Chapter 5

Stability

5.1 Introduction

Systems are designed to perform some tasks or to process signals. If a system is not stable, the system may burn out, disintegrate, or saturate when a signal, no matter how small, is applied. Therefore an unstable system is useless in practice and stability is a basic requirement for all systems. In addition to stability, systems must meet other requirements, such as to track desired signals and to suppress noise, to be really useful in practice.

The response of linear systems can always be decomposed as the zero-state response and the zero-input response. It is customary to study the stabilities of these two responses separately. We will introduce the BIBO (bounded-input bounded-output) stability for the zero-state response and marginal and asymptotic stabilities for the zero-input response. We study first the time-invariant case and then the time-varying case.

5.2 Input–Output Stability of LTI Systems

Consider a SISO linear time-invariant (LTI) system described by

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau = \int_0^t g(\tau)u(t-\tau)\,d\tau \tag{5.1}$$

where $g(t)$ is the impulse response or the output excited by an impulse input applied at $t = 0$. Recall that in order to be describable by (5.1), the system must be linear, time-invariant, and causal. In addition, the system must be initially relaxed at $t = 0$.

An input $u(t)$ is said to be *bounded* if $u(t)$ does not grow to positive or negative infinity or, equivalently, there exists a constant u_m such that

$$|u(t)| \le u_m < \infty \qquad \text{for all } t \ge 0$$

A system is said to be *BIBO stable* (bounded-input bounded-output stable) if every bounded input excites a bounded output. This stability is defined for the zero-state response and is applicable only if the system is initially relaxed.

Theorem 5.1

A SISO system described by (5.1) is BIBO stable if and only if $g(t)$ is absolutely integrable in $[0, \infty)$, or

$$\int_0^\infty |g(t)|dt \le M < \infty$$

for some constant M.

Proof: First we show that if $g(t)$ is absolutely integrable, then every bounded input excites a bounded output. Let $u(t)$ be an arbitrary input with $|u(t)| \le u_m < \infty$ for all $t \ge 0$. Then

$$\begin{aligned} |y(t)| &= \left| \int_0^t g(\tau)u(t-\tau)\, d\tau \right| \le \int_0^t |g(\tau)||u(t-\tau)|\, d\tau \\ &\le u_m \int_0^\infty |g(\tau)|d\tau \le u_m M \end{aligned}$$

Thus the output is bounded. Next we show that if $g(t)$ is not absolutely integrable, then the system is not BIBO stable. If $g(t)$ is not absolutely integrable, then for any arbitrarily large N, there exists a t_1 such that

$$\int_0^{t_1} |g(\tau)|\, d\tau \ge N$$

Let us choose

$$u(t_1 - \tau) = \begin{cases} 1 & \text{if } g(\tau) \ge 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases}$$

Clearly u is bounded. The output excited by this input equals

$$y(t_1) = \int_0^{t_1} g(\tau)u(t_1 - \tau)\, d\tau = \int_0^{t_1} |g(\tau)|\, d\tau \ge N$$

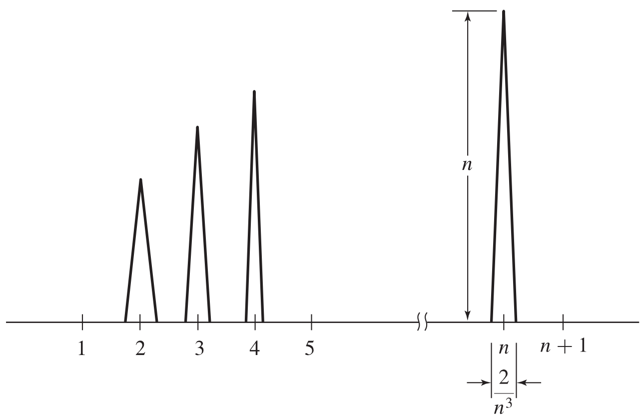
Because $y(t_1)$ can be arbitrarily large, we conclude that a similar bounded input can excite an unbounded output. This completes the proof of Theorem 5.1. Q.E.D.

A function that is absolutely integrable may not be bounded or may not approach zero as $t \to \infty$. Indeed, consider the function defined by

$$f(t-n) = \begin{cases} n + (t-n)n^4 & \text{for } n - 1/n^3 \le t \le n \\ n - (t-n)n^4 & \text{for } n < t \le n + 1/n^3 \end{cases}$$

for $n = 2, 3, \ldots$ and plotted in Fig. 5.1. The area under each triangle is $1/n^2$. Thus the absolute

Figure 5.1 Function.



integration of the function equals $\sum_{n=2}^{\infty}(1/n^2) < \infty$. This function is absolutely integrable but is not bounded and does not approach zero as $t \to \infty$.

Theorem 5.2

If a system with impulse response $g(t)$ is BIBO stable, then, as $t \to \infty$:

1. The output excited by $u(t) = a,\ \text{for } t \geq 0$, approaches $\hat{g}(0) \cdot a$.
2. The output excited by $u(t) = \sin \omega_o t,\ \text{for } t \geq 0$, approaches

$$|\hat{g}(j\omega_o)| \sin(\omega_o t + \measuredangle \hat{g}(j\omega_o))$$

where $\hat{g}(s)$ is the Laplace transform of $g(t)$ or

$$\hat{g}(s) = \int_0^{\infty} g(\tau)e^{-s\tau}\, d\tau \tag{5.2}$$

Proof: If $u(t) = a$ for all $t \geq 0$, then (5.1) becomes

$$y(t) = \int_0^t g(\tau)u(t-\tau)\, d\tau = a \int_0^t g(\tau)\, d\tau$$

which implies

$$y(t) \to a \int_0^{\infty} g(\tau)\, d\tau = a\hat{g}(0) \qquad \text{as } t \to \infty$$

where we have used (5.2) with $s = 0$. This establishes the first part of Theorem 5.2. If $u(t) = \sin \omega_o t$, then (5.1) becomes

$$\begin{aligned} y(t) &= \int_0^t g(\tau) \sin \omega_o (t-\tau)\, d\tau \\ &= \int_0^t g(\tau)\, [\sin \omega_o t \cos \omega_o \tau - \cos \omega_o t \sin \omega_o \tau]\, d\tau \\ &= \sin \omega_o t \int_0^t g(\tau) \cos \omega_o \tau\, d\tau - \cos \omega_o t \int_0^t g(\tau) \sin \omega_o \tau\, d\tau \end{aligned}$$

Thus we have, as $t \to \infty$,

$$y(t) \to \sin \omega_o t \int_0^\infty g(\tau) \cos \omega_o \tau \, d\tau - \cos \omega_o t \int_0^\infty g(\tau) \sin \omega_o \tau \, d\tau \tag{5.3}$$

If $g(t)$ is absolutely integrable, we can replace s by $j\omega$ in (5.2) to yield

$$\hat{g}(j\omega) = \int_0^\infty g(\tau)[\cos \omega\tau - j \sin \omega\tau] \, d\tau$$

The impulse response $g(t)$ is assumed implicitly to be real; thus we have

$$\text{Re}[\hat{g}(j\omega)] = \int_0^\infty g(\tau) \cos \omega\tau \, d\tau$$

and

$$\text{Im}[\hat{g}(j\omega)] = -\int_0^\infty g(\tau) \sin \omega\tau \, d\tau$$

where Re and Im denote, respectively, the real part and imaginary part. Substituting these into (5.3) yields

$$\begin{aligned} y(t) &\to \sin \omega_o t (\text{Re}[\hat{g}(j\omega_o)]) + \cos \omega_o t (\text{Im}[\hat{g}(j\omega_o)]) \\ &= |\hat{g}(j\omega_o)| \sin(\omega_o t + \measuredangle \hat{g}(j\omega_o)) \end{aligned}$$

This completes the proof of Theorem 5.2. Q.E.D.

Theorem 5.2 is a basic result; filtering of signals is based essentially on this theorem. Next we state the BIBO stability condition in terms of proper rational transfer functions.

Theorem 5.3

A SISO system with proper rational transfer function $\hat{g}(s)$ is BIBO stable if and only if every pole of $\hat{g}(s)$ has a negative real part or, equivalently, lies inside the left-half s-plane.

If $\hat{g}(s)$ has pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{s - p_i}, \quad \frac{1}{(s - p_i)^2}, \quad \ldots, \quad \frac{1}{(s - p_i)^{m_i}}$$

Thus the inverse Laplace transform of $\hat{g}(s)$ or the impulse response contains the factors

$$e^{p_i t}, \quad te^{p_i t}, \quad \ldots, \quad t^{m_i - 1} e^{p_i t}$$

It is straightforward to verify that every such term is absolutely integrable if and only if p_i has a negative real part. Using this fact, we can establish Theorem 5.3.

Example 5.1 Consider the positive feedback system shown in Fig. 2.5(a). Its impulse response was computed in (2.9) as

$$g(t) = \sum_{i=1}^\infty a^i \delta(t - i)$$

where the gain a can be positive or negative. The impulse is defined as the limit of the pulse in Fig. 2.3 and can be considered to be positive. Thus we have

$$|g(t)| = \sum_{i=1}^{\infty} |a|^i \delta(t-i)$$

and

$$\int_0^{\infty} |g(t)|\, dt = \sum_{i=1}^{\infty} |a|^i = \begin{cases} \infty & \text{if } |a| \geq 1 \\ |a|/(1-|a|) < \infty & \text{if } |a| < 1 \end{cases}$$

Thus we conclude that the positive feedback system in Fig. 2.5(a) is BIBO stable if and only if the gain a has a magnitude less than 1.

The transfer function of the system was computed in (2.12) as

$$\hat{g}(s) = \frac{ae^{-s}}{1-ae^{-s}}$$

It is an irrational function of s and Theorem 5.3 is not applicable. In this case, it is simpler to use Theorem 5.1 to check its stability.

For multivariable systems, we have the following results.

Theorem 5.M1

A multivariable system with impulse response matrix $\mathbf{G}(t) = [g_{ij}(t)]$ is BIBO stable if and only if every $g_{ij}(t)$ is absolutely integrable in $[0, \infty)$.

Theorem 5.M3

A multivariable system with proper rational transfer matrix $\hat{\mathbf{G}}(s) = [\hat{g}_{ij}(s)]$ is BIBO stable if and only if every pole of every $\hat{g}_{ij}(s)$ has a negative real part.

We now discuss the BIBO stability of state equations. Consider

$$\begin{aligned} \dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t) \end{aligned} \tag{5.4}$$

Its transfer matrix is

$$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I}-\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Thus Equation (5.4) or, to be more precise, the zero-state response of (5.4) is BIBO stable if and only if every pole of $\hat{\mathbf{G}}(s)$ has a negative real part. Recall that every pole of every entry of $\hat{\mathbf{G}}(s)$ is called a pole of $\hat{\mathbf{G}}(s)$.

We discuss the relationship between the poles of $\hat{\mathbf{G}}(s)$ and the eigenvalues of $\mathbf{A}$. Because of

$$\hat{\mathbf{G}}(s) = \frac{1}{\det(s\mathbf{I}-\mathbf{A})} \mathbf{C}[\mathrm{Adj}(s\mathbf{I}-\mathbf{A})]\mathbf{B} + \mathbf{D} \tag{5.5}$$

every pole of $\hat{\mathbf{G}}(s)$ is an eigenvalue of $\mathbf{A}$. Thus if every eigenvalue of $\mathbf{A}$ has a negative real part, then every pole has a negative real part and (5.4) is BIBO stable. On the other hand, because of possible cancellation in (5.5), not every eigenvalue is a pole. Thus, even if $\mathbf{A}$ has some eigenvalues with zero or positive real part, (5.4) may still be BIBO stable, as the next example shows.

EXAMPLE 5.2 Consider the network shown in Fig. 4.2(b). Its state equation was derived in Example 4.4 as

$$\dot{x}(t) = x(t) + 0 \cdot u(t) \qquad y(t) = 0.5x(t) + 0.5u(t) \tag{5.6}$$

The A-matrix is 1 and its eigenvalue is 1. It has a positive real part. The transfer function of the equation is

$$\hat{g}(s) = 0.5(s-1)^{-1} \cdot 0 + 0.5 = 0.5$$

The transfer function equals 0.5. It has no pole and no condition to meet. Thus (5.6) is BIBO stable even though it has an eigenvalue with a positive real part. We mention that BIBO stability does not say anything about the zero-input response, which will be discussed later.

5.2.1 Discrete-Time Case

Consider a discrete-time SISO system described by

$$y[k] = \sum_{m=0}^{k} g[k-m]u[m] = \sum_{m=0}^{k} g[m]u[k-m] \tag{5.7}$$

where $g[k]$ is the impulse response sequence or the output sequence excited by an impulse sequence applied at $k = 0$. Recall that in order to be describable by (5.7), the discrete-time system must be linear, time-invariant, and causal. In addition, the system must be initially relaxed at $k = 0$.

An input sequence $u[k]$ is said to be *bounded* if $u[k]$ does not grow to positive or negative infinity or there exists a constant u_m such that

$$|u[k]| \le u_m < \infty \qquad \text{for } k = 0, 1, 2, \ldots$$

A system is said to be *BIBO stable* (bounded-input bounded-output stable) if every bounded-input sequence excites a bounded-output sequence. This stability is defined for the zero-state response and is applicable only if the system is initially relaxed.

Theorem 5.D1

A discrete-time SISO system described by (5.7) is BIBO stable if and only if $g[k]$ is absolutely summable in $[0, \infty)$ or

$$\sum_{k=0}^{\infty} |g[k]| \le M < \infty$$

for some constant M.

Its proof is similar to the proof of Theorem 5.1 and will not be repeated. We give a discrete counterpart of Theorem 5.2 in the following.

Theorem 5.D2

If a discrete-time system with impulse response sequence $g[k]$ is BIBO stable, then, as $k \to \infty$:

1. The output excited by $u[k] = a$, for $k \geq 0$, approaches $\hat{g}(1) \cdot a$.
2. The output excited by $u[k] = \sin \omega_o k$, for $k \geq 0$, approaches

$$|\hat{g}(e^{j\omega_o})| \sin(\omega_o k + \measuredangle \hat{g}(e^{j\omega_o}))$$

where $\hat{g}(z)$ is the z-transform of $g[k]$ or

$$\hat{g}(z) = \sum_{m=0}^{\infty} g[m] z^{-m} \tag{5.8}$$

Proof: If $u[k] = a$ for all $k \geq 0$, then (5.7) becomes

$$y[k] = \sum_{0}^{k} g[m]u[k-m] = a \sum_{k=0}^{k} g[m]$$

which implies

$$y[k] \to a \sum_{m=0}^{\infty} g[m] = a\hat{g}(1) \qquad \text{as } k \to \infty$$

where we have used (5.8) with $z = 1$. This establishes the first part of Theorem 5.D2. If $u[k] = \sin \omega_o k$, then (5.7) becomes

$$\begin{aligned} y[k] &= \sum_{m=0}^{k} g[m] \sin \omega_o [k-m] \\ &= \sum_{m=0}^{k} g[m](\sin \omega_o k \cos \omega_o m - \cos \omega_o k \sin \omega_o m) \\ &= \sin \omega_o k \sum_{m=0}^{k} g[m] \cos \omega_o m - \cos \omega_o k \sum_{m=0}^{k} g[m] \sin \omega_o m \end{aligned}$$

Thus we have, as $k \to \infty$,

$$y[k] \to \sin \omega_o k \sum_{m=0}^{\infty} g[m] \cos \omega_o m - \cos \omega_o k \sum_{m=0}^{\infty} g[m] \sin \omega_o m \tag{5.9}$$

If $g[k]$ is absolutely summable, we can replace z by $e^{j\omega}$ in (5.8) to yield

$$\hat{g}(e^{j\omega}) = \sum_{m=0}^{\infty} g[m] e^{-j\omega m} = \sum_{m=0}^{\infty} g[m][\cos \omega m - j \sin \omega m]$$

Thus (5.9) becomes

$$y[k] \to \sin \omega_o k(\mathrm{Re}[\hat{g}(e^{j\omega_o})]) + \cos \omega_o k(\mathrm{Im}[\hat{g}(e^{j\omega_o})])$$
$$= |\hat{g}(e^{j\omega_o})| \sin(\omega_o k + \measuredangle \hat{g}(e^{j\omega_o}))$$

This completes the proof of Theorem 5.D2. Q.E.D.

Theorem 5.D2 is a basic result in digital signal processing. Next we state the BIBO stability in terms of discrete proper rational transfer functions.

Theorem 5.D3

A discrete-time SISO system with proper rational transfer function $\hat{g}(z)$ is BIBO stable if and only if every pole of $\hat{g}(z)$ has a magnitude less than 1 or, equivalently, lies inside the unit circle on the z-plane.

If $\hat{g}(z)$ has pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{z - p_i}, \quad \frac{1}{(z - p_i)^2}, \quad \cdots, \quad \frac{1}{(z - p_i)^{m_i}}$$

Thus the inverse z-transform of $\hat{g}(z)$ or the impulse response sequence contains the factors

$$p_i^k, \quad kp_i^k, \quad \cdots, \quad k^{m_i - 1} p_i^k$$

It is straightforward to verify that every such term is absolutely summable if and only if p_i has a magnitude less than 1. Using this fact, we can establish Theorem 5.D3.

In the continuous-time case, an absolutely integrable function $f(t)$, as shown in Fig. 5.1, may not be bounded and may not approach zero as $t \to \infty$. In the discrete-time case, if $g[k]$ is absolutely summable, then it must be bounded and approach zero as $k \to \infty$. However, the converse is not true as the next example shows.

EXAMPLE 5.3 Consider a discrete-time LTI system with impulse response sequence $g[k] = 1/k$, for $k = 1, 2, \ldots,$ and $g[0] = 0$. We compute

$$S := \sum_{k=0}^{\infty} |g[k]| = \sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$
$$= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \left(\frac{1}{9} + \cdots + \frac{1}{16}\right) + \cdots$$

There are two terms, each is $\frac{1}{4}$ or larger, in the first pair of parentheses; therefore their sum is larger than $\frac{1}{2}$. There are four terms, each is $\frac{1}{8}$ or larger, in the second pair of parentheses; therefore their sum is larger than $\frac{1}{2}$. Proceeding forward we conclude

$$S > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots = \infty$$

This impulse response sequence is bounded and approaches 0 as $k \to \infty$ but is not absolutely summable. Thus the discrete-time system is not BIBO stable according to Theorem 5.D1. The transfer function of the system can be shown to equal

$$\hat{g}(z) = -\ln(1 + z^{-1})$$

It is not a rational function of z and Theorem 5.D3 is not applicable.

For multivariable discrete-time systems, we have the following results.

Theorem 5.MD1

A MIMO discrete-time system with impulse response sequence matrix $\mathbf{G}[k] = \left[g_{ij}[k]\right]$ is BIBO stable if and only if every $g_{ij}[k]$ is absolutely summable.

Theorem 5.MD3

A MIMO discrete-time system with discrete proper rational transfer matrix $\hat{\mathbf{G}}(z) = \left[\hat{g}_{ij}(z)\right]$ is BIBO stable if and only if every pole of every $\hat{g}_{ij}(z)$ has a magnitude less than 1.

We now discuss the BIBO stability of discrete-time state equations. Consider

$$\begin{aligned} \mathbf{x}[k+1] &= \mathbf{A}\mathbf{x}[k] + \mathbf{B}\mathbf{u}[k] \\ \mathbf{y}[k] &= \mathbf{C}\mathbf{x}[k] + \mathbf{D}\mathbf{u}[k] \end{aligned} \tag{5.10}$$

Its discrete transfer matrix is

$$\hat{\mathbf{G}}(z) = \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

Thus Equation (5.10) or, to be more precise, the zero-state response of (5.10) is BIBO stable if and only if every pole of $\hat{\mathbf{G}}(z)$ has a magnitude less than 1.

We discuss the relationship between the poles of $\hat{\mathbf{G}}(z)$ and the eigenvalues of $\mathbf{A}$. Because of

$$\hat{\mathbf{G}}(z) = \frac{1}{\det(z\mathbf{I} - \mathbf{A})}\mathbf{C}[\text{Adj}(z\mathbf{I} - \mathbf{A})]\mathbf{B} + \mathbf{D}$$

every pole of $\hat{\mathbf{G}}(z)$ is an eigenvalue of $\mathbf{A}$. Thus if every eigenvalue of $\mathbf{A}$ has a magnitude less than 1, then (5.10) is BIBO stable. On the other hand, even if $\mathbf{A}$ has some eigenvalues with magnitude 1 or larger, (5.10) may, as in the continuous-time case, still be BIBO stable.

5.3 Internal Stability

The BIBO stability is defined for the zero-state response. Now we study the stability of the zero-input response or the response of

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) \tag{5.11}$$

excited by nonzero initial state $\mathbf{x}_o$. Clearly, the solution of (5.11) is

$$\mathbf{x}(t) = e^{\mathbf{A}t}\mathbf{x}_o \tag{5.12}$$

> ***Definition 5.1*** *The zero-input response of (5.4) or the equation* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is* marginally stable *or* stable in the sense of Lyapunov *if every finite initial state* $\mathbf{x}_o$ *excites a bounded response. It is* asymptotically stable *if every finite initial state excites a bounded response, which, in addition, approaches* $\mathbf{0}$ *as* $t \to \infty$.

We mention that this definition is applicable only to linear systems. The definition that is applicable to both linear and nonlinear systems must be defined using the concept of equivalence states and can be found, for example, in Reference [6, pp. 401–403]. This text studies only linear systems; therefore we use the simplified Definition 5.1.

Theorem 5.4

1. The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have zero or negative real parts and those with zero real parts are simple roots of the minimal polynomial of $\mathbf{A}$.

2. The equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have negative real parts.

We first mention that any (algebraic) equivalence transformation will not alter the stability of a state equation. Consider $\bar{\mathbf{x}} = \mathbf{P}\mathbf{x}$, where $\mathbf{P}$ is a nonsingular matrix. Then $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is equivalent to $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}} = \mathbf{P}\mathbf{A}\mathbf{P}^{-1}\bar{\mathbf{x}}$. Because $\mathbf{P}$ is nonsingular, if $\mathbf{x}$ is bounded, so is $\bar{\mathbf{x}}$; if $\mathbf{x}$ approaches $\mathbf{0}$ as $t \to \infty$, so does $\bar{\mathbf{x}}$. Thus we may study the stability of $\mathbf{A}$ by using $\bar{\mathbf{A}}$. Note that the eigenvalues of $\mathbf{A}$ and of $\bar{\mathbf{A}}$ are the same as discussed in Section 4.3.

The response of $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}\bar{\mathbf{x}}$ excited by $\bar{\mathbf{x}}(0)$ equals $\bar{\mathbf{x}}(t) = e^{\bar{\mathbf{A}}t}\bar{\mathbf{x}}(0)$. It is clear that the response is bounded if and only if every entry of $e^{\bar{\mathbf{A}}t}$ is bounded for all $t \geq 0$. If $\bar{\mathbf{A}}$ is in Jordan form, then $e^{\bar{\mathbf{A}}t}$ is of the form shown in (3.48). Using (3.48), we can show that if an eigenvalue has a negative real part, then every entry of (3.48) is bounded and approaches 0 as $t \to \infty$. If an eigenvalue has zero real part and has no Jordan block of order 2 or higher, then the corresponding entry in (3.48) is a constant or is sinusoidal for all t and is, therefore, bounded. This establishes the sufficiency of the first part of Theorem 5.4. If $\bar{\mathbf{A}}$ has an eigenvalue with a positive real part, then every entry in (3.48) will grow without bound. If $\bar{\mathbf{A}}$ has an eigenvalue with zero real part and its Jordan block has order 2 or higher, then (3.48) has at least one entry that grows unbounded. This completes the proof of the first part. To be asymptotically stable, every entry of (3.48) must approach zero as $t \to \infty$. Thus no eigenvalue with zero real part is permitted. This establishes the second part of the theroem.

EXAMPLE 5.4 Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}$$

Its characteristic polynomial is $\Delta(\lambda) = \lambda^2(\lambda + 1)$ and its mimimal polynomial is $\psi(\lambda) = \lambda(\lambda + 1)$. The matrix has eigenvalues 0, 0, and -1. The eigenvalue 0 is a simple root of the minimal polynomial. Thus the equation is marginally stable. The equation

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \mathbf{x}$$

is not marginally stable, however, because its minimal polynomial is $\lambda^2(\lambda + 1)$ and $\lambda = 0$ is not a simple root of the minimal polynomial.

As discussed earlier, every pole of the transfer matrix

$$\hat{\mathbf{G}}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

is an eigenvalue of $\mathbf{A}$. Thus asymptotic stability implies BIBO stability. Note that asymptotic stability is defined for the zero-input response, whereas BIBO stability is defined for the zero-state response. The system in Example 5.2 has eigenvalue 1 and is not asymptotically stable; however, it is BIBO stable. Thus BIBO stability, in general, does not imply asymptotic stability. We mention that marginal stability is useful only in the design of oscillators. Other than oscillators, every physical system is designed to be asymptotically stable or BIBO stable with some additional conditions, as we will discuss in Chapter 7.

5.3.1 Discrete-Time Case

This subsection studies the internal stability of discrete-time systems or the stability of

$$\mathbf{x}[k + 1] = \mathbf{A}\mathbf{x}[k] \tag{5.13}$$

excited by nonzero initial state $\mathbf{x}_o$. The solution of (5.13) is, as derived in (4.20),

$$\mathbf{x}[k] = \mathbf{A}^k \mathbf{x}_o \tag{5.14}$$

Equation (5.13) is said to be *marginally stable* or *stable in the sense of Lyapunov* if every finite initial state $\mathbf{x}_o$ excites a bounded response. It is *asymptotically stable* if every finite initial state excites a bounded response, which, in addition, approaches $\mathbf{0}$ as $k \to \infty$. These definitions are identical to the continuous-time case.

Theorem 5.D4

1. The equation $\mathbf{x}[k + 1] = \mathbf{A}\mathbf{x}[k]$ is marginally stable if and only if all eigenvalues of $\mathbf{A}$ have magnitudes less than or equal to 1 and those equal to 1 are simple roots of the minimal polynomial of $\mathbf{A}$.

2. The equation $\mathbf{x}[k + 1] = \mathbf{A}\mathbf{x}[k]$ is asymptotically stable if and only if all eigenvalues of $\mathbf{A}$ have magnitudes less than 1.

As in the continuous-time case, any (algebraic) equivalence transformation will not alter the stability of a state equation. Thus we can use Jordan form to establish the theorem. The proof is similar to the continuous-time case and will not be repeated. Asymptotic stability implies BIBO stability but not the converse. We mention that marginal stability is useful only in the design of discrete-time oscillators. Other than oscillators, every discrete-time physical system is designed to be asymptotically stable or BIBO stable with some additional conditions, as we will discuss in Chapter 7.

5.4 Lyapunov Theorem

This section introduces a different method of checking asymptotic stability of $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$. For convenience, we call $\mathbf{A}$ stable if every eigenvalue of $\mathbf{A}$ has a negative real part.

Theorem 5.5

All eigenvalues of $\mathbf{A}$ have negative real parts if and only if for any given positive definite symmetric matrix $\mathbf{N}$, the *Lyapunov* equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N} \tag{5.15}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

Corollary 5.5

All eigenvalues of an $n \times n$ matrix $\mathbf{A}$ have negative real parts if and only if for any given $m \times n$ matrix $\bar{\mathbf{N}}$ with $m < n$ and with the property

$$\text{rank } O := \text{rank} \begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{N}}\mathbf{A} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1} \end{bmatrix} = n \quad \text{(full column rank)} \tag{5.16}$$

where O is an $nm \times n$ matrix, the Lyapunov equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\bar{\mathbf{N}}'\bar{\mathbf{N}} =: -\mathbf{N} \tag{5.17}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

For any $\bar{\mathbf{N}}$, the matrix $\mathbf{N}$ in (5.17) is positive semidefinite (Theorem 3.7). Theorem 5.5 and its corollary are valid for any given $\mathbf{N}$; therefore we shall use the simplest possible $\mathbf{N}$. Even so, using them to check stability of $\mathbf{A}$ is not simple. It is much simpler to compute, using MATLAB, the eigenvalues of $\mathbf{A}$ and then check their real parts. Thus the importance of Theorem 5.5 and its corollary is not in checking the stability of $\mathbf{A}$ but rather in studying the stability of nonlinear systems. They are essential in using the so-called second method of Lyapunov. We mention that Corollary 5.5 can be used to prove the Routh–Hurwitz test. See Reference [6, pp. 417–419].

Proof of Theorem 5.5 *Necessity:* Equation (5.15) is a special case of (3.59) with $\mathbf{A} = \mathbf{A}'$ and $\mathbf{B} = \mathbf{A}$. Because $\mathbf{A}$ and $\mathbf{A}'$ have the same set of eigenvalues, if $\mathbf{A}$ is stable, $\mathbf{A}$ has no two eigenvalues such that $\lambda_i + \lambda_j = 0$. Thus the Lyapunov equation is nonsingular and has a unique solution $\mathbf{M}$ for any $\mathbf{N}$. We claim that the solution can be expressed as

$$\mathbf{M} = \int_0^\infty e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}dt \tag{5.18}$$

Indeed, substituting (5.18) into (5.15) yields

$$\begin{aligned}\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} &= \int_0^\infty \mathbf{A}'e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\,dt + \int_0^\infty e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\mathbf{A}\,dt \\ &= \int_0^\infty \frac{d}{dt}\left(e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\right)dt = e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\Big|_{t=0}^{\infty} \\ &= \mathbf{0} - \mathbf{N} = -\mathbf{N}\end{aligned} \tag{5.19}$$

where we have used the fact $e^{\mathbf{A}t} = \mathbf{0}$ at $t = \infty$ for stable $\mathbf{A}$. This shows that the $\mathbf{M}$ in (5.18) is the solution. It is clear that if $\mathbf{N}$ is symmetric, so is $\mathbf{M}$. Let us decompose $\mathbf{N}$ as $\mathbf{N} = \bar{\mathbf{N}}'\bar{\mathbf{N}}$, where $\bar{\mathbf{N}}$ is nonsingular (Theorem 3.7) and consider

$$\mathbf{x}'\mathbf{M}\mathbf{x} = \int_0^\infty \mathbf{x}'e^{\mathbf{A}'t}\bar{\mathbf{N}}'\bar{\mathbf{N}}e^{\mathbf{A}t}\mathbf{x}\,dt = \int_0^\infty ||\bar{\mathbf{N}}e^{\mathbf{A}t}\mathbf{x}||_2^2\,dt \tag{5.20}$$

Because both $\bar{\mathbf{N}}$ and $e^{\mathbf{A}t}$ are nonsingular, for any nonzero $\mathbf{x}$, the integrand of (5.20) is positive for every t. Thus $\mathbf{x}'\mathbf{M}\mathbf{x}$ is positive for any $\mathbf{x} \neq \mathbf{0}$. This shows the positive definiteness of $\mathbf{M}$.

Sufficiency: We show that if $\mathbf{N}$ and $\mathbf{M}$ are positive definite, then $\mathbf{A}$ is stable. Let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{v} \neq \mathbf{0}$ be a corresponding eigenvector; that is, $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$. Even though $\mathbf{A}$ is a real matrix, its eigenvalue and eigenvector can be complex, as shown in Example 3.6. Taking the complex-conjugate transpose of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ yields $\mathbf{v}^*\mathbf{A}^* = \mathbf{v}^*\mathbf{A}' = \lambda^*\mathbf{v}^*$, where the asterisk denotes complex-conjugate transpose. Premultiplying $\mathbf{v}^*$ and postmultiplying $\mathbf{v}$ to (5.15) yields

$$\begin{aligned}-\mathbf{v}^*\mathbf{N}\mathbf{v} &= \mathbf{v}^*\mathbf{A}'\mathbf{M}\mathbf{v} + \mathbf{v}^*\mathbf{M}\mathbf{A}\mathbf{v} \\ &= (\lambda^* + \lambda)\mathbf{v}^*\mathbf{M}\mathbf{v} = 2\mathrm{Re}(\lambda)\mathbf{v}^*\mathbf{M}\mathbf{v}\end{aligned} \tag{5.21}$$

Because $\mathbf{v}^*\mathbf{M}\mathbf{v}$ and $\mathbf{v}^*\mathbf{N}\mathbf{v}$ are, as discussed in Section 3.9, both real and positive, (5.21) implies $\mathrm{Re}(\lambda) < 0$. This shows that every eigenvalue of $\mathbf{A}$ has a negative real part. Q.E.D.

The proof of Corollary 5.5 follows the proof of Theorem 5.5 with some modification. We discuss only where the proof of Theorem 5.5 is not applicable. Consider (5.20). Now $\bar{\mathbf{N}}$ is $m \times n$ with $m < n$ and $\mathbf{N} = \bar{\mathbf{N}}'\bar{\mathbf{N}}$ is positive semidefinite. Even so, $\mathbf{M}$ in (5.18) can still be positive definite if the integrand of (5.20) is not identically zero for all t. Suppose the integrand of (5.20) is identically zero or $\bar{\mathbf{N}}e^{\mathbf{A}t}\mathbf{x} \equiv \mathbf{0}$. Then its derivative with respect to t yields $\bar{\mathbf{N}}\mathbf{A}e^{\mathbf{A}t}\mathbf{x} = \mathbf{0}$. Proceeding forward, we can obtain

$$\begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{N}}\mathbf{A} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1} \end{bmatrix} e^{\mathbf{A}t}\mathbf{x} = \mathbf{0} \tag{5.22}$$

This equation implies that, because of (5.16) and the nonsingularity of $e^{\mathbf{A}t}$, the only $\mathbf{x}$ meeting

(5.22) is $\mathbf{0}$. Thus the integrand of (5.20) cannot be identically zero for any $\mathbf{x} \neq \mathbf{0}$. Thus $\mathbf{M}$ is positive definite under the condition in (5.16). This shows the necessity of Corollary 5.5. Next we consider (5.21) with $\mathbf{N} = \bar{\mathbf{N}}'\bar{\mathbf{N}}$ or[1]

$$2\mathrm{Re}(\lambda)\mathbf{v}^*\mathbf{M}\mathbf{v} = -\mathbf{v}^*\bar{\mathbf{N}}'\bar{\mathbf{N}}\mathbf{v} = -||\bar{\mathbf{N}}\mathbf{v}||_2^2 \tag{5.23}$$

We show that $\bar{\mathbf{N}}\mathbf{v}$ is nonzero under (5.16). Because of $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$, we have $\mathbf{A}^2\mathbf{v} = \lambda\mathbf{A}\mathbf{v} = \lambda^2\mathbf{v}$, $\ldots$, $\mathbf{A}^{n-1}\mathbf{v} = \lambda^{n-1}\mathbf{v}$. Consider

$$\begin{bmatrix} \bar{\mathbf{N}} \\ \bar{\mathbf{N}}\mathbf{A} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1} \end{bmatrix} \mathbf{v} = \begin{bmatrix} \bar{\mathbf{N}}\mathbf{v} \\ \bar{\mathbf{N}}\mathbf{A}\mathbf{v} \\ \vdots \\ \bar{\mathbf{N}}\mathbf{A}^{n-1}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{N}}\mathbf{v} \\ \lambda\bar{\mathbf{N}}\mathbf{v} \\ \vdots \\ \lambda^{n-1}\bar{\mathbf{N}}\mathbf{v} \end{bmatrix}$$

If $\bar{\mathbf{N}}\mathbf{v} = \mathbf{0}$, the rightmost matrix is zero; the leftmost matrix, however, is nonzero under the conditions of (5.16) and $\mathbf{v} \neq \mathbf{0}$. This is a contradiction. Thus $\bar{\mathbf{N}}\mathbf{v}$ is nonzero and (5.23) implies $\mathrm{Re}(\lambda) < 0$. This completes the proof of Corollary 5.5.

In the proof of Theorem of 5.5, we have established the following result. For easy reference, we state it as a theorem.

Theorem 5.6

If all eigenvalues of $\mathbf{A}$ have negative real parts, then the Lyapunov equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N}$$

has a unique solution for every $\mathbf{N}$, and the solution can be expressed as

$$\mathbf{M} = \int_0^\infty e^{\mathbf{A}'t}\mathbf{N}e^{\mathbf{A}t}\,dt \tag{5.24}$$

Because of the importance of this theorem, we give a different proof of the uniqueness of the solution. Suppose there are two solutions $\mathbf{M}_1$ and $\mathbf{M}_2$. Then we have

$$\mathbf{A}'(\mathbf{M}_1 - \mathbf{M}_2) + (\mathbf{M}_1 - \mathbf{M}_2)\mathbf{A} = \mathbf{0}$$

which implies

$$e^{\mathbf{A}'t}[\mathbf{A}'(\mathbf{M}_1 - \mathbf{M}_2) + (\mathbf{M}_1 - \mathbf{M}_2)\mathbf{A}]e^{\mathbf{A}t} = \frac{d}{dt}[e^{\mathbf{A}'t}(\mathbf{M}_1 - \mathbf{M}_2)e^{\mathbf{A}t}] = \mathbf{0}$$

Its integration from 0 to ∞ yields

$$[e^{\mathbf{A}'t}(\mathbf{M}_1 - \mathbf{M}_2)e^{\mathbf{A}t}]\Big|_0^\infty = \mathbf{0}$$

or, using $e^{\mathbf{A}t} \to \mathbf{0}$ as $t \to \infty$,

$$\mathbf{0} - (\mathbf{M}_1 - \mathbf{M}_2) = \mathbf{0}$$

1. Note that if $\mathbf{x}$ is a complex vector, then the Euclidean norm defined in Section 3.2 must be modified as $||\mathbf{x}||_2^2 = \mathbf{x}^*\mathbf{x}$, where $\mathbf{x}^*$ is the complex conjugate transpose of $\mathbf{x}$.

This shows the uniqueness of **M**. Although the solution can be expressed as in (5.24), the integration is not used in computing the solution. It is simpler to arrange the Lyapunov equation, after some transformations, into a standard linear algebraic equation as in (3.60) and then solve the equation. Note that even if **A** is not stable, a unique solution still exists if **A** has no two eigenvalues such that $\lambda_i + \lambda_j = 0$. The solution, however, cannot be expressed as in (5.24); the integration will diverge and is meaningless. If **A** is singular or, equivalently, has at least one zero eigenvalue, then the Lyapunov equation is always singular and solutions may or may not exist depending on whether or not **N** lies in the range space of the equation.

5.4.1 Discrete-Time Case

Before discussing the discrete counterpart of Theorems 5.5 and 5.6, we discuss the discrete counterpart of the Lyapunov equation in (3.59). Consider

$$\mathbf{M} - \mathbf{AMB} = \mathbf{C} \tag{5.25}$$

where **A** and **B** are, respectively, $n \times n$ and $m \times m$ matrices, and **M** and **C** are $n \times m$ matrices. As (3.60), Equation (5.25) can be expressed as $\mathbf{Ym} = \mathbf{c}$, where **Y** is an $nm \times nm$ matrix; $\boldsymbol{m}$ and **c** are $nm \times 1$ column vectors with the m columns of **M** and **C** stacked in order. Thus (5.25) is essentially a set of linear algebraic equations. Let η_k be an eigenvalue of **Y** or of (5.25). Then we have

$$\eta_k = 1 - \lambda_i \mu_j \qquad \text{for } i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m$$

where λ_i and μ_j are, respectively, the eigenvalues of **A** and **B**. This can be established intuitively as follows. Let us define $\mathcal{A}(\mathbf{M}) := \mathbf{M} - \mathbf{AMB}$. Then (5.25) can be written as $\mathcal{A}(\mathbf{M}) = \mathbf{C}$. A scalar η is an eigenvalue of $\mathcal{A}$ if there exists a nonzero **M** such that $\mathcal{A}(\mathbf{M}) = \eta\mathbf{M}$. Let **u** be an $n \times 1$ right eigenvector of **A** associated with λ_i; that is, $\mathbf{Au} = \lambda_i\mathbf{u}$. Let **v** be a $1 \times m$ left eigenvector of **B** associated with μ_j; that is, $\mathbf{vB} = \mathbf{v}\mu_j$. Applying $\mathcal{A}$ to the $n \times m$ nonzero matrix **uv** yields

$$\mathcal{A}(\mathbf{uv}) = \mathbf{uv} - \mathbf{AuvB} = (1 - \lambda_i \mu_j)\mathbf{uv}$$

Thus the eigenvalues of (5.25) are $1 - \lambda_i \mu_j$, for all i and j. If there are no i and j such that $\lambda_i \mu_j = 1$, then (5.25) is nonsingular and, for any **C**, a unique solution **M** exists in (5.25). If $\lambda_i \mu_j = 1$ for some i and j, then (5.25) is singular and, for a given **C**, solutions may or may not exist. The situation here is similar to what was discussed in Section 3.7.

Theorem 5.D5

All eigenvalues of an $n \times n$ matrix **A** have magnitudes less than 1 if and only if for any given positive definite symmetric matrix **N** or for $\mathbf{N} = \bar{\mathbf{N}}'\bar{\mathbf{N}}$, where $\bar{\mathbf{N}}$ is any given $m \times n$ matrix with $m < n$ and with the property in (5.16), the discrete Lyapunov equation

$$\mathbf{M} - \mathbf{A}'\mathbf{MA} = \mathbf{N} \tag{5.26}$$

has a unique symmetric solution **M** and **M** is positive definite.

We sketch briefly its proof for $\mathbf{N} > 0$. If all eigenvalues of $\mathbf{A}$ and, consequently, of $\mathbf{A}'$ have magnitudes less than 1, then we have $|\lambda_i \lambda_j| < 1$ for all i and j. Thus $\lambda_i \lambda_j \neq 1$ and (5.26) is nonsingular. Therefore, for any $\mathbf{N}$, a unique solution exists in (5.26). We claim that the solution can be expressed as

$$\mathbf{M} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m \tag{5.27}$$

Because $|\lambda_i| < 1$ for all i, this infinite series converges and is well defined. Substituting (5.27) into (5.26) yields

$$\begin{aligned}&\sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m - \mathbf{A}' \left(\sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m \right) \mathbf{A} \\ &= \mathbf{N} + \sum_{m=1}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m - \sum_{m=1}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m = \mathbf{N}\end{aligned}$$

Thus (5.27) is the solution. If $\mathbf{N}$ is symmetric, so is $\mathbf{M}$. If $\mathbf{N}$ is positive definite, so is $\mathbf{M}$. This establishes the necessity. To show sufficiency, let λ be an eigenvalue of $\mathbf{A}$ and $\mathbf{v} \neq \mathbf{0}$ be a corresponding eigenvector; that is, $\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$. Then we have

$$\begin{aligned}\mathbf{v}^* \mathbf{N} \mathbf{v} &= \mathbf{v}^* \mathbf{M} \mathbf{v} - \mathbf{v}^* \mathbf{A}' \mathbf{M} \mathbf{A} \mathbf{v} \\ &= \mathbf{v}^* \mathbf{M} \mathbf{v} - \lambda^* \mathbf{v}^* \mathbf{M} \mathbf{v} \lambda = (1 - |\lambda|^2) \mathbf{v}^* \mathbf{M} \mathbf{v}\end{aligned}$$

Because both $\mathbf{v}^* \mathbf{N} \mathbf{v}$ and $\mathbf{v}^* \mathbf{M} \mathbf{v}$ are real and positive, we conclude $(1 - |\lambda|^2) > 0$ or $|\lambda|^2 < 1$. This establishes the theorem for $\mathbf{N} > 0$. The case $\mathbf{N} \geq 0$ can similarly be established.

Theorem 5.D6

If all eigenvalues of $\mathbf{A}$ have magnitudes less than 1, then the discrete Lyapunov equation

$$\mathbf{M} - \mathbf{A}' \mathbf{M} \mathbf{A} = \mathbf{N}$$

has a unique solution for every $\mathbf{N}$, and the solution can be expressed as

$$\mathbf{M} = \sum_{m=0}^{\infty} (\mathbf{A}')^m \mathbf{N} \mathbf{A}^m$$

It is important to mention that even if $\mathbf{A}$ has one or more eigenvalues with magnitudes larger than 1, a unique solution still exists in the discrete Lyapunov equation if $\lambda_i \lambda_j \neq 1$ for all i and j. In this case, the solution cannot be expressed as in (5.27) but can be computed from a set of linear algebraic equations.

Let us discuss the relationships between the continuous-time and discrete-time Lyapunov equations. The stability condition for continuous-time systems is that all eigenvalues lie inside the open left-half s-plane. The stability condition for discrete-time systems is that all eigenvalues lie inside the unit circle on the z-plane. These conditions can be related by the bilinear transformation

$$s = \frac{z-1}{z+1} \qquad z = \frac{1+s}{1-s} \tag{5.28}$$

which maps the left-half s-plane into the interior of the unit circle on the z-plane and vice versa. To differentiate the continuous-time and discrete-time cases, we write

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} = -\mathbf{N} \tag{5.29}$$

and

$$\mathbf{M}_d - \mathbf{A}'_d\mathbf{M}_d\mathbf{A}_d = \mathbf{N}_d \tag{5.30}$$

Following (5.28), these two equations can be related by

$$\mathbf{A} = (\mathbf{A}_d + \mathbf{I})^{-1}(\mathbf{A}_d - \mathbf{I}) \qquad \mathbf{A}_d = (\mathbf{I} + \mathbf{A})(\mathbf{I} - \mathbf{A})^{-1}$$

Substituting the right-hand-side equation into (5.30) and performing a simple manipulation, we find

$$\mathbf{A}'\mathbf{M}_d + \mathbf{M}_d\mathbf{A} = -0.5(\mathbf{I} - \mathbf{A}')\mathbf{N}_d(\mathbf{I} - \mathbf{A})$$

Comparing this with (5.29) yields

$$\mathbf{A} = (\mathbf{A}_d + \mathbf{I})^{-1}(\mathbf{A}_d - \mathbf{I}) \qquad \mathbf{M} = \mathbf{M}_d \qquad \mathbf{N} = 0.5(\mathbf{I} - \mathbf{A}')\mathbf{N}_d(\mathbf{I} - \mathbf{A}) \tag{5.31}$$

These relate (5.29) and (5.30).

The MATLAB function `lyap` computes the Lyapunov equation in (5.29) and `dlyap` computes the discrete Lyapunov equation in (5.30). The function `dlyap` transforms (5.30) into (5.29) by using (5.31) and then calls `lyap`. The result yields $\mathbf{M} = \mathbf{M}_d$.

5.5 Stability of LTV Systems

Consider a SISO linear time-varying (LTV) system described by

$$y(t) = \int_{t_0}^{t} g(t, \tau)u(\tau)\,d\tau \tag{5.32}$$

The system is said to be BIBO stable if every bounded input excites a bounded output. The condition for (5.32) to be BIBO stable is that there exists a finite constant M such that

$$\int_{t_0}^{t} |g(t, \tau)|\,d\tau \le M < \infty \tag{5.33}$$

for all t and t_0 with $t \ge t_0$. The proof in the time-invariant case applies here with only minor modification.

For the multivariable case, (5.32) becomes

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{G}(t, \tau)\mathbf{u}(\tau)\,d\tau \tag{5.34}$$

The condition for (5.34) to be BIBO stable is that every entry of $\mathbf{G}(t, \tau)$ meets the condition in (5.33). For multivariable systems, we can also express the condition in terms of norms. Any norm discussed in Section 3.11 can be used. However, the infinite-norm

$$||\mathbf{u}||_\infty = \max_i |u_i| \qquad ||\mathbf{G}||_\infty = \text{ largest row absolute sum}$$

is probably the most convenient to use in stability study. For convenience, no subscript will be attached to any norm. The necessary and sufficient condition for (5.34) to be BIBO stable is that there exists a finite constant M such that

$$\int_{t_0}^{t} ||\mathbf{G}(t, \tau)||\, d\tau \le M < \infty$$

for all t and t_0 with $t \ge t_0$.

The impulse response matrix of

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(t)\mathbf{x} + \mathbf{B}(t)\mathbf{u} \\ \mathbf{y} &= \mathbf{C}(t)\mathbf{x} + \mathbf{D}(t)\mathbf{u}\end{aligned} \tag{5.35}$$

is

$$\mathbf{G}(t, \tau) = \mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau) + \mathbf{D}(t)\delta(t - \tau)$$

and the zero-state response is

$$\mathbf{y}(t) = \int_{t_0}^{t} \mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)\mathbf{u}(\tau)\, d\tau + \mathbf{D}(t)\mathbf{u}(t)$$

Thus (5.35) or, more precisely, the zero-state response of (5.35) is BIBO stable if and only if there exist constants M_1 and M_2 such that

$$||\mathbf{D}(t)|| \le M_1 < \infty$$

and

$$\int_{t_0}^{t} ||\mathbf{G}(t, \tau)||\, d\tau \le M_2 < \infty$$

for all t and t_0 with $t \ge t_0$.

Next we study the stability of the zero-input response of (5.35). As in the time-invariant case, we define the zero-input response of (5.35) or the equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ to be marginally stable if every finite initial state excites a bounded response. Because the response is governed by

$$\mathbf{x}(t) = \mathbf{\Phi}(t, t_0)\mathbf{x}(t_0) \tag{5.36}$$

we conclude that the response is marginally stable if and only if there exists a finite constant M such that

$$||\mathbf{\Phi}(t, t_0)|| \le M < \infty \tag{5.37}$$

for all t_0 and for all $t \ge t_0$. The equation $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ is asymptotically stable if the response excited by every finite initial state is bounded and approaches zero as $t \to \infty$. The asymptotic stability conditions are the boundedness condition in (5.37) and

$$||\mathbf{\Phi}(t, t_0)|| \to 0 \qquad \text{as } t \to \infty \tag{5.38}$$

A great deal can be said regarding these definitions and conditions. Does the constant M in

(5.37) depend on t_0? What is the rate for the state transition matrix to approach 0 in (5.38)? The interested reader is referred to References [4, 15].

A time-invariant equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if all eigenvalues of $\mathbf{A}$ have negative real parts. Is this also true for the time-varying case? The answer is negative as the next example shows.

Example 5.5 Consider the linear time-varying equation

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix} \mathbf{x} \tag{5.39}$$

The characteristic polynomial of $\mathbf{A}(t)$ is

$$\det(\lambda\mathbf{I} - \mathbf{A}(t)) = \det \begin{bmatrix} \lambda + 1 & -e^{2t} \\ 0 & \lambda + 1 \end{bmatrix} = (\lambda + 1)^2$$

Thus $\mathbf{A}(t)$ has eigenvalues -1 and -1 for all t. It can be verified directly that

$$\mathbf{\Phi}(t, 0) = \begin{bmatrix} e^{-t} & 0.5(e^{t} - e^{-t}) \\ 0 & e^{-t} \end{bmatrix}$$

meets (4.53) and is therefore the state transition matrix of (5.39). See also Problem 4.16. Because the (1,2)th entry of $\mathbf{\Phi}$ grows without bound, the equation is neither asymptotically stable nor marginally stable. This example shows that even though the eigenvalues can be defined for $\mathbf{A}(t)$ at every t, the concept of eigenvalues is not useful in the time-varying case.

All stability properties in the time-invariant case are invariant under any equivalence transformation. In the time-varying case, this is so only for BIBO stability, because the impulse response matrix is preserved. An equivalence transformation can transform, as shown in Theorem 4.3, any $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ into $\dot{\bar{\mathbf{x}}} = \mathbf{A}_o\bar{\mathbf{x}}$, where $\mathbf{A}_o$ is any constant matrix; therefore, in the time-varying case, marginal and asymptotic stabilities are not invariant under any equivalence transformation.

Theorem 5.7

Marginal and asymptotic stabilities of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ are invariant under any Lyapunov transformation.

As discussed in Section 4.6, if $\mathbf{P}(t)$ and $\dot{\mathbf{P}}(t)$ are continuous, and $\mathbf{P}(t)$ is nonsingular for all t, then $\bar{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ is an algebraic transformation. If, in addition, $\mathbf{P}(t)$ and $\mathbf{P}^{-1}(t)$ are bounded for all t, then $\bar{\mathbf{x}} = \mathbf{P}(t)\mathbf{x}$ is a Lyapunov transformation. The fundamental matrix $\mathbf{X}(t)$ of $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$ and the fundamental matrix $\bar{\mathbf{X}}(t)$ of $\dot{\bar{\mathbf{x}}} = \bar{\mathbf{A}}(t)\bar{\mathbf{x}}$ are related by, as derived in (4.71),

$$\bar{\mathbf{X}}(t) = \mathbf{P}(t)\mathbf{X}(t)$$

which implies

$$\begin{aligned} \bar{\mathbf{\Phi}}(t, \tau) &= \bar{\mathbf{X}}(t)\bar{\mathbf{X}}^{-1}(\tau) = \mathbf{P}(t)\mathbf{X}(t)\mathbf{X}^{-1}(\tau)\mathbf{P}^{-1}(\tau) \\ &= \mathbf{P}(t)\mathbf{\Phi}(t, \tau)\mathbf{P}^{-1}(\tau) \end{aligned} \tag{5.40}$$

Because both $\mathbf{P}(t)$ and $\mathbf{P}^{-1}(t)$ are bounded, if $||\mathbf{\Phi}(t, \tau)||$ is bounded, so is $||\bar{\mathbf{\Phi}}(t, \tau)||$; if $||\mathbf{\Phi}(t, \tau)|| \to 0$ as $t \to \infty$, so is $||\bar{\mathbf{\Phi}}(t, \tau)||$. This establishes Theorem 5.7.

In the time-invariant case, asymptotic stability of zero-input responses always implies BIBO stability of zero-state responses. This is not necessarily so in the time-varying case. A time-varying equation is asymptotically stable if

$$||\mathbf{\Phi}(t, t_0)|| \to 0 \qquad \text{as } t \to \infty \tag{5.41}$$

for all t, t_0 with $t \geq t_0$. It is BIBO stable if

$$\int_{t_0}^{t} ||\mathbf{C}(t)\mathbf{\Phi}(t, \tau)\mathbf{B}(\tau)||\, d\tau < \infty \tag{5.42}$$

for all t, t_0 with $t \geq t_0$. A function that approaches 0, as $t \to \infty$, may not be absolutely integrable. Thus asymptotic stability may not imply BIBO stability in the time-varying case. However, if $||\mathbf{\Phi}(t, \tau)||$ decreases to zero rapidly, in particular, exponentially, and if $\mathbf{C}(t)$ and $\mathbf{B}(t)$ are bounded for all t, then asymptotic stability does imply BIBO stability. See References [4, 6, 15].

PROBLEMS

5.1 Is the network shown in Fig. 5.2 BIBO stable? If not, find a bounded input that will excite an unbounded output.

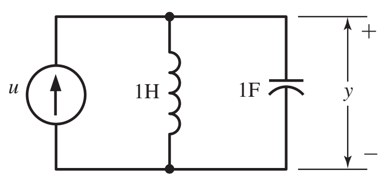


Figure 5.2

5.2 Consider a system with an irrational transfer function $\hat{g}(s)$. Show that a necessary condition for the system to be BIBO stable is that $|\hat{g}(s)|$ is finite for all Re $s \geq 0$.

5.3 Is a system with impulse response $g(t) = 1/(t+1)$ BIBO stable? How about $g(t) = te^{-t}$ for $t \geq 0$?

5.4 Is a system with transfer function $\hat{g}(s) = e^{-2s}/(s+1)$ BIBO stable?

5.5 Show that the negative-feedback system shown in Fig. 2.5(b) is BIBO stable if and only if the gain a has a magnitude less than 1. For $a = 1$, find a bounded input $r(t)$ that will excite an unbounded output.

5.6 Consider a system with transfer function $\hat{g}(s) = (s-2)/(s+1)$. What are the steady-state responses excited by $u(t) = 3$, for $t \geq 0$, and by $u(t) = \sin 2t$, for $t \geq 0$?

5.7 Consider

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix}\mathbf{x} + \begin{bmatrix} -2 \\ 0 \end{bmatrix}u$$
$$y = [-2 \quad 3]\mathbf{x} - 2u$$

Is it BIBO stable?

5.8 Consider a discrete-time system with impulse response sequence

$$g[k] = k(0.8)^k \qquad \text{for } k \geq 0$$

Is the system BIBO stable?

5.9 Is the state equation in Problem 5.7 marginally stable? Asymptotically stable?

5.10 Is the homogeneous state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

marginally stable? Asymptotically stable?

5.11 Is the homogeneous state equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \mathbf{x}$$

marginally stable? Asymptotically stable?

5.12 Is the discrete-time homogeneous state equation

$$\mathbf{x}[k+1] = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k]$$

marginally stable? Asymptotically stable?

5.13 Is the discrete-time homogeneous state equation

$$\mathbf{x}[k+1] = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{x}[k]$$

marginally stable? Asymptotically stable?

5.14 Use Theorem 5.5 to show that all eigenvalues of

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -0.5 & -1 \end{bmatrix}$$

have negative real parts.

5.15 Use Theorem 5.D5 to show that all eigenvalues of the $\mathbf{A}$ in Problem 5.14 have magnitudes less than 1.

5.16 For any distinct negative real λ_i and any nonzero real a_i, show that the matrix

$$\mathbf{M} = \begin{bmatrix} -\frac{a_1^2}{2\lambda_1} & -\frac{a_1 a_2}{\lambda_1+\lambda_2} & -\frac{a_1 a_3}{\lambda_1+\lambda_3} \\ -\frac{a_2 a_1}{\lambda_2+\lambda_1} & -\frac{a_2^2}{2\lambda_2} & -\frac{a_2 a_3}{\lambda_2+\lambda_3} \\ -\frac{a_3 a_1}{\lambda_3+\lambda_1} & -\frac{a_3 a_2}{\lambda_3+\lambda_2} & -\frac{a_3^2}{2\lambda_3} \end{bmatrix}$$

is positive definite. [*Hint*: Use Corollary 5.5 and $\mathbf{A}$=diag$(\lambda_1, \lambda_2, \lambda_3)$.]

5.17 A real matrix $\mathbf{M}$ (not necessarily symmetric) is defined to be positive definite if $\mathbf{x}'\mathbf{M}\mathbf{x} > 0$ for any nonzero $\mathbf{x}$. Is it true that the matrix $\mathbf{M}$ is positive definite if all eigenvalues of $\mathbf{M}$ are real and positive or if all its leading principal minors are positive? If not, how do you check its positive definiteness? [*Hint*: Try

$$\begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 \\ 1.9 & 1 \end{bmatrix}]$$

5.18 Show that all eigenvalues of $\mathbf{A}$ have real parts less than $-\mu < 0$ if and only if, for any given positive definite symmetric matrix $\mathbf{N}$, the equation

$$\mathbf{A}'\mathbf{M} + \mathbf{M}\mathbf{A} + 2\mu\mathbf{M} = -\mathbf{N}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

5.19 Show that all eigenvalues of $\mathbf{A}$ have magnitudes less than ρ if and only if, for any given positive definite symmetric matrix $\mathbf{N}$, the equation

$$\rho^2\mathbf{M} - \mathbf{A}'\mathbf{M}\mathbf{A} = \rho^2\mathbf{N}$$

has a unique symmetric solution $\mathbf{M}$ and $\mathbf{M}$ is positive definite.

5.20 Is a system with impulse response $g(t, \tau) = e^{-2|t|-|\tau|}$, for $t \geq \tau$, BIBO stable? How about $g(t, \tau) = \sin t(e^{-(t-\tau)}) \cos \tau$?

5.21 Consider the time-varying equation

$$\dot{x} = 2tx + u \qquad y = e^{-t^2}x$$

Is the equation BIBO stable? Marginally stable? Asymptotically stable?

5.22 Show that the equation in Problem 5.21 can be transformed by using $\bar{x} = P(t)x$, with $P(t) = e^{-t^2}$, into

$$\dot{\bar{x}} = 0 \cdot \bar{x} + e^{-t^2}u \qquad y = \bar{x}$$

Is the equation BIBO stable? Marginally stable? Asymptotically stable? Is the transformation a Lyapunov transformation?

5.23 Is the homogeneous equation

$$\dot{\mathbf{x}} = \begin{bmatrix} -1 & 0 \\ -e^{-3t} & 0 \end{bmatrix}\mathbf{x}$$

for $t_0 \geq 0$, marginally stable? Asymptotically stable?