EE 547 (PMP): Homework 3

Assigned: Thursday, January 22, 2015; Due: Wednesday, January 28, 2015

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Problem 1 (State Transition Matrix of an LTV Systems) Consider the following system:

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Show the fundamental matrix of system is

$$X = \begin{bmatrix} x_1(0) & x_2(0)[e^{2t}(\frac{t}{2} - \frac{1}{4}) + \frac{1}{4}] \\ 0 & x_2(0)e^{2t} \end{bmatrix} \tag{2}$$

Show that this can be further simplified as

$$X = \begin{bmatrix} 1 & e^{2t}(\frac{t}{2} - \frac{1}{4}) + \frac{1}{4} \\ 0 & e^{2t} \end{bmatrix} \tag{3}$$

b) Show the state transition matrix of system (1) as

$$\varphi(t, t_0) = \begin{bmatrix} 1 & -\frac{e^{-2t_0}}{4}[e^{2t_0}(2t_0 - 1) + 1] + e^{-2t_0}\left[e^{2t}(\frac{t}{2} - \frac{1}{4}) + \frac{1}{4}\right] \\ 0 & e^{2(t-t_0)} \end{bmatrix} \tag{4}$$

(Hint: choose two linearly independent vectors of initial conditions, for examples, $x_{0,1}$ = [$x_1(0)$ 0] and $x_{0,2}$ = [0 $x_2(0)$])

c) Given an input u(t) = sin(t) and an arbitrary condition x_0 = [$x_1(0)$, $x_2(0)$], using the following expression

$$y(t) = C(t)\Phi(t, t_0)x_0 + \int_{t_0}^{t} C(t)\,\Phi(t, \tau)B(\tau)u(\tau)d\tau + D(t)u(t)$$

to show the output is

$$y = x_2(0)e^{2(t-t_0)} + \frac{1}{25}e^{2(t-t_0)}\{4\cos t_0 + 3\sin t_0 + 5t_0\cos t_0 + 10t_0\sin t_0\} - \frac{3}{25}\sin t - \frac{4}{25}\cos t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t \quad (5)$$

Problem 2 (Characteristic Polynomial) Derive by hand the characteristic polynomial of a matrix A as (6).

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 6 & 2 & 7 \\ 4 & 8 & 3 \end{bmatrix} \quad (6)$$

Problem 3 (Matrix Exponential of an LTI System) Consider a system as (7) with initial condition $x_1(0) = 1$ and $x_2(0) = 0.5$.

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix}u(t)$$
$$y = \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (7)$$

a) Show the state transition matrix by Inverse Laplace Transform is

$$\varphi(t) = \begin{bmatrix} e^{-2t}(\cos 2t + \sin 2t) & e^{-2t}\sin 2t \\ -2e^{-2t}\sin 2t & e^{-2t}(\cos 2t - \sin 2t) \end{bmatrix} \quad (8)$$

b) Derive the system output to the step input $u(t) = 1$, $\forall t \geq 0$. You can verify with MATLAB or other software.

Problem 4 (Impulse Response of an LTI System) Consider a system as (9) with initial condition $x_1(0) = 1$ and $x_2(0) = 2$.

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}u(t)$$
$$y = \begin{bmatrix} -4 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \quad (9)$$

a) Derive the state transition matrix by MATLAB.
b) If an impulse input applies over the system, use the MATLAB **impulse** function and the method we learned to plot system's output.

Problem 1.

(a) $\begin{cases} \dot{x}_1 = t x_2 & \cdots\cdots ① \\ \dot{x}_2 = 2 x_2 & \cdots\cdots ② \end{cases}$

From ②, it's clear $x_2 = C_0 e^{2t} = x_2(0) \cdot e^{2t}$

From ①: $\frac{dx_1}{dt} = t x_2$

$$\therefore \int dx_1 = \int t x_2 \, dt$$

$$x_1(t) - x_1(0) = \int_0^t t\, x_2(0) \cdot e^{2t} \, dt$$

$$= x_2(0) \cdot \left[e^{2t}\left(\frac{t}{2} - \frac{1}{4}\right) + \frac{1}{4} \right]$$

$$x_1(t) = x_1(0) + x_2(0) \left[e^{2t}\left(\frac{t}{2} - \frac{1}{4}\right) + \frac{1}{4} \right]$$

If $x(0) = \begin{bmatrix} x_1(0) \\ 0 \end{bmatrix} \longrightarrow x_2(0) = 0$, $x(0) = \begin{bmatrix} x_1(0) \\ 0 \end{bmatrix}$

If $\bar{x}(0) = \begin{bmatrix} 0 \\ x_2(0) \end{bmatrix}$

$$= \begin{bmatrix} x_1(0) + x_2(0) \cdot \left[e^{2\cdot 0}\left(\frac{0}{2} - \frac{1}{4}\right) + \frac{1}{4} \right] \\ x_2(0) \cdot e^{2\cdot 0} \end{bmatrix} = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\Rightarrow x_1(0) = 0$$

$$\therefore \bar{x}(t) = \begin{bmatrix} x_2(0) \left[e^{2t}\left(\frac{t}{2} - \frac{1}{4}\right) + \frac{1}{4} \right] \\ x_2(0) \cdot e^{2t} \end{bmatrix}$$

$\therefore$ Fundamental matrix $\bar{X} = \begin{bmatrix} x(0) & \bar{x}(0) \end{bmatrix}$

$$= \begin{bmatrix} x_1(0) & x_2(0) \cdot \left[e^{2t}\left(\frac{t}{2} - \frac{1}{4}\right) + \frac{1}{4} \right] \\ 0 & x_2(0) \cdot e^{2t} \end{bmatrix}$$

It can be simplified as

$$\overline{X} = \begin{bmatrix} 1 & e^{2t}\left(\frac{t}{2}-\frac{1}{4}\right)+\frac{1}{4} \\ 0 & e^{2t} \end{bmatrix}$$

(b) $\phi(t, t_0)$

$$= \overline{X}(t) \cdot \overline{X}(t_0)^{-1}$$

$$= \begin{bmatrix} 1 & e^{2t}\left(\frac{t}{2}-\frac{1}{4}\right)+\frac{1}{4} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & e^{2t_0}\left(\frac{t_0}{2}-\frac{1}{4}\right)+\frac{1}{4} \\ 0 & e^{2t_0} \end{bmatrix}^{-1}$$

$$= \begin{bmatrix} 1 & e^{2t}\left(\frac{t}{2}-\frac{1}{4}\right)+\frac{1}{4} \\ 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & -\left[\left(\frac{t_0}{2}-\frac{1}{4}\right)+\frac{1}{4}e^{-2t_0}\right] \\ 0 & e^{-2t_0} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -\frac{e^{-2t_0}}{4}\left[e^{2t_0}(2t_0-1)+1\right]+e^{-2t_0}\left[e^{2t}\left(\frac{t}{2}-\frac{1}{4}\right)+\frac{1}{4}\right] \\ 0 & e^{2(t-t_0)} \end{bmatrix}$$

(c) According to the equation:

$$y_n = C \cdot \phi(t, t_0) \cdot X_0$$

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix}$$

$$= \begin{bmatrix} \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} X_{10} \\ X_{20} \end{bmatrix}$$

$$= \overset{0}{\phi_{21}} X_{10} + \phi_{22} X_{20}$$

$$= X_{20} \cdot e^{2(t-t_0)}$$

$$y_p = \int_{t_0}^{t} C\cdot\Phi(t,\tau)\cdot B(\tau)\cdot u(\tau)\,d\tau + \cancel{D(t)}\,u(t)$$

$$= \int_{t_0}^{t} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} \sin(\tau)\,d\tau$$

$$= \int_{t_0}^{t} \begin{bmatrix} \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} 0 \\ \tau \end{bmatrix} \sin\tau \; d\tau$$

$$= \int_{t_0}^{t} \phi_{22}\cdot\tau\cdot\sin\tau \; d\tau$$

$$= \int_{t_0}^{t} e^{2(t-\tau)}\cdot\tau\cdot\sin\tau \; d\tau$$

$$= \frac{1}{25}\exp^{2(t-t_0)}\cdot\left(4\cos t_0 + 3\sin t_0 + 5t_0\cos t_0 + 10t_0\sin t_0\right)$$
$$- \frac{3}{25}\sin t - \frac{4}{25}\cos t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t$$

$$y = y_h + y_p$$
$$= x_{20}e^{2(t-t_0)} + \frac{1}{25}\exp^{2(t-t_0)}\left(4\cos t_0 + 3\sin t_0 + 5t_0\cos t_0 + 10t_0\sin t_0\right)$$
$$- \frac{3}{25}\sin t - \frac{4}{25}\cos t - \frac{1}{5}t\cos t - \frac{2}{5}t\sin t$$

Problem 2

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 6 & 2 & 7 \\ 4 & 8 & 3 \end{bmatrix}$$

characteristic polynomial is $= \det(\lambda I - A)$

$$= \det \begin{bmatrix} \lambda-1 & 0 & -5 \\ -6 & \lambda-2 & -7 \\ -4 & -8 & \lambda-3 \end{bmatrix}$$

$$= (\lambda-1)(\lambda-2)(\lambda-3) + (-6)(-8)(-5) + (-4)\,0\cdot(-7)$$

$$-(-5)(-4)(\lambda-2) - (-6)\cdot 0\cdot(\lambda-3) - (-8)(-7)(\lambda-1)$$

$$= (\lambda^2-3\lambda+2)(\lambda-3) - 240 - 20(\lambda-2) - 56(\lambda-1)$$

$$= \lambda^3 - 3\lambda^2 + 2\lambda - 3\lambda^2 + 9\lambda - 6 - 240 - 20\lambda + 40 - 56\lambda + 56$$

$$= \lambda^3 - 6\lambda^2 - 65\lambda - 150$$

Problem 3

(a) $$A = \begin{bmatrix} 0 & 2 \\ -4 & -4 \end{bmatrix}$$

$$sI - A = \begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 2 \\ -4 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} s & -2 \\ 4 & s+4 \end{bmatrix}$$

$$(sI-A)^{-1} = \begin{bmatrix} s & -2 \\ 4 & s+4 \end{bmatrix}^{-1} = \frac{\begin{bmatrix} s+4 & 2 \\ -4 & s \end{bmatrix}}{s(s+4)-(-8)}$$

$$= \frac{\begin{bmatrix} s+4 & 2 \\ -4 & s \end{bmatrix}}{s(s+4)+8} = \begin{bmatrix} \frac{s+4}{(s+2)^2+4} & \frac{2}{(s+2)^2+4} \\ \frac{-4}{(s+2)^2+4} & \frac{s}{(s+2)^2+4} \end{bmatrix}$$

$$\mathcal{L}^{-1}[sI-A]^{-1}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+4}{(s+2)^2+4} & \frac{2}{(s+2)^2+4} \\ \frac{-4}{(s+2)^2+4} & \frac{s}{(s+2)^2+4} \end{bmatrix}$$

$$= \mathcal{L}^{-1}\begin{bmatrix} \frac{s+2}{(s+2)^2+4} + \frac{2}{(s+2)^2+4} & \frac{2}{(s+2)^2+4} \\ (-2)\frac{2}{(s+2)^2+4} & \frac{s+2}{(s+2)^2+4} - \frac{2}{(s+2)^2+4} \end{bmatrix}$$

From the table:

$$\mathcal{L}^{-1}\left[\frac{b}{(s-a)^2+b^2}\right] = e^{at}\cdot\sin bt$$

$$\mathcal{L}^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cdot\cos bt$$

$$\therefore \mathcal{L}^{-1}[sI-A]^{-1}$$

$$= \begin{bmatrix} e^{-2t}\cos 2t + e^{-2t}\sin 2t & e^{-2t}\cdot\sin 2t \\ -2\cdot e^{-2t}\cdot\sin 2t & e^{-2t}\cos 2t - e^{-2t}\sin 2t \end{bmatrix}$$

$$= \begin{bmatrix} e^{-2t}(\cos 2t + \sin 2t) & e^{-2t}\cdot\sin 2t \\ -2\cdot e^{-2t}\cdot\sin 2t & e^{-2t}(\cos 2t - \sin 2t) \end{bmatrix}$$

$$= \phi(t)$$

(b) $y(t) = c(t)\cdot\phi(t,t_0)x_0 + \int_{t_0}^{t} c(t)\phi(t,\tau)\cdot B(\zeta)u(\zeta)\,d\zeta + D(t)u(t)$

let $t_0 = 0$, $\phi(t,t_0) = \phi(t)$

$c(t) = [1, 2]$, $D(t) = 0$

$u(t) = 1$, $\forall\, t \geq 0$, $x_0 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$

$y(t) = y_n + y_p$

where

$$y_n = c(t)\cdot\phi(t)\cdot x_0$$

$$= [1 \quad 2] \begin{bmatrix} \phi_{11} & \phi_{12} \\ \phi_{21} & \phi_{22} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$= [\phi_{11} + 2\phi_{21} \quad \phi_{12} + 2\phi_{22}] \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$= [\phi_{11} + 2\phi_{21} + \tfrac{1}{2}\phi_{12} + \phi_{22}]$$

$$= e^{-2t}[\cos 2t + \sin 2t + 2\cdot(-2)\sin 2t + \tfrac{1}{2}\sin 2t$$
$$+ \cos 2t - \sin 2t]$$

$$= e^{-2t}[2\cos 2t - \tfrac{7}{2}\sin 2t]$$

$$y_p = \int_{t_0}^{t} C\cdot\phi(t,\tau)B(\tau)u(\tau)\,d\zeta + \cancel{D(t)}^{\,0}u(t)$$

$$= \int_0^t [1, 2] \begin{bmatrix} \phi_{11}(t,\tau) & \phi_{12}(t,\tau) \\ \phi_{21}(t,\tau) & \phi_{22}(t,\tau) \end{bmatrix} \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot 1 \cdot d\zeta$$

$$= \int_0^t [\phi_{11}(t,\zeta) + 2\phi_{21}(t,\zeta) \quad \phi_{12}(t,\zeta) + 2\phi_{22}(t,\zeta)] \begin{bmatrix} 1 \\ -2 \end{bmatrix} d\zeta$$

$$= \int_0^t \phi_{11}(t,\zeta) + 2\phi_{21}(t,\zeta) - 2\phi_{12}(t,\zeta) - 4\phi_{22}(t,\zeta)\,d\zeta$$

(I would recommend software for next step.)

$$y = y_n + y_p$$

Problem 4 Solution:

a) By **expm** function, we have state transition matrix as (5).

$$\Phi = \begin{bmatrix} 4*\exp(-4*\mathrm{t}) - 3*\exp(-3*\mathrm{t}) & 12*\exp(-4*\mathrm{t}) - 12*\exp(-3*\mathrm{t}) \\ \exp(-3*\mathrm{t}) - \exp(-4*\mathrm{t}) & 4*\exp(-3*\mathrm{t}) - 3*\exp(-4*\mathrm{t}) \end{bmatrix} \quad (5)$$

b) For a LTI system, the solution can be derived as (6).

$$y(t) = C(t)\Phi(t,t_0)x_0 + \int_{t_0}^{t} C(t)\,\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \quad (6)$$

The solution is composed of two parts, homogeneous solution and particular solution. The particular solution can be evaluated by **impulse** function of MATLAB because this function assumes zero initial state. The MATLAB code is as following:

```
A4 = [-7 -12; 1 0];
B4 = [1;0];
C4 = [-4 -10];
D4 = [1];
phi4 = expm(A4*t);
xini4 = [1;2];
ss4 = ss(A4,B4,C4,D4);
[ImpulseResponse4,t4] = impulse(ss4,5);

yp4 = ImpulseResponse4;
yh4 = eval(subs(C4*phi4*xini4,t,t4));
y4 = yh4+yp4;
plot(t4,yh4,'r-.',t4,yp4,'g--',t4,y4,'b');
legend('homogeneous solution',' particular solution', 'total solution')
print -djpeg 'HW3_P4.jpeg'
```

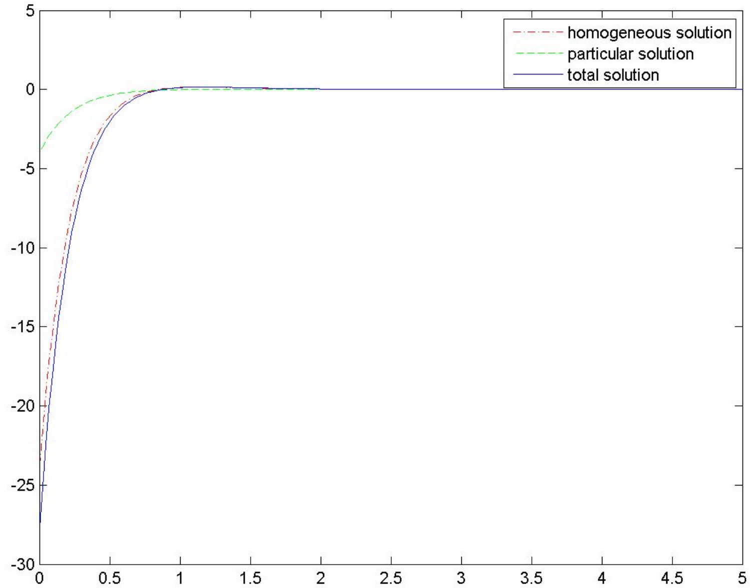

Figure 1 Homogeneous solution, particular solution and total solution of problem 4.