EE 547 (PMP) Lab 6

Wednesday, February 11, 2015

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Demo Problem 1 (Controllability, Observability and Minimal Energy Control) Consider a continuous-time linear time-invariant system as (1). Please write a MATLAB script to:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{s^3 + 9s^2 + 26s + 24}{s^5 + 14s^4 + 73s^3 + 176s^2 + 196s + 80} \tag{1}$$

(a) Convert the transfer function into state-space representation of this system.

(b) Check if the system is asymptotically stable.

(c) Evaluate the controllability and observability matrices of the system, *C* and *O*.

(d) Check if the system is controllable and observable by the ranks of controllability and observability matrices, *C* and *O*.

(e) Find the minimal controllable realization of the system by MATLAB function **minreal**.

(f) In this step, consider only minimal controllable realization of the system from previous step.

$$\begin{aligned} \dot{\bar{x}} &= A_{\mathrm{ctrb}}\bar{x} + B_{\mathrm{ctrb}}u \\ \bar{y} &= C_{\mathrm{ctrb}}\bar{x} + D_{\mathrm{ctrb}}u \end{aligned} \tag{2}$$

Find the controllability and observability gramians, W_c and W_o, of this minimal controllable system.

(g) Please derive minimum energy control input that drives initial state $x_0 = [-50,40,-300]^T$ into $x_1 = [0,0,0]^T$ within 8 seconds.

$$u(t) = -B^T e^{A^T(t_1-t)} W_C^{-1}(t_1)[e^{At_1}x_0 - x_1] \tag{3}$$

where $W_C(t)$ is the controllability gramian.

(h) Use **lsim** function to simulate this minimal controllable system with time span defined as **tspan = 0:0.1:8**. The input is the derived input in previous step. Please plot state variable on one chart and derived input on another chart.

Individual Problem 1 (Controllability, Observability and Minimal Energy Control) Consider a continuous-time linear time-invariant system as (4). Please write a MATLAB script to:

$$G(s) = \frac{s^2 + 3s + 2}{s^5 + 15s^4 + 85s^3 + 225s^2 + 274s + 120} \tag{4}$$

(a) Convert the transfer function into state-space representation of this system.

(b) Check if the system is asymptotically stable.

(c) Evaluate the controllability and observability matrices of the system, *C* and *O*.

(d) Check if the system is controllable and observable by the ranks of controllability and observability matrices, *C* and *O*.

(e) Find the minimal controllable realization of the system by MATLAB function **minreal**.

(f) In this step, consider only minimal controllable realization of the system from previous step.

$$\begin{aligned} \dot{\bar{x}} &= A_{\text{ctrb}}\bar{x} + B_{\text{ctrb}}u \\ \bar{y} &= C_{\text{ctrb}}\bar{x} + D_{\text{ctrb}}u \end{aligned} \tag{5}$$

Find the controllability and observability gramians, W_c and W_o, of this minimal controllable system.

(g) Please derive minimum energy control input that drives initial state $x_0 = [10,20,-30]^T$ into $x_1 = [0,0,0]^T$ within 5 seconds.

$$u(t) = -B^T e^{A^T(t_1-t)} W_C^{-1}(t_1)[e^{At_1}x_0 - x_1] \tag{6}$$

where $W_C(t)$ is the controllability gramian.

(h) Use **lsim** function to simulate this minimal controllable system with time span defined as **tspan = 0:0.1:5**. The input is the derived input in previous step. Please plot state variable on one chart and derived input on another chart.

Solution of Demo Problem:

(a) The state-space representation of the system can be derived by using **tf2ss** function as below:

num = [1 9 26 24];

den = [1 14 73 176 196 80];

[A, B, C, D] = tf2ss(num, den);

(b) By **eig** function, the eigenvalues of matrix A are listed as below:

λ_1 = -5

λ_2 = -4

λ_3 = λ_4 = -2

λ_5 = -1

since all eigenvalues are negative real numbers, the system is asymptotically stable.

(c) The controllability and observability matrices of the system, *C* and *O*, are evaluated as below

$$C = \begin{bmatrix} 1 & -14 & 123 & -876 & 5553 \\ 0 & 1 & -14 & 123 & -876 \\ 0 & 0 & 1 & -14 & 123 \\ 0 & 0 & 0 & 1 & -14 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$O = \begin{bmatrix} 0 & 1 & 9 & 26 & 24 \\ 1 & 9 & 26 & 24 & 0 \\ -5 & -47 & -152 & -196 & -80 \\ 23 & 213 & 684 & 900 & 400 \\ -109 & -995 & -3148 & -4108 & -1840 \end{bmatrix}$$

(d) Since rank of matrix C is 5, the system is controllable. But the rank of matrix O is 3 which is less than the order of matrix A. It is not observable.

(e) The minimal controllable realization of the system can be evaluated by MATLAB function **minreal**.

[Amin, Bmin, Cmin, Dmin] = **minreal**(A, B, C, D);

(f) Then, controllability and observability gramians can be evaluated by **gram**.

Wc = gram(sys_min, '**c**') % c is to return controllability gramian

Wo = gram(sys_min, '**o**') % o is to return observability gramian

(g) From previous step, we have controllability gramian for minimal controllable system. Then, we can use (3) to evaluate input.

u = -Bmin'*expm(Amin'*(tspan(end) - t))*Wc_inv*(expm(Amin*tspan(end))*x0 - x1)

The symbolic expression of input may be too complicated. Therefore, a **for** loop to evaluate input in **double** class is preferred.

(h) The syntax for lsim is as below.

[y, t, x] = **lsim**(sys_min, u, tspan, x0);

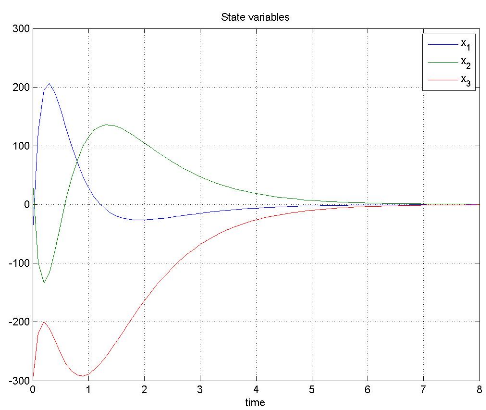

Figure 1 State variables of the DEMO system.

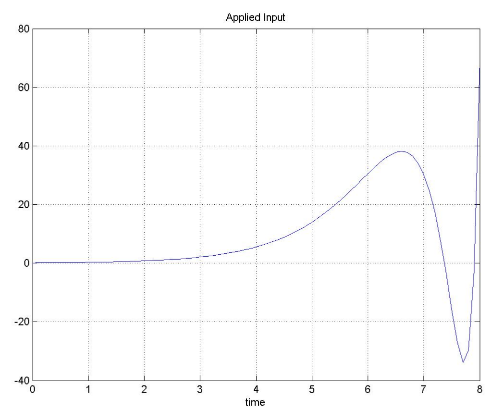

Figure 2 Derived input for the DEMO system.

Solution of Individual Problem:

(a) The state-space representation of the system can be derived by using **tf2ss** function as below:

num2 = [1 3 2];

Den2 = [1 15 85 225 274 120];

[A2, B2, C2, D2] = tf2ss(num2, den2);

(b) By **eig** function, the eigenvalues of matrix A are listed as below:

$\lambda_1 = -5$

$\lambda_2 = -4$

$\lambda_3 = -3$

$\lambda_4 = -2$

$\lambda_5 = -1$

since all eigenvalues are negative real numbers, the system is asymptotically stable.

(c) The controllability and observability matrices of the system, *C* and *O*, are evaluated as below

$$C = \begin{bmatrix} 1 & -15 & 140 & -1050 & 6951 \\ 0 & 1 & -15 & 140 & -1050 \\ 0 & 0 & 1 & -15 & 140 \\ 0 & 0 & 0 & 1 & -15 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$O = \begin{bmatrix} 0 & 0 & 1 & 3 & 2 \\ 0 & 1 & 3 & 2 & 0 \\ 1 & 3 & 2 & 0 & 0 \\ -12 & -83 & -225 & -274 & -120 \\ 97 & 795 & 2426 & 3168 & 1440 \end{bmatrix}$$

(d) Since rank of matrix C is 5, the system is controllable. But the rank of matrix O is 3 which is less than the order of matrix A. It is not observable.

(e) The minimal controllable realization of the system can be evaluated by MATLAB function **minreal**.

[A2min, B2min, C2min, D2min] = **minreal**(A2, B2, C2, D2);

(f) Then, controllability and observability gramians can be evaluated by **gram**.

Wc2 = gram(sys_min, '**c**')   % c is to return controllability gramian

Wo2 = gram(sys_min, '**o**')   % o is to return observability gramian

(g) From previous step, we have controllability gramian for minimal controllable system. Then, we can use (3) to evaluate input.

u2 = -B2min'*expm(A2min'*(tspan2(end) - t))*Wc2_inv*(expm(A2min*tspan2(end))*x02 - x12)

The symbolic expression of input may be too complicated. Therefore, a **for** loop to evaluate input in **double** class is preferred.

(h) The syntax for lsim is as below.

[y2, t2, x2] = **lsim**(sys2_min, u2, tspan2, x02);

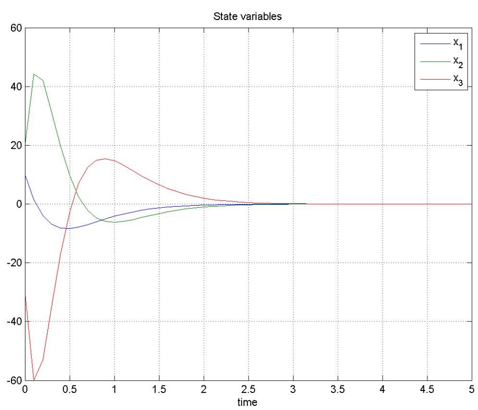

Figure 3 State variables of the INDIVIDUAL system.

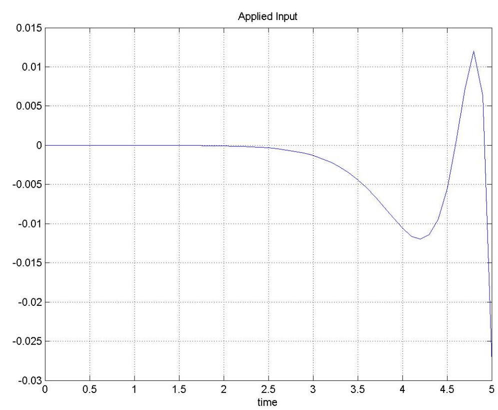

Figure 4 Derived input for the INDIVIDUAL system,

MATLAB Code

```
%% Lab6: Controllability and Observability
close all;
clear all;

%% Define Parameters
syms t
tspan = 0:0.1:8;
tspan2 = 0:0.1:5;
x0 = [-50;40;-300];  %initial state
x02 = [10;20;-30]
x1 = [0;0;0];        %final state
x12 = x1;

%% (a) Formulation of State-space Representation
num = [1 9 26 24];
den = [1 14 73 176 196 80];
[A, B, C, D] = tf2ss(num, den);

num2 = [1 3 2];
den2 = [1 15 85 225 274 120];
[A2,B2,C2,D2] = tf2ss(num2,den2);

%% (b) Eigenvalues and  Stability
eigA = eig(A);
eigA2 = eig(A2);

if(eigA >= 0)
   fprintf('Demo System is not asymptotically stable.\n\n')
else
   fprintf('Demo System is asymptotically stable.\n\n')
end

if(eigA2 >= 0)
   fprintf('Individual System is not asymptotically stable.\n\n')
else
   fprintf('Individual System is asymptotically stable.\n\n')
```

```
end

%% Part (c) - Controllability and Observability matrices
ctrb_demo = ctrb(A,B);
obsv_demo = obsv(A,C);
ctrb_ind = ctrb(A2,B2);
obsv_ind = obsv(A2,C2);

%% Part (d) - Controllability and Observability of the system
ctrb_rank = rank(ctrb_demo);
ctrb_rank2 = rank(ctrb_ind);
obsv_rank = rank(obsv_demo);
obsv_rank2 = rank(obsv_ind);

if(ctrb_rank < length(A))
   fprintf('Demo System is not controllable.\n\n')
else
   fprintf('Demo System is controllable.\n\n')
end

if(obsv_rank < length(A))
   fprintf('Demo System is not observable.\n\n')
else
   fprintf('Demo System is observable.\n\n')
end

% IND
if(ctrb_rank2 < length(A2))
   fprintf('Individual System is not controllable.\n\n')
else
   fprintf('Individual System is controllable.\n\n')
end

if(obsv_rank2 < length(A2))
   fprintf('Individual System is not observable.\n\n')
else
   fprintf('Individual System is observable.\n\n')
```

```
end

%% (e) minimal energy
[Amin, Bmin, Cmin, Dmin] = minreal(A, B, C, D);
[A2min, B2min, C2min, D2min] = minreal(A2, B2, C2, D2);

sys_min = ss(Amin, Bmin, Cmin, Dmin);
sys2_min = ss(A2min, B2min, C2min, D2min);

%% (f) Controllability and Observability Grammians
Wc = gram(sys_min, 'c')
Wo = gram(sys_min, 'o')
Wc2 = gram(sys2_min, 'c')
Wo2 = gram(sys2_min, 'o')

%% (g) Derive Output
Wc_inv = inv(Wc)
Wc2_inv = inv(Wc2)

% tic;
% u = -Bmin'*expm(Amin'*(tspan(end) - t))*Wc_inv*(expm(Amin*tspan(end))*x0 - x1)
% u2 = -B2min'*expm(A2min'*(tspan2(end) - t))*Wc2_inv*(expm(Amin2*tspan2(end))*x02 - x12)
% toc;

% u = eval(sub(u,t,tspan));
% u2 = eval(sub(u2,t,tspan2));

%% (h) Simulation for 8 Seconds
% Input for Demo Problem
for i=1:1:length(tspan)
    t = tspan(i);
    u(i) = -Bmin'*expm(Amin'*(tspan(end) - t))*Wc_inv*...
        (expm(Amin*tspan(end))*x0 - x1);
end

% Input for IND Problem
for i=1:1:length(tspan2)
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```
    t = tspan2(i);
    u2(i) = -B2min'*expm(A2min'*(tspan2(end) - t))*Wc2_inv*...
        (expm(A2min*tspan2(end))*x02 - x12);
end

[y, t, x] = lsim(sys_min, u, tspan, x0);
[y2, t2, x2] = lsim(sys2_min, u2, tspan2, x02);

%% Plot
figure
plot(t, x)
grid on
title('State variables')
xlabel('time')
legend('x_1', 'x_2', 'x_3')
print -djpeg 'StateVariables_Lab6_Demo.jpeg'

figure
plot(t, u)
grid on
title('Applied Input')
xlabel('time')
print -djpeg 'Input_Lab6_Demo.jpeg'

figure
plot(t2, x2)
grid on
title('State variables')
xlabel('time')
legend('x_1', 'x_2', 'x_3')
print -djpeg 'StateVariables_Lab6_IND.jpeg'

figure
plot(t2,u2)
grid on
title('Applied Input')
xlabel('time')
```

```
print -djpeg 'Input_Lab6_IND.jpeg'
```