output set $= \{y(t),\ t \in T\}$

s = state transition function, $s(t_1, t_0, x_0, u) = x(t_1)$

"state at t_1 reached from x_0 at t_0 as a result of u"

r = read out function, $r(t, x(t), u(t))$

"response of $\mathcal{D}$ at t when it is in state $x(t)$ at t and the input is $u(t)$"

Definition

Given $t_0, t_1 \in T$, $t_1 > t_0$

$\mathcal{D}$ is said to be completely controllable on $[t_0, t_1]$ "CC"

iff $\forall\ x_0, x_1 \in \Sigma$, $\exists\ u_{[t_0, t_1]} \in \mathcal{U}$ that drives (x_0, t_0) to (x_1, t_1)

$$x_1 = s(t_1, t_0, x_0, u)$$

Comments

1) The output can be disregarded because it does not play a role in controllability

2) The trajectory $x(t)$ takes from x_0 to x_1 is not specified

3) There is no constraint placed on the input e.g., the magnitude of $u(t)$ can be as large as desired

Special Cases

$\mathcal{D} = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$

1) $x_1 = 0$

$\mathcal{D}$ is completely controllable to the origin on $[t_0, t_1]$

if $\forall\ x_0 \in \Sigma$, $\exists\ u_{[t_0, t_1]}$ such that

$$0 = s(t_1, t_0, x_0, u)$$

u transfers $x(t_0) = x_0$ to $x(t_1) = x_1 = 0$

2) $x_0 = 0$

$\mathcal{D}$ is completely reachable from the origin on $[t_0, t_1]$

if $\forall\ x_1 \in \Sigma$, $\exists\ u_{[t_0, t_1]}$ such that

$$x_1 = s(t_1, t_0, 0, u)$$

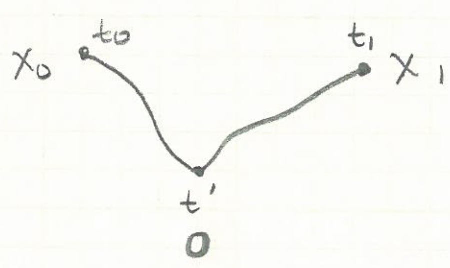
u transfers $x(t_0) = x_0 = 0$ to $x(t_1) = x_1$

Thus, we have

a) cc $\Rightarrow$ cc to 0 on $[t_0, t_1]$

b) cc $\Rightarrow$ cr from 0 on $[t_0, t_1]$

c) $\left.\begin{array}{l} \text{cc to 0 on } [t_0, t'] \\ \text{cr from 0 on } [t', t_1] \end{array}\right\} \Rightarrow$ cc on $[t_0, t_1]$



Controllability of a Linear System

Consider the Linear system $R = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$

$$\text{LTV} \begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \\ x(t_0) = x_0 \end{cases}$$

Theorem CC on $[t_0, t_1]$ iff cc to 0 on $[t_0, t_1]$
iff cr from 0 on $[t_0, t_1]$

Proof CC $\Leftrightarrow$ cc to 0

cc to 0 on $[t_0, t_1]$ $\Leftrightarrow$ $\forall \xi_0$, $\exists u$ s.t.

$$0 = s(t_1, t_0, \xi_0, u)$$
$$= \Phi(t_1, t_0)\xi_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau$$

We need to show $\forall x_0, x_1$, $\exists \tilde{u}$ s.t.

$$x_1 = s(t_1, t_0, x_0, \tilde{u})$$
$$= \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \tilde{u}(\tau)\, d\tau$$

But

$$0 = x_1 - x_1 = \underbrace{\left(\Phi(t_1, t_0) x_0 - x_1\right)}_{\text{if } = \Phi(t_1, t_0)\xi_0,\ \text{we are set}} + \int_{t_0}^{t_1} \Phi(t, \tau) B(\tau) \tilde{u}(\tau)\, d\tau$$

Pick ξ_0 as follows:

$$\Phi(t_1, t_0) \underbrace{\left(x_0 - \Phi^{-1}(t_1, t_0) x_1\right)}_{\xi_0} = \Phi(t_1, t_0)\xi_0$$

Pick $\tilde{u} = u$ to finish proof.

Note: We needed $\Phi(t, t_1)$ to be nonsingular

<u>Proof</u> cc $\Leftrightarrow$ cr from 0

cr from 0 on $[t_0, t_1]$ $\Leftrightarrow$ $\forall \xi_1$, $\exists \bar{u}$ st

$$\xi_1 = s(t_1, t_0, 0, \bar{u})$$
$$= \int_{t_0}^{t_1} \Phi(t_1, \tau)\, B(\tau)\, \bar{u}(\tau)\, d\tau$$

We need to show $\forall x_0, x_1$, $\exists u$ st

$$x_1 = s(t_1, t_0, x_0, u)$$
$$= \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau$$

Pick $\xi_1 = x_1 - \Phi(t_1, t_0) x_0$

$u = \bar{u}$

Then

$$\xi_1 = x_1 - \Phi(t_1, t_0) x_0 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) \bar{u}(\tau)\, d\tau$$

$$x_1 = \Phi(t_1, t_0) x_0 + \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau$$

Which proves the Theorem.

<u>Remark</u> Since for linear systems

cc $\Leftrightarrow$ cr from 0 on $[t_0, t_1]$

we only need to consider the ZSR of the system

$$x(t_0) = 0$$
$$x(t_1) = x_1 = \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau \qquad \text{ZSR}$$

<u>Definition</u>

$$L_c : u_{[t_0, t_1]} \longmapsto \int_{t_0}^{t_1} \Phi(t_1, \tau) B(\tau) u(\tau)\, d\tau \qquad \text{ZSR}$$

The <u>range space</u> (or <u>image</u>) of L_c is defined as the set of all $x_1 \in \mathbb{R}^n$ for which $x_1 = L_c u$ has a solution $u \in \mathbb{R}^k$:

$$\mathcal{R}(L_c) = \{ x_1 \in \mathbb{R}^n : \exists u \in \mathbb{R}^k,\ x_1 = L_c u \}$$

x_1 is the ZSR $\qquad$ $\dim \mathcal{R}(L_c) = \operatorname{rank}(L_c)$

Remark Then we have

1) x is cr from 0 on $[t_0, t_1]$ $\iff$ $x \in \mathcal{R}(L_c)$
2) $\mathcal{R}(L_c) :=$ set of reachable states from 0 on $[t_0, t_1]$
3) cc on $[t_0, t_1]$ $\iff$ $\mathcal{R}(L_c) = \mathbb{R}^n$

Definition Controllability Gramian

$$W_c(t_0, t_1) = \int_{t_0}^{t_1} \underbrace{\Phi(t_1,\tau)}_{n\times n} \underbrace{B(\tau)}_{n\times m} \underbrace{B'(\tau)}_{m\times n} \underbrace{\Phi'(t_1,\tau)}_{n\times n} \, d\tau \quad \in \mathbb{R}^{n\times n}$$

This is an $n\times n$ matrix

Theorem Given $W_c(t_0, t_1)$ as above,

$\mathcal{R}(L_c) = \mathbb{R}^n \iff W_c(t_0, t_1)$ is nonsingular

$\iff$ cc on $[t_0, t_1]$

Construct Input

Assume system is cc on $[t_0, t_1]$

How to find $u(t)$?

We have $W_c(t_0, t_1)$ nonsingular

$$x(t_1) = x_1 = \Phi(t_1, t_0)\, x_0 + \int_{t_0}^{t_1} \Phi(t_1,\tau) B(\tau) u(\tau)\, d\tau$$

Claim

☆ $$u(t) = B'(t)\, \Phi'(t_1, t)\, W_c^{-1}(t_0, t_1)\left(x_1 - \Phi(t_1, t_0)\, x_0\right)$$ control input

Proof - by substitution. ($u(t)$ into $x(t_1)$)

Definition

The null space (or Kernel) of L_c is defined as the set of all $u \in \mathbb{R}^k$ st $L_c(u) = 0$

$$N(L_c) = \{ u \in \mathbb{R}^k : L_c(u) = 0 \}$$

$\dim N(L_c) =$ nullity of L_c

☆ Note $u(t)$ above is unique — prove by showing $N(L_c) = \{0\}$

Controllability of LTI Systems $\quad R = [A, B, C, D]$

Theorem The following are equivalent:

1) (A, B) $\quad A$ $n \times n$, $\quad B$ $n \times k$ is controllable

2) The $n \times n$ matrix

$$W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau = \int_0^t e^{A(t-\tau)} BB' e^{A'(t-\tau)} d\tau$$

is nonsingular $\forall\, t > 0$ (Thus positive definite)

3) The $n \times nk$ matrix

$$\mathcal{C} = [B \quad AB \quad A^2B \ \cdots \ A^{n-1}B]$$ Controllability Matrix

has rank n (full row rank)

4) The $n \times (n+k)$ matrix

$$[A - \lambda I \ \vdots \ B]$$

has full row rank $\forall$ eigenvalue λ of A

5) If all eigenvalues of A have negative real parts (Thus, Asymptotically Stable) then $\exists!$ positive definite solution, W_c, to the Lyapunov equation:

$$AW_c + W_cA' = -BB'$$

The solution can be expressed as

$$W_c = \int_0^\infty e^{A\tau} BB' e^{A'\tau} d\tau$$ Controllability Gramian

$$\text{LTI} \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(t_0) = x_0 \end{cases}$$

Theorem

x is cr from 0 on $[t_0, t_1]$, i.e., $x \in \mathcal{R}(L_c)$ for

$$L_c : u_{[t_0,t_1]} \longmapsto \int_{t_0}^{t_1} \Phi(t_1,\tau) B u(\tau) d\tau \qquad \text{ZSR}$$

$$\Leftrightarrow \quad x \in \mathcal{R}(\mathcal{C}) = \{x \in \mathbb{R}^n : \exists\, v \in \mathbb{R}^{nk},\ x = \mathcal{C} v\}$$

where $\mathcal{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$

Proof ($\Rightarrow$)

Assume $x \in \mathcal{R}(L_c)$

Then $\exists\ u_{[t_0,t_1]}$ s.t. $x = L_c u = \int_{t_0}^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$

Recall $e^{At} = \sum_{k=0}^{\infty} \frac{A^k t^k}{k!}$

Recall Cayley Hamilton Theorem gives

$0 = \Delta(s) = \det(sI - A) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$ characteristic polynomial of A

$0 = \Delta(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$

$\Rightarrow$ (i) $A^n = -\alpha_1 A^{n-1} - \alpha_2 A^{n-2} - \cdots - \alpha_n I$

(ii) $A^k = -\gamma_1 A^{n-1} - \gamma_2 A^{n-2} - \cdots - \gamma_n I \quad \forall\, k \geq n$

(iii) $f(A) = \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_n I$ useful formula

for any polynomial of A

Thus for $f(A) = e^{At}$, $e^{At} = \beta_1 A^{n-1} + \beta_2 A^{n-2} + \cdots + \beta_n I$

Thus

$$x = \sum_{i=0}^{n-1} A^i B \underbrace{\int_{t_0}^{t_1} \beta_i(t_1-\tau) u(\tau) d\tau}_{\text{a vector}}$$

Thus x is a linear combination of $B, AB, \ldots, A^{n-1}B$

Thus $x \in \mathcal{R}(\mathcal{C})$

$\Rightarrow$ $\mathcal{R}(\mathcal{C})$ is the set of reachable states.

Constructing the Input $u(t)$

Claim: the input

$$u(t) = -B' e^{A'(t_1 - t)} W_c^{-1}(t_1) [e^{At_1} x_0 - x_1]$$

"minimum energy control"

will transfer $x(0) = x_0$ to

$$x(t_1) = x_1 = e^{At_1} x_0 + \int_0^{t_1} e^{A(t_1-\tau)} B u(\tau) d\tau$$

Proof. by substitution

$$x(t_1) = e^{At_1} x_0 - \underbrace{\int_0^{t_1} e^{A(t_1-\tau)} B B' e^{A'(t_1-\tau)} d\tau}_{W_c(t_1)} \underbrace{W_c^{-1}(t_1)[e^{At_1} x_0 - x_1]}_{\text{not a function of } t}$$

$$= e^{At_1} x_0 - W_c(t_1) W_c^{-1}(t_1) [e^{At_1} x_0 - x_1]$$

$$= e^{At_1} x_0 - I [e^{At_1} x_0 - x_1]$$

$$= e^{At_1} x_0 - e^{At_1} x_0 + x_1$$

$$= x_1$$

Remark

If W_c nonsingular, W_c^{-1} exists and $u(t)$ transfers $x_0 \to x_1$ $\Rightarrow$ (A, B) is controllable

Remark

Control "energy" $J = \frac{1}{2} \int_{t_0}^{t_1} \| u(t) \|^2 dt$

Minimum energy control = $u(t)$ that expends minimal energy

$u: \underset{t_0}{x_0} \to \underset{t_1}{x_1}$

$$\int_{t_0}^{t_1} \bar{u}^*(t) \bar{u}(t) dt \geq \int_{t_0}^{t_1} u^*(t) u(t) dt$$

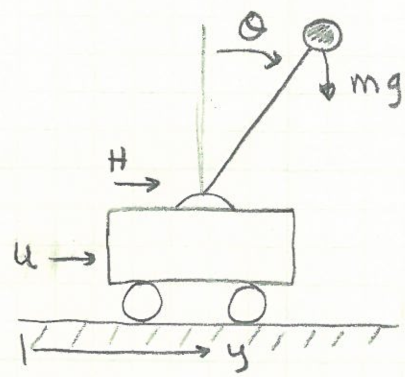
for any other input $\bar{u}(t): \underset{t_0}{x_0} \to \underset{t_1}{x_1}$

Application: minimize fuel consumption on satellite launch

Example 1

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 5 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \\ -2 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} x$$



Inverted Pendulum

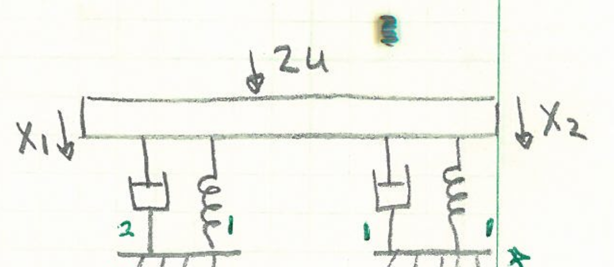
≈ balancing broom on palm

$$\mathcal{C} = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} \qquad n=4$$

$$= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & -2 & 0 & -10 \\ -2 & 0 & -10 & 0 \end{bmatrix}$$

$$\text{rank}(\mathcal{C}) = 4 \quad \Rightarrow \quad cc$$

Example 2



$$\dot{x} = \begin{bmatrix} -0.5 & 0 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} u$$

If $x_1(0) = 10$ and $x_2(0) = -1$, Can we apply a force to bring the platform to equilibrium in 2 seconds?

$$\text{rank}([B \;\; AB]) = \text{rank}(\mathcal{C}) = \text{rank}\left(\begin{bmatrix} 0.5 & -0.25 \\ 1 & -1 \end{bmatrix}\right) = 2 = n$$

Thus cc. Thus $\exists\, u : x(0) \to 0$ in 2 seconds.

$$W_c(2) = \int_0^2 e^{A\tau} BB' e^{A'\tau} d\tau$$

$$= \int_0^2 \begin{bmatrix} e^{-\tau/2} & 0 \\ 0 & e^{-\tau} \end{bmatrix} \begin{bmatrix} 0.5 \\ 1 \end{bmatrix} \begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-\tau/2} & 0 \\ 0 & e^{-\tau} \end{bmatrix} d\tau$$

$$= \begin{bmatrix} 0.2162 & 0.3167 \\ 0.3167 & 0.4908 \end{bmatrix}$$

$$u(t) = -B' e^{A'(2-t)} W_c^{-1}(2) \left[e^{2A} x_0 - x_1 \right]$$

$$= -\begin{bmatrix} 0.5 & 1 \end{bmatrix} \begin{bmatrix} e^{-(2-t)/2} & 0 \\ 0 & e^{-(2-t)} \end{bmatrix} W_c^{-1}(2) \begin{bmatrix} e^{-1} & 0 \\ 0 & e^{-2} \end{bmatrix} \begin{bmatrix} 10 \\ -1 \end{bmatrix}$$

$$= -58.82\, e^{t/2} + 27.96\, e^{t} \qquad t \in [0,2]$$

plot in Matlab

Controllability Indices

$\mathcal{C} = [\,B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\,]$ Controllability Matrix

Case 1 : B $n\times 1$, A $n\times n$ $\Rightarrow$ $\mathcal{C}$ $n\times n$

If rank$(\mathcal{C}) = n$, $\mathcal{C}^{-1}$ exists, (A,B) controllable

$\Rightarrow$ all columns of $\mathcal{C}$ are linearly independant

$\Rightarrow$ can use columns as a basis for $\mathbb{R}^n$ useful fact in control design

Case 2 : B $n\times k$ ← #inputs, A $n\times n$ Multivariable system

$\mathcal{C} = [\,b_1 \; b_2 \cdots b_k \mid Ab_1 \; Ab_2 \cdots Ab_k \mid \cdots \mid A^{n-1}b_1 \; A^{n-1}b_2 \cdots A^{n-1}b_k\,]$

$n\times nk$ matrix

There are many ways to choose n lin. indep. columns

rank$(\mathcal{C}) = n \Rightarrow n$ lin. indep. columns

Method I : Search $\mathcal{C}$ from left to right to find n linearlly independant columns

$\mathcal{C}_j = [\,B \mid AB \mid \cdots \mid A^j B\,]$

rank: $\rho(\mathcal{C}_0) \le \rho(\mathcal{C}_1) \le \cdots \le \rho(\mathcal{C}_p) \le \cdots \le \rho(\mathcal{C}_{n-1}) = \rho(\mathcal{C})$ $= n$ if controllable

Define μ to be the integer s.t.

$\rho(\mathcal{C}_0) < \rho(\mathcal{C}_1) < \cdots < \rho(\mathcal{C}_{\mu-1}) = \rho(\mathcal{C}_\mu) = \cdots = \rho(\mathcal{C}) = n$

μ is called the controllability index

Rearrange n lin. indep. columns of $\mathcal{C}$

$[\,b_1 \; Ab_1 \cdots A^{\mu_1-1}b_1 \mid b_2 \; Ab_2 \cdots A^{\mu_2-1}b_2 \mid \cdots \mid b_k \; Ab_k \cdots A^{\mu_k-1}b_k\,]$

n columns

$\{\mu_1, \mu_2, \ldots, \mu_k\}$ are called the controllability indices of (A,B)

Note $\sum \mu_i = n$

$\mu = \max\limits_i \mu_i$

Theorem $\{\mu_i\}$ $i=1,\dots,k$ is invariant under any equivalence transformation and any reordering of the columns of B.

Method II: Rearrange $\mathcal{C}$ as follows

$$\underset{n\times np}{\mathcal{C}} = [\underbrace{b_1\ Ab_1 \cdots A^{n-1}b_1}_{\bar{\mu}_1 \text{ LI colums}} \mid \underbrace{b_2\ Ab_2 \cdots A^{n-1}b_2}_{\bar{\mu}_2 \text{ additional LI columns}} \mid \cdots \mid \underbrace{b_k\ Ab_k \cdots A^{n-1}b_k}_{\bar{\mu}_k \text{ additional LI cols}}]$$

and search left to right for $\underline{n}$ LI columns.

$\Rightarrow \sum_{1}^{k} \bar{\mu}_i = n$ if controllable

$\bar{\mu}_i$ depends on ordering of $\{b_1, b_2, \dots, b_k\}$

$\bar{\mu} = \max_i \bar{\mu}_i \geq \mu$

Example

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{array}{c} \begin{matrix} b_1 & b_2 \end{matrix} \\ \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \end{array} \qquad \begin{matrix} n = 3 \\ k = 2 \end{matrix} \qquad \begin{matrix} x \in \mathbb{R}^3 \\ u \in \mathbb{R}^2 \end{matrix}$$

$$AB = \begin{bmatrix} 0 & 1 \\ 2 & 0 \\ 1 & 1 \end{bmatrix} \qquad A^2B = \begin{bmatrix} 1 & 3 \\ 4 & 0 \\ 3 & 1 \end{bmatrix}$$

Method I:

$$\mathcal{C} = \begin{array}{c} \left[\begin{array}{cc|cc|cc} 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & 0 & 2 & 0 & 4 & 0 \\ 0 & 1 & 1 & 1 & 3 & 1 \end{array}\right] \\ \begin{matrix} b_1 & b_2 & Ab_1 & Ab_2 & A^2b_1 & A^2b_2 \end{matrix} \end{array}$$

LI

$\mu = 2$
$= \#$ blocks needed to find $n=3$ LI columns

Rearrange $\mathcal{C} = \begin{array}{c} \left[\begin{array}{c|cc} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{array}\right] \\ \begin{matrix} b_1 & b_2 & Ab_2 \end{matrix} \end{array}$ $\quad \mu_1 = 1 \quad \mu_2 = 2 \quad \mu_1 + \mu_2 = 3 \quad \mu = \max(\mu_1, \mu_2) = 2$

Method II:

$$\mathcal{C} = \begin{array}{c} \left[\begin{array}{ccc|ccc} 0 & 0 & 1 & 0 & 1 & 3 \\ 1 & 2 & 4 & 0 & 0 & 0 \\ 0 & 1 & 3 & 1 & 1 & 1 \end{array}\right] \\ \begin{matrix} b_1 & Ab_1 & A^2b_1 & b_2 & Ab_2 & A^2b_2 \end{matrix} \end{array}$$

LI

$\bar{\mu}_1 = 3$

$\bar{\mu}_2 = 0$

$\bar{\mu} = \max(\bar{\mu}_1, \bar{\mu}_2) = 3 > \mu = 2$

Theorem

The controllability property is invariant under any equivalence transform.

Proof

Consider (A, B) with controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B]$$

And its equivalent pair $(\bar{A}, \bar{B})$ with

$$\bar{A} = PAP^{-1}$$
$$\bar{B} = PB$$

The controllability matrix of $(\bar{A}, \bar{B})$ is

$$\bar{\mathcal{C}} = [\bar{B} \quad \bar{A}\bar{B} \quad \cdots \quad \bar{A}^{n-1}\bar{B}]$$

$(MN)^{-1} = N^{-1}M^{-1}$

$$= [PB \quad PAP^{-1}PB \quad \cdots \quad PA^{n-1}P^{-1}PB]$$
$$= [PB \quad PAB \quad \cdots \quad PA^{n-1}B]$$
$$= P[B \quad AB \quad \cdots \quad A^{n-1}B]$$
$$= P\mathcal{C}$$

Because P is nonsingular, $\rho(\mathcal{C}) = \rho(\bar{\mathcal{C}})$

This proves the theorem. $\boxplus$

Theorem

(A, B) is controllable iff the matrix

$$\mathcal{C}_{n-k+1} = [B \quad AB \quad \cdots \quad A^{n-k}B]$$

where $\rho(B) = k$, has rank n.

$\Rightarrow$ don't need to check $n \times nk$ matrix $\mathcal{C}$

Outline

Observability of a Dynamical System

Observability of a Linear System

Observability of LTI Systems

Goal estimate the state from the output

Observability of a Dynamical System

$\mathcal{D} = \{\mathcal{U}, \Sigma, \mathcal{Y}, s, r\}$

$t_0, t_1 \in T \subset \mathbb{R} \qquad t_1 > t_0$

Definition

$\mathcal{D}$ is said to be completely observable on $[t_0, t_1]$ "CO"
iff given any $u_{[t_0,t_1]}$ and corresponding $y_{[t_0,t_1]}$,
the initial state $x_0 = x(t_0)$ is uniquely determined.

Comment

Once x_0 is found, knowing s, we can find $x(t)$
for any time $t \in [t_0, t_1]$ uniquely:

$$x(t) = s(t, t_0, x_0, u_{[t_0, x_0]})$$

Observability of a Linear System

$R = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$

$$\text{LTV}\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \\ x(t_0) = x_0 \end{cases}$$

Theorem co on $[t_0, t_1]$ $\Leftrightarrow$ ZIR co on $[t_0, t_1]$

Proof

$$y(t_1) = C(t_1)\Phi(t_1,t_0)\,x_0 + \int_{t_0}^{t_1} C(t_1)\,\Phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

y and u are known. x_0 is unknown.

Thus, we can write

$$C(t_1)\,\Phi(t_1,t_0)\,x_0 = \bar{y}(t_1) \qquad \text{ZIR}$$

Where

$$\bar{y}(t_1) = y(t_1) - \int_{t_0}^{t_1} C(t_1)\,\Phi(t_1,\tau)\,B(\tau)\,u(\tau)\,d\tau \qquad \text{known}$$

Note - the input doesn't affect observability

Thus, x_0 can be uniquely determined from the ZIR

Definition

$$L_0 : x_0 \longmapsto C(\cdot)\,\Phi(\cdot, t_0)\,x_0 \qquad \text{ZIR}$$

Remarks

1) co on $[t_0, t_1]$ $\Leftrightarrow$ $N(L_0) = \{0\}$ Nullspace

2) $N(L_0) :=$ set of unobservable x_0

Recall $N(L_0) = \{x_0 \in \mathbb{R}^n : L_0 x_0 = 0\}$

Definition Observability Gramian

$$W_o(t_0, t_1) = \int_{t_0}^{t_1} \Phi'(\tau, t_0)\, C'(\tau)\, C(\tau)\, \Phi(\tau, t_0)\, d\tau$$

This is an $n \times n$ matrix

Theorem Given $W_o(t_0, t_1)$ as above,

co on $[t_0, t_1]$ $\iff$ $W_o(t_0, t_1)$ is nonsingular

Construct Initial State

Assume system is co on $[t_0, t_1]$.

How to find $x(t_0) = x_0$?

We have $W_o(t_0, t_1)$ nonsingular

Claim

$$x_0 = W_o^{-1}(t_0, t_1) \int_{t_0}^{t_1} \Phi'(t_1, t_0)\, C'(t)\, y(t)\, dt$$

is the unique initial state

Observability of LTI Systems $\quad R = [A, B, C, D]$

$$\text{LTI} \begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n \\ y = Cx + Du & y \in \mathbb{R}^m \\ x(t_0) = x_0 & u \in \mathbb{R}^k \end{cases}$$

Theorem The LTI system is observable iff

$$W_o(t) = \int_0^t e^{A'\tau} C'C e^{A\tau} \, d\tau$$ Observability Gramian

is nonsingular $\forall \; t > 0$.

Theorem Duality

(A, B) is controllable $\Leftrightarrow$ (A', B') is observable

Proof

(A, B) controllable $\Leftrightarrow$ $W_c(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau$ nonsingular $\quad$ || identical

(A', B') observable $\Leftrightarrow$ $W_o(t) = \int_0^t e^{A\tau} BB' e^{A'\tau} d\tau$ nonsingular

Theorem The following are equivalent:

1) (A, C) is observable $\quad A_{n\times n}, \quad C_{m\times n}$

2) The $n \times n$ matrix

$$W_o(t) = \int_0^t e^{A'\tau} C' C e^{A\tau} \, d\tau$$

is nonsingular $\forall\, t > 0$

3) The $nm \times n$ matrix

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

Observability Matrix

has rank n (full column rank)

4) The $(n+m) \times n$ matrix

$$\begin{bmatrix} A - \lambda I \\ C \end{bmatrix}$$

has full column rank $\forall$ eigenvalue λ of A

5) If all eigenvalues of A have negative real parts ← Thus Asymptotically Stable
then $\exists!$ positive definite solution W_o to the Lyapunov Equation

$$A' W_o + W_o A = -C'C$$

The solution can be expressed as

$$W_o = \int_0^\infty e^{A'\tau} C' C e^{A\tau} \, d\tau$$

Observability Gramian

Theorem

The observability property is invariant under any equivalence transformation.

Outline

Decomposition of State Equations

- Separation of Controllable Part
- Separation of Observable Part
- Kalman Decomposition

Goal: Establish a relationship between the state-space description and the transfer function matrix $\hat{G}(s)$

Motivation: Suppose system is not cc ($\bar{c}$), i.e., $\mathcal{R}(\mathcal{C}) \neq \mathbb{R}^n$. We will show that we can then change coordinates (states) to decompose the system states into reachable, unreachable, observable & unobservable parts

Chen 6.4
(LTI)

Separation of Controllable Part

Recall the equivalence transformation:

$$① \begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad \overset{\bar{x} = Px}{\Longrightarrow} \quad ② \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + \bar{D}u \end{cases}$$

$x \in \mathbb{R}^n$
$y \in \mathbb{R}^m$
$u \in \mathbb{R}^p$

P nonsingular $\quad \bar{A} = PAP^{-1} \quad \bar{C} = CP^{-1}$

$\bar{B} = PB \quad \bar{D} = D$

Remarks

1) ① is equivalent to ②

2) All properties of ①, including stability, controllability & observability are preserved in ②

3) We also have

$$\bar{\mathcal{C}} = P\,\mathcal{C} \quad \text{and} \quad \bar{\mathcal{O}} = \mathcal{O}P^{-1}$$

Controllability Matrix — Observability Matrix

Theorem For LTI system ① with

$$\text{rank}(\mathcal{C}) = \rho(\mathcal{C}) = \rho([B \ \ AB \ \cdots \ A^{n-1}B]) = n_1 < n$$

Meaning $\dim \mathcal{R}(\mathcal{C}) = n_1 < n$

Meaning (A, B) and $(\bar{A}, \bar{B})$ are not controllable

We form the $n \times n$ matrix

$$P^{-1} = [q_1 \ \ q_2 \ \cdots \ q_{n_1} \ \cdots \ q_n]$$

Where the first n_1 columns are <u>any</u> n_1 linearly independent columns of $\mathcal{C}$, and the remaining columns can be arbitrarily chosen as long as P is nonsingular.

Theorem (continued)

Then the equivalence transformation $\bar{x} = Px$ or $x = P^{-1}\bar{x}$ will transform the LTI system ① into

$$\begin{cases} \begin{bmatrix} \dot{\bar{x}}_c \\ \dot{\bar{x}}_{\bar{c}} \end{bmatrix} = \begin{bmatrix} \bar{A}_c & \bar{A}_{12} \\ 0 & \bar{A}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} \bar{x}_c \\ \bar{x}_{\bar{c}} \end{bmatrix} + Du \end{cases}$$

Where $\bar{A}_c$ $n_1 \times n_1$

$\bar{A}_{\bar{c}}$ $(n-n_1) \times (n-n_1)$

Furthermore, the n_1-dimensional subsystem

$$② \begin{cases} \dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u \\ \bar{y} = \bar{C}_c \bar{x}_c + Du \end{cases}$$

is controllable and has the same transfer function matrix as the original system ①

Proof

1) Show ② and ① have same transfer function

$$\hat{G}_1(s) = C(sI-A)^{-1}B + \cancel{D} \overset{?}{=} \hat{G}_2(s) = \bar{C}_c(sI-\bar{A}_c)^{-1}\bar{B}_c + \cancel{D}$$

Substitute

$$A = P^{-1}\bar{A}P, \quad B = P^{-1}\bar{B}, \quad C = \bar{C}P$$

$$\begin{aligned} C(sI-A)^{-1}B &= \bar{C}P(sI - P^{-1}\bar{A}P)^{-1}P^{-1}B \\ &= \bar{C}(PsP^{-1} - \bar{A})^{-1}B \\ &= \bar{C}(sI - \bar{A})^{-1}B \\ &= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} sI-\bar{A}_c & -\bar{A}_{12} \\ 0 & sI-\bar{A}_{\bar{c}} \end{bmatrix}^{-1} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} \bar{C}_c & \bar{C}_{\bar{c}} \end{bmatrix} \begin{bmatrix} (sI-\bar{A}_c)^{-1} & X \\ 0 & X \end{bmatrix} \begin{bmatrix} \bar{B}_c \\ 0 \end{bmatrix} \\ &= \bar{C}_c(sI-\bar{A}_c)^{-1}\bar{B}_c \\ &= \hat{G}_2(s) \end{aligned}$$

⊞

Proof (continued)

2) Show ② is controllable

$$\bar{\mathcal{C}} = [\, \bar{B}_c \quad \bar{A}_c \bar{B}_c \quad \cdots \quad \bar{A}_c^{\,n_1-1} \bar{B}_c \quad \cdots \quad \bar{A}_c^{\,n-1} \bar{B}_c \,]$$

Need to show rank $\rho(\bar{\mathcal{C}}) = n_1$

$$\mathcal{C} = [\, B \quad AB \quad \cdots \quad A^{n-1}B \,]$$ from original system ①

$$P\mathcal{C} = [\, PB \quad PAB \quad \cdots \quad PA^{n-1}B \,]$$

$$= [\, \bar{B} \quad \bar{A}\bar{B} \quad \cdots \quad \bar{A}^{n-1}\bar{B} \,]$$

$$= \left[\begin{array}{c|c|c|c} \bar{B}_c & \bar{A}_c \bar{B}_c & \cdots & \bar{A}_c^{\,n-1} \bar{B}_c \\ 0 & 0 & & 0 \end{array} \right] \begin{array}{l} n_1 \text{ rows} \\ n-n_1 \text{ rows} \end{array}$$

$$\rho(P\mathcal{C}) = n_1$$

$\rho(\bar{\mathcal{C}}) = n_1$ because P is invertible

Thus for

$$\bar{\mathcal{C}}_1 = [\, \bar{B}_c \quad \bar{A}_c \bar{B}_c \quad \cdots \quad \bar{A}_c^{\,n_1-1} \bar{B}_c \,]$$

$$\rho(\bar{\mathcal{C}}_1) = n_1$$

Which means the system ② is controllable. ∎

Example (Chen 6.8)

$$(1)\begin{cases} \dot{x} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} x + \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x \end{cases}$$

$n = 3$
$m = 1$
$k = 2$

$\rho(B) = 2$

Therefore, we can use the matrix

$$\mathcal{C}_2 = [B \quad AB] \qquad n - \rho(B) = 3 - 2 = 1$$

instead of the controllability matrix

$$\mathcal{C} = [B \quad AB \quad A^2B]$$

to check for controllability

$$\rho(\mathcal{C}_2) = \rho([B \ AB]) = \rho\left(\left[\begin{array}{cc|cc} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \end{array}\right]\right) = 2 < 3$$

Thus, the state equation is not controllable.

Choose

$$P^{-1} = Q = \begin{bmatrix} \underbrace{\begin{matrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{matrix}}_{\substack{n_1 = 2 \\ \text{LI columns} \\ \text{of } \mathcal{C}_2}} & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \end{bmatrix}$$

choose to have full rank

Let $\bar{x} = Px$, Then:

$$\bar{A} = PAP^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{c|c} \bar{A}_c & \bar{A}_{12} \\ \hline 0 & \bar{A}_{\bar{c}} \end{array}\right]$$

$$\bar{B} = PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline 0 & 0 \end{array}\right] = \left[\begin{array}{c} \bar{B}_c \\ \hline 0 \end{array}\right]$$

$$\bar{C} = CP^{-1} = [1 \ 1 \ 1] \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|c} 1 & 2 & 1 \end{array}\right] = \left[\begin{array}{c|c} \bar{C}_c & \bar{C}_{\bar{c}} \end{array}\right]$$

Example (continued)

Thus the original system ① can be reduced to

$$\begin{cases} \dot{\bar{x}}_c = \bar{A}_c \bar{x}_c + \bar{B}_c u \\ \bar{y} = \bar{C}_c \bar{x}_c \end{cases}$$

or

$$\begin{cases} \dot{\bar{x}}_c = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \bar{x}_c + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 2 \end{bmatrix} \bar{x}_c \end{cases}$$

This equation is controllable and has the same transfer function matrix $\hat{G}(s)$ as system ① has.

Separation of Observable Part (LTI)

$$(1)\begin{cases}\dot{x} = Ax + Bu \\ y = Cx + Du\end{cases}\qquad \begin{matrix} x \in \mathbb{R}^n \\ y \in \mathbb{R}^m \\ u \in \mathbb{R}^q \end{matrix}$$

Theorem For the LTI system (1) with

$$\rho(\mathcal{O}) = \rho\begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1}\end{bmatrix} = n_2 < n$$

Meaning (A, C) is not observable

We form the $n \times n$ matrix

$$P = \begin{bmatrix} p_1 \\ \vdots \\ p_{n_2} \\ \vdots \\ p_n \end{bmatrix}$$

Where the first n_2 rows are any n_2 linearlly independent rows of $\mathcal{O}$, and the remaining rows can be chosen arbitrarily as long as P is nonsingular.

Then the equivalence transformation $\bar{x} = Px$ will transform (1) into

$$\begin{cases}\begin{bmatrix} \dot{\bar{x}}_o \\ \dot{\bar{x}}_{\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_o & 0 \\ \bar{A}_{21} & \bar{A}_{\bar{o}} \end{bmatrix}\begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_o \\ \bar{B}_{\bar{o}} \end{bmatrix} u \\ y = \begin{bmatrix} \bar{C}_o & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_o \\ \bar{x}_{\bar{o}} \end{bmatrix} + Du \end{cases}\qquad \begin{matrix} \bar{A}_o \;\; n_2 \times n_2 \\ \bar{A}_{\bar{o}} \;\; (n-n_2)\times(n-n_2) \end{matrix}$$

And the n_2-dimensional subsystem

$$(2)\begin{cases}\dot{\bar{x}}_o = \bar{A}_o \bar{x}_o + \bar{B}_o u \\ y = \bar{C}_o \bar{x}_o + Du\end{cases}$$

is observable <u>and</u> has the same transfer function matrix as the original system (1).

Kalman Decomposition Theorem (LTI)

Every state space equation can be transformed by an equivalence transformation into the following Kalman Canonical Form:

$$\begin{bmatrix} \dot{\bar{x}}_{co} \\ \dot{\bar{x}}_{c\bar{o}} \\ \dot{\bar{x}}_{\bar{c}o} \\ \dot{\bar{x}}_{\bar{c}\bar{o}} \end{bmatrix} = \begin{bmatrix} \bar{A}_{co} & 0 & \bar{A}_{13} & 0 \\ \bar{A}_{21} & \bar{A}_{c\bar{o}} & \bar{A}_{23} & \bar{A}_{24} \\ 0 & 0 & \bar{A}_{\bar{c}o} & 0 \\ 0 & 0 & \bar{A}_{43} & \bar{A}_{\bar{c}\bar{o}} \end{bmatrix} \begin{bmatrix} \bar{x}_{co} \\ \bar{x}_{c\bar{o}} \\ \bar{x}_{\bar{c}o} \\ \bar{x}_{\bar{c}\bar{o}} \end{bmatrix} + \begin{bmatrix} \bar{B}_{co} \\ \bar{B}_{c\bar{o}} \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} \bar{C}_{co} & 0 & \bar{C}_{\bar{c}o} & 0 \end{bmatrix} \bar{x} + Du$$

where

$\bar{x}_{co}$ = controllable & observable states

$\bar{x}_{c\bar{o}}$: controllable & unobservable states

$\bar{x}_{\bar{c}o}$ = uncontrollable & observable states

$\bar{x}_{\bar{c}\bar{o}}$ = uncontrollable & unobservable states

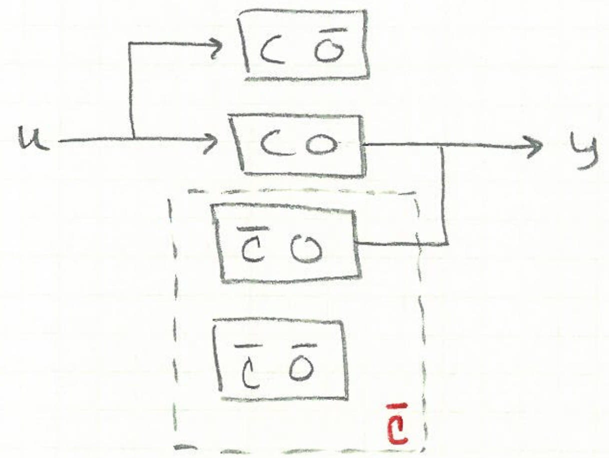
Furthermore, the state equation is zero-state equivalent (meaning ZSR same) to the controllable & observable state equation

$$\begin{cases} \dot{\bar{x}}_{co} = \bar{A}_{co}\bar{x}_{co} + \bar{B}_{co}u \\ y = \bar{C}_{co}\bar{x}_{co} + Du \end{cases}$$

And has the transfer function matrix

$$\hat{G}(s) = \bar{C}_{co}(sI - \bar{A}_{co})^{-1}\bar{B}_{co} + D$$

Remark



1) The transfer function matrix only depends on the C & O part of the equation

$\Rightarrow$ CO is only part connected to the input & output

2) The transfer function, or I/O, description of the system and the state-space description are not necessarily equivalent.

3) Thus, the I/O description of a system is sometimes insufficient to describe a system

b/c $\bar{C}$ & $\bar{O}$ parts don't appear in the transfer function description