EE547 - HW3 - Problems 1 - 3

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In [1]:
```
import warnings
warnings.filterwarnings('ignore')
from sympy import *
init_printing()
from IPython.display import Image
```

Problem 1.a

In [2]:
```
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```

Out[2]:

Problem 1 (State Transition Matrix of an LTV Systems) Consider the following system:

$$\dot{x} = \begin{bmatrix} 0 & t \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ t \end{bmatrix} u \tag{1}$$

$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

a) Show the fundamental matrix of system is

$$X = \begin{bmatrix} x_1(0) & x_2(0)[e^{2t}(\frac{t}{2} - \frac{1}{4}) + \frac{1}{4}] \\ 0 & x_2(0)e^{2t} \end{bmatrix} \tag{2}$$

Show that this can be further simplified as

$$X = \begin{bmatrix} 1 & e^{2t}(\frac{t}{2} - \frac{1}{4}) + \frac{1}{4} \\ 0 & e^{2t} \end{bmatrix} \tag{3}$$

In [29]:
```
# create symbols
t, x, xdot, x1, x2, u, y = symbols("t, x, \dot{x}, x_1, x_2, u, y")
X, Phi = symbols("X, Phi")
fm1, fm2, t0 = symbols("fm1, fm2, t0")
x0, x0_1, x0_2 = symbols("x0, x0_1, x0_2")
```

In [30]:
```
# create and display state matrices
A = Matrix([[0, t], [0, 2]])
B = Matrix([[0], [t]])
C = Matrix([0, 1]).transpose()
D = Matrix([0])
A, B, C, D
```

Out[30]:

$\left(\begin{bmatrix} 0 & t \end{bmatrix}, \begin{bmatrix} 0 \end{bmatrix}, [\quad], [\quad] \right)$

$$\begin{matrix} 0 & 2 \end{matrix} \quad t \qquad \begin{matrix} 0 & 1 \end{matrix} \quad 0$$

In [31]:
```
# Compute and display (1)
x1 = Function('x1')(t)
x2 = Function('x2')(t)
xdot = Function('\dot{x}')(t)
x = Matrix([x1, x2])
Eq(xdot, MatAdd(A*x, B*u))
```

Out[31]:

$$\dot{x}(t) = \begin{bmatrix} t\mathrm{x}_2(t) \\ 2\mathrm{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ tu \end{bmatrix}$$

In [32]:
```
# For the ZIR, u(t) = 0
Eq(xdot, MatAdd(A*x, B*u).subs(u, 0))
```

Out[32]:

$$\dot{x}(t) = \begin{bmatrix} t\mathrm{x}_2(t) \\ 2\mathrm{x}_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

In [33]:
```
#Solve the system of ODEs, solve for x2 first
dx2 = diff(x2)
eq2 = Eq(dx2, 2*x2)
dsolve(eq2)
```

Out[33]: $\mathrm{x}_2(t) = C_1 e^{2t}$

In [34]:
```
# Subsitute in the solution for x2 and solve x1
c1, c2, tau = symbols("c1, c2, tau")
x2 = c1*exp(2*t)
Eq(x1, c2 + integrate(tau*x2.subs(t, tau), (tau, 0, t)))
```

Out[34]:

$$\mathrm{x}_1(t) = \frac{c_1}{4} + c_2 + \frac{e^{2t}}{4}(2c_1 t - c_1)$$

In [35]:
```
# Put x together
x1 = c2 + c1/4 + exp(2*t)*(2*c1*t - c1)/4
Eq(x, Matrix([x1, x2]))
```

Out[35]:

$$\begin{bmatrix} \mathrm{x}_1(t) \\ \mathrm{x}_2(t) \end{bmatrix} = \begin{matrix} \frac{c_1}{4} + c_2 + \frac{e^{2t}}{4}(2c_1 t - c_1) \\ c_1 e^{2t} \end{matrix}$$

Solution 1.a

In [36]:
```
# look at initial conditions [0, 1] and [1, 0]
X = Function('X')(t)
fm1 = Matrix([x1, x2]).subs(c1, 1).subs(c2, 0)
fm2 = Matrix([x1, x2]).subs(c1, 0).subs(c2, 1)
X = fm2.row_join(fm1)
X
```

Out[36]:

$$\begin{matrix} 1 & \frac{e^{2t}}{4}(2t-1) + \frac{1}{4} \\ 0 & e^{2t} \end{matrix}$$

Problem 1.b

In [37]:
```
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```

Out[37]:

b) Show the state transition matrix of system (1) as

$$\varphi(t,t_0)=\begin{bmatrix}1 & -\frac{e^{-2t_0}}{4}[e^{-2t_0}(2t_0-1)+1]+e^{-2t_0}\left[e^{2t}(\frac{t}{2}-\frac{1}{4})+\frac{1}{4}\right]\\ 0 & e^{2(t-t_0)}\end{bmatrix} \quad (4)$$

(Hint: choose two linearly independent vectors of initial conditions, for examples, $x_{0,1}$ = [x_1(0) 0] and $x_{0,2}$ = [0 x_2(0)])

Solution 1.b

In [38]:
```
# Find STM (State Transition Matrix) by multiplying FM with it's inverse
X = fm1.row_join(fm2)
Phi = simplify(X*X.inv().subs(t, t0))
Phi
```

Out[38]:

$$\begin{matrix}1 & \frac{1}{4}\left((2t-1)e^{2t}+(-2t_0+1)e^{2t_0}\right)e^{-2t_0}\\ 0 & e^{2t-2t_0}\end{matrix}$$

Problem 1.c

In [39]:
```
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```

Out[39]:

c) Given an input u(t) = sin(t) and an arbitrary condition x_0 = [x_1(0), x_2(0)], using the following expression

$$y(t)=C(t)\Phi(t,t_0)x_0+\int_{t_0}^{t} C(t)\,\Phi(t,\tau)B(\tau)u(\tau)d\tau+D(t)u(t)$$

to show the output is

In [40]:
```
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```

Out[40]:

$$y=x_2(0)e^{2(t-t_0)}+\frac{1}{25}e^{2(t-t_0)}\{4\cos t_0+3\sin t_0+5t_0\cos t_0+10t_0\sin t_0\}-\frac{3}{25}\sin t-\frac{4}{25}\cos t-\frac{1}{5}t\cos t-\frac{2}{5}t\sin t \quad (5)$$

Solution 1.c

In [55]:
```
# Compute the terms of expression y
x0 = Matrix([x0_1, x0_2])
u = sin(t)
C*Phi*x0
```

Out[55]: $[x_{02}e^{2t-2t_0}]$

In [56]:
```
# the integrand
integrate(C*Phi.subs(t0, tau)*B*u, (tau, t0, t))
```

Out[56]: $\left[\frac{t}{2}e^{2t-2t_0}\sin(t) - \frac{t}{2}\sin(t)\right]$

In [57]:
```
D*u
```

Out[57]: $[0]$

In [58]:
```
# Put it all together
Eq(y, (C*Phi*x0 + integrate(C*Phi.subs(t0, tau)*B*u, (tau, t0, t)) +
D*u)[0])
```

Out[58]: $y = \frac{t}{2}e^{2t-2t_0}\sin(t) - \frac{t}{2}\sin(t) + x_{02}e^{2t-2t_0}$

Problem 2

In [59]:
```
Image(filename='/home/adamspr/Pictures/ee547/wk3/Selection_007.png')
```

Out[59]:

Problem 2 (Characteristic Polynomial) Derive by hand the characteristic polynomial of a matrix A as (6).

$$A = \begin{bmatrix} 1 & 0 & 5 \\ 6 & 2 & 7 \\ 4 & 8 & 3 \end{bmatrix} \tag{6}$$

In [60]:
```
s = symbols("s")
A = Matrix([[1, 0, 5], [6, 2, 7], [4, 8, 3]])
det(s*eye(3) - A)
```

Out[60]: $s^3 - 6s^2 - 65s - 150$

Problem 3

In [61]:
```
Image(filename='/home/adamspr/Pictures/ee547/wk3/Selection_008.png')
```

Out[61]:

Problem 3 (Matrix Exponential of an LTI System) Consider a system as (7) with initial condition $x_1(0) = 1$ and $x_2(0) = 0.5$.

$$\dot{x} = \begin{bmatrix} 0 & 2 \\ -4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ -2 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{7}$$

a) Show the state transition matrix by Inverse Laplace Transform is

$$\varphi(t) = \begin{bmatrix} e^{-2t}(\cos 2t + \sin 2t) & e^{-2t} \sin 2t \\ -2e^{-2t} \sin 2t & e^{-2t}(\cos 2t + \sin 2t) \end{bmatrix} \tag{8}$$

b) Derive the system output to the step input $u(t) = 1$, $\forall t \geq 0$. You can verify with MATLAB or other software.

In [78]:
```
# initialize
Phi = Function("Phi")(t)
A, B, C, D = symbols("A, B, C, D")
A = Matrix([[0, 2], [-4, -4]])
B = Matrix([1, -2])
C = Matrix([1, 2]).T
x0 = Matrix([1, 0.5])
u = 1
A, B, C
```

Out[78]:

$$\left(\begin{bmatrix} 0 & 2 \\ -4 & -4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 \end{bmatrix}\right)$$

Solution 3.a

The state transition matrix is $\mathcal{L}^{-1}[(s\mathbf{I}_n - \mathbf{A})^{-1}]$

In [80]:
```
sI_A = s*eye(2) - A
inverse_laplace_transform(sI_A.inv(), s, t)
```

Out[80]:

$$\begin{bmatrix} (\sin(2t) + \cos(2t)) e^{-2t} \theta(t) & e^{-2t} \sin(2t)\theta(t) \\ -2e^{-2t} \sin(2t)\theta(t) & (-\sin(2t) + \cos(2t)) e^{-2t} \theta(t) \end{bmatrix}$$

Solution 3.b

In [81]:
```
Phi = Matrix([[(sin(2*t)+cos(2*t))*exp(-2*t), sin(2*t)*exp(-2*t)],
              [-2*exp(-2*t)*sin(2*t), sin(2*t)+cos(2*t)*exp(-2*t)]])
Phi
```

Out[81]:

$$\begin{bmatrix} (\sin(2t) + \cos(2t)) e^{-2t} & e^{-2t} \sin(2t) \\ -2e^{-2t} \sin(2t) & \sin(2t) + e^{-2t} \cos(2t) \end{bmatrix}$$

In [25]:
```
# term 1
simplify(C*Phi*x0)
```

Out[25]:

$$\left[\left(-1.0e^{2t} \sin(2t) - 2.5 \sin(2t) + 2.0 \cos(2t)\right) e^{-2t} \right]$$

In [26]:
```
# term 2
C*integrate(Phi.subs(t, tau)*B*u, (tau, 0, t))
```

Out[26]:

$$\left[-2\cos(2t) + \frac{1}{2} e^{-2t} \sin(2t) + 2e^{-2t} \cos(2t) \right]$$

In [83]:
```
Eq(y, simplify(C*Phi*x0 + C*integrate(Phi.subs(t, tau)*B*u, (tau, 0, t)))[0])
```

Out[83]:

$$y = \left((1.0 \sin(2t) + 2\cos(2t) - 4) e^{2t} - 2.0 \sin(2t) + 4.0 \cos(2t)\right) e^{-2t}$$

In [85]:
```
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```

Out[85]:

Problem 4 (Impulse Response of an LTI System) Consider a system as (9) with initial condition $x_1(0) = 1$ and $x_2(0) = 2$.

$$\dot{x} = \begin{bmatrix} -7 & -12 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$

$$y = \begin{bmatrix} -4 & -10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + u(t) \qquad (9)$$

a) Derive the state transition matrix by MATLAB.

b) If an impulse input applies over the system, use the MATLAB **impulse** function and the method we learned to plot system's output.

Please see attached Matlab html for solution to Problem 4