EE 547 (PMP): Homework 1

Assigned: Thursday January 8, 2015. Due: Wednesday January 14, 2015.

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Problem 1 Consider a MinSeg model as below. The geometric parameters and some constants are labeled on the figure. A system of equations is derived as (1) to describe the movement and angle of rotation of MinSeg body.

a) Let's focus on the translation of MinSeg wheel (x) and rotation of MinSeg pendulum (α). Please linearize (1) around equilibrium point $x^{eq} = \alpha^{eq} = 0$. (Hint: high order terms shall be ignored.)

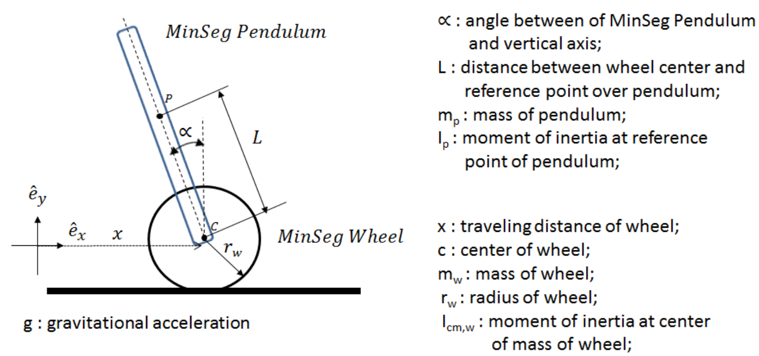

Figure 1 MinSeg system

$$\begin{bmatrix} -(I_p + m_p L^2) & m_p L\cos\alpha \\ m_p L\, r_w^2 \cos\alpha & -(I_{cm,w} + m_w r_w^2 + m_p r_w^2) \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} T_m - m_p L g \sin\alpha \\ T_m r_w + m_p L\, r_w^2 \dot{\alpha}^2 \cos\alpha \end{bmatrix} \tag{1}$$

where T_m is the torque from the DC motor, which is not explicitly demonstrated in Figure 1.

b) A torque equation is given as (2). Please rewrite the derived linear model with (2).

$$T_m = \frac{k_t}{R} V + \frac{k_t k_b}{R r_w} \dot{x} + \frac{k_t k_b}{R} \dot{\alpha} \tag{2}$$

where V is applied voltage, R is the resistance of DC motor, k_t is torque constant and k_b is back-EMF constant.

c) If state variables are defined as $X = [\alpha\ \dot{\alpha}\ x\ \dot{x}]^T$ and input variable is $u = V$, please derive state space matrices A and B. Now we are interested in $Y = [\alpha\ x]^T$ and there is no disturbance over the system. Please derive matrices C and D.

HW hint:

$$\begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} -(I_p + m_p L^2) & m_p L \cos\alpha \\ m_p L\, r_w^2 \cos\alpha & -(I_{cm,w} + m_w r_w^2 + m_p r_w^2) \end{bmatrix}^{-1} \begin{bmatrix} T_m - m_p L g \sin\alpha \\ T_m r_w + m_p L\, r_w^2 \dot{\alpha}^2 \cos\alpha \end{bmatrix} = \begin{bmatrix} f_2(\alpha, \dot{\alpha}, x, \dot{x}, V) \\ f_4(\alpha, \dot{\alpha}, x, \dot{x}, V) \end{bmatrix} \tag{3}$$

$$\dot{X} = \frac{\partial}{\partial t} \begin{bmatrix} \alpha \\ \dot{\alpha} \\ x \\ \dot{x} \end{bmatrix} = \begin{bmatrix} f_1(\alpha, \dot{\alpha}, x, \dot{x}, V) \\ f_2(\alpha, \dot{\alpha}, x, \dot{x}, V) \\ f_3(\alpha, \dot{\alpha}, x, \dot{x}, V) \\ f_4(\alpha, \dot{\alpha}, x, \dot{x}, V) \end{bmatrix} \tag{4}$$

Problem 1 solution:

a) Around equilibrium point $x^{eq} = \alpha^{eq} = 0$, we can assume

$\cos\alpha \sim 1;\ \sin\alpha \sim \alpha;\ \dot{\alpha}^2 \sim 0$

Then, (1) can be simplified as (5).

$$\begin{bmatrix} -(I_p + m_p L^2) & m_p L \\ m_p L r_w^2 & -(I_{cm,w} + m_w r_w^2 + m_p r_w^2) \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} T_m - m_p L g \alpha \\ T_m r_w \end{bmatrix} \tag{5}$$

(5) can be rewritten as (6).

$$\begin{bmatrix} -(I_p + m_p L^2) & m_p L \\ m_p L & -(\frac{I_{cm,w}}{r_w^2} + m_w + m_p) \end{bmatrix} \begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} T_m - m_p L g \alpha \\ \frac{T_m}{r_w} \end{bmatrix} \tag{6}$$

$$\begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} -(I_p + m_p L^2) & m_p L \\ m_p L & -(\frac{I_{cm,w}}{r_w^2} + m_w + m_p) \end{bmatrix}^{-1} \begin{bmatrix} T_m - m_p L g \alpha \\ \frac{T_m}{r_w} \end{bmatrix} \tag{7}$$

b) Simply substitute (2) into (6), the equations become

$$\begin{bmatrix} \ddot{\alpha} \\ \ddot{x} \end{bmatrix} = \begin{bmatrix} -(I_p + m_p L^2) & m_p L \\ m_p L & -(\frac{I_{cm,w}}{r_w^2} + m_w + m_p) \end{bmatrix}^{-1} \begin{bmatrix} \frac{k_t}{R} V + \frac{k_t k_b}{R r_w} \dot{x} + \frac{k_t k_b}{R} \dot{\alpha} - m_p L g \alpha \\ \frac{k_t}{R r_w} V + \frac{k_t k_b}{R r_w^2} \dot{x} + \frac{k_t k_b}{R r_w} \dot{\alpha} \end{bmatrix} \tag{8}$$

c) Since state variable is defined as $X = [\,\alpha\ \dot{\alpha}\ x\ \dot{x}]^T$, we could write the state-space representation as

$$\dot{X} = \begin{bmatrix}\dot{\alpha}\\ \ddot{\alpha}\\ \dot{x}\\ \ddot{x}\end{bmatrix} = [A]\begin{bmatrix}\alpha\\ \dot{\alpha}\\ x\\ \dot{x}\end{bmatrix} + \ [B]V \tag{9}$$

The first row and third row of matrix A are simple row vectors [0 1 0 0] and [0 0 0 1], respectively. To have second row and fourth row, we have to solve (8) by taking the inversion of coefficient matrix. The final matrix A and matrix B are shown in (10) and (11).

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{gLm_p(i_{\text{cmw}}+(m_p+m_w)r_w^2)}{i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2} & -\frac{k_bk_t(i_{\text{cmw}}+r_w(m_wr_w+m_p(L+r_w)))}{R(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} & 0 & -\frac{k_bk_t(i_{\text{cmw}}+r_w(m_wr_w+m_p(L+r_w)))}{Rr_w(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} \\ 0 & 0 & 0 & 1 \\ \frac{gL^2m_p^2r_w^2}{i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2} & -\frac{k_bk_tr_w(i_p+Lm_p(L+r_w))}{R(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} & 0 & -\frac{k_bk_t(i_p+Lm_p(L+r_w))}{R(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} \end{bmatrix} \tag{10}$$

$$B=\begin{bmatrix} 0 \\ -\frac{k_t(i_{\text{cmw}}+r_w(m_wr_w+m_p(L+r_w)))}{R(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} \\ 0 \\ -\frac{k_tr_w(i_p+Lm_p(L+r_w))}{R(i_{\text{cmw}}(i_p+L^2m_p)+(L^2m_pm_w+i_p(m_p+m_w))r_w^2)} \end{bmatrix} \tag{11}$$

$$C=\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{12}$$

$$D=\begin{bmatrix}0\\0\\0\\0\end{bmatrix} \tag{13}$$

MATLAB reference code:

```
%% MATLAB reference code for EE547 homework1 Witner 2015
%% Developer: HRLin
% Description: This is an example to show MATLAB's way to evaluate
% A B C D matrices through jacobian function.
clear all;
close all;
clc;
%% Definition and Initialization
% g:        gravity constant
% t:        time
% x:        position of wheel center
% alpha:    rotation angle
% L:        distance between wheel center and reference point over pendulum
% mp:       mass of pendulum
% Ip:       moment of inertia of pendulum
% mw:       mass of wheel
% rw:       wheel radius of wheel
% Icmw:     moment of inertia at wheel center of mass
% Tm:       torque from DC motor
% V:        applied voltage
% kt:       torque constant
% kb:       back EMF constant
% R:        resisrance of DC motor
% xi:       state variables

syms g t x xdot alpha alphadot L mp Ip mw rw Icmw Tm V kt kb R x1 x2 x3 x4

C1 = [  -(Ip + mp*L^2),           mp*L*cos(alpha);
        mp*L*rw^2*cos(alpha),   -(Icmw+mw*rw^2+mp*rw^2)];

C2 = [  Tm-mp*L*g*sin(alpha);
        Tm*rw+mp*L*rw^2*alphadot^2*cos(alpha)];

%Tm = kt/R*V+kt*kb/R/rw*xdot+kt*kb/R*alphadot;
%%
% state variable vector = [alpha alphadot x xdot] = [x1 x2 x3 x4]';
% time derivative of state variable vector =
% [alphadot alpha_doubledot xdot x_doubledot] = [f1 f2 f3 f4]
```

```
C3 = inv(C1)*C2;
C3 = subs(C3,'Tm',kt/R*V+kt*kb/R/rw*xdot+kt*kb/R*alphadot);

f1 = alphadot;
f2 = C3(1,1);
f3 = xdot;
f4 = C3(2,1);

f = [f1;f2;f3;f4];
xvec = [alpha alphadot x xdot]; % xvec is state variable, not position.
u = V;

JA = jacobian(f,xvec);
JB = jacobian(f,u);
A = subs(JA,xvec, [0 0 0 0]);   % Two of equilibrium values are also zeros.
B = subs(JB,xvec, [0 0 0 0]);
C = eye(size(A));
D = zeros(size(C,1),size(B,2));
```