Homework 5 - EE 547 (PMP) - Winter 2015

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Initialization

```
function hw5()
```

```
close all
digits(3);
format shortG
set(0, 'defaultTextInterpreter', 'latex');
numerical_precision = 1e-9;
syms s xdot x x_1 x_2 x_3 x_4 x_5 u_1 u_2 u_3 u_4 u_5 u t t_0
```

Problem 1 (Controllability and Observability)

```
x = [x_1; x_2; x_3; x_4; x_5];
u = [u_1; u_2; u_3; u_4; u_5];
A = [-9 -31 -51 -40 -12;
     1 0 0 0 0;
     0 1 0 0 0;
     0 0 1 0 0;
     0 0 0 1 0];
B = [1; 0; 0; 0; 0];
C = [95 92 -3 60 -72];
D = 0;
render_latex(['\dot{x} = ' latex(sym(A)) '\mathbf{x} + ' latex(sym(B)) 'u'], 12, 1
render_latex(['y = ' latex(sym(B)) '\mathbf{x} + ' latex(sym(D)) 'u'], 12, 1.3)
```

$$\dot{x} = \begin{bmatrix} -9 & -31 & -51 & -40 & -12 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \mathbf{x} + 0u$$

System is asymptotically stable if $\Re\{\lambda_i\} < 0$ for all eigenvalues

```
if all(real(eig(A)) < 0)
    disp('System is Asymptotically stable')
else
    disp('System is Not Asymptotically stable')
end
```

System is Asymptotically stable

```
cm = ctrb(A, B);
om = obsv(A, C);
render_latex(['\mathcal{C} = ' latex(sym(cm))], 12, 1.3)
render_latex(['\mathcal{O} = ' latex(sym(om))], 12, 1.3)
```

$$\mathcal{C} = \begin{bmatrix} 1 & -9 & 50 & -222 & 867 \\ 0 & 1 & -9 & 50 & -222 \\ 0 & 0 & 1 & -9 & 50 \\ 0 & 0 & 0 & 1 & -9 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} 95 & 92 & -3 & 60 & -72 \\ -763 & -2948 & -4785 & -3872 & -1140 \\ 3919 & 18868 & 35041 & 29380 & 9156 \\ -16403 & -86448 & -170489 & -147604 & -47028 \\ 61179 & 338004 & 688949 & 609092 & 196836 \end{bmatrix}$$

The system is controllable and observable if the rank (linearly independant columns) of the controllability and observability matrices are at least equal to n, respectively.

```
n = size(A, 1);
if rank(cm) >= n
    disp('System is controllable')
else
    disp('System is not controllable')
end

if rank(om) >= n
    disp('System is observable')
else
    disp('System is not observable')
end
```

System is controllable
System is observable

Utilize the Matlab function gram

```
sys = ss(A, B, C, D);
Wc = gram(sys, 'c')
Wo = gram(sys, 'o')
```

```
Wc =

    0.072894  -2.9387e-18   -0.0032176   1.4403e-18    0.0006713
 -2.9387e-18    0.0032176  -8.4337e-19   -0.0006713   1.2181e-18
  -0.0032176  -8.4337e-19    0.0006713   2.8571e-20  -0.00043981
  1.4403e-18   -0.0006713   2.8571e-20   0.00043981   1.1539e-18
   0.0006713   1.2181e-18  -0.00043981   1.1539e-18    0.0013657


Wo =

      678.82       1596.8       1227.7        683.3          216
      1596.8        25447        45270        27386        16930
      1227.7        45270        92387        62607        32482
       683.3        27386        62607        51653        25532
         216        16930        32482        25532        21160
```

Implement the minimum energy control input, given by

$$u(t) = \mathbf{B}^T \exp\left(\mathbf{A}^T(t_1 - t)\right) W_c^{-1} \left[\exp\left(\mathbf{A}t_1\right)x_0 - x_1\right]$$

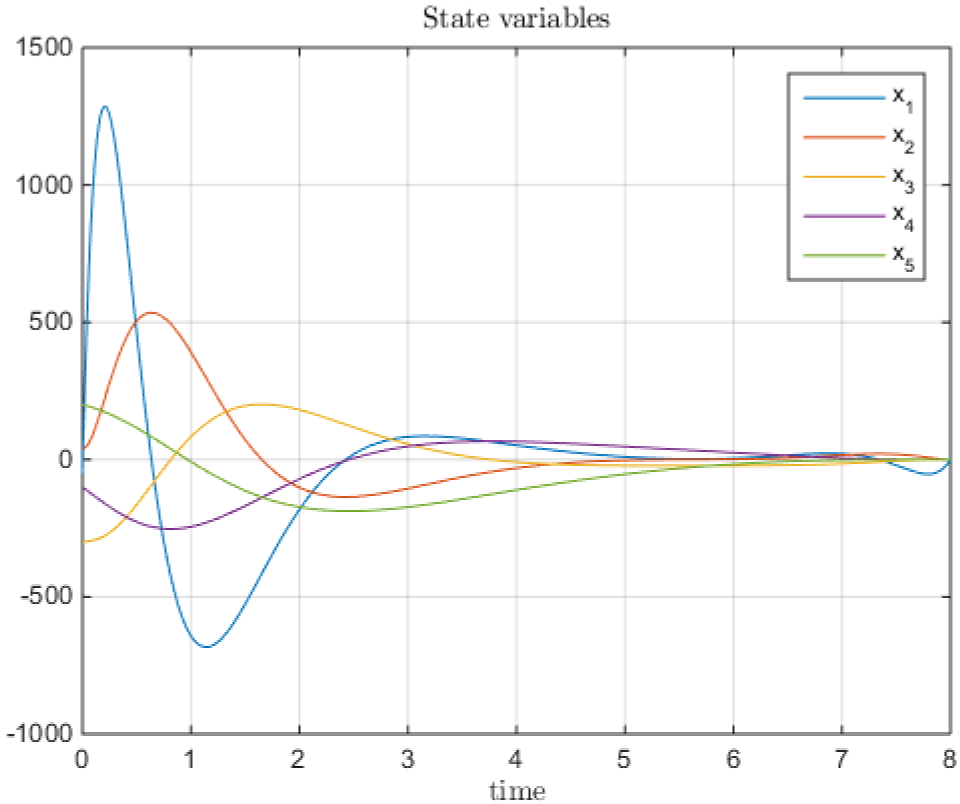
```
x0 = [-50; 40; -300; -100; 200];
x1 = zeros(5, 1);
t1 = 8;
syms t
u = vpa(-B'*expm(A'*(t1-t))*inv(Wc)*(expm(A*t1)*x0 - x1), 4);
render_latex(['u = ' latex(u)], 11, 0.5)
```

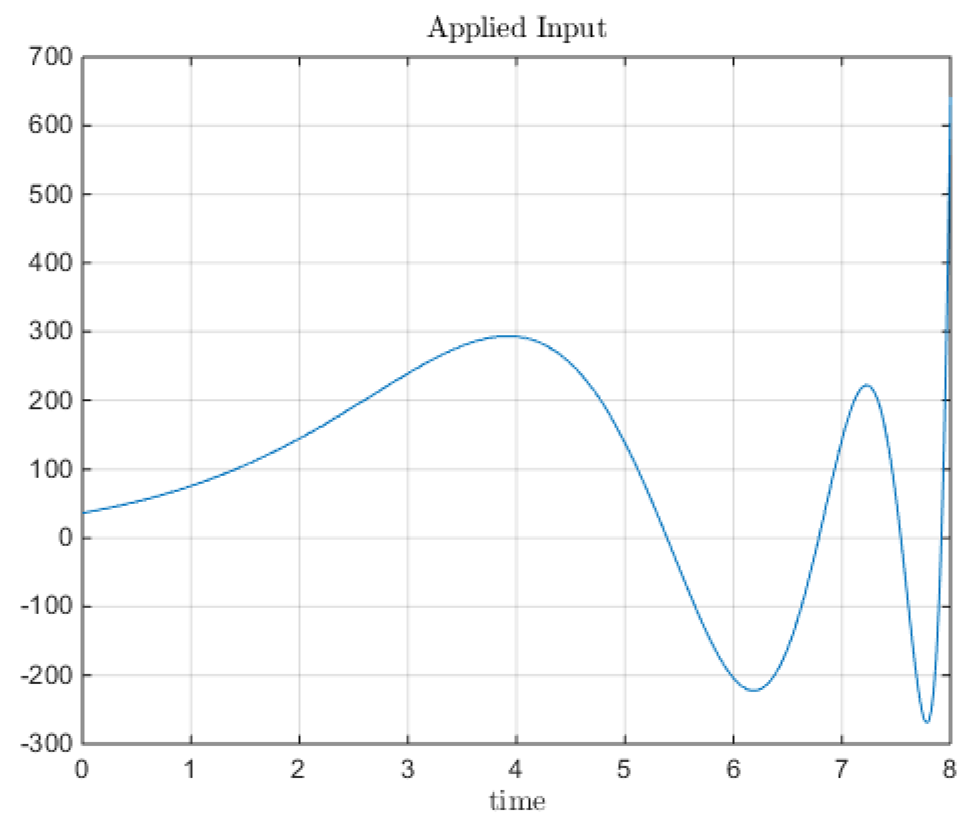
$$u = \left(1.34 \cdot 10^{6}\right)\, \mathrm{e}^{2.0\,t-16.0} + \left(1.03 \cdot 10^{5}\right)\, \mathrm{e}^{3.0\,t-24.0} + \left(1.08 \cdot 10^{5}\right)\, \mathrm{e}^{t-8.0} - \left(1.68 \cdot 10^{5}\right)\, t\,\mathrm{e}^{2.0\,t-16.0} - \left(2.61 \cdot 10^{4}\right)\, t\,\mathrm{e}^{t-8.0}$$

```
tspan = 0:0.01:t1;
u = eval(subs(u, t, tspan));
[y, t, x] = lsim(ss(sys), u, tspan, x0);

figure,
plot(t, x)
grid on
title('State variables')
xlabel('time')
legend('x_1', 'x_2', 'x_3', 'x_4', 'x_5')

figure
grid on
plot(t, u)
title('Applied Input')
xlabel('time')
```


State variables
1500
1000
500
0
-500
-1000
0
1
2
3
4
5
6
7
8
time
x_1
x_2
x_3
x_4
x_5


Applied Input
700
600
500
400
300
200
100
0
-100
-200
-300
0
1
2
3
4
5
6
7
8
time

Problem 2 (Lyapunov Controllability Test)

```
x = [x_1; x_2; x_3];
u = [u_1; u_2; u_3];
A = [-9 9 -5;
     7 -9 7;
     2 2 -6];
B = eye(3);
C = eye(3);
D = zeros(3);
render_latex(['\dot{x} = ' latex(sym(A)) '\mathbf{x} + ' latex(sym(B)) 'u'], 12, 0
render_latex(['y = ' latex(sym(B)) '\mathbf{x} + ' latex(sym(D)) 'u'], 12, 0.8)
```

$$\dot{x} = \begin{bmatrix} -9 & 9 & -5 \\ 7 & -9 & 7 \\ 2 & 2 & -6 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} u$$

System is asymptotically stable if $\Re\{\lambda_i\} < 0$ for all eigenvalues

```
if all(real(eig(A)) < 0)
    disp('System is Asymptotically stable')
else
    disp('System is Not Asymptotically stable')
end
```

```
System is Asymptotically stable
```

System is controllable if there does not exist a positive definite solution to the Lyapnuov equation, given by

$$\mathbf{A}\mathbf{W_c} + \mathbf{W_c}\mathbf{A}^T = -\mathbf{B}\mathbf{B}^T$$

```
try
    R = lyapchol(A, B);
    Wc = R*R';
    if all(eig(Wc)) > 0
        disp('There does exist a solution to the Lyapunov equation. The system is
    else
        disp('There does not exist a solution to the Lyapunov equation. The system
    end
catch err
    disp('There does not exist a solution to the Lyapunov equation. The system is
end
```

```
There does exist a solution to the Lyapunov equation. The system is controllable.
```

The controllability matrix is defined as

$\mathcal{C} = \left[\mathbf{B}, \mathbf{AB}, \mathbf{A}^2\mathbf{B}, \cdots, \mathbf{A}^{n-1}\mathbf{B}\right]$ The obsevability matrix is defined as

$$\mathcal{C} = \left[\mathbf{C}, \mathbf{CA}, \mathbf{CA}^2, \cdots, \mathbf{CA}^{n-1}\right]^T$$

```
cm = [];
for n = 1:size(A,1)
    cm = [cm, A^(n - 1)*B];
end

om = [];
for n = 1:size(A,1)
    om = [om; C*A^(n - 1)];
end
render_latex(['\mathcal{C} = ' latex(sym(cm))], 12, 1)
render_latex(['\mathcal{O} = ' latex(sym(om))], 12, 2)
```

$$\mathcal{C} = \begin{bmatrix} 1 & 0 & 0 & -9 & 9 & -5 & 134 & -172 & 138 \\ 0 & 1 & 0 & 7 & -9 & 7 & -112 & 158 & -140 \\ 0 & 0 & 1 & 2 & 2 & -6 & -16 & -12 & 40 \end{bmatrix}$$

$$\mathcal{O} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -9 & 9 & -5 \\ 7 & -9 & 7 \\ 2 & 2 & -6 \\ 134 & -172 & 138 \\ -112 & 158 & -140 \\ -16 & -12 & 40 \end{bmatrix}$$

The system is controllable and observable if the rank (linearly independant columns) of the controllability and observability matrices are at least equal to n, respectively.

```
n = size(A, 1);
if rank(cm) >= n
    disp('System is controllable')
else
    disp('System is not controllable')
end

if rank(om) >= n
    disp('System is observable')
else
    disp('System is not observable')
end
```

```
System is controllable
System is observable
```

Utilize the Matlab function gram

```
sys = ss(A, B, C, D);
Wc = gram(sys, 'c')
Wo = gram(sys, 'o')


Wc =

       1.0582        1.45     0.80529
         1.45       2.119       1.203
      0.80529       1.203     0.75275


Wo =

       1.5385      1.6978     0.73077
       1.6978      1.9396     0.83791
      0.73077     0.83791     0.45192
```

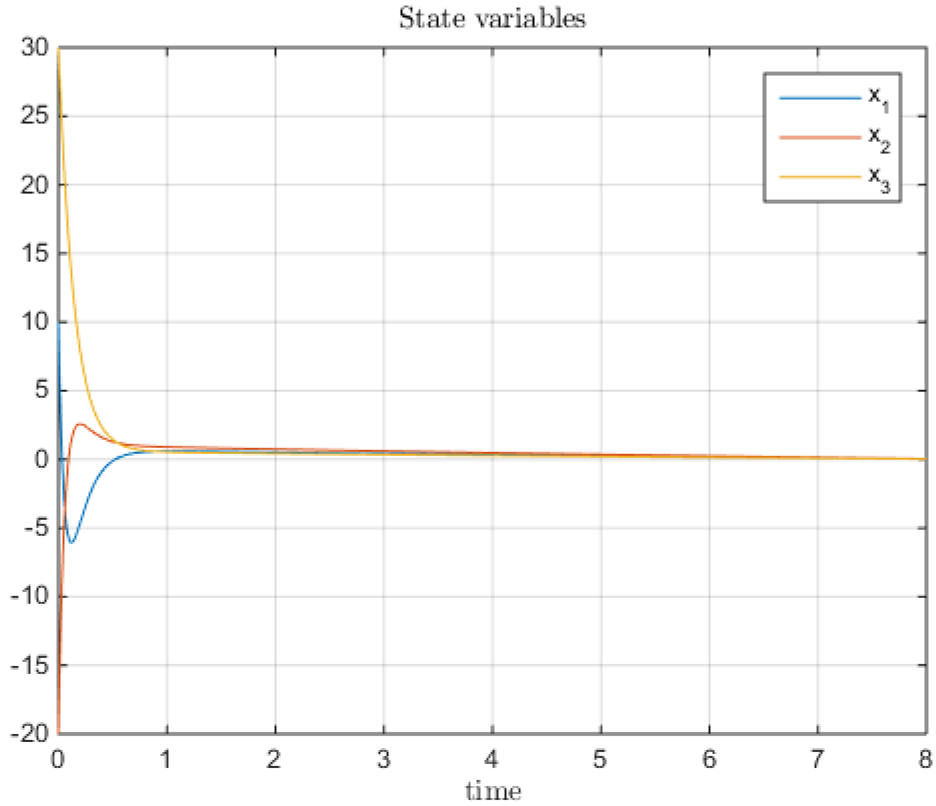
Implement the minimum energy control input, given by

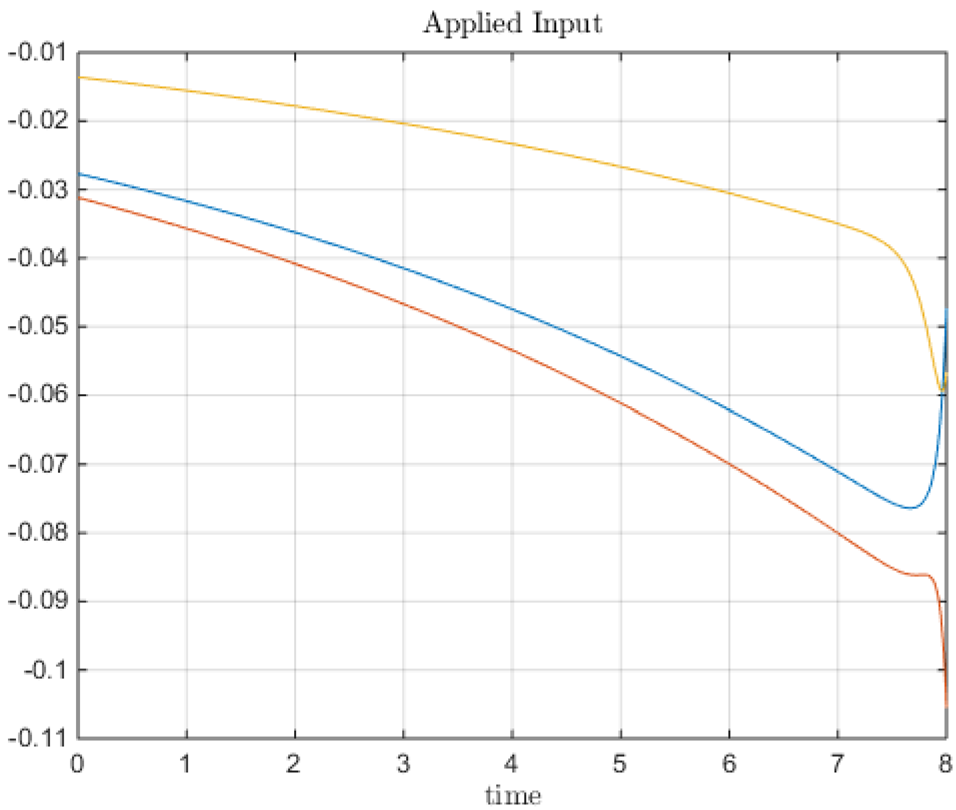
$$u(t) = \mathbf{B}^T \exp\left(\mathbf{A}^T(t_1 - t)\right) W_c^{-1} \left[\exp\left(\mathbf{A}t_1\right)x_0 - x_1\right]$$

```
x0 = [10; -20; 30];
x1 = zeros(3, 1);
t1 = 8;
syms tau
% Wc = int(expm(A*(t1-tau))*(B*B')*expm(A'*(t1-tau)), tau, 0, t1);
t = 0:0.01:t1;
u = zeros(length(t), n);
for i = 1:length(t)
    u(i, :) = -B'*expm(A'*(t1-t(i)))*inv(Wc)*(expm(A*t1)*x0 - x1);
end


[y, tout, x] = lsim(ss(sys), u, t, x0);
% lsim(ss(sys), u, t, x0);
figure,
plot(tout, x)
grid on
title('State variables')
xlabel('time')
legend('x_1', 'x_2', 'x_3')

figure
grid on
plot(tout, u)
title('Applied Input')
xlabel('time')
```


State variables
x1
x2
x3
time


Applied Input
time

```
close all
```

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