Class 1

Lecture 1

- Fundamental Concepts
- Terminology
- Block Diagrams
- State Space Representation of a Linear System

Lecture 2

- Linearization
- Mechanical Systems & Examples

Outline

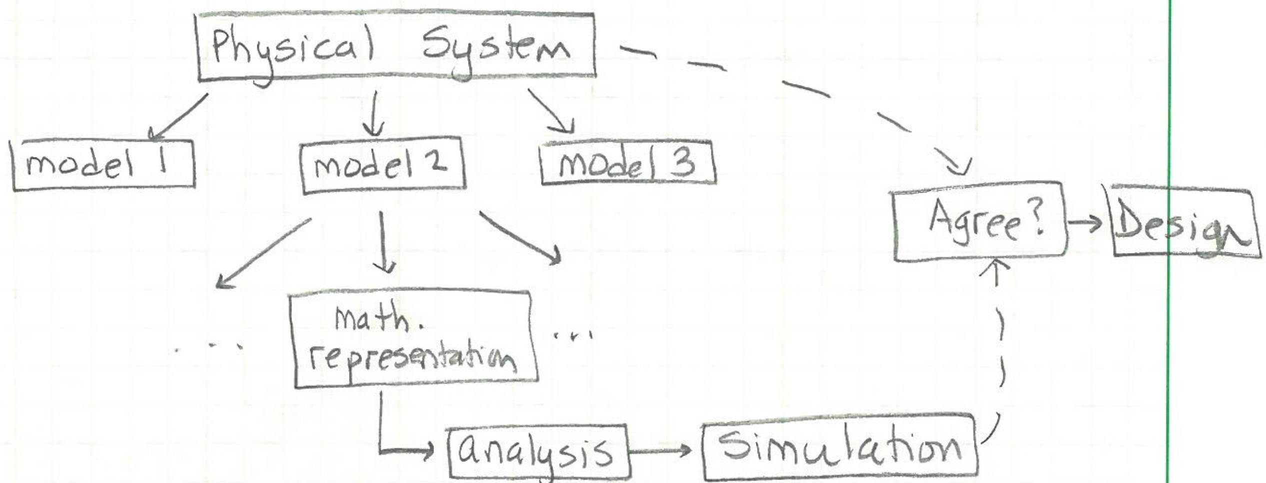
Fundamental Concepts

Linear Systems - continuous & discrete time

Terminology / Definitions

Block Diagrams & Interconnections

Fundamental Concepts



Physical System: e.g., satellite, robot, vehicle, circuit, airplane

Model: eg, model satellite as a particle, rigid body or elastic body

☆ Selecting a model is a difficult but important step!
– close to physical system but simple enough to analyze analytically

Mathematical Representation (of the model):
e.g., use KVL, KCL, Newton's Laws, Euler-Lagrange equations to write equations.

Analysis: quantitative – response of system to certain inputs $u \to \boxed{} \to y$
vs
qualitative – general properties such as
controllability
observability
stability
robustness

The Study of Systems: consists of four parts

Modeling (we assume these are given)
Setting up mathematical equations
Analysis
Design

Definition: Linear Systems Theory is the study of the properties, capabilities & limitations of representations of linear models of physical systems.

Example: RLC Circuit
Model #1: model passive elements as LTI elements (passive: consumes, but doesn't produce energy; elements: R, L, C)
Model #2: take into account nonlinearities of elements.

⇒ Each gives own state equation for the system

Linear Systems (Continuous Time)

Every linear system can be described as the input-output

$$u \longrightarrow \boxed{\text{System}} \longrightarrow y$$

$$y(t) = \int_{t_0}^{t} G(t,\tau)\, u(\tau)\, d\tau$$

[will see later
G is impulse response
matrix of system]

If system has finite number of state variables, then

$$\text{LTV}\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) & \text{state equation} \quad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k \\ y(t) = C(t)\,x(t) + D(t)\,u(t) & \text{output equation} \quad y \in \mathbb{R}^m \end{cases}$$

$u : [0, \infty) \to \mathbb{R}^k$ input

$x : [0, \infty) \to \mathbb{R}^n$ state

$y : [0, \infty) \to \mathbb{R}^m$ output

These equations are referred to as "state-space"

If system is Time-Invariant,

$$\text{LTI}\left\{\begin{array}{l} y(t) = \int_0^t G(t-\tau)\, u(\tau)\, d\tau \\ \begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) \\ y(t) = C\,x(t) + D\,u(t) \end{cases} \end{array}\right\} \text{time domain}$$

shorthand $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$

Recalling Laplace Transform, we also get

$$\hat{y}(s) = \hat{G}(s)\,\hat{u}(s) \quad \Big\} \text{ frequency domain}$$

[will see later
$\hat{G}(s)$ is the
transfer function]

Linear Systems (Discrete Time)

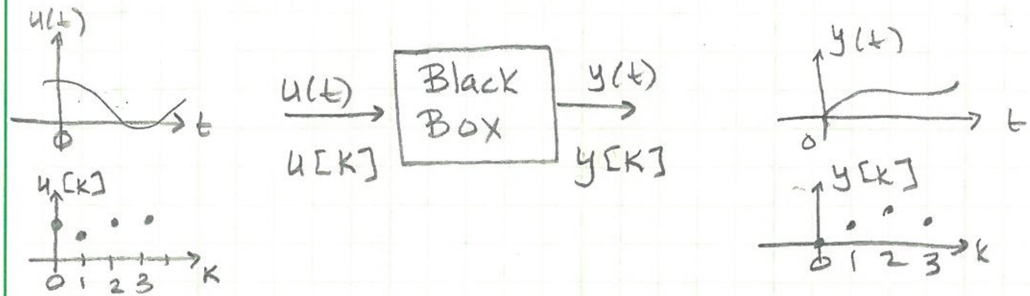
$$x(t+1) = A(t)\,x(t) + B(t)\,u(t) \qquad x \in \mathbb{R}^n,\ u \in \mathbb{R}^k$$

$$y(t) = C(t)\,x(t) + D(t)\,u(t) \qquad y \in \mathbb{R}^m$$

Domain of signals on $\mathbb{N} = \{0, 1, 2, \dots\}$

shorthand: $\begin{cases} x^+ = Ax + Bu \\ y = Cx + Du \end{cases}$

Terminology



input $\longrightarrow$ unique response = output

SISO - single-input, single-output

MIMO - multi-input, multi-output

SIMO - single-input, multi-output

Continuous Time system: $u(t) \rightsquigarrow y(t)$ $t \in \mathbb{R}$

Discrete Time system: $u[k] \rightsquigarrow y[k]$ k Integer

Memoryless system: $y(t_0)$ depends only on $u(t_0)$
independent of input applied before or after t_0

System with Memory: $y(t_0)$ depends on $u(t)$ for
$t < t_0$, $t = t_0$ and $t > t_0$
i.e., past current future inputs

Causal System: $y(t_0)$ depends on $u(t)$ for
$t < t_0$ & $t = t_0$ ONLY
past current

* No physical system can be non-causal, i.e., predict future inputs $\Rightarrow$ causality is necessary condition in order to build/implement system in real world

Difficulty: $u(t)$ for $t = -\infty$ to $t = t_0$ affects $y(t_0)$
BUT - tracking $u(t)$ from $-\infty$ to t_0 is hard if not impossible

Solution: Concept of a state

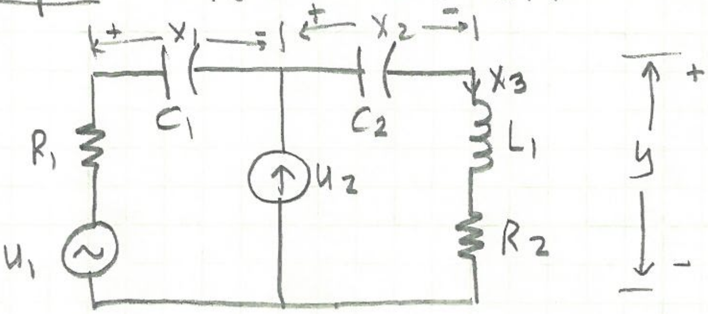
Terminology (continued)

Definition: The <u>state</u> $x(t_0)$ of a system at time t_0 is the information at t_0, that together with the input $u(t)$ for $t \geq t_0$, determines <u>uniquely</u> the output $y(t)$ $\forall$ $t \geq t_0$.

$\Rightarrow$ if we know $x(t_0)$, no need to know $u(t)$ applied before t_0 to determine $y(t)$ after t_0.

$\Rightarrow$ state summarizes effect of past inputs on future outputs.

<u>Example</u> RLC Circuit



If we know $x_1(t_0)$ voltage, $x_2(t_0)$ voltage, $x_3(t_0)$ current

Then $\forall$ $u(t)$ $t \geq t_0$ we can uniquely determine $y(t)$ $t \geq t_0$

$$x(t_0) = \begin{bmatrix} x_1(t_0) \\ x_2(t_0) \\ x_3(t_0) \end{bmatrix}$$ state variables

Initial state = initial conditions

$$\left.\begin{matrix} x(t_0) \\ u(t), t \geq t_0 \end{matrix}\right\} \rightarrow y(t) \quad t \geq t_0$$

Lumped System: # state variables is <u>finite</u>

Distributed System: ∞ number of state variables
e.g., transmission line

Block Diagrams & Interconnections

Compact way to represent system

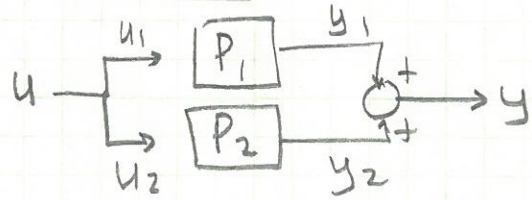
Single Block $\quad u \rightarrow \boxed{P} \rightarrow y$

Let LTI Systems

$$P_1 : \begin{cases} \dot{x}_1 = A_1 x_1 + B_1 u_1 & x \in \mathbb{R}^{n_1},\ u \in \mathbb{R}^{k_1} \\ y_1 = C_1 x_1 + D_1 u_1 & y \in \mathbb{R}^{m_1} \end{cases}$$

$$P_2 : \begin{cases} \dot{x}_2 = A_2 x_2 + B_2 u_2 & x \in \mathbb{R}^{n_2},\ u \in \mathbb{R}^{k_2} \\ y_2 = C_2 x_2 + D_2 u_2 & y \in \mathbb{R}^{m_2} \end{cases}$$

Parallel Interconnection



$u = u_1 = u_2 \qquad\qquad y = y_1 + y_2$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u \qquad x \in \mathbb{R}^{n_1+n_2}$$

$$y = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + (D_1 + D_2) u$$

Cascade Interconnection

$$u \xrightarrow{u_1} \boxed{P_1} \xrightarrow[y_1]{u_2} \boxed{P_2} \xrightarrow[y_2]{} y$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \qquad x \in \mathbb{R}^{n_1+n_2}$$

$$y = \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 D_1 u$$

ICP Derive This (5 minutes)

Solution: $y = y_2 = C_2 x_2 + D_2 u_2$

$u_2 = y_1 = C_1 x_1 + D_1 u_1$

$$\Rightarrow y = C_2 x_2 + D_2 (C_1 x_1 + D_1 u_1) = D_2 C_1 x_1 + C_2 x_2 + D_2 D_1 u_1$$

$$= \begin{bmatrix} D_2 C_1 & C_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 D_1 u_1$$

$$u = u_1 \Rightarrow y = \begin{bmatrix} D_2 C_1 & C_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + D_2 D_1 u \ \checkmark$$

Interconnections (continued)

ICP (continued)

$$\dot{x}_1 = A_1 x_1 + B_1 u_1$$

$$u_1 = u \Rightarrow \dot{x}_1 = A_1 x_1 + B_1 u \quad \checkmark$$

$$\dot{x}_2 = A_2 x_2 + B_2 \underbrace{u_2}$$

$$u_2 = y_1 = C_1 x_1 + D_1 u_1$$

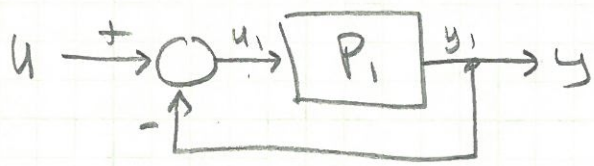
$$\Rightarrow \dot{x}_2 = A_2 x_2 + B_2 (C_1 x_1 + D_1 u_1)$$

$$= B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u_1$$

$$u_1 = u \Rightarrow \dot{x}_2 = B_2 C_1 x_1 + A_2 x_2 + B_2 D_1 u \quad \checkmark$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A_1 & 0 \\ B_2 C_1 & A_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 D_1 \end{bmatrix} u \quad \checkmark$$

Negative Feedback Interconnection



$$u_1 = u - y_1$$
$$y = y_1$$

ICP Derive LTI Equations (10 min)

Solution:

$$y = y_1 = C_1 x_1 + D_1 \underbrace{u_1}$$

$$u_1 = u - y_1 = u - y$$

$$\Rightarrow y = C_1 x_1 + D_1 (u - y) = C_1 x_1 + D_1 u - D_1 y$$

$$y + D_1 y = C_1 x_1 + D_1 u$$

$$(I + D_1) y = C_1 x_1 + D_1 u$$

$$y = (I + D_1)^{-1} C_1 x_1 + (I + D_1)^{-1} D_1 u \quad \checkmark$$

Interconnections (continued)

ICP (continued)

$$\dot{x}_1 = A_1 x_1 + B_1 \underbrace{u_1}$$

$$u_1 = u - y_1 = u - y$$

$$\Rightarrow \quad \dot{x}_1 = A_1 x_1 + B_1 (u - y)$$

$$= A_1 x_1 + B_1 u - B_1 \underbrace{y}$$

$$y = (I + D_1)^{-1} C_1 x_1 + (I + D_1)^{-1} D_1 u$$

$$\Rightarrow \dot{x}_1 = A_1 x_1 + B_1 u - B_1 (I + D_1)^{-1} C_1 x_1 - B_1 (I + D_1)^{-1} D_1 u$$

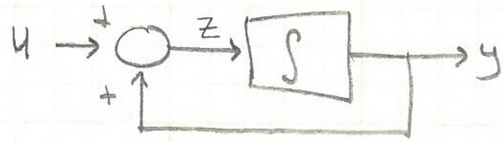
$$= \left(A_1 - B_1 (I + D_1)^{-1} C_1\right) x_1 + B_1 \left(I - (I + D_1)^{-1} D_1\right) u \checkmark$$

System Decomposition

LTI complex equation $\longrightarrow$ simple block diagram

Example Integrator

$$\begin{cases} \dot{x} = x + u \\ y = x \end{cases}$$



input: $z = \dot{x} = x + u = y + u$

Example LTI

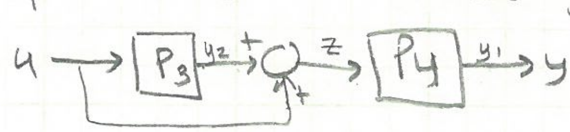
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 5 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \quad P_3 : \begin{cases} \dot{x}_2 = 3x_2 + 5u \\ y_2 = x_2 \end{cases}$$

$$P_4 : \begin{cases} \dot{x}_1 = x + x_2 + u = x_1 + z \\ y_1 = y = x_1 \end{cases}$$

Represent as 2 cascade systems



Let $y_2 = x_2$

$z = y_2 + u = x_2 + u$

$$\Rightarrow \quad P_3 : u \to y_2 \qquad P_4 : z \to y$$

two systems

Outline

Linearization

Example: Nonlinear system → LTI System

Mechanical Systems

Examples

Goal : Write mathematical equation

Linearize and/or approximations

State-space form (LTI System)

Linearization

Nonlinear system: $\begin{cases} \dot{x}(t) = f(x(t), u(t)) \\ y(t) = g(x(t), u(t)) \end{cases}$

Some NL systems can be approximated by linear equations

Linearization around an Equilibrium Point

Define: $(x^{eq}, u^{eq}) \in \mathbb{R}^n \times \mathbb{R}^k$ is an equilibrium point of the NL system above if $f(x^{eq}, u^{eq}) = 0$

i.e., $\left.\begin{aligned} u(t) &= u^{eq} \\ x(t) &= x^{eq} \\ y(t) &= y^{eq} = g(x^{eq}, u^{eq}) \end{aligned}\right\}$ Solution to NL system equations

Suppose we perturb the input and initial condition:

$$x(0) = x^{eq} + \delta x^{eq}$$

$$u(t) = u^{eq} + \delta u(t)$$

Where $(\delta x^{eq})^i$ and $(\delta u(t))^i$ $i>0$ are very small compared to δx^{eq} and $\delta u(t)$.

How much are $x(t)$ and $y(t)$ perturbed by $\delta u(\cdot)$ & δx^{eq}?

Let $\delta x(t) = x(t) - x^{eq}$

$\delta y(t) = y(t) - y^{eq}$

Then

$$\delta \dot{x} = \dot{x} = f(x,u) = f(x^{eq} + \delta x, u^{eq} + \delta u)$$

$$\delta y = g(x,u) - y^{eq} = g(x^{eq} + \delta x, u^{eq} + \delta u) - g(x^{eq}, u^{eq})$$

Expand $f(\cdot)$ and $g(\cdot)$ as Taylor Series around (x^{eq}, u^{eq})

$$\delta \dot{x} = \underbrace{\cancel{f(x^{eq}, u^{eq})}}_{0} + \left.\frac{\partial f}{\partial x}\right|_{x^{eq}, u^{eq}} \delta x + \left.\frac{\partial f}{\partial u}\right|_{x^{eq}, u^{eq}} \delta u + \text{H.O.T.}$$

$$\delta y = \underbrace{\cancel{g(x^{eq}, u^{eq})}}_{0} + \left.\frac{\partial g}{\partial x}\right|_{x^{eq}, u^{eq}} \delta x + \left.\frac{\partial g}{\partial u}\right|_{x^{eq}, u^{eq}} \delta u + \text{H.O.T.}$$

Linearization (continued)

where
$$\left.\frac{\partial f}{\partial x}\right|_{x^{eq},u^{eq}} = \left[\left(\frac{\partial f_i(x,u)}{\partial x_j}\right)_{ij}\right]\Bigg|_{x^{eq},u^{eq}} \in \mathbb{R}^{n\times n} \quad \text{"Jacobian"}$$

$n \times n$ Matrix

$$\left.\frac{\partial f}{\partial u}\right|_{x^{eq},u^{eq}} = \left[\left(\frac{\partial f_i(x,u)}{\partial u_j}\right)_{ij}\right]\Bigg|_{x^{eq},u^{eq}} \in \mathbb{R}^{n\times k}$$

expanding,

$$\left.\frac{\partial f}{\partial x}\right|_{x^{eq},u^{eq}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right|_{x^{eq},u^{eq}} = A(t) \in \mathbb{R}^{n\times n}$$

$$\left.\frac{\partial f}{\partial u}\right|_{x^{eq},u^{eq}} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_k} \\ \vdots & & & \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_k} \end{bmatrix}\right|_{x^{eq},u^{eq}} = B(t) \in \mathbb{R}^{n\times k}$$

The linearized equations around the equilibrium point (x^{eq}, u^{eq}) are then

$$\left.\begin{aligned} \delta \dot{x}(t) &= A(t)\,\delta x(t) + B(t)\,\delta u(t) \\ &\text{and similarly,} \\ \delta y(t) &= C(t)\,\delta x(t) + D(t)\,\delta u(t) \end{aligned}\right\} \text{Linearized Equations}$$

* See p. 14 text for Linearization around a trajectory

- similar process

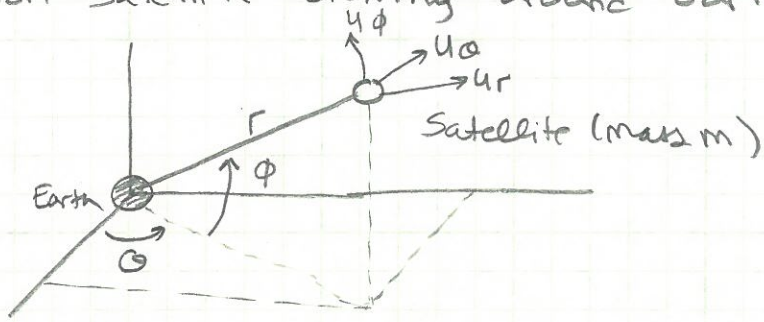
Note $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \vdots \\ \dot{x}_n \end{bmatrix} := \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ \vdots \\ f_n \end{bmatrix}$

* do $\frac{\partial f_i}{\partial x_j}$ above on all $i = 1, \dots n$ and $j = 1, \dots, n$ to get $n \times n$ Jacobian matrix

i.e., $\dot{x}_1(t) = f_1(x(t), u(t))$ nonlinear function

Example (Linearization)

Communication satellite orbiting around Earth



$$x(t) = \begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \\ \phi(t) \\ \dot{\phi}(t) \end{bmatrix} \qquad u(t) = \begin{bmatrix} u_r(t) \\ u_\theta(t) \\ u_\phi(t) \end{bmatrix} \qquad y(t) = \begin{bmatrix} r(t) \\ \theta(t) \\ \phi(t) \end{bmatrix}$$

state   orthogonal thrusts = controller   altitude

$$\dot{x}(t) = f(x,u) = \begin{bmatrix} \dot{r} \\ r\dot{\theta}^2\cos^2\phi + r\dot{\phi}^2 - K/r^2 + u_r/m \\ \dot{\theta} \\ -2\dot{r}\dot{\theta}/r + 2\dot{\theta}\dot{\phi}\sin\phi/\cos\phi + u_\theta/mr\cos\phi \\ \dot{\phi} \\ -\dot{\theta}^2\cos\phi\sin\phi - 2\dot{r}\dot{\phi}/r + u_\phi/mr \end{bmatrix}$$

A solution corresponds to the satellite being in a circular equatorial orbit:

$$x_0(t) = [\, r_0 \quad 0 \quad w_0 t \quad w_0 \quad 0 \quad 0 \,]'$$

$$u_0(t) = 0$$

where radius r_0 & angular velocity w_0 are s.t. $r_0^3 w_0^2 = K$ constant

Satellite will deviate from (x_0, u_0) due to disturbances.

Of interest to consider linearized equations around (x_0, u_0),

Use Jacobians to find linear equations

By definition $y = Cx = \left[\begin{array}{cccc|cc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right] x$

Example (continued)

$$B = \left.\frac{\partial f}{\partial u}\right|_{x_0,u_0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \frac{\partial f_1}{\partial u_3} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \frac{\partial f_2}{\partial u_3} \\ \vdots & & \\ \frac{\partial f_6}{\partial u_1} & \frac{\partial f_6}{\partial u_2} & \frac{\partial f_6}{\partial u_3} \end{bmatrix}\right|_{x_0,u_0} = \begin{bmatrix} 0 & 0 & 0 \\ 1/m & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1/mr_0 & 0 \\ \hline 0 & 0 & 0 \\ 0 & 0 & 1/mr_0 \end{bmatrix}$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{x_0,u_0} = \left.\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_6} \\ \vdots & & & \\ \vdots & & & \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdots & \frac{\partial f_6}{\partial x_6} \end{bmatrix}\right|_{x_0,u_0}$$

$$= \left[\begin{array}{cccc|cc} 0 & 1 & 0 & 0 & 0 & 0 \\ 3w_0^2 & 0 & 0 & 2w_0r_0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -2w_0/r_0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -w_0^2 & 0 \end{array}\right]$$

Calculations

$x_1 = r$

$$\left.\frac{\partial f_2}{\partial x_1}\right|_{x_0,u_0} = \left.\dot\theta^2\cos^2\phi + \phi^2 + 2K/r^3\right|_{x_0,u_0} = w_0^2\cos 0 + \frac{2\cancel{r_0^3}w_0^2}{\cancel{r_0^3}} = 3w_0^2$$

$$\dot\theta = w_0,\ \phi = 0,\ r = r_0,\ K = r_0^3 w_0^2$$

$$\left.\frac{\partial f_2}{\partial x_4}\right|_{x_0,u_0} = \left.\frac{\partial f_2}{\partial \dot\theta}\right|_{x_0,u_0} = \left.2r\dot\theta\cos^2\phi\right|_{x_0,u_0} = 2r_0w_0$$

ICP

Derive $\left.\frac{\partial f_4}{\partial x_2}\right|_{x_0,u_0} = \left.\frac{\partial f_4}{\partial \dot r}\right|_{x_0,u_0} = \left.-\frac{2\dot\theta}{r}\right|_{x_0,u_0} = -\frac{2w_0}{r_0}$

$$\left.\frac{\partial f_6}{\partial x_5}\right|_{x_0,u_0} = \left.\frac{\partial f_6}{\partial \phi}\right|_{x_0,u_0} = \left.-\dot\theta^2\frac{\partial}{\partial\phi}(\cos\phi\sin\phi)\right|_{x_0,u_0} = -w_0^2$$

<u>Example</u> (continued)

$$\begin{cases} x(t) = x_0(t) + \delta x(t) \\ u(t) = u_0(t) + \delta u(t) \\ y(t) = y_0 + \delta y(t) \end{cases}$$

Small perturbations

$\Rightarrow$ Linearized equations

$$\begin{cases} \delta \dot{x} = A\,\delta x + B\,\delta u \\ \delta y = C\,\delta x \end{cases}$$

with A, B, C as derived above

Note - can be decoupled into 2 subsystems
$\{r, \dot{r}, \theta, \dot{\theta}\}$ and $\{\phi, \dot{\phi}\}$

Mechanical Systems

$$M(q)\,\ddot{q} + B(q,\dot{q})\,\dot{q} + G(q) = F \qquad \text{equation of motion}$$

where

$q \in \mathbb{R}^k$ generalized coordinates vector

$M(q)$ $k \times k$ positive-definite matrix called the mass matrix

Note A symmetric matrix M is PD if

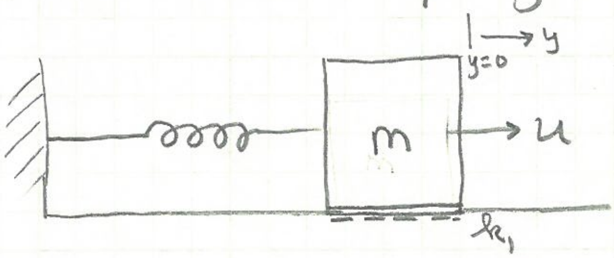
$$x' M x > 0 \quad \forall\, x \neq 0$$

$F \in \mathbb{R}^k$ applied forces / torques

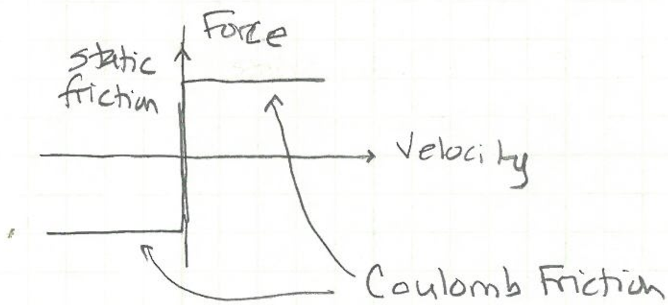
$G(q) \in \mathbb{R}^k$ conservative forces (e.g., gravity, spring forces)

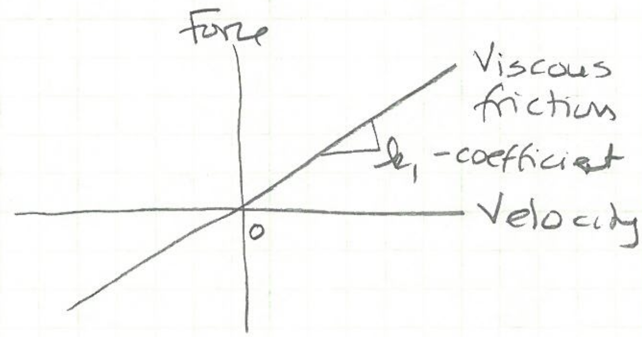
$B(q,\dot{q}) \in \mathbb{R}^{k \times k}$ centrifugal / Coriolis / friction matrix

Example 1 (Mass-Spring) (Chen 2.5)



output $= y =$ displacement from equilibrium

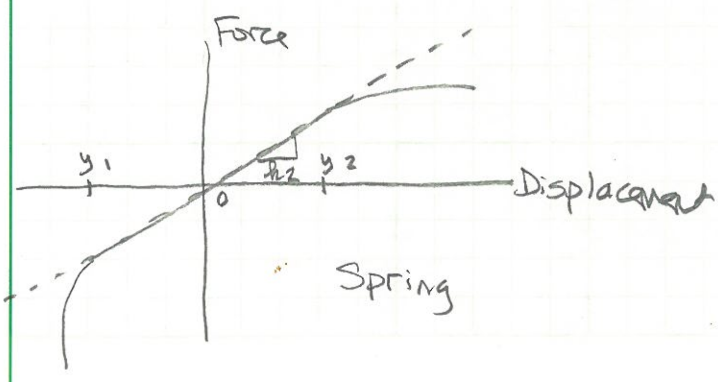




$\frac{dy}{dt} =$ velocity

Static + Coulomb friction are NL $\Rightarrow$ disregard

Viscous friction $= k_1 \dot{y}(t)$



Spring displacement $\in (y_1, y_2)$ is approx. linear

Spring force $= k_2 y$

Example 1 (continued)

Model mechanical system as linear under linearization & simplification:

Newton's Law: $F = ma$

applied force, u, must overcome viscous friction & spring force & also accelerate the block

$$\underset{\substack{\uparrow \\ M(q)}}{m}\,\underset{\substack{\uparrow \\ \ddot{q}}}{\ddot{y}} = \underset{\substack{\uparrow \\ F}}{u} - \underset{\substack{\uparrow \\ B(q,\dot{q})\dot{q}}}{k_1 \dot{y}} - \underset{\substack{\uparrow \\ G(q)}}{k_2 y}$$

Apply Laplace transform & assume zero i.c.

$$m s^2 \hat{y}(s) = \hat{u}(s) - k_1 s \hat{y}(s) - k_2 \hat{y}(s)$$

$$\Rightarrow \quad \hat{y}(s) = \underbrace{\frac{1}{ms^2 + k_1 s + k_2}}_{\hat{G}(s) \text{ transfer function}} \hat{u}(s)$$

State Space Equations:

Let $x_1 = y$

$x_2 = \dot{y}$

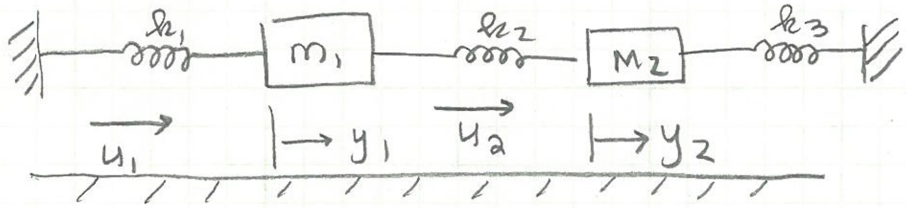
Than $\dot{x}_1 = x_2$

$$\dot{x}_2 = \frac{1}{m}(u - k_1 x_2 - k_2 x_1)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_2/m & -k_1/m \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/m \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

Example 2 (Spring- Mass System) (Chen 2.5)



Vibration example

Assume – no friction between blocks & floor

Applied force u_1 must overcome spring forces & accelerate block

$$\Rightarrow \quad u_1 - k_1 y_1 - k_2(y_1 - y_2) = m_1 \ddot{y}_1$$

or

$$m_1 \ddot{y}_1 + (k_1 + k_2) y_1 - k_2 y_2 = u_1$$

For 2nd block,

$$m_2 \ddot{y}_2 - k_2 y_1 + (k_1 + k_2) y_2 = u_2$$

$$\Rightarrow \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{y}_1 \\ \ddot{y}_2 \end{bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

"normal form" standard vibration equation

Define $x_1 = y_1 \qquad x_3 = y_2$

$x_2 = \dot{y}_1 \qquad x_4 = \dot{y}_2$ state variables

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -(k_1+k_2)/m_1 & 0 & k_2/m_1 & 0 \\ 0 & 0 & 0 & 1 \\ k_2/m_2 & 0 & -(k_1+k_2)/m_1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/m_1 & 0 \\ 0 & 0 \\ 1 & 1/m_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$y := \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x$$

Laplace Transform: (zero i.c.)

$$m_1 s^2 \hat{y}_1(s) + (k_1 + k_2)\hat{y}_1(s) - k_2 \hat{y}_2(s) = \hat{u}_1(s)$$

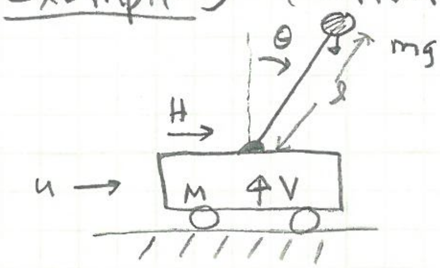
$$m_2 s^2 \hat{y}_2(s) - k_2 \hat{y}_1(s) + (k_1 + k_2)\hat{y}_2(s) = \hat{u}_2(s)$$

Transfer Function

$$\Rightarrow \begin{bmatrix} \hat{y}_1(s) \\ \hat{y}_2(s) \end{bmatrix} = \begin{bmatrix} \dfrac{m_2 s^2 + k_1 + k_2}{d(s)} & k_2/d(s) \\ k_2/d(s) & \dfrac{m_1 s^2 + k_1 + k_2}{d(s)} \end{bmatrix} \begin{bmatrix} \hat{u}_1(s) \\ \hat{u}_2(s) \end{bmatrix}$$

$$d(s) = (m_1 s^2 + k_1 + k_2)(m_2 s^2 + k_1 + k_2) - k_2^2$$

Example 3 (Inverted Pendulum on a Cart) (Chen 2.5)



Assume cart & pendulum move in one plane.

Disregard friction, mass of stick, gust of wind

Problem: maintain pendulum at vertical position

Simple model for a space vehicle at take-off

Let H = horizontal force, V = vertical force } exerted by cart on pendulum

Apply Newton's Law to linear movements:

$$\begin{cases} M\ddot{y} = u - H \\ H = m\frac{d^2}{dt^2}(y + l\sin\theta) = m\ddot{y} + ml\ddot{\theta}\cos\theta - ml\dot{\theta}^2\sin\theta \\ mg - V = m\frac{d^2}{dt^2}(l\cos\theta) = ml(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta) \end{cases}$$

Apply Newton's Law to rotational movement of pendulum around hinge:

$$mgl\sin\theta = ml\ddot{\theta}\cdot l + m\ddot{y}l\cos\theta$$

Nonlinear equations.

Assume $\theta, \dot{\theta}$ small (want to keep pendulum vertical)

$$\Rightarrow \quad \sin\theta = \theta, \quad \cos\theta = 1$$

Drop H.O.T., e.g., $\theta^2, \dot{\theta}^2, \theta\dot{\theta}, \theta\ddot{\theta}$

$$\Rightarrow \begin{cases} H = m\ddot{y} + ml\ddot{\theta} \\ V = mg \\ M\ddot{y} = u - m\ddot{y} - ml\ddot{\theta} \\ g\theta = l\ddot{\theta} + \ddot{y} \end{cases}$$

} linear movement

} rotational movement

$\Rightarrow$ rewrite

$$\begin{cases} M\ddot{y} = u - mg\theta \\ Ml\ddot{\theta} = (M+m)g\theta - u \end{cases}$$

} Linearized Equations

Example 3 (Continued)

Laplace Transform (zero i.c.)

$$M s^2 \hat{y}(s) = \hat{u}(s) - mg\,\hat{\theta}(s) \qquad \text{eqt 1}$$

$$M\ell s^2 \hat{\theta}(s) = (M+m)\,g\,\hat{\theta}(s) - \hat{u}(s) \qquad \text{eqt 2}$$

Transfer Functions

$u \to y$:

$$y = \frac{u - mg\theta}{Ms^2} \qquad \text{from eq. 1}$$

$$\theta = \frac{-u}{M\ell s^2 - (M+m)g} \qquad \text{from eq 2}$$

eq. 2 → 1

$$y = \frac{u - mg\left(\frac{-u}{M\ell s^2 - (M+m)g}\right)}{Ms^2}$$

$$= \frac{u}{Ms^2}\left(1 + \frac{mg}{M\ell s^2 - (M+m)g}\right)$$

$$= \frac{u}{Ms^2}\left(\frac{M\ell s^2 - Mg}{M\ell s^2 - (M+m)g}\right)$$

$$\hat{y}(s) = \frac{\ell s^2 - g}{s^2\left(M\ell s^2 - (M+m)g\right)} \cdot \hat{u}(s)$$

$u \to \theta$:

$$\theta = \frac{-u}{M\ell s^2 - (M+m)g}$$

$$\hat{\theta}(s) = \frac{-1}{M\ell s^2 - (M+m)g} \cdot \hat{u}(s)$$

State Space

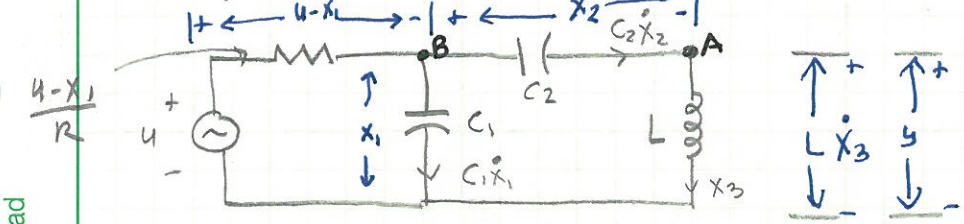
Let $x_1 = y$ $\quad x_3 = \theta$

$\quad x_2 = \dot{y}$ $\quad x_4 = \dot{\theta}$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -mg/M & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/M \\ 0 \\ -1/M\ell \end{bmatrix} u$$

for small $\theta, \dot{\theta}$

Example 4 (RLC Circuit) (Chen 2.5)



Set States as capacitor voltages & inductor current:

$x_1 = C_1$ capacitor voltage

$x_2 = C_2$ capacitor voltage — with polarities as shown

$x_3 =$ inductor current

$\Rightarrow$ currents are $C_1\dot{x}_1$ & $C_2\dot{x}_2$

voltage is $L\dot{x}_3$

$\Rightarrow$ voltage across resistor R is $u - x_1$

$V = IR \Rightarrow$ current through R is $\frac{u - x_1}{R}$

Kirchoff's Current Law (KCL)

Sum of currents flowing into a node = sum flowing out

KCL @ node A:

$$\frac{u - x_1}{R} = C_1\dot{x}_1 + \underbrace{C_2\dot{x}_2}_{= x_3} = C_1\dot{x}_1 + x_3$$

$$\Rightarrow \begin{cases} \dot{x}_1 = \frac{x_1}{RC_1} - \frac{x_3}{C_1} + \frac{u}{RC_1} \\ \dot{x}_2 = \frac{x_3}{C_2} \end{cases}$$

Kirchoff's Voltage Law (KVL)

sum of voltages around any closed circuit is zero

KVL on right-hand-side loop

$$L\dot{x}_3 = x_1 - x_2$$

$$\Rightarrow \dot{x}_3 = \frac{x_1}{L} - \frac{x_2}{L}$$

Example 4 (continued)

Output $y = L\dot{x}_3 = x_1 - x_2$

State space form:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -1/RC_1 & 0 & -1/C_1 \\ 0 & 0 & 1/C_2 \\ 1/L & -1/L & 0 \end{bmatrix} x + \begin{bmatrix} 1/RC_1 \\ 0 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & -1 & 0 \end{bmatrix} x + 0 \cdot u$$

Review

(see Handout #1)

Physical System $\longrightarrow$ Model $\longrightarrow$ Mathematical Equations
given given derive
"representation of system"

State Space Representation

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases}$$

Linearization around an equilibrium point

Given NL system

$$\begin{cases} \dot{x}(t) = f(x(t), u(t)) & x \in \mathbb{R}^n,\ u \in \mathbb{R}^k \\ y(t) = g(x(t), u(t)) \end{cases}$$

s.t.
$$\dot{x}(t) = \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(x(t), u(t)) \\ f_2(x(t), u(t)) \\ \vdots \\ f_n(x(t), u(t)) \end{bmatrix}$$

Given x^{eq}, u^{eq} st $f(x^{eq}, u^{eq}) = 0$ equilibrium point

Derive Jacobian matrices

$$A = \frac{\partial f}{\partial x}\bigg|_{x^{eq}, u^{eq}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\Bigg|_{x^{eq}, u^{eq}} \quad n \times n \text{ matrix}$$

$$B = \frac{\partial f}{\partial u}\bigg|_{x^{eq}, u^{eq}}, \quad C = \frac{\partial g}{\partial x}\bigg|_{x^{eq}, u^{eq}}, \quad D = \frac{\partial g}{\partial u}\bigg|_{x^{eq}, u^{eq}}$$

To get linear approximation of NL system around x^{eq}, u^{eq}:

$$\begin{cases} \delta\dot{x} = A\,\delta x + B\,\delta u \\ y = C\,\delta x + D\,\delta u \end{cases}$$

Review of Class 1 (Continued)

(See Handout #2)

Concept of State (Kailath 2.2.3)

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & x \in \mathbb{R}^n \quad u \in \mathbb{R}^q \\ y(t) = Cx(t) + Du(t) & y \in \mathbb{R}^m \end{cases}$$

$$u \longrightarrow \boxed{\begin{matrix} A, B, \\ C, D \;\; x \end{matrix}} \longrightarrow y$$

$x(t)$ describes evolution of basic internal variables

If we know $x(t_0)$ for any given $t = t_0$,
Then given $u(t)$ for $t \geq t_0$, we can calculate all present & future values of $y(t)$

Note, if we were given $u(t)$ for all time $-\infty < t$ we could calculate $y(t)$ $\forall t$

BUT We may only start looking at the system at $t = 0$, and we may not know the input $u(t)$ for $-\infty < t < t_0$

SO $x(t_0)$ gives sufficient information to be able to calculate the future response $t \geq t_0$ to a new input $u(t)$, $t \geq t_0$, without worrying about what $u(t)$ was in the past.

Note – more than one past input can lead to $x(t_0)$, so $x(t_0)$ is a minimal sufficient statistic for the system, containing just enough information to calculate future responses

State vector gives knowledge of state/condition of system