Chapter 4

State-Space Solutions and Realizations

4.1 Introduction

We showed in Chapter 2 that linear systems can be described by convolutions and, if lumped, by state-space equations. This chapter discusses how to find their solutions. First we discuss briefly how to compute solutions of the input-output description. There is no simple analytical way of computing the convolution

$$y(t) = \int_{\tau=t_0}^{t} g(t, \tau)u(\tau)\, d\tau$$

The easiest way is to compute it numerically on a digital computer. Before doing so, the equation must be discretized. One way is to discretize it as

$$y(k\Delta) = \sum_{m=k_0}^{k} g(k\Delta, m\Delta)u(m\Delta)\Delta \tag{4.1}$$

where Δ is called the integration step size. This is basically the discrete convolution discussed in (2.34). This discretization is the easiest but yields the least accurate result for the same integration step size. For other integration methods, see, for example, Reference [17].

For the linear time-invariant (LTI) case, we can also use $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$ to compute the solution. If a system is distributed, $\hat{g}(s)$ will not be a rational function of s. Except for some special cases, it is simpler to compute the solution directly in the time domain as in (4.1). If the system is lumped, $\hat{g}(s)$ will be a rational function of s. In this case, if the Laplace transform of $u(t)$ is also a rational function of s, then the solution can be obtained by taking the inverse Laplace transform of $\hat{g}(s)\hat{u}(s)$. This method requires computing poles, carrying out partial fraction expansion, and then using a Laplace transform table. These can be carried out using the MATLAB functions `roots` and `residue`. However, when there are repeated poles, the computation may become very sensitive to small changes in the data, including roundoff errors; therefore computing solutions using the Laplace transform is not a viable method on digital computers. A better method is to transform transfer functions into state-space equations and then compute the solutions. This chapter discusses solutions of state equations, how to transform transfer functions into state equations, and other related topics. We discuss first the time-invariant case and then the time-varying case.

4.2 Solution of LTI State Equations

Consider the linear time-invariant (LTI) state-space equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.2}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.3}$$

where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, and $\mathbf{D}$ are, respectively, $n \times n$, $n \times p$, $q \times n$, and $q \times p$ constant matrices. The problem is to find the solution excited by the initial state $\mathbf{x}(0)$ and the input $\mathbf{u}(t)$. The solution hinges on the exponential function of $\mathbf{A}$ studied in Section 3.6. In particular, we need the property in (3.55) or

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

to develop the solution. Premultiplying $e^{-\mathbf{A}t}$ on both sides of (4.2) yields

$$e^{-\mathbf{A}t}\dot{\mathbf{x}}(t) - e^{-\mathbf{A}t}\mathbf{A}\mathbf{x}(t) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

which implies

$$\frac{d}{dt}\left(e^{-\mathbf{A}t}\mathbf{x}(t)\right) = e^{-\mathbf{A}t}\mathbf{B}\mathbf{u}(t)$$

Its integration from 0 to t yields

$$e^{-\mathbf{A}\tau}\mathbf{x}(\tau)\Big|_{\tau=0}^{t} = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

Thus we have

$$e^{-\mathbf{A}t}\mathbf{x}(t) - e^{\mathbf{0}}\mathbf{x}(0) = \int_0^t e^{-\mathbf{A}\tau}\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{4.4}$$

Because the inverse of $e^{-\mathbf{A}t}$ is $e^{\mathbf{A}t}$ and $e^{\mathbf{0}} = \mathbf{I}$ as discussed in (3.54) and (3.52), (4.4) implies

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{4.5}$$

This is the solution of (4.2).

It is instructive to verify that (4.5) is the solution of (4.2). To verify this, we must show that (4.5) satisfies (4.2) and the initial condition $\mathbf{x}(t) = \mathbf{x}(0)$ at $t = 0$. Indeed, at $t = 0$, (4.5) reduces to

$$\mathbf{x}(0) = e^{\mathbf{A}\cdot 0}\mathbf{x}(0) = e^{\mathbf{0}}\mathbf{x}(0) = \mathbf{I}\mathbf{x}(0) = \mathbf{x}(0)$$

Thus (4.5) satisfies the initial condition. We need the equation

$$\frac{\partial}{\partial t}\int_{t_0}^{t} f(t,\tau)\,d\tau = \int_{t_0}^{t}\left(\frac{\partial}{\partial t} f(t,\tau)\right)d\tau + f(t,\tau)|_{\tau=t} \tag{4.6}$$

to show that (4.5) satisfies (4.2). Differentiating (4.5) and using (4.6), we obtain

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \frac{d}{dt}\left[e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau\right] \\ &= \mathbf{A}e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t \mathbf{A}e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\big|_{\tau=t} \\ &= \mathbf{A}\left(e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau\right) + e^{\mathbf{A}\cdot 0}\mathbf{B}\mathbf{u}(t)\end{aligned}$$

which becomes, after substituting (4.5),

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Thus (4.5) meets (4.2) and the initial condition $\mathbf{x}(0)$ and is the solution of (4.2).

Substituting (4.5) into (4.3) yields the solution of (4.3) as

$$\mathbf{y}(t) = \mathbf{C}e^{\mathbf{A}t}\mathbf{x}(0) + \mathbf{C}\int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau + \mathbf{D}\mathbf{u}(t) \tag{4.7}$$

This solution and (4.5) are computed directly in the time domain. We can also compute the solutions by using the Laplace transform. Applying the Laplace transform to (4.2) and (4.3) yields, as derived in (2.14) and (2.15),

$$\begin{aligned}\hat{\mathbf{x}}(s) &= (s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)] \\ \hat{\mathbf{y}}(s) &= \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}[\mathbf{x}(0) + \mathbf{B}\hat{\mathbf{u}}(s)] + \mathbf{D}\hat{\mathbf{u}}(s)\end{aligned}$$

Once $\hat{\mathbf{x}}(s)$ and $\hat{\mathbf{y}}(s)$ are computed algebraically, their inverse Laplace transforms yield the time-domain solutions.

We now give some remarks concerning the computation of $e^{\mathbf{A}t}$. We discussed in Section 3.6 three methods of computing functions of a matrix. They can all be used to compute $e^{\mathbf{A}t}$:

1. Using Theorem 3.5: First, compute the eigenvalues of $\mathbf{A}$; next, find a polynomial $h(\lambda)$ of degree $n-1$ that equals $e^{\lambda t}$ on the spectrum of $\mathbf{A}$; then $e^{\mathbf{A}t} = h(\mathbf{A})$.
2. Using Jordan form of $\mathbf{A}$: Let $\mathbf{A} = \mathbf{Q}\hat{\mathbf{A}}\mathbf{Q}^{-1}$; then $e^{\mathbf{A}t} = \mathbf{Q}e^{\hat{\mathbf{A}}t}\mathbf{Q}^{-1}$, where $\hat{\mathbf{A}}$ is in Jordan form and $e^{\hat{\mathbf{A}}t}$ can readily be obtained by using (3.48).
3. Using the infinite power series in (3.51): Although the series will not, in general, yield a closed-form solution, it is suitable for computer computation, as discussed following (3.51).

In addition, we can use (3.58) to compute $e^{\mathbf{A}t}$, that is,

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}(s\mathbf{I} - \mathbf{A})^{-1} \tag{4.8}$$

The inverse of $(s\mathbf{I}-\mathbf{A})$ is a function of $\mathbf{A}$; therefore, again, we have many methods to compute it:

1. Taking the inverse of $(s\mathbf{I} - \mathbf{A})$.
2. Using Theorem 3.5.
3. Using $(s\mathbf{I} - \mathbf{A})^{-1} = \mathbf{Q}(s\mathbf{I} - \hat{\mathbf{A}})^{-1}\mathbf{Q}^{-1}$ and (3.49).
4. Using the infinite power series in (3.57).
5. Using the Leverrier algorithm discussed in Problem 3.26.

Example 4.1 We use Methods 1 and 2 to compute $(s\mathbf{I} - \mathbf{A})^{-1}$, where

$$\mathbf{A} = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}$$

Method 1: We use (3.20) to compute

$$(s\mathbf{I} - \mathbf{A})^{-1} = \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s^2+2s+1}\begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$
$$= \begin{bmatrix} (s+2)/(s+1)^2 & -1/(s+1)^2 \\ 1/(s+1)^2 & s/(s+1)^2 \end{bmatrix}$$

Method 2: The eigenvalues of $\mathbf{A}$ are $-1, -1$. Let $h(\lambda) = \beta_0 + \beta_1\lambda$. If $h(\lambda)$ equals $f(\lambda) := (s-\lambda)^{-1}$ on the spectrum of $\mathbf{A}$, then

$$f(-1) = h(-1): \qquad (s+1)^{-1} = \beta_0 - \beta_1$$
$$f'(-1) = h'(-1): \qquad (s+1)^{-2} = \beta_1$$

Thus we have

$$h(\lambda) = [(s+1)^{-1} + (s+1)^{-2}] + (s+1)^{-2}\lambda$$

and

$$(s\mathbf{I} - \mathbf{A})^{-1} = h(\mathbf{A}) = [(s+1)^{-1} + (s+1)^{-2}]\mathbf{I} + (s+1)^{-2}\mathbf{A}$$
$$= \begin{bmatrix} (s+2)/(s+1)^2 & -1/(s+1)^2 \\ 1/(s+1)^2 & s/(s+1)^2 \end{bmatrix}$$

Example 4.2 Consider the equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

Its solution is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0) + \int_0^t e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)\,d\tau$$

The matrix function $e^{\mathbf{A}t}$ is the inverse Laplace transform of $(s\mathbf{I} - \mathbf{A})^{-1}$, which was computed in the preceding example. Thus we have

$$e^{\mathbf{A}t} = \mathcal{L}^{-1}\begin{bmatrix} \dfrac{s+2}{(s+1)^2} & \dfrac{-1}{(s+1)^2} \\ \dfrac{1}{(s+1)^2} & \dfrac{s}{(s+)^2} \end{bmatrix} = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

and

$$\mathbf{x}(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}\mathbf{x}(0) + \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)}u(\tau)\,d\tau \\ \int_0^t [1-(t-\tau)]e^{-(t-\tau)}u(\tau)\,d\tau \end{bmatrix}$$

We discuss a general property of the zero-input response $e^{\mathbf{A}t}\mathbf{x}(0)$. Consider the second matrix in (3.39). Then we have

$$e^{\mathbf{A}t} = \mathbf{Q}\begin{bmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} & t^2e^{\lambda_1 t}/2 & 0 & 0 \\ 0 & e^{\lambda_1 t} & te^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & e^{\lambda_1 t} & 0 & 0 \\ 0 & 0 & 0 & e^{\lambda_1 t} & 0 \\ 0 & 0 & 0 & 0 & e^{\lambda_2 t} \end{bmatrix}\mathbf{Q}^{-1}$$

Every entry of $e^{\mathbf{A}t}$ and, consequently, of the zero-input response is a linear combination of terms $\{e^{\lambda_1 t},\ te^{\lambda_1 t},\ t^2e^{\lambda_1 t},\ e^{\lambda_2 t}\}$. These terms are dictated by the eigenvalues and their indices. In general, if $\mathbf{A}$ has eigenvalue λ_1 with index $\bar{n}_1$, then every entry of $e^{\mathbf{A}t}$ is a linear combination of

$$e^{\lambda_1 t} \quad te^{\lambda_1 t} \quad \cdots \quad t^{\bar{n}_1-1}e^{\lambda_1 t}$$

Every such term is *analytic* in the sense that it is infinitely differentiable and can be expanded in a Taylor series at every t. This is a nice property and will be used in Chapter 6.

If every eigenvalue, simple or repeated, of $\mathbf{A}$ has a negative real part, then every zero-input response will approach zero as $t \to \infty$. If $\mathbf{A}$ has an eigenvalue, simple or repeated, with a positive real part, then most zero-input responses will grow unbounded as $t \to \infty$. If $\mathbf{A}$ has some eigenvalues with zero real part and all with index 1 and the remaining eigenvalues all have negative real parts, then no zero-input response will grow unbounded. However, if the index is 2 or higher, then some zero-input response may become unbounded. For example, if $\mathbf{A}$ has eigenvalue 0 with index 2, then $e^{\mathbf{A}t}$ contains the terms $\{1,\ t\}$. If a zero-input response contains the term t, then it will grow unbounded.

4.2.1 Discretization

Consider the continuous-time state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \tag{4.9}$$

$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \tag{4.10}$$

If the set of equations is to be computed on a digital computer, it must be discretized. Because

$$\dot{\mathbf{x}}(t) = \lim_{T\to 0}\frac{\mathbf{x}(t+T)-\mathbf{x}(t)}{T}$$

we can approximate (4.9) as

$$\mathbf{x}(t+T) = \mathbf{x}(t) + \mathbf{A}\mathbf{x}(t)T + \mathbf{B}\mathbf{u}(t)T \tag{4.11}$$

If we compute $\mathbf{x}(t)$ and $\mathbf{y}(t)$ only at $t = kT$ for $k = 0,\ 1,\ \ldots$, then (4.11) and (4.10) become

$$\mathbf{x}((k+1)T) = (\mathbf{I} + T\mathbf{A})\mathbf{x}(kT) + T\mathbf{B}\mathbf{u}(kT)$$
$$\mathbf{y}(kT) = \mathbf{C}\mathbf{x}(kT) + \mathbf{D}\mathbf{u}(kT)$$

This is a discrete-time state-space equation and can easily be computed on a digital computer. This discretization is the easiest to carry out but yields the least accurate results for the same T. We discuss next a different discretization.

If an input $\mathbf{u}(t)$ is generated by a digital computer followed by a digital-to-analog converter, then $\mathbf{u}(t)$ will be piecewise constant. This situation often arises in computer control of control systems. Let

$$\mathbf{u}(t) = \mathbf{u}(kT) =: \mathbf{u}[k] \qquad \text{for } kT \le t < (k+1)T \tag{4.12}$$

for $k = 0,\ 1,\ 2,\ \ldots$. This input changes values only at discrete-time instants. For this input, the solution of (4.9) still equals (4.5). Computing (4.5) at $t = kT$ and $t = (k+1)T$ yields

$$\mathbf{x}[k] := \mathbf{x}(kT) = e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{4.13}$$

and

$$\mathbf{x}[k+1] := \mathbf{x}((k+1)T) = e^{\mathbf{A}(k+1)T}\mathbf{x}(0) + \int_0^{(k+1)T} e^{\mathbf{A}((k+1)T-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau \tag{4.14}$$

Equation (4.14) can be written as

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\left[e^{\mathbf{A}kT}\mathbf{x}(0) + \int_0^{kT} e^{\mathbf{A}(kT-\tau)}\mathbf{B}\mathbf{u}(\tau)d\tau\right]$$
$$+ \int_{kT}^{(k+1)T} e^{\mathbf{A}(kT+T-\tau)}\mathbf{B}\mathbf{u}(\tau)\,d\tau$$

which becomes, after substituting (4.12) and (4.13) and introducing the new variable $\alpha := kT + T - \tau$,

$$\mathbf{x}[k+1] = e^{\mathbf{A}T}\mathbf{x}[k] + \left(\int_0^{T} e^{\mathbf{A}\alpha}d\alpha\right)\mathbf{B}\mathbf{u}[k]$$

Thus, if an input changes value only at discrete-time instants kT and if we compute only the responses at $t = kT$, then (4.9) and (4.10) become

$$\mathbf{x}[k+1] = \mathbf{A}_d\mathbf{x}[k] + \mathbf{B}_d\mathbf{u}[k] \tag{4.15}$$

$$\mathbf{y}[k] = \mathbf{C}_d\mathbf{x}[k] + \mathbf{D}_d\mathbf{u}[k] \tag{4.16}$$

with

$$\mathbf{A}_d = e^{\mathbf{A}T} \qquad \mathbf{B}_d = \left(\int_0^T e^{\mathbf{A}\tau} d\tau\right)\mathbf{B} \qquad \mathbf{C}_d = \mathbf{C} \qquad \mathbf{D}_d = \mathbf{D} \tag{4.17}$$

This is a discrete-time state-space equation. Note that there is no approximation involved in this derivation and (4.15) yields the exact solution of (4.9) at $t = kT$ if the input is piecewise constant.

We discuss the computation of $\mathbf{B}_d$. Using (3.51), we have

$$\int_0^T \left(\mathbf{I} + \mathbf{A}\tau + \mathbf{A}^2\frac{\tau^2}{2!} + \cdots\right) d\tau$$
$$= T\mathbf{I} + \frac{T^2}{2!}\mathbf{A} + \frac{T^3}{3!}\mathbf{A}^2 + \frac{T^4}{4!}\mathbf{A}^3 + \cdots$$

This power series can be computed recursively as in computing (3.51). If $\mathbf{A}$ is nonsingular, then the series can be written as, using (3.51),

$$\mathbf{A}^{-1}\left(T\mathbf{A} + \frac{T^2}{2!}\mathbf{A}^2 + \frac{T^3}{3!}\mathbf{A}^3 + \cdots + \mathbf{I} - \mathbf{I}\right) = \mathbf{A}^{-1}(e^{\mathbf{A}T} - \mathbf{I})$$

Thus we have

$$\mathbf{B}_d = \mathbf{A}^{-1}(\mathbf{A}_d - \mathbf{I})\mathbf{B} \qquad \text{(if } \mathbf{A} \text{ is nonsingular)} \tag{4.18}$$

Using this formula, we can avoid computing an infinite series.

The MATLAB function `[ad,bd]=c2d(a,b,T)` transforms the continuous-time state equation in (4.9) into the discrete-time state equation in (4.15).

4.2.2 Solution of Discrete-Time Equations

Consider the discrete-time state-space equation

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k] \end{aligned} \tag{4.19}$$

where the subscript d has been dropped. It is understood that if the equation is obtained from a continuous-time equation, then the four matrices must be computed from (4.17). The two equations in (4.19) are algebraic equations. Once $\mathbf{x}[0]$ and $\mathbf{u}[k]$, $k = 0, 1, \ldots$, are given, the response can be computed recursively from the equations.

The MATLAB function `dstep` computes unit-step responses of discrete-time state-space equations. It also computes unit-step responses of discrete transfer functions; internally, it first transforms the transfer function into a discrete-time state-space equation by calling `tf2ss`, which will be discussed later, and then uses `dstep`. The function `dlsim`, an acronym for discrete linear simulation, computes responses excited by any input. The function `step` computes unit-step responses of continuous-time state-space equations. Internally, it first uses the function `c2d` to transform a continuous-time state equation into a discrete-time equation and then carries out the computation. If the function `step` is applied to a continuous-time transfer function, then it first uses `tf2ss` to transform the transfer function into a continuous-time state equation and then discretizes it by using `c2d` and then uses `dstep` to compute the response. Similar remarks apply to `lsim`, which computes responses of continuous-time state equations or transfer functions excited by any input.

In order to discuss the general behavior of discrete-time state equations, we will develop a general form of solutions. We compute

$$\mathbf{x}[1] = \mathbf{A}\mathbf{x}[0] + \mathbf{B}\mathbf{u}[0]$$
$$\mathbf{x}[2] = \mathbf{A}\mathbf{x}[1] + \mathbf{B}\mathbf{u}[1] = \mathbf{A}^2\mathbf{x}[0] + \mathbf{A}\mathbf{B}\mathbf{u}[0] + \mathbf{B}\mathbf{u}[1]$$

Proceeding forward, we can readily obtain, for $k > 0$,

$$\mathbf{x}[k] = \mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m] \tag{4.20}$$

$$\mathbf{y}[k] = \mathbf{C}\mathbf{A}^k\mathbf{x}[0] + \sum_{m=0}^{k-1} \mathbf{C}\mathbf{A}^{k-1-m}\mathbf{B}\mathbf{u}[m] + \mathbf{D}\mathbf{u}[k] \tag{4.21}$$

They are the discrete counterparts of (4.5) and (4.7). Their derivations are considerably simpler than the continuous-time case.

We discuss a general property of the zero-input response $\mathbf{A}^k\mathbf{x}[0]$. Suppose $\mathbf{A}$ has eigenvalue λ_1 with multiplicity 4 and eigenvalue λ_2 with multiplicity 1 and suppose its Jordan form is as shown in the second matrix in (3.39). In other words, λ_1 has index 3 and λ_2 has index 1. Then we have

$$\mathbf{A}^k = \mathbf{Q}\begin{bmatrix} \lambda_1^k & k\lambda_1^{k-1} & k(k-1)\lambda_1^{k-2}/2 & 0 & 0 \\ 0 & \lambda_1^k & k\lambda_1^{k-1} & 0 & 0 \\ 0 & 0 & \lambda_1^k & 0 & 0 \\ 0 & 0 & 0 & \lambda_1^k & 0 \\ 0 & 0 & 0 & 0 & \lambda_2^k \end{bmatrix}\mathbf{Q}^{-1}$$

which implies that every entry of the zero-input response is a linear combination of $\{\lambda_1^k,\ k\lambda_1^{k-1},\ k^2\lambda_1^{k-2},\ \lambda_2^k\}$. These terms are dictated by the eigenvalues and their indices.

If every eigenvalue, simple or repeated, of $\mathbf{A}$ has magnitude less than 1, then every zero-input response will approach zero as $k \to \infty$. If $\mathbf{A}$ has an eigenvalue, simple or repeated, with magnitude larger than 1, then most zero-input responses will grow unbounded as $k \to \infty$. If $\mathbf{A}$ has some eigenvalues with magnitude 1 and all with index 1 and the remaining eigenvalues all have magnitudes less than 1, then no zero-input response will grow unbounded. However, if the index is 2 or higher, then some zero-state response may become unbounded. For example, if $\mathbf{A}$ has eigenvalue 1 with index 2, then $\mathbf{A}^k$ contains the terms $\{1,\ k\}$. If a zero-input response contains the term k, then it will grow unbounded as $k \to \infty$.

4.3 Equivalent State Equations

The example that follows provides a motivation for studying equivalent state equations.

Example 4.3 Consider the network shown in Fig. 4.1. It consists of one capacitor, one inductor, one resistor, and one voltage source. First we select the inductor current x_1 and capacitor voltage x_2 as state variables as shown. The voltage across the inductor is $\dot{\mathbf{x}}_1$ and the current through the capacitor is $\dot{\mathbf{x}}_2$. The voltage across the resistor is x_2; thus its current is $x_2/1 = x_2$. Clearly we have $x_1 = x_2 + \dot{x}_2$ and $\dot{x}_1 + x_2 - u = 0$. Thus the network is described by the following state equation:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$$
$$y = [0 \ \ 1]\mathbf{x} \tag{4.22}$$

If, instead, the loop currents $\bar{x}_1$ and $\bar{x}_2$ are chosen as state variables as shown, then the voltage across the inductor is $\dot{\bar{x}}_1$ and the voltage across the resistor is $(\bar{x}_1 - \bar{x}_2) \cdot 1$. From the left-hand-side loop, we have

$$u = \dot{\bar{x}}_1 + \bar{x}_1 - \bar{x}_2 \quad \text{or} \quad \dot{\bar{x}}_1 = -\bar{x}_1 + \bar{x}_2 + u$$

The voltage across the capacitor is the same as the one across the resistor, which is $\bar{x}_1 - \bar{x}_2$. Thus the current through the capacitor is $\dot{\bar{x}}_1 - \dot{\bar{x}}_2$, which equals $\bar{x}_2$ or

$$\dot{\bar{x}}_2 = \dot{\bar{x}}_1 - \bar{x}_2 = -\bar{x}_1 + u$$

Thus the network is also described by the state equation

$$\begin{bmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \tag{4.23}$$
$$y = [1 \ \ -1]\bar{\mathbf{x}}$$

The state equations in (4.22) and (4.23) describe the same network; therefore they must be closely related. In fact, they are equivalent as will be established shortly.

Consider the n-dimensional state equation

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{4.24}$$

where $\mathbf{A}$ is an $n \times n$ constant matrix mapping an n-dimensional real space $\mathcal{R}^n$ into itself. The state $\mathbf{x}$ is a vector in $\mathcal{R}^n$ for all t; thus the real space is also called the state space. The state equation in (4.24) can be considered to be associated with the orthonormal basis in (3.8). Now we study the effect on the equation by choosing a different basis.

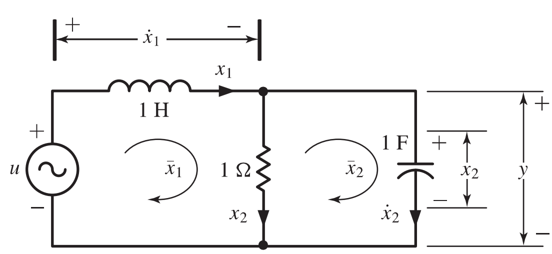


Figure 4.1 Network with two different sets of state variables.

Definition 4.1 *Let* $\mathbf{P}$ *be an* $n \times n$ *real nonsingular matrix and let* $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$. *Then the state equation,*

$$\begin{aligned} \dot{\bar{\mathbf{x}}}(t) &= \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \bar{\mathbf{B}}\mathbf{u}(t) \\ \mathbf{y}(t) &= \bar{\mathbf{C}}\bar{\mathbf{x}}(t) + \bar{\mathbf{D}}\mathbf{u}(t) \end{aligned} \tag{4.25}$$

where

$$\bar{\mathbf{A}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1} \qquad \bar{\mathbf{B}} = \mathbf{P}\mathbf{B} \qquad \bar{\mathbf{C}} = \mathbf{C}\mathbf{P}^{-1} \qquad \bar{\mathbf{D}} = \mathbf{D} \tag{4.26}$$

is said to be (algebraically) equivalent *to (4.24) and* $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ *is called an* equivalence transformation.

Equation (4.26) is obtained from (4.24) by substituting $\mathbf{x}(t) = \mathbf{P}^{-1}\bar{\mathbf{x}}(t)$ and $\dot{\mathbf{x}}(t) = \mathbf{P}^{-1}\dot{\bar{\mathbf{x}}}(t)$. In this substitution, we have changed, as in Equation (3.7), the basis vectors of the state space from the orthonormal basis to the columns of $\mathbf{P}^{-1} =: \mathbf{Q}$. Clearly $\mathbf{A}$ and $\bar{\mathbf{A}}$ are similar and $\bar{\mathbf{A}}$ is simply a different representation of $\mathbf{A}$. To be precise, let $\mathbf{Q} = \mathbf{P}^{-1} = [\mathbf{q}_1 \ \ \mathbf{q}_2 \ \ \cdots \ \ \mathbf{q}_n]$. Then the ith column of $\bar{\mathbf{A}}$ is, as discussed in Section 3.4, the representation of $\mathbf{A}\mathbf{q}_i$ with respect to the basis $\{\mathbf{q}_1, \ \mathbf{q}_2, \ \cdots, \ \mathbf{q}_n\}$. From the equation $\bar{\mathbf{B}} = \mathbf{P}\mathbf{B}$ or $\mathbf{B} = \mathbf{P}^{-1}\bar{\mathbf{B}} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \cdots \ \mathbf{q}_n]\bar{\mathbf{B}}$, we see that the ith column of $\bar{\mathbf{B}}$ is the representation of the ith column of $\mathbf{B}$ with respect to the basis $\{\mathbf{q}_1, \ \mathbf{q}_2, \ \cdots, \ \mathbf{q}_n\}$. The matrix $\bar{\mathbf{C}}$ is to be computed from $\mathbf{C}\mathbf{P}^{-1}$. The matrix $\mathbf{D}$, called the *direct transmission part* between the input and output, has nothing to do with the state space and is not affected by the equivalence transformation.

We show that (4.24) and (4.25) have the same set of eigenvalues and the same transfer matrix. Indeed, we have, using $\det(\mathbf{P})\det(\mathbf{P}^{-1}) = 1$,

$$\begin{aligned} \bar{\Delta}(\lambda) &= \det(\lambda\mathbf{I} - \bar{\mathbf{A}}) = \det(\lambda\mathbf{P}\mathbf{P}^{-1} - \mathbf{P}\mathbf{A}\mathbf{P}^{-1}) = \det[\mathbf{P}(\lambda\mathbf{I} - \mathbf{A})\mathbf{P}^{-1}] \\ &= \det(\mathbf{P})\det(\lambda\mathbf{I} - \mathbf{A})\det(\mathbf{P}^{-1}) = \det(\lambda\mathbf{I} - \mathbf{A}) = \Delta(\lambda) \end{aligned}$$

and

$$\begin{aligned} \hat{\bar{\mathbf{G}}}(s) &= \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}} + \bar{\mathbf{D}} = \mathbf{C}\mathbf{P}^{-1}[\mathbf{P}(s\mathbf{I} - \mathbf{A})\mathbf{P}^{-1}]^{-1}\mathbf{P}\mathbf{B} + \mathbf{D} \\ &= \mathbf{C}\mathbf{P}^{-1}\mathbf{P}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{P}^{-1}\mathbf{P}\mathbf{B} + \mathbf{D} = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \hat{\mathbf{G}}(s) \end{aligned}$$

Thus equivalent state equations have the same characteristic polynomial and, consequently, the same set of eigenvalues and same transfer matrix. In fact, all properties of (4.24) are preserved or invariant under any equivalence transformation.

Consider again the network shown in Fig. 4.1, which can be described by (4.22) and (4.23). We show that the two equations are equivalent. From Fig. 4.1, we have $x_1 = \bar{x}_1$. Because the voltage across the resistor is x_2, its current is $x_2/1$ and equals $\bar{x}_1 - \bar{x}_2$. Thus we have

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

or

$$\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}^{-1} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{4.27}$$

Note that, for this **P**, its inverse happens to equal itself. It is straightforward to verify that (4.22) and (4.23) are related by the equivalence transformation in (4.26).

The MATLAB function `[ab,bb,cb,db]=ss2ss(a,b,c,d,p)` carries out equivalence transformations.

Two state equations are said to be *zero-state equivalent* if they have the same transfer matrix or

$$\mathbf{D} + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \bar{\mathbf{D}} + \bar{\mathbf{C}}(s\mathbf{I} - \bar{\mathbf{A}})^{-1}\bar{\mathbf{B}}$$

This becomes, after substituting (3.57),

$$\mathbf{D} + \mathbf{C}\mathbf{B}s^{-1} + \mathbf{C}\mathbf{A}\mathbf{B}s^{-2} + \mathbf{C}\mathbf{A}^2\mathbf{B}s^{-3} + \cdots = \bar{\mathbf{D}} + \bar{\mathbf{C}}\bar{\mathbf{B}}s^{-1} + \bar{\mathbf{C}}\bar{\mathbf{A}}\bar{\mathbf{B}}s^{-2} + \bar{\mathbf{C}}\bar{\mathbf{A}}^2\bar{\mathbf{B}}s^{-3} + \cdots$$

Thus we have the theorem that follows.

Theorem 4.1

Two linear time-invariant state equations $\{\mathbf{A},\ \mathbf{B},\ \mathbf{C},\ \mathbf{D}\}$ and $\{\bar{\mathbf{A}},\ \bar{\mathbf{B}},\ \bar{\mathbf{C}},\ \bar{\mathbf{D}}\}$ are zero-state equivalent or have the same transfer matrix if and only if $\mathbf{D} = \bar{\mathbf{D}}$ and

$$\mathbf{C}\mathbf{A}^m\mathbf{B} = \bar{\mathbf{C}}\bar{\mathbf{A}}^m\bar{\mathbf{B}} \qquad m = 0,\ 1,\ 2,\ \ldots$$

It is clear that (algebraic) equivalence implies zero-state equivalence. In order for two state equations to be equivalent, they must have the same dimension. This is, however, not the case for zero-state equivalence, as the next example shows.

Example 4.4 Consider the two networks shown in Fig. 4.2. The capacitor is assumed to have capacitance -1 F. Such a negative capacitance can be realized using an op-amp circuit. For the circuit in Fig. 4.2(a), we have $y(t) = 0.5 \cdot u(t)$ or $\hat{y}(s) = 0.5\hat{u}(s)$. Thus its transfer function is 0.5. To compute the transfer function of the network in Fig. 4.2(b), we may assume the initial voltage across the capacitor to be zero. Because of the symmetry of the four resistors, half of the current will go through each resistor or $i(t) = 0.5u(t)$, where $i(t)$ denotes the right upper resistor's current. Consequently, $y(t) = i(t) \cdot 1 = 0.5u(t)$ and the transfer function also equals 0.5. Thus the two networks, or more precisely their state equations, are zero-state equivalent. This fact can also be verified by using Theorem 4.1. The network in Fig. 4.2(a) is described by the zero-dimensional state equation $y(t) = 0.5u(t)$ or $\mathbf{A} = \mathbf{B} = \mathbf{C} = 0$ and $\mathbf{D} = 0.5$. To

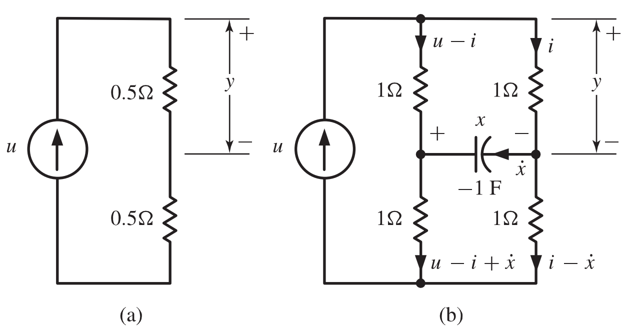


Figure 4.2 Two zero-state equivalent networks.

develop a state equation for the network in Fig. 4.2(b), we assign the capacitor voltage as state variable x with polarity shown. Its current is $\dot{x}$ flowing from the negative to positive polarity because of the negative capacitance. If we assign the right upper resistor's current as $i(t)$, then the right lower resistor's current is $i - \dot{x}$, the left upper resistor's current is $u - i$, and the left lower resistor's current is $u - i + \dot{x}$. The total voltage around the upper right-hand loop is 0:

$$i - x - (u - i) = 0 \quad \text{or} \quad i = 0.5(x + u)$$

which implies

$$y = 1 \cdot i = i = 0.5(x + u)$$

The total voltage around the lower right-hand loop is 0:

$$x + (i - \dot{x}) - (u - i + \dot{x}) = 0$$

which implies

$$2\dot{x} = 2i + x - u = x + u + x - u = 2x$$

Thus the network in Fig. 4.2(b) is described by the one-dimensional state equation

$$\begin{aligned} \dot{x}(t) &= x(t) \\ y(t) &= 0.5x(t) + 0.5u(t) \end{aligned}$$

with $\bar{\mathbf{A}} = 1$, $\bar{\mathbf{B}} = 0$, $\bar{\mathbf{C}} = 0.5$, and $\bar{\mathbf{D}} = 0.5$. We see that $\mathbf{D} = \bar{\mathbf{D}} = 0.5$ and $\mathbf{CA}^m\mathbf{B} = \bar{\mathbf{C}}\bar{\mathbf{A}}^m\bar{\mathbf{B}} = 0$ for $m = 0, 1, \ldots$. Thus the two equations are zero-state equivalent.

4.3.1 Canonical Forms

MATLAB contains the function `[ab,bb,cb,db,P]=canon(a,b,c,d, 'type')`. If `type=companion`, the function will generate an equivalent state equation with $\bar{\mathbf{A}}$ in the companion form in (3.24). This function works only if $\mathbf{Q} := [\mathbf{b}_1 \ \mathbf{Ab}_1 \ \cdots \ \mathbf{A}^{n-1}\mathbf{b}_1]$ is nonsingular, where $\mathbf{b}_1$ is the first column of $\mathbf{B}$. This condition is the same as $\{\mathbf{A}, \ \mathbf{b}_1\}$ controllable, as we will discuss in Chapter 6. The $\mathbf{P}$ that the function `canon` generates equals $\mathbf{Q}^{-1}$. See the discussion in Section 3.4.

We discuss a different canonical form. Suppose $\mathbf{A}$ has two real eigenvalues and two complex eigenvalues. Because $\mathbf{A}$ has only real coefficients, the two complex eigenvalues must be complex conjugate. Let λ_1, λ_2, $\alpha + j\beta$, and $\alpha - j\beta$ be the eigenvalues and $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$, and $\mathbf{q}_4$ be the corresponding eigenvectors, where λ_1, λ_2, α, β, $\mathbf{q}_1$, and $\mathbf{q}_2$ are all real and $\mathbf{q}_4$ equals the complex conjugate of $\mathbf{q}_3$. Define $\mathbf{Q} = [\mathbf{q}_1 \ \mathbf{q}_2 \ \mathbf{q}_3 \ \mathbf{q}_4]$. Then we have

$$\mathbf{J} := \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix} = \mathbf{Q}^{-1}\mathbf{AQ}$$

Note that $\mathbf{Q}$ and $\mathbf{J}$ can be obtained from `[q,j]=eig(a)` in MATLAB as shown in Examples 3.5 and 3.6. This form is useless in practice but can be transformed into a real matrix by the following similarity transformation

$$\bar{\mathbf{Q}}^{-1}\mathbf{J}\bar{\mathbf{Q}} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & j & -j \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha + j\beta & 0 \\ 0 & 0 & 0 & \alpha - j\beta \end{bmatrix}$$
$$\cdot \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\beta & \alpha \end{bmatrix} =: \bar{\mathbf{A}}$$

We see that this transformation transforms the complex eigenvalues on the diagonal into a block with the real part of the eigenvalues on the diagonal and the imaginary part on the off-diagonal. This new A-matrix is said to be in *modal* form. The MATLAB function `[ab,bb,cb,db,P]=canon(a,b,c,d, 'modal')` or `canon(a,b,c,d)` with no type specified will yield an equivalent state equation with $\bar{\mathbf{A}}$ in modal form. Note that there is no need to transform $\mathbf{A}$ into a diagonal form and then to a modal form. The two transformations can be combined into one as

$$\mathbf{P}^{-1} = \mathbf{Q}\bar{\mathbf{Q}} = [\mathbf{q}_1 \;\; \mathbf{q}_2 \;\; \mathbf{q}_3 \;\; \mathbf{q}_4] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0.5 & -0.5j \\ 0 & 0 & 0.5 & 0.5j \end{bmatrix}$$
$$= [\mathbf{q}_1 \; \mathbf{q}_2 \; \mathrm{Re}(\mathbf{q}_3) \; \mathrm{Im}(\mathbf{q}_3)]$$

where Re and Im stand, respectively, for the real part and imaginary part and we have used in the last equality the fact that $\mathbf{q}_4$ is the complex conjugate of $\mathbf{q}_3$. We give one more example. The modal form of a matrix with real eigenvalue λ_1 and two pairs of distinct complex conjugate eigenvalues $\alpha_i \pm j\beta_i$, for $i = 1, 2$, is

$$\bar{\mathbf{A}} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & \beta_1 & 0 & 0 \\ 0 & -\beta_1 & \alpha_1 & 0 & 0 \\ 0 & 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & -\beta_2 & \alpha_2 \end{bmatrix} \tag{4.28}$$

It is block diagonal and can be obtained by the similarity transformation

$$\mathbf{P}^{-1} = [\mathbf{q}_1 \;\; \mathrm{Re}(\mathbf{q}_2) \;\; \mathrm{Im}(\mathbf{q}_2) \;\; \mathrm{Re}(\mathbf{q}_4) \;\; \mathrm{Im}(\mathbf{q}_4)]$$

where $\mathbf{q}_1$, $\mathbf{q}_2$, and $\mathbf{q}_4$ are, respectively, eigenvectors associated with λ_1, $\alpha_1 + j\beta_1$, and $\alpha_2 + j\beta_2$. This form is useful in state-space design.

4.3.2 Magnitude Scaling in Op-Amp Circuits

As discussed in Section 2.3.1, every LTI state equation can be implemented using an op-amp circuit.[1] In actual op-amp circuits, all signals are limited by power supplies. If we use ± 15-volt

1. This subsection may be skipped without loss of continuity.

power supplies, then all signals are roughly limited to ± 13 volts. If any signal goes outside the range, the circuit will saturate and will not behave as the state equation dictates. Therefore saturation is an important issue in actual op-amp circuit implementation.

Consider an LTI state equation and suppose all signals must be limited to $\pm M$. For linear systems, if the input magnitude increases by α, so do the magnitudes of all state variables and the output. Thus there must be a limit on input magnitude. Clearly it is desirable to have the admissible input magnitude as large as possible. One way to achieve this is to use an equivalence transformation so that

$$|x_i(t)| \le |y(t)| \le M$$

for all i and for all t. The equivalence transformation, however, will not alter the relationship between the input and output; therefore we can use the original state equation to find the input range to achieve $|y(t)| \le M$. In addition, we can use the same transformation to amplify some state variables to increase visibility or accuracy. This is illustrated in the next example.

Example 4.5 Consider the state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.1 & 2 \\ 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 10 \\ 0.1 \end{bmatrix} u$$
$$y = [0.2 \quad -1]\mathbf{x}$$

Suppose the input is a step function of various magnitude and the equation is to be implemented using an op-amp circuit in which all signals must be limited to ± 10. First we use MATLAB to find its unit-step response. We type

```
a=[-0.1 2;0 -1];b=[10;0.1];c=[0.2 -1];d=0;
[y,x,t]=step(a,b,c,d);
plot(t,y,t,x)
```

which yields the plot in Fig. 4.3(a). We see that $|x_1|_{max} = 100 > |y|_{max} = 20$ and $|x_2| << |y|_{max}$. The state variable x_2 is hardly visible and its largest magnitude is found to be 0.1 by plotting it separately (not shown). From the plot, we see that if $|u(t)| \le 0.5$, then the output will not saturate but $x_1(t)$ will.

Let us introduce new state variables as

$$\bar{x}_1 = \frac{20}{100} x_1 = 0.2 x_1 \qquad \bar{x}_2 = \frac{20}{0.1} x_2 = 200 x_2$$

With this transformation, the maximum magnitudes of $\bar{x}_1(t)$ and $\bar{x}_2(t)$ will equal $|y|_{max}$. Thus if $y(t)$ does not saturate, neither will all the state variables $\bar{x}_i$. The transformation can be expressed as $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$ with

$$\mathbf{P} = \begin{bmatrix} 0.2 & 0 \\ 0 & 200 \end{bmatrix} \qquad \mathbf{P}^{-1} = \begin{bmatrix} 5 & 0 \\ 0 & 0.005 \end{bmatrix}$$

Then its equivalent state equation can readily be computed from (4.26) as

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} -0.1 & 0.002 \\ 0 & -1 \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} 2 \\ 20 \end{bmatrix} u$$
$$y = [1 \quad -0.005]\bar{\mathbf{x}}$$

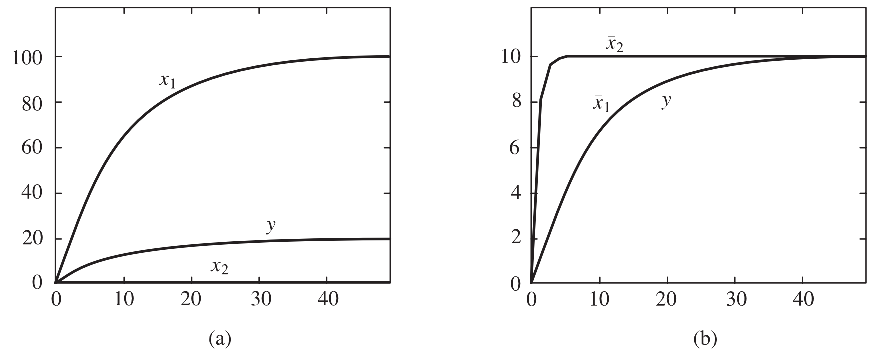


Figure 4.3 Time responses.

Its step responses due to $u(t) = 0.5$ are plotted in Fig. 4.3(b). We see that all signals lie inside the range ± 10 and occupy the full scale. Thus the equivalence state equation is better for op-amp circuit implementation or simulation.

The magnitude scaling is important in using op-amp circuits to implement or simulate continuous-time systems. Although we discuss only step inputs, the idea is applicable to any input. We mention that analog computers are essentially op-amp circuits. Before the advent of digital computers, magnitude scaling in analog computer simulation was carried out by trial and error. With the help of digital computer simulation, the magnitude scaling can now be carried out easily.

4.4 Realizations

Every linear time-invariant (LTI) system can be described by the input–output description

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$$

and, if the system is lumped as well, by the state-space equation description

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{4.29}$$

If the state equation is known, the transfer matrix can be computed as $\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B}+\mathbf{D}$. The computed transfer matrix is unique. Now we study the converse problem, that is, to find a state-space equation from a given transfer matrix. This is called the *realization* problem. This terminology is justified by the fact that, by using the state equation, we can build an op-amp circuit for the transfer matrix.

A transfer matrix $\hat{\mathbf{G}}(s)$ is said to be *realizable* if there exists a finite-dimensional state equation (4.29) or, simply, $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ such that

$$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

and $\{\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}\}$ is called a *realization* of $\hat{\mathbf{G}}(s)$. An LTI distributed system can be described by a transfer matrix, but not by a finite-dimensional state equation. Thus not every $\hat{\mathbf{G}}(s)$ is realizable. If $\hat{\mathbf{G}}(s)$ is realizable, then it has infinitely many realizations, not necessarily of the same dimension. Thus the realization problem is fairly complex. We study here only the realizability condition. The other issues will be studied in later chapters.

Theorem 4.2

A transfer matrix $\hat{\mathbf{G}}(s)$ is realizable if and only if $\hat{\mathbf{G}}(s)$ is a proper rational matrix.

We use (3.19) to write

$$\hat{\mathbf{G}}_{sp}(s) := \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} = \frac{1}{\det(s\mathbf{I} - \mathbf{A})}\mathbf{C}[\mathrm{Adj}\ (s\mathbf{I} - \mathbf{A})]\mathbf{B} \tag{4.30}$$

If $\mathbf{A}$ is $n \times n$, then det $(s\mathbf{I} - \mathbf{A})$ has degree n. Every entry of Adj $(s\mathbf{I} - \mathbf{A})$ is the determinant of an $(n-1) \times (n-1)$ submatrix of $(s\mathbf{I} - \mathbf{A})$; thus it has at most degree $(n-1)$. Their linear combinations again have at most degree $(n-1)$. Thus we conclude that $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$ is a strictly proper rational matrix. If $\mathbf{D}$ is a nonzero matrix, then $\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$ is proper. This shows that if $\hat{\mathbf{G}}(s)$ is realizable, then it is a proper rational matrix. Note that we have

$$\hat{\mathbf{G}}(\infty) = \mathbf{D}$$

Next we show the converse; that is, if $\hat{\mathbf{G}}(s)$ is a $q \times p$ proper rational matrix, then there exists a realization. First we decompose $\hat{\mathbf{G}}(s)$ as

$$\hat{\mathbf{G}}(s) = \hat{\mathbf{G}}(\infty) + \hat{\mathbf{G}}_{sp}(s) \tag{4.31}$$

where $\hat{\mathbf{G}}_{sp}$ is the strictly proper part of $\hat{\mathbf{G}}(s)$. Let

$$d(s) = s^r + \alpha_1 s^{r-1} + \cdots + \alpha_{r-1}s + \alpha_r \tag{4.32}$$

be the least common denominator of all entries of $\hat{\mathbf{G}}_{sp}(s)$. Here we require $d(s)$ to be monic; that is, its leading coefficient is 1. Then $\hat{\mathbf{G}}_{sp}(s)$ can be expressed as

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{d(s)}[\mathbf{N}(s)] = \frac{1}{d(s)}\left[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_{r-1}s + \mathbf{N}_r\right] \tag{4.33}$$

where $\mathbf{N}_i$ are $q \times p$ constant matrices. Now we claim that the set of equations

$$\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} -\alpha_1\mathbf{I}_p & -\alpha_2\mathbf{I}_p & \cdots & -\alpha_{r-1}\mathbf{I}_p & -\alpha_r\mathbf{I}_p \\ \mathbf{I}_p & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_p & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_p & \mathbf{0} \end{bmatrix}\mathbf{x} + \begin{bmatrix} \mathbf{I}_p \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}\mathbf{u} \\
\mathbf{y} &= \begin{bmatrix} \mathbf{N}_1 & \mathbf{N}_2 & \cdots \mathbf{N}_{r-1} & \mathbf{N}_r \end{bmatrix}\mathbf{x} + \hat{\mathbf{G}}(\infty)\mathbf{u}
\end{aligned} \tag{4.34}$$

is a realization of $\hat{\mathbf{G}}(s)$. The matrix $\mathbf{I}_p$ is the $p \times p$ unit matrix and every $\mathbf{0}$ is a $p \times p$ zero matrix. The A-matrix is said to be in block companion form; it consists of r rows and r columns of $p \times p$ matrices; thus the A-matrix has order $rp \times rp$. The B-matrix has order $rp \times p$. Because the C-matrix consists of r number of $\mathbf{N}_i$, each of order $q \times p$, the C-matrix has order $q \times rp$. The realization has dimension rp and is said to be in *controllable canonical form*.

We show that (4.34) is a realization of $\hat{\mathbf{G}}(s)$ in (4.31) and (4.33). Let us define

$$\mathbf{Z} := \begin{bmatrix} \mathbf{Z}_1 \\ \mathbf{Z}_2 \\ \vdots \\ \mathbf{Z}_r \end{bmatrix} := (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \tag{4.35}$$

where $\mathbf{Z}_i$ is $p \times p$ and $\mathbf{Z}$ is $rp \times p$. Then the transfer matrix of (4.34) equals

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \hat{\mathbf{G}}(\infty) = \mathbf{N}_1\mathbf{Z}_1 + \mathbf{N}_2\mathbf{Z}_2 + \cdots + \mathbf{N}_r\mathbf{Z}_r + \hat{\mathbf{G}}(\infty) \tag{4.36}$$

We write (4.35) as $(s\mathbf{I} - \mathbf{A})\mathbf{Z} = \mathbf{B}$ or

$$s\mathbf{Z} = \mathbf{A}\mathbf{Z} + \mathbf{B} \tag{4.37}$$

Using the shifting property of the companion form of $\mathbf{A}$, from the second to the last block of equations in (4.37), we can readily obtain

$$s\mathbf{Z}_2 = \mathbf{Z}_1, \quad s\mathbf{Z}_3 = \mathbf{Z}_2, \quad \cdots, \quad s\mathbf{Z}_r = \mathbf{Z}_{r-1}$$

which implies

$$\mathbf{Z}_2 = \frac{1}{s}\mathbf{Z}_1, \quad \mathbf{Z}_3 = \frac{1}{s^2}\mathbf{Z}_1, \quad \cdots, \quad \mathbf{Z}_r = \frac{1}{s^{r-1}}\mathbf{Z}_1$$

Substituting these into the first block of equations in (4.37) yields

$$\begin{aligned} s\mathbf{Z}_1 &= -\alpha_1\mathbf{Z}_1 - \alpha_2\mathbf{Z}_2 - \cdots - \alpha_r\mathbf{Z}_r + \mathbf{I}_p \\ &= -\left(\alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right)\mathbf{Z}_1 + \mathbf{I}_p \end{aligned}$$

or, using (4.32),

$$\left(s + \alpha_1 + \frac{\alpha_2}{s} + \cdots + \frac{\alpha_r}{s^{r-1}}\right)\mathbf{Z}_1 = \frac{d(s)}{s^{r-1}}\mathbf{Z}_1 = \mathbf{I}_p$$

Thus we have

$$\mathbf{Z}_1 = \frac{s^{r-1}}{d(s)}\mathbf{I}_p, \quad \mathbf{Z}_2 = \frac{s^{r-2}}{d(s)}\mathbf{I}_p, \quad \cdots, \quad \mathbf{Z}_r = \frac{1}{d(s)}\mathbf{I}_p$$

Substituting these into (4.36) yields

$$\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \hat{\mathbf{G}}(\infty) = \frac{1}{d(s)}[\mathbf{N}_1 s^{r-1} + \mathbf{N}_2 s^{r-2} + \cdots + \mathbf{N}_r] + \hat{\mathbf{G}}(\infty)$$

This equals $\hat{\mathbf{G}}(s)$ in (4.31) and (4.33). This shows that (4.34) is a realization of $\hat{\mathbf{G}}(s)$.

Example 4.6 Consider the proper rational matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \dfrac{-12}{2s+1} & \dfrac{3}{s+2} \\ \dfrac{1}{(2s+1)(s+2)} & \dfrac{s+1}{(s+2)^2} \end{bmatrix} \tag{4.38}$$

where we have decomposed $\hat{\mathbf{G}}(s)$ into the sum of a constant matrix and a strictly proper rational matrix $\hat{\mathbf{G}}_{sp}(s)$. The monic least common denominator of $\hat{\mathbf{G}}_{sp}(s)$ is $d(s) = (s+0.5)(s+2)^2 = s^3 + 4.5s^2 + 6s + 2$. Thus we have

$$\hat{\mathbf{G}}_{sp}(s) = \frac{1}{s^3+4.5s^2+6s+2} \begin{bmatrix} -6(s+2)^2 & 3(s+2)(s+0.5) \\ 0.5(s+2) & (s+1)(s+0.5) \end{bmatrix}$$
$$= \frac{1}{d(s)} \left(\begin{bmatrix} -6 & 3 \\ 0 & 1 \end{bmatrix} s^2 + \begin{bmatrix} -24 & 7.5 \\ 0.5 & 1.5 \end{bmatrix} s + \begin{bmatrix} -24 & 3 \\ 1 & 0.5 \end{bmatrix} \right)$$

and a realization of (4.38) is

$$\dot{\mathbf{x}} = \left[\begin{array}{cc:cc:cc} -4.5 & 0 & -6 & 0 & -2 & 0 \\ 0 & -4.5 & 0 & -6 & 0 & -2 \\ \hdashline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right] \mathbf{x} + \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ \hdashline 0 & 0 \\ 0 & 0 \\ \hdashline 0 & 0 \\ 0 & 0 \end{array}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\mathbf{y} = \left[\begin{array}{cc:cc:cc} -6 & 3 & -24 & 7.5 & -24 & 3 \\ 0 & 1 & 0.5 & 1.5 & 1 & 0.5 \end{array}\right] \mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{4.39}$$

This is a six-dimensional realization.

We discuss a special case of (4.31) and (4.34) in which $p = 1$. To save space, we assume $r = 4$ and $q = 2$. However, the discussion applies to any positive integers r and q. Consider the 2×1 proper rational matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} + \frac{1}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4} \cdot \begin{bmatrix} \beta_{11}s^3 + \beta_{12}s^2 + \beta_{13}s + \beta_{14} \\ \beta_{21}s^3 + \beta_{22}s^2 + \beta_{23}s + \beta_{24} \end{bmatrix} \tag{4.40}$$

Then its realization can be obtained directly from (4.34) as

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 & -\alpha_2 & -\alpha_3 & -\alpha_4 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} u$$
$$\mathbf{y} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix} \mathbf{x} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} u \tag{4.41}$$

This controllable-canonical-form realization can be read out from the coefficients of $\hat{\mathbf{G}}(s)$ in (4.40).

There are many ways to realize a proper transfer matrix. For example, Problem 4.9 gives a different realization of (4.33) with dimension rq. Let $\hat{\mathbf{G}}_{ci}(s)$ be the ith column of $\hat{\mathbf{G}}(s)$ and let u_i be the ith component of the input vector $\mathbf{u}$. Then $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$ can be expressed as

$$\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}_{c1}(s)\hat{u}_1(s) + \hat{\mathbf{G}}_{c2}(s)\hat{u}_2(s) + \cdots =: \hat{\mathbf{y}}_{c1}(s) + \hat{\mathbf{y}}_{c2}(s) + \cdots$$

as shown in Fig. 4.4(a). Thus we can realize each column of $\hat{\mathbf{G}}(s)$ and then combine them to yield a realization of $\hat{\mathbf{G}}(s)$. Let $\hat{\mathbf{G}}_{ri}(s)$ be the ith row of $\hat{\mathbf{G}}(s)$ and let y_i be the ith component of the output vector $\mathbf{y}$. Then $\hat{\mathbf{y}}(s) = \hat{\mathbf{G}}(s)\hat{\mathbf{u}}(s)$ can be expressed as

$$\hat{y}_i(s) = \hat{\mathbf{G}}_{ri}(s)\hat{\mathbf{u}}(s)$$

as shown in Fig. 4.4(b). Thus we can realize each row of $\hat{\mathbf{G}}(s)$ and then combine them to obtain a realization of $\hat{\mathbf{G}}(s)$. Clearly we can also realize each entry of $\hat{\mathbf{G}}(s)$ and then combine them to obtain a realization of $\hat{\mathbf{G}}(s)$. See Reference [6, pp. 158–160].

The MATLAB function `[a,b,c,d]=tf2ss(num,den)` generates the controllable-canonical-form realization shown in (4.41) for any single-input multiple-output transfer matrix $\hat{\mathbf{G}}(s)$. In its employment, there is no need to decompose $\hat{\mathbf{G}}(s)$ as in (4.31). But we must compute its least common denominator, not necessarily monic. The next example will apply `tf2ss` to each column of $\hat{\mathbf{G}}(s)$ in (4.38) and then combine them to form a realization of $\hat{\mathbf{G}}(s)$.

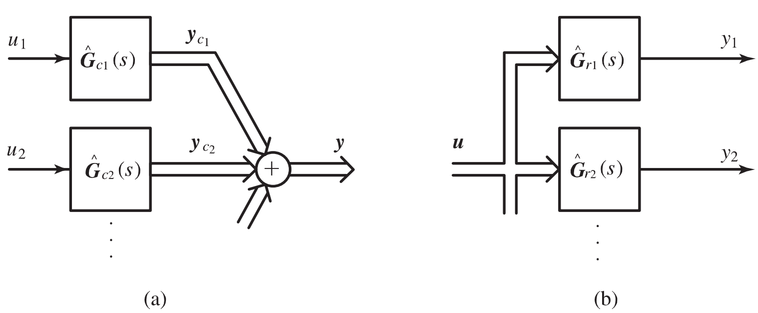


Figure 4.4 Realizations of $\hat{\mathbf{G}}(s)$ by columns and by rows.

EXAMPLE 4.7 Consider the proper rational matrix in (4.38). Its first column is

$$\hat{\mathbf{G}}_{c1}(s) = \begin{bmatrix} \dfrac{4s-10}{2s+1} \\ \dfrac{1}{(2s+1)(s+2)} \end{bmatrix} = \begin{bmatrix} \dfrac{(4s-10)(s+2)}{(2s+1)(s+2)} \\ \dfrac{1}{2s^2+5s+2} \end{bmatrix} = \begin{bmatrix} \dfrac{4s^2-2s-20}{2s^2+5s+2} \\ \dfrac{1}{2s^2+5s+2} \end{bmatrix}$$

Typing

```
n1=[4 -2 -20;0 0 1];d1=[2 5 2]; [a,b,c,d]=tf2ss(n1,d1)
```

yields the following realization for the first column of $\hat{\mathbf{G}}(s)$:

$$\begin{aligned} \dot{\mathbf{x}}_1 &= \mathbf{A}_1\mathbf{x}_1 + \mathbf{b}_1 u_1 = \begin{bmatrix} -2.5 & -1 \\ 1 & 0 \end{bmatrix}\mathbf{x}_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1 \\ \mathbf{y}_{c1} &= \mathbf{C}_1\mathbf{x}_1 + \mathbf{d}_1 u_1 = \begin{bmatrix} -6 & -12 \\ 0 & 0.5 \end{bmatrix}\mathbf{x}_1 + \begin{bmatrix} 2 \\ 0 \end{bmatrix} u_1 \end{aligned} \tag{4.42}$$

Similarly, the function `tf2ss` can generate the following realization for the second column of $\hat{\mathbf{G}}(s)$:

$$\begin{aligned} \dot{\mathbf{x}}_2 &= \mathbf{A}_2\mathbf{x}_2 + \mathbf{b}_2 u_2 = \begin{bmatrix} -4 & -4 \\ 1 & 0 \end{bmatrix}\mathbf{x}_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_2 \\ \mathbf{y}_{c2} &= \mathbf{C}_2\mathbf{x}_2 + \mathbf{d}_2 u_2 = \begin{bmatrix} 3 & 6 \\ 1 & 1 \end{bmatrix}\mathbf{x}_2 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} u_2 \end{aligned} \tag{4.43}$$

These two realizations can be combined as

$$\begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\mathbf{x}}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix}\begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} + \begin{bmatrix} \mathbf{b}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{b}_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$
$$\mathbf{y} = \mathbf{y}_{c1} + \mathbf{y}_{c2} = [\mathbf{C}_1 \;\; \mathbf{C}_2]\mathbf{x} + [\mathbf{d}_1 \;\; \mathbf{d}_2]\mathbf{u}$$

or

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} -2.5 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -4 & -4 \\ 0 & 0 & 1 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}\mathbf{u} \\ \mathbf{y} &= \begin{bmatrix} -6 & -12 & 3 & 6 \\ 0 & 0.5 & 1 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{u} \end{aligned} \tag{4.44}$$

This is a different realization of the $\hat{\mathbf{G}}(s)$ in (4.38). This realization has dimension 4, two less than the one in (4.39).

The two state equations in (4.39) and (4.44) are zero-state equivalent because they have the same transfer matrix. They are, however, not algebraically equivalent. More will be said in Chapter 7 regarding realizations. We mention that all discussion in this section, including `tf2ss`, applies without any modification to the discrete-time case

4.5 Solution of Linear Time-Varying (LTV) Equations

Consider the linear time-varying (LTV) state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \tag{4.45}$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t) \tag{4.46}$$

It is assumed that, for every initial state $\mathbf{x}(t_0)$ and any input $\mathbf{u}(t)$, the state equation has a unique solution. A sufficient condition for such an assumption is that every entry of $\mathbf{A}(t)$ is a continuous function of t. Before considering the general case, we first discuss the solutions of $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t)$ and the reasons why the approach taken in the time-invariant case cannot be used here.

The solution of the time-invariant equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ can be extended from the scalar equation $\dot{x} = ax$. The solution of $\dot{x} = ax$ is $x(t) = e^{at}x(0)$ with $d(e^{at})/dt = ae^{at} = e^{at}a$. Similarly, the solution of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is $\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}(0)$ with

$$\frac{d}{dt}e^{\mathbf{A}t} = \mathbf{A}e^{\mathbf{A}t} = e^{\mathbf{A}t}\mathbf{A}$$

where the commutative property is crucial. Note that, in general, we have $\mathbf{AB} \neq \mathbf{BA}$ and $e^{(\mathbf{A}+\mathbf{B})t} \neq e^{\mathbf{A}t}e^{\mathbf{B}t}$.

The solution of the scalar time-varying equation $\dot{x} = a(t)x$ due to $x(0)$ is

$$x(t) = e^{\int_0^t a(\tau)d\tau}x(0)$$

with

$$\frac{d}{dt}e^{\int_0^t a(\tau)d\tau} = a(t)e^{\int_0^t a(\tau)d\tau} = e^{\int_0^t a(\tau)d\tau}a(t)$$

Extending this to the matrix case becomes

$$\mathbf{x}(t) = e^{\int_0^t \mathbf{A}(\tau)d\tau}\mathbf{x}(0) \tag{4.47}$$

with, using (3.51),

$$e^{\int_0^t \mathbf{A}(\tau)d\tau} = \mathbf{I} + \int_0^t \mathbf{A}(\tau)d\tau + \frac{1}{2}\left(\int_0^t \mathbf{A}(\tau)d\tau\right)\left(\int_0^t \mathbf{A}(s)ds\right) + \cdots$$

This extension, however, is not valid because

$$\begin{aligned}\frac{d}{dt}e^{\int_0^t \mathbf{A}(\tau)d\tau} &= \mathbf{A}(t) + \frac{1}{2}\mathbf{A}(t)\left(\int_0^t \mathbf{A}(s)ds\right) + \frac{1}{2}\left(\int_0^t \mathbf{A}(\tau)d\tau\right)\mathbf{A}(t) + \cdots \\ &\neq \mathbf{A}(t)e^{\int_0^t \mathbf{A}(\tau)d\tau}\end{aligned} \tag{4.48}$$

Thus, in general, (4.47) is not a solution of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. In conclusion, we cannot extend the solution of scalar time-varying equations to the matrix case and must use a different approach to develop the solution.

Consider

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} \tag{4.49}$$

where $\mathbf{A}$ is $n \times n$ with continuous functions of t as its entries. Then for every initial state $\mathbf{x}_i(t_0)$, there exists a unique solution $\mathbf{x}_i(t)$, for $i = 1, 2, \ldots, n$. We can arrange these n solutions as $\mathbf{X} = [\mathbf{x}_1 \ \mathbf{x}_2 \ \cdots \ \mathbf{x}_n]$, a square matrix of order n. Because every $\mathbf{x}_i$ satisfies (4.49), we have

$$\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t) \tag{4.50}$$

If $\mathbf{X}(t_0)$ is nonsingular or the n initial states are linearly independent, then $\mathbf{X}(t)$ is called a *fundamental matrix* of (4.49). Because the initial states can arbitrarily be chosen, as long as they are linearly independent, the fundamental matrix is not unique.

Example 4.8 Consider the homogeneous equation

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} \mathbf{x}(t) \tag{4.51}$$

or

$$\dot{x}_1(t) = 0 \qquad \dot{x}_2(t) = t x_1(t)$$

The solution of $\dot{x}_1(t) = 0$ for $t_0 = 0$ is $x_1(t) = x_1(0)$; the solution of $\dot{x}_2(t) = t x_1(t) = t x_1(0)$ is

$$x_2(t) = \int_0^t \tau x_1(0) d\tau + x_2(0) = 0.5t^2 x_1(0) + x_2(0)$$

Thus we have

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}$$

and

$$\mathbf{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{x}(t) = \begin{bmatrix} 1 \\ 0.5t^2 + 2 \end{bmatrix}$$

The two initial states are linearly independent; thus

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix} \tag{4.52}$$

is a fundamental matrix.

A very important property of the fundamental matrix is that $\mathbf{X}(t)$ is nonsingular for all t. For example, $\mathbf{X}(t)$ in (4.52) has determinant $0.5t^2 + 2 - 0.5t^2 = 2$; thus it is nonsingular for all t. We argue intuitively why this is the case. If $\mathbf{X}(t)$ is singular at some t_1, then there exists a nonzero vector $\mathbf{v}$ such that $\mathbf{x}(t_1) := \mathbf{X}(t_1)\mathbf{v} = \mathbf{0}$, which, in turn, implies $\mathbf{x}(t) := \mathbf{X}(t)\mathbf{v} \equiv \mathbf{0}$ for all t, in particular, at $= t_0$. This is a contradiction. Thus $\mathbf{X}(t)$ is nonsingular for all t.

Definition 4.2 *Let* $\mathbf{X}(t)$ *be any fundamental matrix of* $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. *Then*

$$\mathbf{\Phi}(t, t_0) := \mathbf{X}(t)\mathbf{X}^{-1}(t_0)$$

is called the state transition matrix *of* $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. *The state transition matrix is also the unique solution of*

$$\frac{\partial}{\partial t}\mathbf{\Phi}(t, t_0) = \mathbf{A}(t)\mathbf{\Phi}(t, t_0) \tag{4.53}$$

with the initial condition $\mathbf{\Phi}(t_0, t_0) = \mathbf{I}$.

Because $\mathbf{X}(t)$ is nonsingular for all t, its inverse is well defined. Equation (4.53) follows directly from (4.50). From the definition, we have the following important properties of the state transition matrix:

$$\mathbf{\Phi}(t, t) = \mathbf{I} \tag{4.54}$$

$$\mathbf{\Phi}^{-1}(t, t_0) = [\mathbf{X}(t)\mathbf{X}^{-1}(t_0)]^{-1} = \mathbf{X}(t_0)\mathbf{X}^{-1}(t) = \mathbf{\Phi}(t_0, t) \tag{4.55}$$

$$\mathbf{\Phi}(t, t_0) = \mathbf{\Phi}(t, t_1)\mathbf{\Phi}(t_1, t_0) \tag{4.56}$$

for every t, t_0, and t_1.

Example 4.9 Consider the homogeneous equation in Example 4.8. Its fundamental matrix was computed as

$$\mathbf{X}(t) = \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix}$$

Its inverse is, using (3.20),

$$\mathbf{X}^{-1}(t) = \begin{bmatrix} 0.25t^2 + 1 & -0.5 \\ -0.25t^2 & 0.5 \end{bmatrix}$$

Thus the state transition matrix is given by

$$\begin{aligned}\mathbf{\Phi}(t, t_0) &= \begin{bmatrix} 1 & 1 \\ 0.5t^2 & 0.5t^2 + 2 \end{bmatrix} \begin{bmatrix} 0.25t_0^2 + 1 & -0.5 \\ -0.25t_0^2 & 0.5 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0.5(t^2 - t_0^2) & 1 \end{bmatrix}\end{aligned}$$

It is straightforward to verify that this transition matrix satisfies (4.53) and has the three properties listed in (4.54) through (4.56).

Now we claim that the solution of (4.45) excited by the initial state $\mathbf{x}(t_0) = \mathbf{x}_0$ and the input $\mathbf{u}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \tag{4.57}$$

$$= \mathbf{\Phi}(t, t_0)\left[\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t_0, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\right] \tag{4.58}$$

where $\mathbf{\Phi}(t, \tau)$ is the state transition matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. Equation (4.58) follows from (4.57) by using $\mathbf{\Phi}(t, \tau) = \mathbf{\Phi}(t, t_0)\mathbf{\Phi}(t_0, \tau)$. We show that (4.57) satisfies the initial condition and the state equation. At $t = t_0$, we have

$$\mathbf{x}(t_0) = \mathbf{\Phi}(t_0, t_0)\mathbf{x}_0 + \int_{t_0}^{t_0} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau = \mathbf{I}\mathbf{x}_0 + \mathbf{0} = \mathbf{x}_0$$

Thus (4.57) satisfies the initial condition. Using (4.53) and (4.6), we have

$$\begin{aligned}
\frac{d}{dt}\mathbf{x}(t) &= \frac{\partial}{\partial t}\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \frac{\partial}{\partial t}\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau \\
&= \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \left(\frac{\partial}{\partial t}\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\right) d\tau + \mathbf{\Phi}(t, t)\mathbf{B}(t)\mathbf{u}(t) \\
&= \mathbf{A}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{A}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{B}(t)\mathbf{u}(t) \\
&= \mathbf{A}(t)\left[\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau\right] + \mathbf{B}(t)\mathbf{u}(t) \\
&= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)
\end{aligned}$$

Thus (4.57) is the solution. Substituting (4.57) into (4.46) yields

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{\Phi}(t, t_0)\mathbf{x}_0 + \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t) \tag{4.59}$$

If the input is identically zero, then Equation (4.57) reduces to

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}_0$$

This is the zero-input response. Thus the state transition matrix governs the unforced propagation of the state vector. If the initial state is zero, then (4.59) reduces to

$$\begin{aligned}
\mathbf{y}(t) &= \mathbf{C}(t)\int_{t_0}^{t} \mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t) \\
&= \int_{t_0}^{t} \left[\mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau) + \mathbf{D}\delta(t - \tau)\right] \mathbf{u}(\tau)\, d\tau
\end{aligned} \tag{4.60}$$

This is the zero-state response. As discussed in (2.5), the zero-state response can be described by

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t, \tau)\mathbf{u}(\tau)\, d\tau \tag{4.61}$$

where $\mathbf{G}(t, \tau)$ is the impulse response matrix and is the output at time t excited by an impulse input applied at time τ. Comparing (4.60) and (4.61) yields

$$\begin{aligned}
\mathbf{G}(t, \tau) &= \mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t - \tau) \\
&= \mathbf{C}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t - \tau)
\end{aligned} \tag{4.62}$$

This relates the input–output and state-space descriptions.

The solutions in (4.57) and (4.59) hinge on solving (4.49) or (4.53). If $\mathbf{A}(t)$ is triangular such as

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} a_{11}(t) & 0 \\ a_{21}(t) & a_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

we can solve the scalar equation $\dot{x}_1(t) = a_{11}(t)x_1(t)$ and then substitute it into

$$\dot{x}_2(t) = a_{22}(t)x_2(t) + a_{21}(t)x_1(t)$$

Because $x_1(t)$ has been solved, the preceding scalar equation can be solved for $x_2(t)$. This is what we did in Example 4.8. If $\mathbf{A}(t)$, such as $\mathbf{A}(t)$ diagonal or constant, has the commutative property

$$\mathbf{A}(t)\left(\int_{t_0}^{t} \mathbf{A}(\tau)\, d\tau\right) = \left(\int_{t_0}^{t} \mathbf{A}(\tau)\, d\tau\right)\mathbf{A}(t)$$

for all t_0 and t, then the solution of (4.53) can be shown to be

$$\mathbf{\Phi}(t, t_0) = e^{\int_{t_0}^{t} \mathbf{A}(\tau)\, d\tau} = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\int_{t_0}^{t} \mathbf{A}(\tau)\, d\tau\right)^k \tag{4.63}$$

For $\mathbf{A}(t)$ constant, (4.63) reduces to

$$\mathbf{\Phi}(t, \tau) = e^{\mathbf{A}(t-\tau)} = \mathbf{\Phi}(t - \tau)$$

and $\mathbf{X}(t) = e^{\mathbf{A}t}$. Other than the preceding special cases, computing state transition matrices is generally difficult.

4.5.1 Discrete-Time Case

Consider the discrete-time state equation

$$\mathbf{x}[k + 1] = \mathbf{A}[k]\mathbf{x}[k] + \mathbf{B}[k]\mathbf{u}[k] \tag{4.64}$$

$$\mathbf{y}[k] = \mathbf{C}[k]\mathbf{x}[k] + \mathbf{D}[k]\mathbf{u}[k] \tag{4.65}$$

The set consists of algebraic equations and their solutions can be computed recursively once the initial state $\mathbf{x}[k_0]$ and the input $\mathbf{u}[k]$, for $k \geq k_0$, are given. The situation here is much simpler than the continuous-time case.

As in the continuous-time case, we can define the discrete state transition matrix as the solution of

$$\mathbf{\Phi}[k + 1, k_0] = \mathbf{A}[k]\mathbf{\Phi}[k, k_0] \qquad \text{with } \mathbf{\Phi}[k_0, k_0] = \mathbf{I}$$

for $k = k_0,\ k_0 + 1,\ \ldots$. This is the discrete counterpart of (4.53) and its solution can be obtained directly as

$$\mathbf{\Phi}[k, k_0] = \mathbf{A}[k - 1]\mathbf{A}[k - 2] \cdots \mathbf{A}[k_0] \tag{4.66}$$

for $k > k_0$ and $\mathbf{\Phi}[k_0, k_0] = \mathbf{I}$. We discuss a significant difference between the continuous- and discrete-time cases. Because the fundamental matrix in the continuous-time case is nonsingular for all t, the state transition matrix $\mathbf{\Phi}(t, t_0)$ is defined for $t \geq t_0$ and $t < t_0$ and can govern the propagation of the state vector in the positive-time and negative-time directions. In the discrete-time case, the A-matrix can be singular; thus the inverse of $\mathbf{\Phi}[k, k_0]$ may not be defined. Thus $\mathbf{\Phi}[k, k_0]$ is defined only for $k \geq k_0$ and governs the propagation of the state vector in only the positive-time direction. Therefore the discrete counterpart of (4.56) or

$$\mathbf{\Phi}[k, k_0] = \mathbf{\Phi}[k, k_1]\mathbf{\Phi}[k_1, k_0]$$

holds only for $k \geq k_1 \geq k_0$.

Using the discrete state transition matrix, we can express the solutions of (4.64) and (4.65) as, for $k > k_0$,

$$\begin{aligned}\mathbf{x}[k] &= \mathbf{\Phi}[k, k_0]\mathbf{x}_0 + \sum_{m=k_0}^{k-1} \mathbf{\Phi}[k, m+1]\mathbf{B}[m]\mathbf{u}[m] \\ \mathbf{y}[k] &= \mathbf{C}[k]\mathbf{\Phi}[k, k_0]\mathbf{x}_0 + \mathbf{C}[k]\sum_{m=k_0}^{k-1} \mathbf{\Phi}[k, m+1]\mathbf{B}[m]\mathbf{u}[m] + \mathbf{D}[k]\mathbf{u}[k]\end{aligned} \tag{4.67}$$

Their derivations are similar to those of (4.20) and (4.21) and will not be repeated.

If the initial state is zero, Equation (4.67) reduces to

$$\mathbf{y}[k] = \mathbf{C}[k]\sum_{m=k_0}^{k-1} \mathbf{\Phi}[k, m+1]\mathbf{B}[m]\mathbf{u}[m] + \mathbf{D}[k]\mathbf{u}[k] \tag{4.68}$$

for $k > k_0$. This describes the zero-state response of (4.65). If we define $\mathbf{\Phi}[k, m] = \mathbf{0}$ for $k < m$, then (4.68) can be written as

$$\mathbf{y}[k] = \sum_{m=k_0}^{k} \left(\mathbf{C}[k]\mathbf{\Phi}[k, m+1]\mathbf{B}[m] + \mathbf{D}[m]\delta[k-m]\right)\mathbf{u}[m]$$

where the impulse sequence $\delta[k-m]$ equals 1 if $k = m$ and 0 if $k \neq m$. Comparing this with the multivariable version of (2.34), we have

$$\mathbf{G}[k, m] = \mathbf{C}[k]\mathbf{\Phi}[k, m+1]\mathbf{B}[m] + \mathbf{D}[m]\delta[k-m]$$

for $k \geq m$. This relates the impulse response sequence and the state equation and is the discrete counterpart of (4.62).

4.6 Equivalent Time-Varying Equations

This section extends the equivalent state equations discussed in Section 4.3 to the time-varying case. Consider the n-dimensional linear time-varying state equation

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{y} &= \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u}\end{aligned} \tag{4.69}$$

Let $\mathbf{P}(t)$ be an $n \times n$ matrix. It is assumed that $\mathbf{P}(t)$ is nonsingular and both $\mathbf{P}(t)$ and $\dot{\mathbf{P}}(t)$ are continuous for all t. Let $\bar{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$. Then the state equation

$$\begin{aligned} \dot{\bar{\mathbf{x}}} &= \bar{\mathbf{A}}(t)\bar{\mathbf{x}} + \bar{\mathbf{B}}(t)\mathbf{u} \\ \mathbf{y} &= \bar{\mathbf{C}}(t)\bar{\mathbf{x}} + \bar{\mathbf{D}}(t)\mathbf{u} \end{aligned} \tag{4.70}$$

where

$$\begin{aligned} \bar{\mathbf{A}}(t) &= [\mathbf{P}(t)\mathbf{A}(t) + \dot{\mathbf{P}}(t)]\mathbf{P}^{-1}(t) \\ \bar{\mathbf{B}}(t) &= \mathbf{P}(t)\mathbf{B}(t) \\ \bar{\mathbf{C}}(t) &= \mathbf{C}(t)\mathbf{P}^{-1}(t) \\ \bar{\mathbf{D}}(t) &= \mathbf{D}(t) \end{aligned}$$

is said to be (algebraically) equivalent to (4.69) and $\mathbf{P}(t)$ is called an (*algebraic*) *equivalence* transformation.

Equation (4.70) is obtained from (4.69) by substituting $\bar{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ and $\dot{\bar{\mathbf{x}}} = \dot{\mathbf{P}}(t)\mathbf{x} + \mathbf{P}(t)\dot{\mathbf{x}}$. Let $\mathbf{X}$ be a fundamental matrix of (4.69). Then we claim that

$$\bar{\mathbf{X}}(t) := \mathbf{P}(t)\mathbf{X}(t) \tag{4.71}$$

is a fundamental matrix of (4.70). By definition, $\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)$ and $\mathbf{X}(t)$ is nonsingular for all t. Because the rank of a matrix will not change by multiplying a nonsingular matrix, the matrix $\mathbf{P}(t)\mathbf{X}(t)$ is also nonsingular for all t. Now we show that $\mathbf{P}(t)\mathbf{X}(t)$ satisfies the equation $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}(t)\bar{\mathbf{x}}$. Indeed, we have

$$\begin{aligned} \frac{d}{dt}[\mathbf{P}(t)\mathbf{X}(t)] &= \dot{\mathbf{P}}(t)\mathbf{X}(t) + \mathbf{P}(t)\dot{\mathbf{X}}(t) = \dot{\mathbf{P}}(t)\mathbf{X}(t) + \mathbf{P}(t)\mathbf{A}(t)\mathbf{X}(t) \\ &= [\dot{\mathbf{P}}(t) + \mathbf{P}(t)\mathbf{A}(t)][\mathbf{P}^{-1}(t)\mathbf{P}(t)]\mathbf{X}(t) = \bar{\mathbf{A}}(t)[\mathbf{P}(t)\mathbf{X}(t)] \end{aligned}$$

Thus $\mathbf{P}(t)\mathbf{X}(t)$ is a fundamental matrix of $\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}(t)\bar{\mathbf{x}}(t)$.

Theorem 4.3

Let $\mathbf{A}_o$ be an arbitrary constant matrix. Then there exists an equivalence transformation that transforms (4.69) into (4.70) with $\bar{\mathbf{A}}(t) = \mathbf{A}_o$.

Proof: Let $\mathbf{X}(t)$ be a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. The differentiation of $\mathbf{X}^{-1}(t)$ $\mathbf{X}(t) = \mathbf{I}$ yields

$$\dot{\mathbf{X}}^{-1}(t)\mathbf{X}(t) + \mathbf{X}^{-1}\dot{\mathbf{X}}(t) = \mathbf{0}$$

which implies

$$\dot{\mathbf{X}}^{-1}(t) = -\mathbf{X}^{-1}(t)\mathbf{A}(t)\mathbf{X}(t)\mathbf{X}^{-1}(t) = -\mathbf{X}^{-1}(t)\mathbf{A}(t) \tag{4.72}$$

Because $\bar{\mathbf{A}}(t) = \mathbf{A}_o$ is a constant matrix, $\bar{\mathbf{X}}(t) = e^{\mathbf{A}_o t}$ is a fundamental matrix of $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}(t)\bar{\mathbf{x}} = \mathbf{A}_o\bar{\mathbf{x}}$. Following (4.71), we define

$$\mathbf{P}(t) := \bar{\mathbf{X}}(t)\mathbf{X}^{-1}(t) = e^{\mathbf{A}_o t}\mathbf{X}^{-1}(t) \tag{4.73}$$

and compute

$$\bar{\mathbf{A}}(t) = [\mathbf{P}(t)\mathbf{A}(t) + \dot{\mathbf{P}}(t)]\mathbf{P}^{-1}(t)$$
$$= [e^{\mathbf{A}_o t}\mathbf{X}^{-1}(t)\mathbf{A}(t) + \mathbf{A}_o e^{\mathbf{A}_o t}\mathbf{X}^{-1}(t) + e^{\mathbf{A}_o t}\dot{\mathbf{X}}^{-1}(t)]\mathbf{X}(t)e^{-\mathbf{A}_o t}$$

which becomes, after substituting (4.72),

$$\bar{\mathbf{A}}(t) = \mathbf{A}_o e^{\mathbf{A}_o t}\mathbf{X}^{-1}(t)\mathbf{X}(t)e^{-\mathbf{A}_o t} = \mathbf{A}_o$$

This establishes the theorem. Q.E.D.

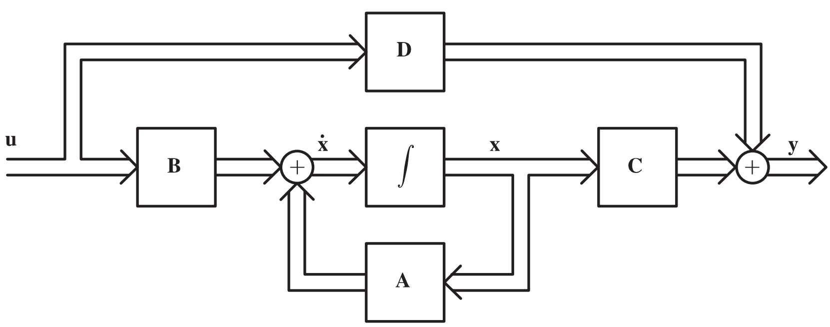
If $\mathbf{A}_o$ is chosen as a zero matrix, then $\mathbf{P}(t) = \mathbf{X}^{-1}(t)$ and (4.70) reduces to

$$\bar{\mathbf{A}}(t) = \mathbf{0} \qquad \bar{\mathbf{B}}(t) = \mathbf{X}^{-1}(t)\mathbf{B}(t) \qquad \bar{\mathbf{C}}(t) = \mathbf{C}(t)\mathbf{X}(t) \qquad \bar{\mathbf{D}}(t) = \mathbf{D}(t) \tag{4.74}$$

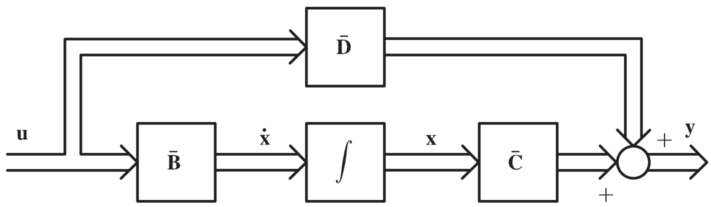
The block diagrams of (4.69) with $\mathbf{A}(t) \neq \mathbf{0}$ and $\mathbf{A}(t) = \mathbf{0}$ are plotted in Fig. 4.5. The block diagram with $\mathbf{A}(t) = \mathbf{0}$ has no feedback and is considerably simpler. Every time-varying state equation can be transformed into such a block diagram. However, in order to do so, we must know its fundamental matrix.

The impulse response matrix of (4.69) is given in (4.62). The impulse response matrix of (4.70) is, using (4.71) and (4.72),

$$\bar{\mathbf{G}}(t,\tau) = \bar{\mathbf{C}}(t)\bar{\mathbf{X}}(t)\bar{\mathbf{X}}^{-1}(\tau)\bar{\mathbf{B}}(\tau) + \bar{\mathbf{D}}(t)\delta(t-\tau)$$



(a)



(b)

Figure 4.5 Block daigrams with feedback and without feedback.

$$= \mathbf{C}(t)\mathbf{P}^{-1}(t)\mathbf{P}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{P}^{-1}(\tau)\mathbf{P}(\tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t-\tau)$$
$$= \mathbf{C}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t-\tau) = \mathbf{G}(t,\tau)$$

Thus the impulse response matrix is invariant under any equivalence transformation. The property of the A-matrix, however, may not be preserved in equivalence transformations. For example, every A-matrix can be transformed, as shown in Theorem 4.3, into a constant or a zero matrix. Clearly the zero matrix does not have any property of $\mathbf{A}(t)$. In the time-invariant case, equivalence transformations will preserve all properties of the original state equation. Thus the equivalence transformation in the time-invariant case is not a special case of the time-varying case.

> ***Definition 4.3*** *A matrix* $\mathbf{P}(t)$ *is called a* Lyapunov transformation *if* $\mathbf{P}(t)$ *is nonsingular,* $\mathbf{P}(t)$ *and* $\dot{\mathbf{P}}(t)$ *are continuous, and* $\mathbf{P}(t)$ *and* $\mathbf{P}^{-1}(t)$ *are bounded for all* t*. Equations (4.69) and (4.70) are said to be* Lyapunov equivalent *if* $\mathbf{P}(t)$ *is a Lyapunov transformation.*

It is clear that if $\mathbf{P}(t) = \mathbf{P}$ is a constant matrix, then it is a Lyapunov transformation. Thus the (algebraic) transformation in the time-invariant case is a special case of the Lyapunov transformation. If $\mathbf{P}(t)$ is required to be a Lyapunov transformation, then Theorem 4.3 does not hold in general. In other words, not every time-varying state equation can be Lyapunov equivalent to a state equation with a constant A-matrix. However, this is true if $\mathbf{A}(t)$ is periodic.

Periodic state equations Consider the linear time-varying state equation in (4.69). It is assumed that

$$\mathbf{A}(t+T) = \mathbf{A}(t)$$

for all t and for some positive constant T. That is, $\mathbf{A}(t)$ is periodic with period T. Let $\mathbf{X}(t)$ be a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ or $\dot{\mathbf{X}}(t) = \mathbf{A}(t)\mathbf{X}(t)$ with $\mathbf{X}(0)$ nonsingular. Then we have

$$\dot{\mathbf{X}}(t+T) = \mathbf{A}(t+T)\mathbf{X}(t+T) = \mathbf{A}(t)\mathbf{X}(t+T)$$

Thus $\mathbf{X}(t+T)$ is also a fundamental matrix. Furthermore, it can be expressed as

$$\mathbf{X}(t+T) = \mathbf{X}(t)\mathbf{X}^{-1}(0)\mathbf{X}(T) \tag{4.75}$$

This can be verified by direct substitution. Let us define $\mathbf{Q} = \mathbf{X}^{-1}(0)\mathbf{X}(T)$. It is a constant nonsingular matrix. For this $\mathbf{Q}$ there exists a constant matrix $\bar{\mathbf{A}}$ such that $e^{\bar{\mathbf{A}}T} = \mathbf{Q}$ (Problem 3.24). Thus (4.75) can be written as

$$\mathbf{X}(t+T) = \mathbf{X}(t)e^{\bar{\mathbf{A}}T} \tag{4.76}$$

Define

$$\mathbf{P}(t) := e^{\bar{\mathbf{A}}t}\mathbf{X}^{-1}(t) \tag{4.77}$$

We show that $\mathbf{P}(t)$ is periodic with period T:

$$\mathbf{P}(t+T) = e^{\bar{\mathbf{A}}(t+T)}\mathbf{X}^{-1}(t+T) = e^{\bar{\mathbf{A}}t}e^{\bar{\mathbf{A}}T}[e^{-\bar{\mathbf{A}}T}\mathbf{X}^{-1}(t)]$$
$$= e^{\bar{\mathbf{A}}t}\mathbf{X}^{-1}(t) = \mathbf{P}(t)$$

Theorem 4.4

Consider (4.69) with $\mathbf{A}(t) = \mathbf{A}(t + T)$ for all t and some $T > 0$. Let $\mathbf{X}(t)$ be a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. Let $\bar{\mathbf{A}}$ be the constant matrix computed from $e^{\bar{\mathbf{A}}T} = \mathbf{X}^{-1}(0)\mathbf{X}(T)$. Then (4.69) is Lyapunov equivalent to

$$\dot{\bar{\mathbf{x}}}(t) = \bar{\mathbf{A}}\bar{\mathbf{x}}(t) + \mathbf{P}(t)\mathbf{B}(t)\mathbf{u}(t)$$
$$\bar{\mathbf{y}}(t) = \mathbf{C}(t)\mathbf{P}^{-1}(t)\bar{\mathbf{x}}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

where $\mathbf{P}(t) = e^{\bar{\mathbf{A}}t}\mathbf{X}^{-1}(t)$.

The matrix $\mathbf{P}(t)$ in (4.77) satisfies all conditions in Definition 4.3; thus it is a Lyapunov transformation. The rest of the theorem follows directly from Theorem 4.3. The homogeneous part of Theorem 4.4 is the *theory of Floquet*. It states that if $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ and if $\mathbf{A}(t+T) = \mathbf{A}(t)$ for all t, then its fundamental matrix is of the form $\mathbf{P}^{-1}(t)e^{\bar{\mathbf{A}}t}$, where $\mathbf{P}^{-1}(t)$ is a periodic function. Furthermore, $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is Lyapunov equivalent to $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}}$.

4.7 Time-Varying Realizations

We studied in Section 4.4 the realization problem for linear time-invariant systems. In this section, we study the corresponding problem for linear time-varying systems. The Laplace transform cannot be used here; therefore we study the problem directly in the time domain.

Every linear time-varying system can be described by the input–output description

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t, \tau)\mathbf{u}(\tau)\, d\tau$$

and, if the system is lumped as well, by the state equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned} \tag{4.78}$$

If the state equation is available, the impulse response matrix can be computed from

$$\mathbf{G}(t, \tau) = \mathbf{C}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t - \tau) \qquad \text{for } t \geq \tau \tag{4.79}$$

where $\mathbf{X}(t)$ is a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$. The converse problem is to find a state equation from a given impulse response matrix. An impulse response matrix $\mathbf{G}(t, \tau)$ is said to be *realizable* if there exists $\{\mathbf{A}(t),\ \mathbf{B}(t),\ \mathbf{C}(t),\ \mathbf{D}(t)\}$ to meet (4.79).

Theorem 4.5

A $q \times p$ impulse response matrix $\mathbf{G}(t, \tau)$ is realizable if and only if $\mathbf{G}(t, \tau)$ can be decomposed as

$$\mathbf{G}(t, \tau) = \mathbf{M}(t)\mathbf{N}(\tau) + \mathbf{D}(t)\delta(t - \tau) \tag{4.80}$$

for all $t \geq \tau$, where $\mathbf{M}$, $\mathbf{N}$, and $\mathbf{D}$ are, respectively, $q \times n$, $n \times p$, and $q \times p$ matrices for some integer n.

Proof: If $\mathbf{G}(t, \tau)$ is realizable, there exists a realization that meets (4.79). Identifying $\mathbf{M}(t) = \mathbf{C}(t)\mathbf{X}(t)$ and $\mathbf{N}(\tau) = \mathbf{X}^{-1}(\tau)\mathbf{B}(\tau)$ establishes the necessary part of the theorem.

If $\mathbf{G}(t, \tau)$ can be decomposed as in (4.80), then the n-dimensional state equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{N}(t)\mathbf{u}(t)\\ \mathbf{y}(t) &= \mathbf{M}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)\end{aligned} \tag{4.81}$$

is a realization. Indeed, a fundamental matrix of $\dot{\mathbf{x}} = \mathbf{0} \cdot \mathbf{x}$ is $\mathbf{X}(t) = \mathbf{I}$. Thus the impulse response matrix of (4.81) is

$$\mathbf{M}(t)\mathbf{I} \cdot \mathbf{I}^{-1}\mathbf{N}(\tau) + \mathbf{D}(t)\delta(t - \tau)$$

which equals $\mathbf{G}(t, \tau)$. This shows the sufficiency of the theorem. Q.E.D.

Although Theorem 4.5 can also be applied to time-invariant systems, the result is not useful in practical implementation, as the next example illustrates.

Example 4.10 Consider $g(t) = te^{\lambda t}$ or

$$g(t, \tau) = g(t - \tau) = (t - \tau)e^{\lambda(t-\tau)}$$

It is straightforward to verify

$$g(t - \tau) = [e^{\lambda t} \quad te^{\lambda t}]\begin{bmatrix} -\tau e^{-\lambda\tau} \\ e^{-\lambda\tau} \end{bmatrix}$$

Thus the two-dimensional time-varying state equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -te^{-\lambda t} \\ e^{-\lambda t} \end{bmatrix}u(t)\\ y(t) &= [e^{\lambda t} \quad te^{\lambda t}]\mathbf{x}(t)\end{aligned} \tag{4.82}$$

is a realization of the impulse response $g(t) = te^{\lambda t}$.

The Laplace transform of the impulse response is

$$\hat{g}(s) = \mathcal{L}[te^{\lambda t}] = \frac{1}{(s - \lambda)^2} = \frac{1}{s^2 - 2\lambda s + \lambda^2}$$

Using (4.41), we can readily obtain

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \begin{bmatrix} 2\lambda & -\lambda^2 \\ 1 & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)\\ y(t) &= [0 \quad 1]\mathbf{x}(t)\end{aligned} \tag{4.83}$$

This LTI state equation is a different realization of the same impulse response.This realization is clearly more desirable because it can readily be implemented using an op-amp circuit. The implementation of (4.82) is much more difficult in practice.

PROBLEMS

4.1 An oscillation can be generated by

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \mathbf{x}$$

Show that its solution is

$$\mathbf{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \mathbf{x}(0)$$

4.2 Use two different methods to find the unit-step response of

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = [2 \;\; 3]\mathbf{x}$$

4.3 Discretize the state equation in Problem 4.2 for $T = 1$ and $T = \pi$.

4.4 Find the companion-form and modal-form equivalent equations of

$$\dot{\mathbf{x}} = \begin{bmatrix} -2 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & -2 & -2 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u$$
$$y = [1 \;\; -1 \;\; 0]\mathbf{x}$$

4.5 Find an equivalent state equation of the equation in Problem 4.4 so that all state variables have their largest magnitudes roughly equal to the largest magnitude of the output. If all signals are required to lie inside ± 10 volts and if the input is a step function with magnitude a, what is the permissible largest a?

4.6 Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ \bar{b}_1 \end{bmatrix} u \qquad y = [c_1 \quad \bar{c}_1]\mathbf{x}$$

where the overbar denotes complex conjugate. Verify that the equation can be transformed into

$$\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} + \bar{\mathbf{b}}u \qquad y = \bar{\mathbf{c}}\bar{\mathbf{x}}$$

with

$$\bar{\mathbf{A}} = \begin{bmatrix} 0 & 1 \\ -\lambda\bar{\lambda} & \lambda + \bar{\lambda} \end{bmatrix} \qquad \bar{\mathbf{b}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \bar{\mathbf{c}} = [-2\mathrm{Re}(\bar{\lambda} b_1 c_1) \quad 2\mathrm{Re}(b_1 c_1)]$$

by using the transformation $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$ with

$$\mathbf{Q} = \begin{bmatrix} -\bar{\lambda} b_1 & b_1 \\ -\lambda \bar{b}_1 & \bar{b}_1 \end{bmatrix}$$

4.7 Verify that the Jordan-form equation

$$\dot{\mathbf{x}} = \begin{bmatrix} \lambda & 1 & 0 & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\lambda} & 1 & 0 \\ 0 & 0 & 0 & 0 & \bar{\lambda} & 1 \\ 0 & 0 & 0 & 0 & 0 & \bar{\lambda} \end{bmatrix} \mathbf{x} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \bar{b}_1 \\ \bar{b}_2 \\ \bar{b}_3 \end{bmatrix} u$$
$$y = [c_1 \ \ c_2 \ \ c_3 \ \ \bar{c}_1 \ \ \bar{c}_2 \ \ \bar{c}_3]\mathbf{x}$$

can be transformed into

$$\dot{\bar{\mathbf{x}}} = \begin{bmatrix} \bar{\mathbf{A}} & \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0} & \bar{\mathbf{A}} & \mathbf{I}_2 \\ \mathbf{0} & \mathbf{0} & \bar{\mathbf{A}} \end{bmatrix} \bar{\mathbf{x}} + \begin{bmatrix} \bar{\mathbf{b}} \\ \bar{\mathbf{b}} \\ \bar{\mathbf{b}} \end{bmatrix} u \qquad y = [\bar{\mathbf{c}}_1 \ \ \bar{\mathbf{c}}_2 \ \ \bar{\mathbf{c}}_3]\bar{\mathbf{x}}$$

where $\bar{\mathbf{A}}$, $\bar{\mathbf{b}}$, and $\bar{\mathbf{c}}_i$ are defined in Problem 4.6 and $\mathbf{I}_2$ is the unit matrix of order 2. [*Hint*: Change the order of the state variables from $[x_1 \ x_2 \ x_3 \ x_4 \ x_5 \ x_6]'$ to $[x_1 \ x_4 \ x_2 \ x_5 \ x_3 \ x_6]'$ and then apply the equivalence transformation $\mathbf{x} = \mathbf{Q}\bar{\mathbf{x}}$ with $\mathbf{Q} = \text{diag}(\mathbf{Q}_1, \mathbf{Q}_2, \mathbf{Q}_3)$.]

4.8 Are the two sets of state equations

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 2 \\ 0 & 2 & 2 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = [1 \ -1 \ 0]\mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} u \qquad y = [1 \ -1 \ 0]\mathbf{x}$$

equivalent? Are they zero-state equivalent?

4.9 Verify that the transfer matrix in (4.33) has the following realization:

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 \mathbf{I}_q & \mathbf{I}_q & \mathbf{0} & \cdots & \mathbf{0} \\ -\alpha_2 \mathbf{I}_q & \mathbf{0} & \mathbf{I}_q & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & & \vdots \\ -\alpha_{r-1} \mathbf{I}_q & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{I}_q \\ -\alpha_r \mathbf{I}_q & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix} \mathbf{x} + \begin{bmatrix} \mathbf{N}_1 \\ \mathbf{N}_2 \\ \vdots \\ \mathbf{N}_{r-1} \\ \mathbf{N}_r \end{bmatrix} \mathbf{u}$$
$$\mathbf{y} = [\mathbf{I}_q \ \ \mathbf{0} \ \ \mathbf{0} \ \ \cdots \ \ \mathbf{0}]\mathbf{x}$$

This is called the *observable canonical form realization* and has dimension rq. It is dual to (4.34).

4.10 Consider the 1×2 proper rational matrix

$$\hat{\mathbf{G}}(s) = [d_1 \quad d_2] + \frac{1}{s^4 + \alpha_1 s^3 + \alpha_2 s^2 + \alpha_3 s + \alpha_4}$$

$$\times\ [\beta_{11}s^3 + \beta_{21}s^2 + \beta_{31}s + \beta_{41} \quad \beta_{12}s^3 + \beta_{22}s^2 + \beta_{32}s + \beta_{42}]$$

Show that its observable canonical form realization can be deduced from Problem 4.9 as

$$\dot{\mathbf{x}} = \begin{bmatrix} -\alpha_1 & 1 & 0 & 0 \\ -\alpha_2 & 0 & 1 & 0 \\ -\alpha_3 & 0 & 0 & 1 \\ -\alpha_4 & 0 & 0 & 0 \end{bmatrix} \mathbf{x} + \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \\ \beta_{31} & \beta_{32} \\ \beta_{41} & \beta_{42} \end{bmatrix} \mathbf{u}$$

$$y = [1\ \ 0\ \ 0\ \ \ 0]\mathbf{x} + [d_1\ \ d_2]\mathbf{u}$$

4.11 Find a realization for the proper rational matrix

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{2}{s+1} & \dfrac{2s-3}{(s+1)(s+2)} \\ \dfrac{s-2}{s+1} & \dfrac{s}{s+2} \end{bmatrix}$$

4.12 Find a realization for each column of $\hat{\mathbf{G}}(s)$ in Problem 4.11 and then connect them, as shown in Fig. 4.4(a), to obtain a realization of $\hat{\mathbf{G}}(s)$. What is the dimension of this realization? Compare this dimension with the one in Problem 4.11.

4.13 Find a realization for each row of $\hat{\mathbf{G}}(s)$ in Problem 4.11 and then connect them, as shown in Fig. 4.4(b), to obtain a realization of $\hat{\mathbf{G}}(s)$. What is the dimension of this realization? Compare this dimension with the ones in Problems 4.11 and 4.12.

4.14 Find a realization for

$$\hat{\mathbf{G}}(s) = \begin{bmatrix} \dfrac{-(12s+6)}{3s+34} & \dfrac{22s+23}{3s+34} \end{bmatrix}$$

4.15 Consider the n-dimensional state equation

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}u \qquad y = \mathbf{c}\mathbf{x}$$

Let $\hat{g}(s)$ be its transfer function. Show that $\hat{g}(s)$ has m zeros or, equivalently, the numerator of $\hat{g}(s)$ has degree m if and only if

$$\mathbf{c}\mathbf{A}^i\mathbf{b} = 0 \qquad \text{for } i = 0, 1, 2, \ldots, n-m-2$$

and $\mathbf{c}\mathbf{A}^{n-m-1}\mathbf{b} \neq 0$. Or, equivalently, the difference between the degrees of the denominator and numerator of $\hat{g}(s)$ is $\alpha = n - m$ if and only if

$$\mathbf{c}\mathbf{A}^{\alpha-1}\mathbf{b} \neq 0 \quad \text{and} \quad \mathbf{c}\mathbf{A}^i\mathbf{b} = 0$$

for $i = 0, 1, 2, \ldots, \alpha - 2$.

4.16 Find fundamental matrices and state transition matrices for

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 0 & t \end{bmatrix} \mathbf{x}$$

and

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \mathbf{x}$$

4.17 Show $\partial \mathbf{\Phi}(t_0, t)/\partial t = -\mathbf{\Phi}(t_0, t)\mathbf{A}(t)$.

4.18 Given

$$\mathbf{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$$

show

$$\det \mathbf{\Phi}(t, t_0) = \exp\left[\int_{t_0}^{t} (a_{11}(\tau) + a_{22}(\tau))\, d\tau\right]$$

4.19 Let

$$\mathbf{\Phi}(t, t_0) = \begin{bmatrix} \mathbf{\Phi}_{11}(t, t_0) & \mathbf{\Phi}_{12}(t, t_0) \\ \mathbf{\Phi}_{21}(t, t_0) & \mathbf{\Phi}_{22}(t, t_0) \end{bmatrix}$$

be the state transition matrix of

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} \mathbf{A}_{11}(t) & \mathbf{A}_{12}(t) \\ \mathbf{0} & \mathbf{A}_{22}(t) \end{bmatrix} \mathbf{x}(t)$$

Show that $\mathbf{\Phi}_{21}(t, t_0) = \mathbf{0}$ for all t and t_0 and that $(\partial/\partial t)\mathbf{\Phi}_{ii}(t, t_0) = \mathbf{A}_{ii}\mathbf{\Phi}_{ii}(t, t_0)$, for $i = 1, 2$.

4.20 Find the state transition matrix of

$$\dot{\mathbf{x}} = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} \mathbf{x}$$

4.21 Verify that $\mathbf{X}(t) = e^{\mathbf{A}t}\mathbf{C}e^{\mathbf{B}t}$ is the solution of

$$\dot{\mathbf{X}} = \mathbf{AX} + \mathbf{XB} \qquad \mathbf{X}(0) = \mathbf{C}$$

4.22 Show that if $\dot{\mathbf{A}}(t) = \mathbf{A}_1\mathbf{A}(t) - \mathbf{A}(t)\mathbf{A}_1$, then

$$\mathbf{A}(t) = e^{\mathbf{A}_1 t}\mathbf{A}(0)e^{-\mathbf{A}_1 t}$$

Show also that the eigenvalues of $\mathbf{A}(t)$ are independent of t.

4.23 Find an equivalent time-invariant state equation of the equation in Problem 4.20.

4.24 Transform a time-invariant $(\mathbf{A}, \mathbf{B}, \mathbf{C})$ into $(\mathbf{0}, \bar{\mathbf{B}}(t), \bar{\mathbf{C}}(t))$ by a time-varying equivalence transformation.

4.25 Find a time-varying realization and a time-invariant realization of the impulse response $g(t) = t^2 e^{\lambda t}$.

4.26 Find a realization of $g(t, \tau) = \sin t(e^{-(t-\tau)}) \cos \tau$. Is it possible to find a time-invariant state equation realization?