Class 2

Lecture 3

Properties of Systems

Impulse Response of a Linear System

Lecture 4

Impulse Response of a LTI System

Transfer Function

Equivalent Systems

Concept: state space $\longleftrightarrow$ transfer function

Outline

Basic Properties of LTI Systems
- Memoryless
- Causal
- Time-invariant
- Linear

Impulse Response

Transfer Function

Lecture 3

Basic Properties of LTI & LTV Systems

① Memoryless

A system is <u>memoryless</u> if the output at time t depends only on the input at time t.

$$u(t) \longrightarrow \boxed{\text{System}} \longrightarrow y(t)$$

<u>Examples:</u>

$y(t) = 4u(t)$ memoryless

$y(t) = u(t-3)$ not memoryless - needs past value

$y(t) = u(3t)$ not mem. - speed up

$y(t) = (t+5)u(t)$ mem.

$y(t) = u(-t)$ not mem. - time reversal
$y(-1) = u(1)$ needs future time

$y(t) = \cos(t^2 + u(t)^2)$ mem.

② Causal

A system is causal if the output at time t depends only on the input at time t plus past values of the input.

$$u(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$$

$y(t+a)$ depends only on $u(t+b)$ for $b \le a$

The system doesn't anticipate the input
"non-anticipatory" system

Examples:

$y(t) = 4u(t)$ causal - amplification

$y(t) = u(t-3)$ causal - delay

$y(t) = u(t+5)$ noncausal - time shift forward

$y(t) = u(3t)$ noncausal - speed up

$y(t) = (t+5)u(t)$ causal - ramp times input

$y(t) = u(-t)$ non causal - time reversal

$y(t) = u(t+t_0)$ causal for $t_0 \le 0$

$y(t) = \int_{-\infty}^{t} u(\tau+a)\,d\tau$ causal for $a \le 0$

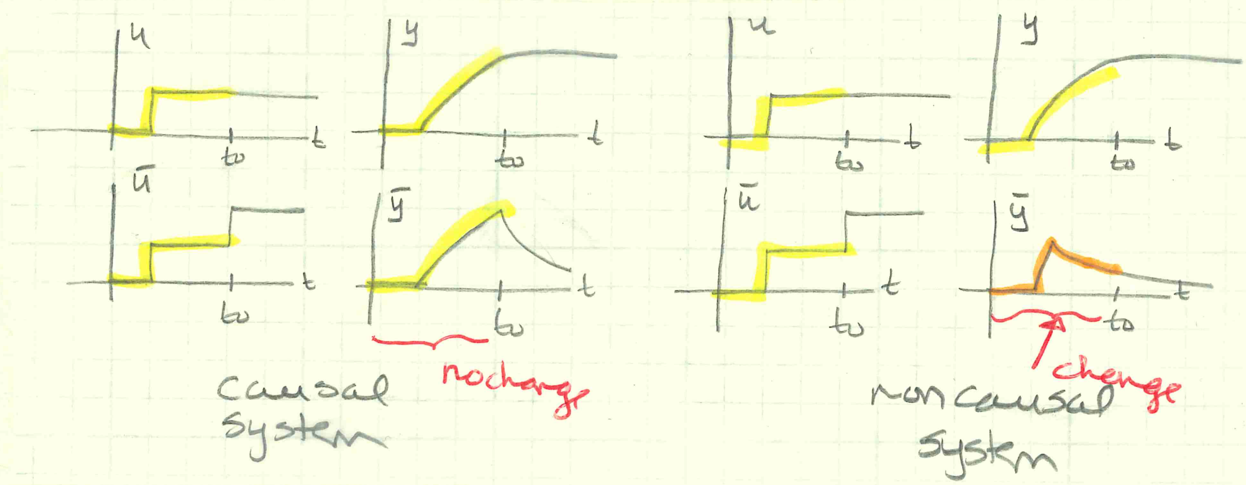
☆ All memoryless systems are causal

Formal definition of <u>causality</u> for continuous LTV System:

$$u(t) \rightarrow \boxed{\text{LTV}} \rightarrow y(t)$$

The system is causal if $\forall$ other input $\bar{u}(t)$ for which $\bar{u}(t) = u(t) \;\; \forall \; 0 \le t < T$, $T > 0$ the system exhibits at least one output $\bar{y}(t)$ that satisfies $\bar{y}(t) = y(t) \;\; \forall \; 0 \le t < T$.

<u>Example</u>



(3) Time - Invariant

A system is time-invariant if a time shift in the input results in the same time shift in the output.

$$u(t) \rightarrow \boxed{\text{System}} \rightarrow y(t)$$

$$u(t-t_0) \rightarrow \boxed{\text{System}} \rightarrow y(t-t_0)$$

Examples:

$y(t) = 2u(t)$ T.I. – amplification

$y(t) = (u(t))^2$ T.I. $y(t-t_0) = (u(t-t_0))^2$

$y(t) = t\,u(t)$ Time-varying $y(t-t_0) = (t-t_0)u(t-t_0)$
but $u(t-t_0) \rightsquigarrow t\,u(t-t_0)$

$y(t) = 3u(t-5)$ T.I. $y(t-t_0) = 3u(t-t_0-5)$
$u(t-t_0) \rightsquigarrow 3u(t-t_0-5)$

$y(t) = \int_{-\infty}^{t} u(\tau)\,d\tau$ T.I. $y(t-t_0) = \int_{-\infty}^{t-t_0} u(\tau)\,d\tau$
$u(t-t_0) \rightsquigarrow \int_{-\infty}^{t} u(\tau-t_0)\,d\tau$
$= \int_{-\infty}^{t-t_0} u(v)\,dv$

$y(t) = u(5t)$ Time-varying $y(t-t_0) = u(5(t-t_0)) = u(5t-5t_0)$
$u(t-t_0) \rightsquigarrow u(5t-t_0)$

$y(t) = u(-t)$ Time-varying $y(t-t_0) = u(-(t-t_0)) = u(-t+t_0)$
$u(t-t_0) \rightsquigarrow u(-t-t_0)$

$y(t) = \sin(u(t))$ T.I.

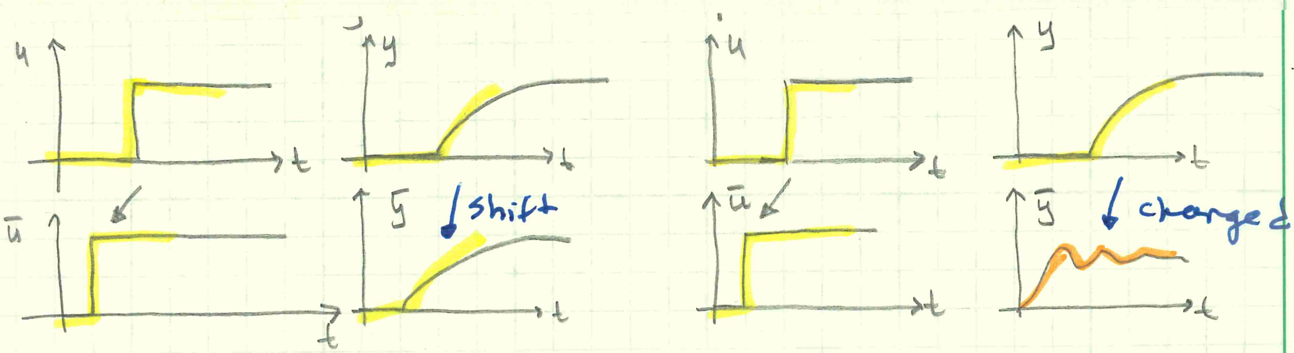
Formal definition of <u>time-invariance</u> for continuous LTV System:

$$u(t) \rightarrow \boxed{LTV} \rightarrow y(t)$$

The system is time-invariant if

$$u \rightsquigarrow y \quad \Rightarrow \quad u(t+T) \rightsquigarrow y(t+T) \quad \forall T > 0$$
$$\forall t \geq 0$$

<u>Example</u>:



Time-invariant system　　　　Time-varying system

<u>Comment on Initial Condition</u>

$$\left.\begin{array}{l} X(t_0) \\ u(t),\ t \geq t_0 \end{array}\right\} \rightsquigarrow y(t),\ t \geq t_0$$

$$\Downarrow$$

$$\left.\begin{array}{ll} X(t_0+T) & \forall T \\ u(t-T), & t \geq t_0+T \end{array}\right\} \rightsquigarrow y(t-T),\ t \geq t_0 + T$$

initial state shifted to $t_0 + T$
same input waveform applied at $t_0 + T$ (vs t_0)
$\Rightarrow$ output waveform is the same except it starts to appear at time $t_0 + T$

4) Linear

A system is <u>linear</u> if it has the following three properties:

1) <u>Additivity</u>

$$\left.\begin{array}{l} x_i(t_0) \\ u_i(t),\ t \ge t_0 \end{array}\right\} \rightsquigarrow y_i(t),\ t \ge t_0$$

$$\Rightarrow \left.\begin{array}{l} \sum_i x_i(t_0) \\ \sum_i u_i(t),\ t \ge t_0 \end{array}\right\} \rightsquigarrow \sum_i y_i(t),\ t \ge t_0$$

2) <u>Scale / Homogeneity</u>

$$\left.\begin{array}{l} x(t_0) \\ u(t),\ t \ge t_0 \end{array}\right\} \rightsquigarrow y(t),\ t \ge t_0 \Rightarrow \left.\begin{array}{l} \alpha x(t_0) \\ \alpha u(t_0),\ t \ge t_0 \end{array}\right\} \rightsquigarrow \alpha y(t),\ t \ge t_0$$

$\alpha \in \mathbb{R}$

3) <u>Superposition</u>

$$\left.\begin{array}{l} x_i(t_0) \\ u_i(t),\ t \ge t_0 \end{array}\right\} \rightsquigarrow y_i(t),\ t \ge t_0$$

$$\Rightarrow \left.\begin{array}{l} \sum_i \alpha_i x_i(t_0) \\ \sum_i \alpha_i u_i(t) \\ \quad t \ge t_0 \end{array}\right\} \rightsquigarrow \sum_i \alpha_i y_i(t),\ t \ge t_0$$

Examples:

$y(t) = c\,u(t)$ linear - amplification

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = c(\alpha_1 u_1(t) + \alpha_2 u_2(t))$$

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \rightsquigarrow c(\alpha_1 u_1(t) + \alpha_2 u_2(t))$$

$y(t) = (u(t))^2$ nonlinear

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) \neq (\alpha_1 u_1(t) + \alpha_2 u_2(t))^2$$

$y(t) = u(t) + 5$ nonlinear

$$\alpha y(t) = \alpha u(t) + 5\alpha \neq \alpha u(t) \rightsquigarrow \alpha u(t) + 5$$

$y(t) = \int_{-2}^{2} u(t+\tau)\,d\tau$ linear

$y(t) = \int_{t-1/2}^{t+1/2} u(\tau)\,d\tau$ linear

$y(t) = \dfrac{du(t)}{dt}$ linear

$y(t) = \dfrac{(u(t))^2}{u(t-1)}$ nonlinear

$$\alpha_1 u_1(t) + \alpha_2 u_2(t) \not\rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

Complete Response to a Given Input

For every linear system:

Response = zero-input response + zero-state response

⇒ Can study two responses separately then sum for complete response

Zero-input Response y_{zi}

$$\left.\begin{array}{l} x(t_0) \\ u(t) \equiv 0 \quad t \ge t_0 \end{array}\right\} \longrightarrow y_{zi}(t) \quad t \ge t_0$$

Zero-state Response y_{zs}

$$\left.\begin{array}{l} x(t_0) = 0 \\ u(t) \quad t \ge t_0 \end{array}\right\} \longrightarrow y_{zs}(t) \quad t \ge t_0$$

Additivity property ⇒

$$\text{Response to } \left\{\begin{array}{l} x(t_0) \\ u(t) \quad t \ge t_0 \end{array}\right. = \text{Response to } \left\{\begin{array}{l} x(t_0) \\ u(t) \equiv 0 \quad t \ge t_0 \end{array}\right.$$

$$+ \text{ Response to } \left\{\begin{array}{l} x(t_0) = 0 \\ u(t) \quad t \ge t_0 \end{array}\right.$$

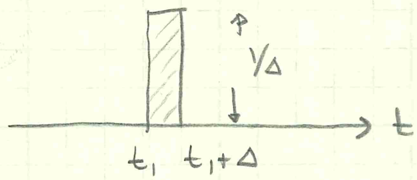
Impulse Response

Fundamental Result: Any LTI system can be completely characterized by its impulse response.

Consider a SISO system first.

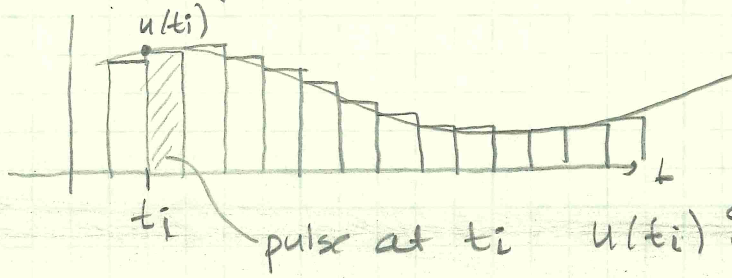
Consider the zero-state response – output is excited exclusively by the input.

Let $\delta_\Delta(t-t_1)$ be the unit pulse



width Δ
height $1/\Delta$
located at t_1

Then, every input $u(t)$ can be approximated by a series of pulses



$u(t_i)\,\delta_\Delta(t-t_i)\,\Delta$

$\Rightarrow \quad u(t) \approx \sum u(t_i)\,\delta_\Delta(t-t_i)\,\Delta$

Let $g_\Delta(t,t_i)$ be the output at time t with impulse at t_i from input $u(t) = \delta_\Delta(t-t_i)$ applied at t_i

$\Rightarrow$

$$\delta_\Delta(t-t_i) \longrightarrow g_\Delta(t,t_i)$$

$$\delta_\Delta(t-t_i)u(t_i)\Delta \longrightarrow g_\Delta(t,t_i)\,u(t_i)\Delta \quad \text{homogeneity}$$

$$\sum_i \delta_\Delta(t-t_i)u(t_i)\Delta \longrightarrow \sum_i g_\Delta(t,t_i)\,u(t_i)\Delta \quad \text{additivity}$$

$\Rightarrow$ The output $y(t)$ excited by the input $u(t)$ can be approximated by

$$y(t) \approx \sum_i g_\Delta(t,t_i)\,u(t_i)\,\Delta \qquad \star$$

No. 937 811E Engineer's Computation Pad STAEDTLER

Impulse Response (continued)

Let $\Delta \to 0$

$\Rightarrow$ $\delta_\Delta(t-t_i)$ pulse $\longrightarrow$ impulse $\delta(t-t_i)$ at t_i

$g_\Delta(t,t_i) \longrightarrow g(t,t_i)$ output

$y(t) \approx \sum_i g_\Delta(t,t_i)\,u(t_i)\Delta \longrightarrow y(t) = \int_{-\infty}^{\infty} g(t,\tau)\,u(\tau)\,d\tau$

summation becomes integral; t_i becomes continuum; Δ written as $d\tau$

Note - $g(t,\tau)$ is function of 2 variables
τ = time at which impulse input applied
t = time at which output is observed

& is called the **impulse response**

If causal system, $g(t,\tau)=0$ for $t<\tau$
(output doesn't appear before input applied)

If $x(t_0)=0$, output excited by $u(t)$, $t \geq t_0$ only

"relaxed at t_0"

Then $$y(t) = \int_{t_0}^{t} g(t,\tau)\,u(\tau)\,d\tau$$

limits changed

Define **Impulse Response Matrix**

p inputs, q outputs (MIMO)

$$y(t) = \int_{t_0}^{t} G(t,\tau)\,u(\tau)\,d\tau$$

$$G(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & g_{12}(t,\tau) & \cdots & g_{1p}(t,\tau) \\ \vdots & & & \\ g_{q1}(t,\tau) & g_{q2}(t,\tau) & \cdots & g_{qp}(t,\tau) \end{bmatrix}$$

Where $g_{ij}(t,\tau)$ = response at time t at i^{th} output terminal due to impulse applied at time τ at j^{th} input terminal.

If time-invariant,

$$G(t+T, \tau+T) = G(t,\tau) \quad \forall t, \tau, T \geq 0$$

Then for $\tau = 0$, $t_1 = T$, $t_2 = t+T$

$$G(t_2, t_1) = G(t_2 - t_1, 0) \quad \forall t_2 \geq t_1 \geq 0$$

$\Rightarrow$ $G(t_2, t_1)$ is a function of $t_2 - t_1$

If causal & time-invariant

Then

$$y(t) = \int_0^t G(t-\tau)\, u(\tau)\, d\tau$$

$$= (G * u)(t) \quad \forall t \geq 0$$

Convolution Operator

Transfer Function

Recall Laplace Transform

- $\mathcal{L}[x(t)] = \hat{x}(s) = \int_0^\infty e^{-st} x(t)\, dt$ $\quad s \in \mathbb{C}$ complex number
- $\mathcal{L}[\dot{x}(t)] = s\hat{x}(s) - x(0)$
- $\mathcal{L}[(x * y)(t)] = \hat{x}(s)\, \hat{y}(s)$

Thus

$$\hat{y}(s) = \hat{G}(s)\, \hat{u}(s)$$

Define Transfer Function of a continuous time, causal, LTI system is

$$\hat{G}(s) = \mathcal{L}[G(t)] = \int_0^\infty e^{-st} G(t)\, dt$$

Transfer function = Laplace transform of the impulse response

Easier to use transform vs convolution

Recall: circuit analogy

Outline

Impulse Response for LTI System

Transfer Function for LTI System

Converting State Space $\longleftrightarrow$ Transfer Function

Equivalent Systems

LTI State Space equation

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & x \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) & u \in \mathbb{R}^{\ell} \quad y \in \mathbb{R}^m \end{cases}$$

Apply Laplace transform

$$s\hat{x}(s) - x(0) = A\hat{x}(s) + B\hat{u}(s)$$

$$\hat{y}(s) = C\hat{x}(s) + D\hat{u}(s)$$

Combining terms

$$s\hat{x}(s) - A\hat{x}(s) = x(0) + B\hat{u}(s)$$

$$(sI - A)\hat{x}(s) = x(0) + B\hat{u}(s)$$

$$\hat{x}(s) = (sI-A)^{-1}x(0) + (sI-A)^{-1}B\hat{u}(s)$$

$$\hat{y}(s) = C(sI-A)^{-1}x(0) + C(sI-A)^{-1}B\hat{u}(s) + D\hat{u}(s)$$

$$= \underbrace{C(sI-A)^{-1}x(0)}_{\text{zero-input response}} + \underbrace{\left(C(sI-A)^{-1}B + D\right)}_{\text{zero-state response}}\hat{u}(s)$$

Define $\hat{y}(s) = \hat{\Psi}(s)x(0) + \hat{G}(s)\hat{u}(s)$

$$\hat{\Psi}(s) = C(sI-A)^{-1}$$

frequency domain $\hat{G}(s) = C(sI-A)^{-1}B + D$ Transfer Function

Unique

time domain $G(t) = \mathcal{L}^{-1}[G(s)]$ Impulse Response

Example (p. 89 Chen)

$$A = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

Recall $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $A^{-1} = \dfrac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$

$$(sI-A)^{-1} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} = \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

Motivation

Given $y(t) = \int_{t_0}^{t} G(t-\tau)\, u(\tau)\, d\tau$

No simple way to compute convolution

Easiest way would be numerically on computer on discretized equation

But – least accurate result because of step size

For LTI Systems, we can use

$$\hat{y}(s) = \hat{G}(s)\, \hat{u}(s)$$

to compute the solution via $\mathcal{L}^{-1}(\hat{y}(s))$

But – complicated due to sensitivity in data, round-off errors etc.

Better Method: Transform transfer function into state-space equations & compute solution

Today – see connection between steady-state & transfer function

$\uparrow$ 1st order differential equations

$\uparrow$ higher-order differential equations

Realization

State Space $\leftarrow$ Transfer Function

LTI System $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

is a realization of $\hat{G}(s)$ if

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

Equivalent Systems : Zero-State Equivalent

Two state-space systems are zero-state equivalent if they realize the same transfer function

i.e., they have the same I/O pairs.

Equivalent Systems: Algebraically Equivalent

Two LTI systems

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\bar{u} \\ \bar{y} = \bar{C}\bar{x} + \bar{D}\bar{u} \end{cases}$$

are algebraically equivalent if $\exists$ nonsingular matrix P s.t.

$$\bar{x} = Px$$

$$\bar{A} = PAP^{-1}$$

$$\bar{B} = PB$$

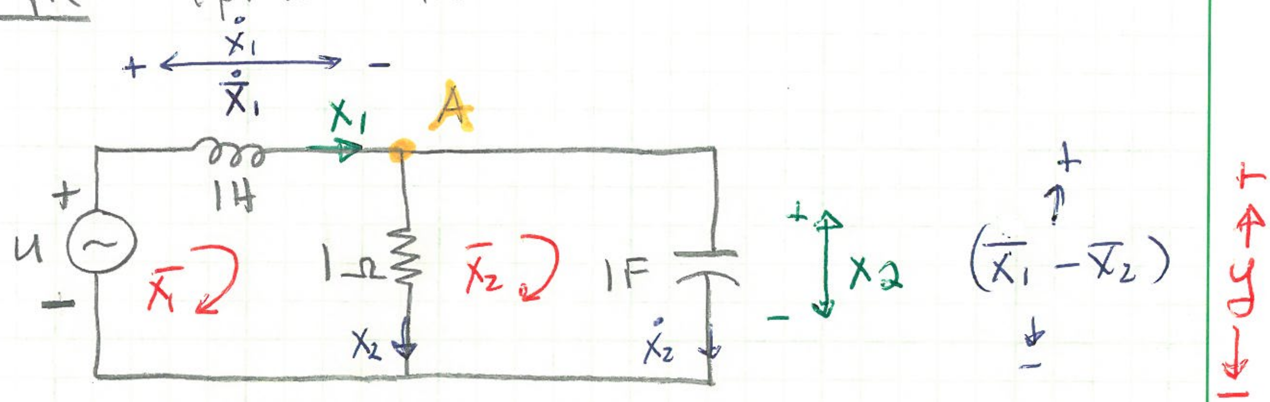
$$\bar{C} = CP^{-1}$$

$$\bar{D} = D$$

☆ Algebraic Equivalence $\Rightarrow$ Zero-State Equivalence

but not vice versa

Example (p. 93 Chen)



State Variable Set #1

x_1 = inductor current

x_2 = capacitor voltage

Note

$$i = C\frac{dv}{dt} = C\dot{x}_2 = \dot{x}_2$$

$$v = L\frac{di}{dt} = L\dot{x}_1 = \dot{x}_1$$

$$i = \frac{V}{R} = \frac{x_2}{R} = x_2$$

KCL @ A

$$x_1 = x_2 + \dot{x}_2 \quad \text{or} \quad \dot{x}_2 = x_1 - x_2$$

KVL @ LHS

$$\dot{x}_1 + x_2 - u = 0 \quad \text{or} \quad \dot{x}_1 = -x_2 + u$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$

$$y = [\,0 \quad 1\,]\, x$$

State Variable Set #2

$\bar{x}_1$ = loop current LHS

$\bar{x}_2$ = loop current RHS

Note

$$V = L\frac{di}{dt} = L\dot{\bar{x}}_1 = \dot{\bar{x}}_1$$

$$V = iR = (\bar{x}_1 - \bar{x}_2)R = (\bar{x}_1 - \bar{x}_2)$$

KVL @ LHS

$$u = \dot{\bar{x}}_1 + (\bar{x}_1 - \bar{x}_2)$$

or

$$\dot{\bar{x}}_1 = -\bar{x}_1 + \bar{x}_2 + u$$

$$i = C\frac{dv}{dt} = C(\dot{\bar{x}}_1 - \dot{\bar{x}}_2) = \bar{x}_2$$

or

$$\dot{\bar{x}}_2 = \dot{\bar{x}}_1 - \bar{x}_2 = -\bar{x}_1 + u$$

$$\Rightarrow \begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [\,1 \quad -1\,]\, \bar{x}$$

Two different equations for same network

Are they equivalent?

Example (continued)

Check if Algebraically Equivalent:

Need to find P s.t. $\boxed{\bar{x} = Px}$

By definition, $\bar{x}_1 = x_1$

Voltage across resistor $= x_2 \cdot R = x_2$

Current through resistor $= x_2 = \bar{x}_1 - \bar{x}_2$

$$\Rightarrow \quad \bar{x}_2 = \bar{x}_1 - x_2 = x_1 - x_2$$

$$\Rightarrow \quad P = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

nonsingular ✓

Check

✓ $\bar{A} = PAP^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$

where $P^{-1} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{-1} \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$

note $P = P^{-1}$

✓ $\bar{B} = PB = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

✓ $\bar{C} = CP^{-1} = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \end{bmatrix}$

✓ $\bar{D} = D = 0$

So, Algebraically Equivalent

Example (Continued)

Check if Zero-State Equivalent

Need to see if $\hat{G}(s) = \hat{\bar{G}}(s)$

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \left[\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \right]^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} s & 1 \\ -1 & s+1 \end{bmatrix}^{-1} = \frac{1}{s(s+1)+1} \begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix}$$

$$= \frac{1}{s^2+s+1} \begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+1}{s^2+s+1} & \frac{-1}{s^2+s+1} \\ \frac{1}{s^2+s+1} & \frac{s}{s^2+s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{s+1}{s^2+s+1} & \frac{-1}{s^2+s+1} \\ \frac{1}{s^2+s+1} & \frac{s}{s^2+s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{s^2+s+1} & \frac{s}{s^2+s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s^2+s+1}$$

$$\hat{\bar{G}}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B} + \bar{D}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} s+1 & -1 \\ 1 & s \end{bmatrix}^{-1} = \frac{1}{s(s+1)+1} \begin{bmatrix} s & 1 \\ -1 & s+1 \end{bmatrix} = \begin{bmatrix} \frac{s}{s^2+s+1} & \frac{1}{s^2+s+1} \\ \frac{-1}{s^2+s+1} & \frac{s+1}{s^2+s+1} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \end{bmatrix} \begin{bmatrix} \frac{s}{s^2+s+1} & \frac{1}{s^2+s+1} \\ \frac{-1}{s^2+s+1} & \frac{s+1}{s^2+s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{s+1}{s^2+s+1} & \frac{-s}{s^2+s+1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{s^2+s+1}$$

So Zero-State Equivalent

Theorem A transfer function $\hat{G}(s)$ can be realized by an LTI state-space system iff $\hat{G}(s)$ is a proper rational function

Concept:

$$\hat{G}(s) = \frac{\text{num}}{\text{den}} = \frac{\text{polynomial in } s}{\text{polynomial in } s}$$

If degree (den) $\geq$ degree (num)

Then $\hat{G}(s)$ is a proper rational function.

What does $\hat{G}(s)$ look like in polynomial form?

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

$$M^{-1} = \frac{1}{\det M}\left(\operatorname{adj} M\right)' \quad , \quad \operatorname{adj} M = \left[\operatorname{cof}_{ij} M\right]$$

transpose

Adjoint Matrix of M

ij^{th} cofactor of M = determinant of submatrix of M obtained by removing row i and column j multiplied by $(-1)^{i+j}$

Thus,

$$\hat{G}(s) = \frac{1}{\det(sI-A)} C\left[\operatorname{adj}(sI-A)'\right]B + D$$

denominator := characteristic polynomial of A

roots = eigenvalues of A

Note, if A diagonal, $A = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{bmatrix}$

then $\det(sI-A) = (s-a_1)(s-a_2)(s-a_3)$

roots = eigenvalues for diagonal A

So $\det(sI-A)$ is a polynomial of A of degree n.

$\operatorname{adj}(sI-A)$ will have degree $(n-1)$ or smaller

$C\,[\operatorname{adj}(sI-A)']\,B$ will be a linear combination of polynomials of degree $(n-1)$ or smaller.

$\Rightarrow$ $\hat{G}(s)$ is a proper rational function

Examples

$$\hat{G}(s) = \frac{1}{s^2+5s+2}$$

$$\hat{G}(s) = \frac{s+2}{s^2+2s+4}$$

$$\hat{G}(s) = \frac{s^2-2s-10}{s^2+5s+4}$$

Given $\hat{G}(s)$, Find State-Space Equations

SISO

$$\dot{x} = Ax + Bu$$

$$y = Cx + Du$$

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \hat{G}(s) = \frac{\beta_1 s^{n-1} + \beta_2 s^{n-2} + \cdots + \beta_{n-1} s + \beta_n}{s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} + \cdots + \alpha_{n-1} s + \alpha_n}$$

Controllable Canonical Form:

$$A = \begin{bmatrix} -\alpha_1 & -\alpha_2 & \cdots & -\alpha_{n-1} & -\alpha_n \\ 1 & & & 0 & 0 \\ & 1 & & & 0 \\ & 0 & \ddots & & \vdots \\ & & & 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \\ | \\ 0 \end{bmatrix}$$

$$C = [\beta_1 \ \beta_2 \ \cdots \ \beta_{n-1} \ \beta_n]$$

Observable Canonical Form:

$$\bar{A} = \begin{bmatrix} -\alpha_1 & 1 & & 0 \\ -\alpha_2 & & 1 & \\ \vdots & & & \ddots \\ & & 0 & \ddots \\ -\alpha_n & & & 1 \end{bmatrix} \qquad \bar{B} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{n-1} \\ \beta_n \end{bmatrix}$$

$$\bar{C} = [1 \ 0 \ \cdots \ 0]$$

note $A = \bar{A}'$ $\quad B = \bar{C}'$ $\quad C = \bar{B}'$

Create A, B, C, D by reading coefficients of $\hat{G}(s)$

MIMO

Given $\hat{G}(s)$

Decompose $\hat{G}(s) = C(sI-A)^{-1}B + D$

$$= \underbrace{\hat{G}_{sp}(s)}_{\text{strictly proper}} + \underbrace{\lim_{s\to\infty} \hat{G}(s)}_{\text{constant}}$$

Find monic least common denominator of all entries of $\hat{G}_{sp}(s)$.

monic lcd = monic polynomial (lead coefficient of 1) of smallest order that can be divided by all others

$$d(s) = s^n + \alpha_1 s^{n-1} + \alpha_2 s^{n-2} \dots + \alpha_{n-1} s + \alpha_n$$

Expand $\hat{G}_{sp}(s) = \frac{1}{d(s)}\left[N_1 s^{n-1} + N_2 s^{n-2} + \dots + N_{n-1} s + N_n\right]$

N_i — $m \times k$ matrices

Set

$$A = \begin{bmatrix} -\alpha_1 I_{k\times k} & -\alpha_2 I_{k\times k} & \cdots & -\alpha_n I_{k\times k} \\ I_{k\times k} & & 0 & 0_{k\times k} \\ & \ddots & & \vdots \\ 0 & & I_{k\times k} & 0_{k\times k} \end{bmatrix}_{nk\times nk}$$

$$B = \begin{bmatrix} I_{k\times k} \\ 0_{k\times k} \\ \vdots \\ 0_{k\times k} \end{bmatrix}_{nk\times k} \qquad C = \begin{bmatrix} N_1 & N_2 & \cdots & N_{n-1} & N_n \end{bmatrix}_{m\times nk}$$

This gives a realization of $\hat{G}(s)$.

Chen 4.6

Example Two Zero-State Equivalent Realizations (but not Algebraically Equivalent)

1) Given $\hat{G}(s) = \begin{bmatrix} \frac{4s-10}{2s+1} & \frac{3}{s+2} \\ \frac{1}{(2s+1)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$

Rewrite as constant matrix + strictly proper rational matrix

$$\hat{G}(s) = \lim_{s\to\infty} \hat{G}(s) + \hat{G}_{sp}(s)$$

$$= \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \frac{-6}{s+1/2} & \frac{3}{s+2} \\ \frac{1/2}{(s+1/2)(s+2)} & \frac{s+1}{(s+2)^2} \end{bmatrix}$$

$$\frac{-12}{2s+1} = \frac{-6}{s+1/2}$$

$$\lim_{s\to\infty}\left(\frac{4s-10}{2s+1}\right) = \lim_{s\to\infty}\left(\frac{4s}{2s}\right) = 2$$

$$\frac{4s-10}{2s+1} = \frac{2(2s+1)}{2s+1} - \frac{12}{2s+1}$$

$$d(s) = (s+1/2)(s+2)^2 = s^3 + 4.5s^2 + 6s + 2$$

$$\hat{G}_{sp}(s) = \frac{1}{s^3+4.5s^2+6s+2}\begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+1/2) \\ \frac{1}{2}(s+2) & (s+1)(s+1/2) \end{bmatrix}$$

$$= \frac{1}{d(s)}\begin{bmatrix} (-6s^2 - 24s - 24) & (3s^2 + 7.5s + 3) \\ (1/2\, s + 1) & (s^2 + 1.5s + 1/2) \end{bmatrix}$$

$$= \frac{1}{d(s)}\left(\underbrace{\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix}}_{N_1} s^2 + \underbrace{\begin{bmatrix} -24 & 7.5 \\ 1/2 & 1.5 \end{bmatrix}}_{N_2} s + \underbrace{\begin{bmatrix} -24 & 3 \\ 1 & 1/2 \end{bmatrix}}_{N_3} \right)$$

The realization is

(1)

$$\dot{x} = \left[\begin{array}{cc|cc|cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hline 1 & 0 & & & & \\ 0 & 1 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c|}{0} & 1 & 0 & \multicolumn{2}{c}{0} \\ & & 0 & 1 & & \end{array}\right] x + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hline \multicolumn{2}{c}{0} \\ \hline \multicolumn{2}{c}{0} \end{array}\right] u$$

$$y = \left[\begin{array}{cc|cc|cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 1/2 & 1.5 & 1 & 1/2 \end{array}\right] x + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} u$$

Chap 4.7

Example (Continued)

2) Look at 1^{st} column of $\hat{G}(s)$:

$$\hat{G}_1(s) = \begin{bmatrix} \frac{4s-10}{2s+1} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{(4s-10)(s+2)}{(2s+1)(s+2)} \\ \frac{1}{(2s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \frac{4s^2-2s+20}{2s^2+5s+2} \\ \frac{1}{2s^2+5s+2} \end{bmatrix}$$

Matlab $n_1 = [4, -2, -20; 0, 0, 1]$

$d_1 = [2, 5, 2]$

$[A_1, B_1, C_1, D_1] = tf2ss(n_1, d_1)$

$$\Rightarrow \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \\ y_1 = C_1 x_1 + D_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix} x_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1 \end{cases}$$

Do same for 2^{nd} column:

$$\hat{G}_2(s) = \begin{bmatrix} \frac{3}{s+2} \\ \frac{s+1}{(s+2)^2} \end{bmatrix}$$ & use tf2ss

$$\Rightarrow \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2 \\ y_2 = C_2 x_2 + D_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix} x_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2 \end{cases}$$

Combine

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y = [C_1 \quad C_2]\, x + [D_1 \quad D_2]\, u$$

$$(2) \begin{cases} \dot{x} = \left[\begin{array}{cc|cc} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{array}\right] x + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} u \\ y = \left[\begin{array}{cc|cc} -6 & -12 & 3 & 6 \\ 0 & 1/2 & 1 & 1 \end{array}\right] x + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} u \end{cases}$$

Why?

★ BUT not algebraically equivalent

(1) & (2) Zero-State Equivalent b/c same transfer function