Class 5

Lecture 8

Internal or Lyapunov Stability

Lecture 9

Input - Output Stability (BIBO Stability)

Usage - Internal stability for ZIR

I-O stability for ZSR

$y = ZIR + ZSR$

Motivation: If system is unstable, the system may burn out, disintigrate or saturate when a signal is applied. $\Rightarrow$ useless in practice.

Stability is a basic requirement for all systems

Introduction to the concept of Stability

Dynamical systems usually function in some specific mode.

operating points

equilibrium points

This Lecture: study the qualitative behavior of operating points (equilibrium points).

(1) Asymptotic Stability of an equilibrium/operating point will be most important.

⇒ When a state of a system is displaced or disturbed from its desired operating value (equilibrium), we expect the state to return to the equilibrium.

Example: Car using cruise control at 50mph desired speed (operating point).

Perturbations: hill climbing, descending

speed ↓   speed ↑

Car will return to 50mph.

(2) Another stability characterization of dynamical systems is the expectation that bounded-inputs will result in bounded-outputs, and small changes in inputs will result in small changes in outputs.

⇒ Input-Output Stability

Example: Tracking systems

Output is expected to follow a desired input

Lecture 8

Outline

Lyapunov Stability

Quadratic Form & Positive Definiteness

Lyapunov Functions

Lyapunov Stability Theorem

Indirect Method of Lyapunov

Lyapunov Stability

Consider an LTI System

Study the stability of the homogeneous LTI system:

$$① \begin{cases} \dot{x}(t) = A x(t) \\ y(t) = C x(t) \\ x(t_0) = x_0 \end{cases}$$

Without loss of generality, assume $x^{eq} = 0$
(shift to origin)

Solution is

$$x(t) = e^{A(t-t_0)} x_0 = \Phi(t, t_0) x_0$$

Definition The system ① is said to be

1) Marginally stable or internally stable or stable i.s.L. (in the sense of Lyapunov) if for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, the solution $x(t) = \Phi(t, t_0) x_0$ is bounded.

every finite x_0 excites a bounded response

2) Asymptotically stable if above is true plus for every initial condition $x(t_0) = x_0 \in \mathbb{R}^n$, the solution $x(t) \longrightarrow \underset{x^{eq}}{0}$ as $t \to \infty$.

3) Exponentially stable if above is true plus $\exists$ constants $c, \lambda > 0$ such that

$$\|x(t)\| \le c\, e^{\lambda(t-t_0)} \|x_0\| \quad \forall t \ge 0$$

i.e., $x(t) \to \underset{x^{eq}}{0}$ as $t \to \infty$ very fast

Theorem The homogeneous LTI System is

1) Marginally stable iff all eigenvalues of A have zero or negative real parts.
   Those with zero real parts are simple roots of the minimal polynomial of A.
   $\Rightarrow$ this means the e-value with zero real part has no Jordan Block of order 2 or higher

roots of $\Delta(\lambda) = \det(\lambda I -$

poles of $\hat{G}(s) = C(sI-A)$ $+D$

2) Asymptotically stable if all eigenvalues of A have negative real parts.
3) Exponentially stable if all eigenvalues of A have negative real parts.

Notes

1) Algebraic equivalence transformation will not alter the stability of the state equation

$\bar{x} = Px$ P nonsingular matrix

$\dot{x} = Ax \iff \dot{\bar{x}} = \bar{A}\bar{x} = PAP^{-1}\bar{x}$

Since P is nonsingular, if x is bounded, then $\bar{x}$ is bounded

then if $x \xrightarrow{t\to\infty} 0$, $\bar{x} \xrightarrow{t\to\infty} 0$

$\Rightarrow$ study stability of A by using $\bar{A}$

use Jordan Form for $\bar{A}$

2) eigenvalues of A = eigenvalues of $\bar{A}$

3) Exponential Stability $\Rightarrow$ BIBO stability (defined later)
   (but not vice versa)

4) Exponential Stability $\Rightarrow$ Asymptotic Stability by definition

Examples

$$\begin{cases} \dot{x} = Ax \\ x(t_0) = x_0 \end{cases}$$

1) $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ $x^{eq} = 0$

$\Delta(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda \end{bmatrix} = \lambda^2 + 1 = 0$

$\Rightarrow \lambda_1 = j, \quad \lambda_2 = -j$ $j = \sqrt{-1}$
distinct

The eigenvalues have zero real parts, but since they are distinct, the corresponding Jorden Blocks are of order 1.

Check: $J = \begin{bmatrix} j & 0 \\ 0 & -j \end{bmatrix}$ by using $Q = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix}$, $Q\bar{A} = AQ$, $\bar{A} = J$

$\Rightarrow x^{eq}$ equilibrium of the system is marginably stable

2) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $x^{eq} = 0$

$\Delta(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ 0 & \lambda \end{bmatrix} = \lambda^2 = 0 \Rightarrow \lambda_1 = 0, \lambda_2 = 0$

A already in Jordan Form with Jordan Block of order 2

$\Rightarrow x^{eq}$ equilibrium of the system is unstable

3) $A = \begin{bmatrix} 2.8 & 9.6 \\ 9.6 & -2.8 \end{bmatrix}$ $x^{eq} = 0$ $\lambda_1 = 10, \lambda_2 = -10$

$\Rightarrow x^{eq}$ equilibrium of the system is unstable

4) $A = \begin{bmatrix} -1 & 0 \\ -1 & -2 \end{bmatrix}$ $x^{eq} = 0$ $\lambda_1 = -1, \lambda_2 = -2$

$\Rightarrow x^{eq}$ equilibrium of the system is exponentially stable

Quadratic Form

M $n \times n$ real matrix

<u>Definition</u> M is symmetric if $M^T = M$

<u>Definition</u> $x^T M x$ for $x \in \mathbb{R}^n$, $M^T = M$

is called a quadratic form

<u>Fact</u> All eigenvalues of a symmetric M are real.

Positive Definiteness

M $n \times n$ real, symmetric matrix

<u>Definition</u> M is positive definite if

$x^T M x > 0 \quad \forall$ nonzero $x \in \mathbb{R}^n$

<u>Definition</u> M is positive semidefinite if

$x^T M x \geq 0 \quad \forall$ nonzero $x \in \mathbb{R}^n$

ie, $\exists\, x$ st $x^T M x = 0$

<u>Theorem</u> A symmetric $n \times n$ matrix M is positive definite (positive semi-definite) iff any one of the following is true:

1) every eigenvalue of M is positive (0 or positive)
2) all leading principle minors of M are positive (all principle minors of M are 0 or positive)
3) $\exists$ $n \times n$ nonsingular matrix N ($\exists$ $n \times n$ singular matrix N or $m \times n$ N with $m < n$) such that $M = N^T N$

<u>Definition</u> Leading principle minors of $M = [m_{ij}]$ are

m_{11}, $\det \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$, . . . , $\det M$

Background on Lyapunov

Alexandr Lyapunov 1857-1918

PhD 1892 "The general problem of the stability of motion"

Main idea: If the total energy is dissipative, then the system must be stable

Application: Look at how an energy-like function, V (called the Lyapunov Function) changes over time

$\Rightarrow$ may be able to conclude that a system is stable (or asymptotically stable) without solving DEs.

Question: How to find a Lyapunov Function, V?

Possibilities: Let V be a measurement of

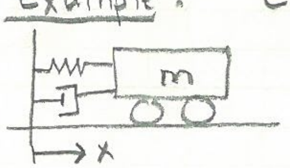
- size (norm) of state or output error
- size of deviation
- energy difference from desired equilibrium
- weighted combination of above

Analysis: Check if V is decreasing in time:

$\frac{dV}{dt} < 0$ continuous time

$V(k+1) - V(k) < 0$ discrete time

Example: Cart



$$m\ddot{x} = -b\dot{x}|\dot{x}| - k_0 x - k_1 x^3 \qquad b, k_0, k_1 > 0$$

$$\text{Total energy} = KE + PE = V = \frac{mv^2}{2} + \int_0^x F_{spring}\, ds$$

$$\Rightarrow V(x,\dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}k_0 x^2 + \frac{1}{4}k_1 x^4 > 0 \ , \ V(0,0) = 0$$

$$\frac{d}{dt}V = m\ddot{x}\dot{x} + k_0 x\dot{x} + k_1 x^3\dot{x} = -b|\dot{x}|^3 < 0 \text{ for } \dot{x} \neq 0$$

What does this mean?

Lyapunov Functions

"Direct Method" of Lyapunov

Idea: use to prove stability of certain fixed points in a dynamical system

Most importantly – near an equilibrium point

All solutions that start off near x^{eq} stay near x^{eq}

Theorem

$$\dot{x} = f(x)$$

$$f(0) = 0 \qquad x = 0 \in \Omega \subset \mathbb{R}^n \quad \text{(equilibrium point)}$$

Let $V : \Omega \to \mathbb{R}$ be a C^1 function. "Smooth function"

If (1) $V(0) = 0$

(2) $V(x) > 0 \quad \forall x \in \Omega, x \neq 0$ "positive definite function"

(3) $\dot{V}(x) \leq 0$ along the trajectories of the system in Ω "energy reduced"

Then $x = 0$ is locally stable

If, furthermore

(4) $\dot{V}(x) < 0 \quad \forall x \in \Omega, x \neq 0$

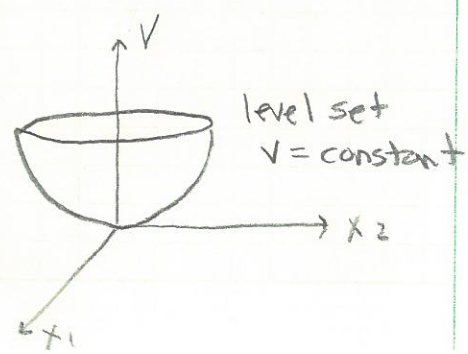
Then $x = 0$ is locally asymptotically stable

And V is called the Lyapunov function

A look closer at condition #(3):

$$\dot{V}(x) = \frac{\partial V}{\partial x}\dot{x} = \sum_i \frac{\partial V}{\partial x_i} f_i(x) \leq 0$$

where $\frac{\partial V}{\partial x} = \left[\frac{\partial V}{\partial x_1} \; \frac{\partial V}{\partial x_2} \cdots \frac{\partial V}{\partial x_n} \right] = \nabla V$ gradient



Constructing a Lyapunov Function

$\dot{x} = Ax$

Case 1 $\quad V(x) = x^T x = \|x\|^2$ $\quad$ distance between x and $x^{eq} = 0$

$$\dot{V}(x) = \dot{x}^T x + x^T \dot{x}$$
$$= (Ax)^T x + x^T (Ax)$$
$$= x^T A^T x + x^T A x$$
$$= x^T (A^T + A) x$$

Case 2 $\quad V(x) = x^T P x \quad , \quad P = P^T > 0$ $\quad$ Scalar

$$\dot{V}(x) = \dot{x}^T P x + x^T P \dot{x}$$
$$= (Ax)^T P x + x^T P (Ax)$$
$$= x^T A^T P x + x^T P A x$$
$$= x^T (A^T P + PA) x$$
$$= -x^T Q x \qquad Q = Q^T > 0$$

Note

$V(x) = x^T P x$ is generalized energy function

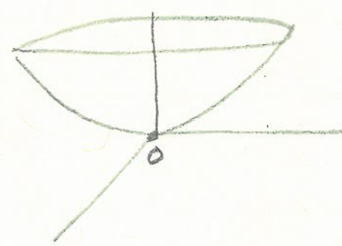
$\dot{V}(x) = -x^T Q x$ is associated energy dissipation

Thus

$\dot{V}(x) < 0 \quad \forall x \neq 0$

$V(0) = 0$

$V(x) > 0 \quad \forall x$



$\Rightarrow \quad x \xrightarrow{t \to \infty} x^{eq} = 0$

$\Rightarrow$ asymptotic stability by above theorem

Result

Don't have to calculate $x(t)$, but know it converges to x_{eq}

Lyapunov Stability Theorem

Alternate way to check asymptotic stability

Given the continuous-time LTI system

$$① \begin{cases} \dot{x} = Ax & x \in \mathbb{R}^n \\ x(t_0) = x_0 \\ x^{eq} = 0 \end{cases}$$

The following conditions are equivalent:

1) The system ① is asymptotically stable
2) The system ① is exponentially stable
3) $\mathrm{Re}\{\lambda_i(A)\} < 0 \quad \forall i$ open LHP
4) For any $Q = Q^T > 0$ (positive definite, symmetric)

   $\exists!\ P = P^T > 0$ solution to the Lyapunov equation

   $$\boxed{A^T P + PA = -Q}$$

5) $\exists \quad P = P^T > 0$ s.t.

   $$A^T P + PA < 0$$

Example

Lyapunov function for a linear system

$$\dot{x} = Ax = \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Find eigenvalues:

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & -4 \\ 0 & \lambda+3 \end{bmatrix} = (\lambda+1)(\lambda+3) = 0$$

$\lambda = -1, -3$ negative real parts

Thus, system is asymptotically stable

Find a quadratic Lyapunov function for the system:

$$V(x) = x^T P x = [x_1 \ x_2] \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad P = P^T > 0$$

Take any $Q = Q^T > 0$, say $Q = I_{2\times 2}$

Solve $A^T P + PA = -Q$ for P:

$$A^T P + PA = -I$$

$$\begin{bmatrix} -1 & 0 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\begin{bmatrix} -2p_{11} & -4p_{12} + 4p_{11} \\ -4p_{12} + 4p_{11} & 8p_{12} - 6p_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 5/6 \end{bmatrix} > 0$$

By the Theorem, $x^{eq} = 0$ is exponentially stable

Example (Vidyasagar p. 176)

$$A = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} p_1 & p_2 \\ p_3 & p_4 \end{bmatrix}$$

$$A^T P + PA = -Q \quad \Rightarrow \quad P = \begin{bmatrix} -1/2 & 3/2 \\ 3/2 & -2 \end{bmatrix}$$

Unique P

But P is not p.d.

→ Not stable

Thus, not all e-values of A have negative real parts

Stability of Locally Linearized Systems

as in Fundamental Theorem

Assume $Z\ \mathbb{R}$, $u(t) \equiv 0$

$\dot{x}(t) = f(x(t))$ nonlinear system $x \in \mathbb{R}^n$, f piecewise continuous in t

$x^{eq} \in \mathbb{R}^n$ equilibrium point $f(x^{eq}) = 0 \ \forall t$

We can linearize this nonlinear system around the equilibrium point:

$$\delta\dot{x} = A\,\delta x$$

where $\delta x = x - x^{eq}$ is the deviation from the equilibrium point.

Questions

1) With small perturbation, do we get a small deviation?
2) Will the deviation eventually die out?
3) If the linearized system is stable, is the nonlinear system stable?

Definition The equilibrium x^{eq} (or zero solution $x(t) = x^{eq}$) for $t \geq 0$ of the nonlinear system $\dot{x}(t) = f(x(t))$ is said to be stable i.s.L. (in the sense of Lyapunov) iff

$\forall\, t_0 \geq 0$ and $\forall\, \varepsilon > 0$, $\exists\, \gamma(\varepsilon, t_0)$ s.t.

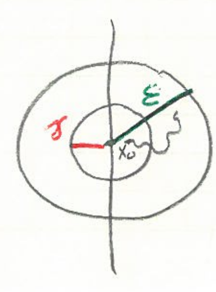
$$\|x_0\| \leq \gamma(\varepsilon, t_0) \Rightarrow \|\phi(t, t_0, x_0)\| < \varepsilon,\ \forall\, t \geq t_0$$

Recall from Lecture 5 p.4 "state at t reached from x_0 at t_0"

i.e., if initial disturbance $< \gamma < \varepsilon$ then all future states $< \varepsilon$

Geometrically

Trajectory remains forever in the ball $\|x\| \leq \varepsilon$



Example Nonlinear System

$$\text{①} \begin{cases} \dot{x}_1 = x_1(x_1^2 - 3x_1 + 3) - x_2 \\ \dot{x}_2 = x_1 - x_2 \end{cases}$$

Equilibrium point is $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ So it will not move

A cubic equation $\Rightarrow$ 3 equilibrium points

$\dot{x}_2 = 0 \Rightarrow x_1 = x_2$

$\dot{x}_1 = 0 \Rightarrow x_1(x_1^2 - 3x_1 + 3) = x_2$

Solve to get x^{eq}: $\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ 3 equilibrium points

Linearize equations ① at each x^{eq}

$f_1(x_1, x_2) = x_1^3 - 3x_1^2 + 3x_1 - x_2$ $\qquad$ $f_2(x_1, x_2) = x_1 - x_2$

$\frac{\partial f_1}{\partial x_1} = 3x_1^2 - 6x_1 + 3$ $\qquad$ $\frac{\partial f_2}{\partial x_1} = 1$

$\frac{\partial f_1}{\partial x_2} = -1$ $\qquad$ $\frac{\partial f_2}{\partial x_2} = -1$

We get the linearized system

$\delta \dot{x} = A\, \delta x$

Where $A = \frac{\partial f}{\partial x}\Big|_{x^{eq}} = \begin{bmatrix} 3x_1^2 - 6x_1 + 3 & -1 \\ 1 & -1 \end{bmatrix}\Big|_{x^{eq}}$ for 3 x^{eq}

$A\Big|_{\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}, A\Big|_{\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}, A\Big|_{\left[\begin{smallmatrix}2\\2\end{smallmatrix}\right]} = \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$

Look at the eigenvalues to see if stable:

$x_1^{eq} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$: $\lambda_{1,2} = \frac{1}{2}(2 \pm \sqrt{12})$ $\left.\begin{matrix}\text{One positive} \\ \text{One negative}\end{matrix}\right\} \Rightarrow$ Unstable system

$x_2^{eq} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$: $\lambda_{1,2} = \frac{1}{2}(-1 \pm \sqrt{-3})$ $Re(\lambda) < 0 \Rightarrow$ Stable equilibrium point

$x_3^{eq} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$: $\lambda_{1,2}$ = same as x_1^{eq} case

asymptotic & exponential Stability

$x \xrightarrow{t \to \infty} x_2^{eq}$ exponentially fast

Outline

Norms

- of vectors
- of matrices

BIBO Stability for LTI Systems

- time-domain condition
- frequency domain condition

Note: BIBO Stability for ZSR only

$X(t_0)=0$ I/O Analysis

Take away - check location of poles of $\hat{g}(s)$ transfer function to determine BIBO Stability

$\rightarrow$ poles of $\hat{g}(s)$

$=$ eigenvalues of A for LTI System

Norms of Vectors (Chen 3.2)

Concept of a norm is a generalization of length or magnitude.

Any real-valued function of x, denoted $f(x) = \|x\|$ can be defined as a norm if it has the following properties:

1) $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$

   $\|x\| = 0 \quad \text{iff} \quad x = 0$

2) $\|\alpha x\| = |\alpha| \cdot \|x\| \quad \forall \alpha \in \mathbb{R}$

3) $\|x + y\| \leq \|x\| + \|y\| \quad \forall x, y \in \mathbb{R}^n$ triangular inequality

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$

Types of different norms:

$$\|x\|_1 := \sum_{i=1}^{n} |x_i| \qquad \text{1-norm}$$

$$\|x\|_2 := \sqrt{x'x} = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} \qquad \text{2-norm or Euclidean norm}$$

$$\|x\|_\infty := \max_i |x_i| \qquad \infty\text{-norm}$$

☆ We will use the Euclidean norm (& drop subscript) = "length of a vector from the origin"

Norms of Matrices

Extend concept of a norm of a vector to matrices.

A n x n matrix

The norm of A is defined as

$$\|A\| = \sup_{x \neq 0} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$$

where "sup" stands for "supremum" or "least upper bound"

☆ For different $\|x\|$, we have different $\|A\|$

Common Matrix Norms

$\|x\|_1$: $\|A\|_1 = \max_j \left(\sum_{i=1}^{m} |a_{ij}| \right)$ = largest column absolute sum

a_{ij}: ijth element of A

$\|x\|_2$: $\|A\|_2 = \sigma_{max}(A)$ = largest singular value of A
$= (\text{largest eigenvalue of } A'A)^{1/2}$

$\|x\|_\infty$: $\|A\|_\infty = \max_i \left(\sum_{j=1}^{n} |a_{ij}| \right)$ = largest row absolute sum

Example

$$A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$$

$\|A\|_1 = 3 + |-1| = 4$

$\|A\|_2 = 3.7$ → ☆ see next page for derivation

$\|A\|_\infty = 3 + 2 = 5$

Properties

$$\|Ax\| \leq \|A\| \, \|x\|$$

$$\|A+B\| \leq \|A\| + \|B\|$$

$$\|AB\| \leq \|A\| \, \|B\|$$

Example

Derivation of $\|A\|_2$ for $A = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}$

$$\|A\|_2 = (\text{largest eigenvalue of } A'A)^{1/2}$$

$$A'A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 4 \end{bmatrix}$$

The characteristic polynomial is

$$\Delta(s) = \det(sI - A) = \det \begin{bmatrix} s-10 & -6 \\ -6 & s-4 \end{bmatrix}$$

$$= (s-10)(s-4) - 36$$

$$= s^2 - 14s + 40 - 36$$

$$= s^2 - 14s + 4$$

To find the roots of the characteristic polynomial, use the quadratic formula

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{14 \pm \sqrt{(-14)^2 - 4 \cdot 1 \cdot 4}}{2 \cdot 1}$$

$$= 7 \pm \frac{1}{2}\sqrt{196 - 16}$$

$$= 7 \pm \frac{1}{2}(13.416)$$

2 eigenvalues

$$= 7 \pm 6.708 \Rightarrow \text{roots } 13.708,\ 0.292$$

Take square-root of largest eigenvalue

$$\sqrt{13.708} = 3.7 = \|A\|_2$$

Bounded-Input, Bounded-Output Stability

Motivation

$$u \longrightarrow \boxed{\begin{array}{l}\dot{x} = Ax + Bu \\ y = Cx + Du\end{array}} \longrightarrow y$$

For any function $f : \mathbb{R}$ or $\mathbb{C} \longrightarrow V$ vector space, think $\mathbb{R}^n$
we say f is bounded iff
$\exists\, M < \infty$ such that $\|f(t)\| < M \quad \forall\, t \in \mathbb{R}$ (or $\mathbb{C}$)

Suppose $u(t)$ is bounded. Is $y(t)$ bounded?

$$|u(t)| \leq u_m < \infty \quad \forall\, t \geq 0.$$

for some constant u_m

$u(t)$ does not grow to positive or negative infinity

Definition: A system is said to be BIBO Stable if every bounded input produces a bounded output.

Note: BIBO stability is only defined for the ZSR
i.e., $x(t_0) = 0$ (system is initially at rest

Consider SISO LTI System ZSR

$$y(t) = \int_0^t g(t-\tau)\, u(\tau)\, d\tau$$

g - impulse response

$$= \int_0^t g(\tau)\, u(t-\tau)\, d\tau$$

because time-invariant

Theorem A SISO LTI System is BIBO Stable iff $g(t)$ is absolutely integrable in $[0, \infty)$, or

$$\int_0^\infty |g(t)|\, dt \leq M < \infty$$

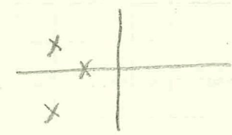
time-domain condition

for some constant M.

Theorem A SISO LTI System with proper rational transfer function $\hat{g}(s)$ is BIBO Stable iff every pole of $\hat{g}(s)$ has a negative real part, or equivalently, lies inside the left-half s-plane.

Look at $\hat{g}(s)$: frequency-domain Condition

$$\hat{g}(s) = \frac{\alpha_0 s^q + \alpha_1 s^{q-1} + \cdots + \alpha_q}{(s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}}$$



It's Partial Fraction Expansion (PFE) contains the factors

$$\frac{1}{s-\lambda_i}, \quad \frac{1}{(s-\lambda_i)^2}, \quad \frac{1}{(s-\lambda_i)^3} \quad \cdots \quad \frac{1}{(s-\lambda_i)^{m_i}}$$

The inverse Laplace Transform of $\hat{g}(s)$, which is the impulse response, $g(t)$ contains the factors

$$e^{\lambda_i t}, \quad t e^{\lambda_i t}, \quad \cdots \quad t^{m_i - 1} e^{\lambda_i t}$$

If all λ_i have strictly negative real parts, then these terms $\rightarrow 0$ via property of the exponential

MIMO LTI System

$$G(t) = [g_{ij}(t)]$$ impulse response matrix

Theorem A MIMO LTI System is BIBO Stable iff every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$

Theorem A MIMO LTI System with $\hat{G}(s) = [\hat{g}_{ij}(s)]$ proper rational transfer function is BIBO Stable iff every pole of $\hat{g}_{ij}(s)$ has a negative real part.

Example

$$\begin{cases} \dot{x} = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 1 \end{bmatrix} x \end{cases}$$

Is the system BIBO stable?

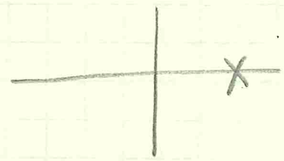
$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s & -3 \\ -3 & s \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{1}{s^2-9} \begin{bmatrix} s & 3 \\ 3 & s \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{s+3}{s^2-9} = \frac{1}{s-3}$$

pole of $\hat{G}(s) = 3$

$\Rightarrow$ not BIBO Stable



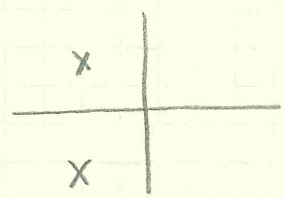
Example

LTI system with $\hat{G}(s) = \dfrac{s+1}{s^2+2s+3}$

poles of $\hat{G}(s) = -1 \pm i\sqrt{2}$

$\operatorname{Re}\{\lambda_1\} = \operatorname{Re}\{\lambda_2\} = -1 < 0$

$\Rightarrow$ BIBO stable

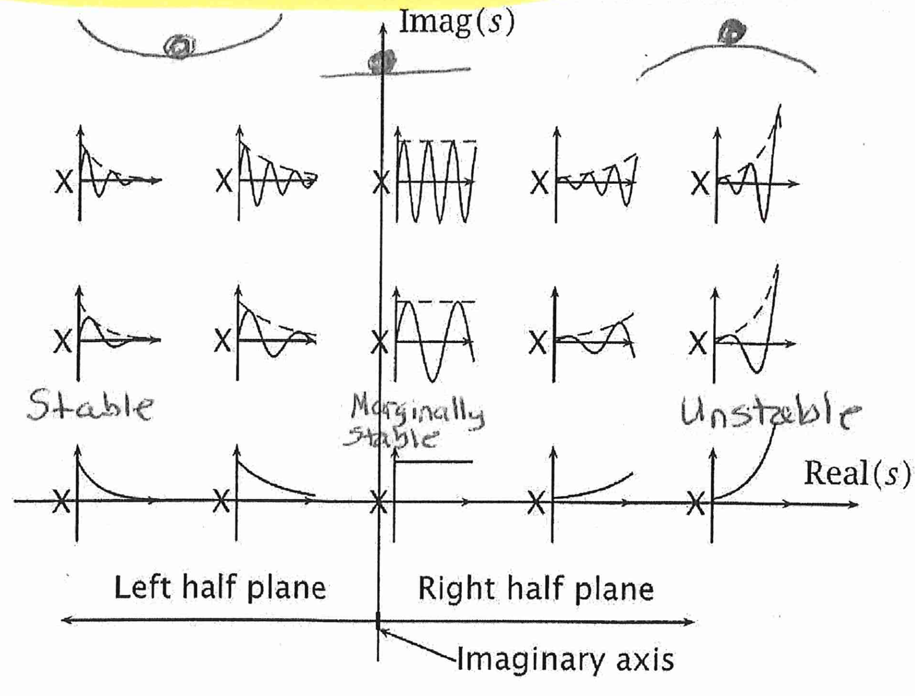


Transfer Function & Stability of Response

$\frac{1}{s+a}$

$a = \sigma \pm j\omega$

complex roots



Pole locations and corresponding transient responses

on Re{s} axis

$\frac{1}{s+a}$

$a = \sigma$

real roots

$$H(s) = \frac{N(s)}{D(s)} = \frac{K(s+z_1)\cdots(s+z_m)}{(s+p_1)\cdots(s+p_n)} = \frac{K_1}{s+p_1} + \frac{K_2}{s+p_2} + \cdots + \frac{K_n}{s+p_n}$$

$$V_{in}(s) \longrightarrow \boxed{H(s)} \longrightarrow V_{out}(s) \qquad V_{out}(s) = H(s) \cdot V_{in}(s)$$

Depending on the nature of the poles, $V_{out}(t)$ will have terms

Terms in $V_{out}(s)$	Terms in $V_{out}(t)$
$\frac{K}{s+a}$	$K e^{-at} u(t)$
$\frac{K}{(s+a)^2}$	$Kt e^{-at} u(t)$
$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$	$2\lvert K\rvert e^{-\alpha t} \cos(\beta t + \theta) u(t)$
$\frac{K}{(s+\alpha-j\beta)^2} + \frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\lvert K\rvert e^{-\alpha t} \cos(\beta t + \theta) u(t)$

$K = \lvert K\rvert \angle\theta$

Thus, $V_{out}(t)$ could be a combination of these

See how the terms in $V_{out}(t)$ correspond to the plot of the response in the s-plane above.

Relationship Between Asymptotic & BIBO Stability

$\dot{x} = Ax$ is asymptotically stable

if $Re\{\lambda_k(A)\} < 0 \ \forall k$

$\hat{G}(s) = C(sI-A)^{-1}B + D$ is BIBO stable

if all poles of $\hat{G}(s)$ have negative real parts

BUT poles of $\hat{G}(s)$ = eigenvalues of A

Thus Asymptotic stability $\Rightarrow$ BIBO Stability

However: Opposite is not true.

Why?

Some eigenvalues of A may not appear as poles of $\hat{G}(s)$

& these e-values may have $Re\{\lambda_k(A)\} \geq 0$

$\Rightarrow$ BIBO stable, but not asymptotically stable

Example

Stabilize an unstable system via a cascade compensator

$G_p(s) = \frac{1}{s-1}$ not stable

$G_c(s) = \frac{s-1}{s+1}$ compensator

$G_p(s)\,G_c(s) = \frac{1}{s+1}$ BIBO stable

Is it asymptotically stable?

Find a state model of the system

$u \rightarrow \boxed{G_c(s)} \xrightarrow{v} \boxed{G_p(s)} \rightarrow y$

Let $G_p(s): \begin{cases} \dot{x}_1 = x_1 + v \\ y = x_1 \end{cases}$ $\quad G_c(s): \begin{cases} \dot{x}_2 = -x_2 + u \\ v = -2x_2 + u \end{cases}$

$\Rightarrow \dot{x}_1 = x_1 - 2x_2 + u$

$\dot{x} = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$ "hidden mode"

$A = \begin{bmatrix} 1 & -2 \\ 0 & -1 \end{bmatrix}$ $\quad \lambda_1 = 1, \ \lambda_2 = -1$ Not asymptotically stable