Lecture 7

More Methods to Compute LTI Solutions

Matrix Representations

Functions of a Matrix

Outline

Matrix Representations for LTI Systems
- Similarity Transformation
- Eigenvalues & Eigenvectors
- Jordan Normal Form

Functions of a Matrix

Goal: Find easier way to compute e^{At} or A^t for solution $x(t)$ to LTI System

e^{At} = STM, A^t = STM — cts & discrete time

Review

Homogeneous
$$\begin{cases} \dot{x}(t) = A\,x(t) \\ y(t) = C\,x(t) \end{cases}$$

$$x(t) = \Phi(t,t_0)\,x_0 \qquad \text{unique solution}$$
$$\Phi(t,t_0) = e^{A(t-t_0)} \qquad \text{STM}$$

Non-homogeneous
$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = C\,x(t) + Du(t) \end{cases}$$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}\,B\,u(\tau)\,d\tau \qquad \text{unique solution}$$
$$\Phi(t,t_0) = e^{A(t-t_0)} \qquad \text{STM}$$

$$e^{At} = \mathcal{L}^{-1}\left[(sI-A)^{-1}\right]$$

Review p. 6.6 Notes – Summary

Similarity Transformation

Definition Square matrices A $n\times n$ and $\bar{A}$ $n\times n$ are said to be similar if $\exists$ $n\times n$ nonsingular matrix Q such that

$$AQ = Q\bar{A}$$

Writing this differently,

$$A = Q\bar{A}Q^{-1} \quad \text{or} \quad \bar{A} = Q^{-1}AQ$$ Similarity transformation

Question What is the "representation of A w.r.t. a certain basis"? (This denoted as $\bar{A}$ above)

Recall: a basis of a linear space V is a set of linearlly independant vectors that spans the space. i.e., any vector $x \in V$ can be uniquely written as a linear combination of the basis vectors,

$$x = \sum_{i=1}^{n} \alpha_i v_i \qquad \alpha_i \in \mathbb{R} \text{ constants}$$

Consider $A: \mathbb{R}^n \to \mathbb{R}^n$, $n\times n$ square matrix

Let $\{i_1, i_2, \dots, i_n\}$ be the orthonormal basis or "standard basis" of $\mathbb{R}^n$.

$$i_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad i_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad \dots \quad i_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

For any vector $x \in \mathbb{R}^n$, we can write

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 i_1 + x_2 i_2 + \dots + x_n i_n = I \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = I\bar{x}$$

where I is the Identity Matrix $\left[\, i_1 \;\vdots\; i_2 \;\vdots\; \dots \; i_n \,\right]$

Thus, the "representation of x" wrt the orthonormal basis $\{i_1, i_2, \ldots, i_n\}$, $\bar{x}$, is just itself, x.

Consider now some other, general basis of $\mathbb{R}^n$

$$\{q_1, q_2, \ldots q_n\} \qquad q_i \in \mathbb{R}^n \text{ vector}$$

Then every $x \in \mathbb{R}^n$ can be written uniquely as

$$x = \alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_n q_n$$

Define $n \times n$ square matrix

$$Q := [\, q_1 \mid q_2 \mid \cdots \mid q_n \,]$$

Then $x = Q \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix} = Q\bar{x}$

We call $\bar{x} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ the "representation of x wrt the basis $\{q_1, q_2, \ldots q_n\}$"

Example $\mathbb{R}^2$ $\quad x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$\{i_1, i_2\}$ basis

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\, i_1 + 3\, i_2$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \end{bmatrix} = I\bar{x} \qquad \Rightarrow x = \bar{x}$$

$\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ basis $\quad \{q_1, q_2\}$

$$x = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 \cdot \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1\, q_1 + 2\, q_2$$

$$= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = Q\bar{x}$$

$\Rightarrow \bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is the representation of $x = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ wrt the basis $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Back to our Question What is the representation of a $n \times n$ matrix A wrt a basis? (what is $\bar{A}$?)

Answer

1) For $\{i_1, i_2, \dots, i_n\}$ orthonormal basis for $\mathbb{R}^n$

1^{st} column of $\bar{A}$ $:= \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}$

is the representation of $\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = A i_1$ wrt orthonormal basis

i.e.,

$$A i_1 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} = a_{11} i_1 + a_{21} i_2 + \dots + a_{n1} i_n$$

This is true for every column, thus

$$A I = I \bar{A} \quad , \text{ or } \quad A = \bar{A}$$

2) For $\{q_1, q_2, \dots, q_n\}$ arbitrary basis for $\mathbb{R}^n$

Then A has a different representation $\bar{A}$.

i.e., i^{th} column of $\bar{A}$ is the representation of $A q_i$ with respect to the basis $\{q_1, q_2, \cdots q_n\}$

or

$$[A q_1 \quad A q_2 \cdots A q_n] = [q_1 \; q_2 \cdots q_n] \bar{A}$$

or

$$A [q_1 \quad q_2 \cdots q_n] = [q_1 \; q_2 \cdots q_n] \bar{A}$$

or

$$\boxed{A Q = Q \bar{A}}$$

which is just the Similarity Transformation

$$\boxed{A = Q \bar{A} Q^{-1} \quad \text{or} \quad \bar{A} = Q^{-1} A Q}$$

Example (Chen p. 53)

$$A = \begin{bmatrix} 3 & 2 & -1 \\ -2 & 3 & 0 \\ 4 & 1 & 1 \end{bmatrix}$$

Step 1 Find a basis for $\mathbb{R}^3$

Choose a basis as follows: (not unique)

Let $q_1 = b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$q_2 = Ab = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$q_3 = A^2 b = A(Ab) = \begin{bmatrix} -4 \\ 2 \\ -3 \end{bmatrix}$

Check that $Q = [q_1 \ q_2 \ q_3] = \begin{bmatrix} 0 & -1 & -4 \\ 0 & 0 & 2 \\ 1 & 1 & -3 \end{bmatrix}$ is nonsingular

one method: Q^{-1} exists ✓

Step 2 Compute $\bar{A}$ = representation of A wrt $\{q_1, q_2, q_3\}$

1st column $\bar{A}$ = representation of Aq_1 wrt $\{q_1, q_2, q_3\}$

$$Aq_1 = Ab = [q_1 \ q_2 \ q_3] \begin{bmatrix} \alpha_{11} \\ \alpha_{21} \\ \alpha_{31} \end{bmatrix}$$

$$= [b \ \ Ab \ \ A^2 b] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ 1st column of $\bar{A}$

2nd column $\bar{A}$: $Aq_2 = A(Ab) = A^2 b = [b \ \ Ab \ \ A^2 b] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ 2nd column of $\bar{A}$

3rd column $\bar{A}$: $Aq_3 = A(A^2 b) = A^3 b = [b \ \ Ab \ \ A^2 b] \begin{bmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \end{bmatrix}$

$$A^3 b = A(A^2 b) = \begin{bmatrix} -5 \\ 10 \\ -13 \end{bmatrix} = 17b - 15Ab + 5A^2 b \Rightarrow \begin{bmatrix} \alpha_{13} \\ \alpha_{23} \\ \alpha_{33} \end{bmatrix} = \begin{bmatrix} 17 \\ -15 \\ 5 \end{bmatrix}$$

Thus

$$\bar{A} = \begin{bmatrix} 0 & 0 & 17 \\ 1 & 0 & -15 \\ 0 & 1 & 5 \end{bmatrix}$$

Check $AQ = Q\bar{A}$ ✓

Eigenvalues & Eigenvectors

A $n \times n$ square matrix has different representations with respect to different sets of basis.

We now consider a basis for $\mathbb{R}^n$ that makes the representation diagonal or block diagonal. easier!

Definition A real or complex number λ is called an eigenvalue of the $n \times n$ real matrix A if $\exists$ nonzero vector $x \in \mathbb{R}^n$ s.t.

$$\boxed{Ax = \lambda x}$$

(left ev $xA = \lambda x$)

x is called a (right) eigenvector of A associated with eigenvalue λ.

Question: How to find λ and x?

Write the above equation as

$$Ax = \lambda x = \lambda I x \quad \Rightarrow \quad (A - \lambda I)x = 0$$

e-vector of A

x is solution

This is a homogeneous equation.

Recall Theorem For $Ax = y$ with A square:

1) If A is nonsingular, then the equation has a unique solution $\forall y$: $x = A^{-1}y$

The only solution of $Ax = 0$ is $x = 0$.

2) $Ax = 0$ has nonzero solutions iff A is singular.

The number of linearly independent solutions

$=$ nullity of A = maximum # lin. indep. null vectors of A, i.e., # x s.t. $Ax = 0$

$=$ # columns of A $-$ # linearly indep. columns of A

$=$ # columns of A $-$ rank(A)

Thus, if $(A-\lambda I)$ is nonsingular, the only solution to $(A-\lambda I)x = 0$ is $x = 0$.

Thus, in order for $(A-\lambda I)x = 0$ to have a nonzero solution x, $(A-\lambda I)$ must be singular. or have determinant zero.

$\dot{x} = Ax$
$sx = Ax$
$(sI-A)x = 0$

<u>Recall</u> $\Delta(\lambda) = \det(\lambda I - A)$ characteristic polynomial of A of degree n

If λ is a root of $\Delta(\lambda)$, then $\Delta(\lambda) = 0$ and $(A-\lambda I)x = 0$ has at least one nonzero solution.

☆ Thus, every root of $\Delta(\lambda)$ is an eigenvalue of A ☆

$(A-\lambda I)x = 0 \Rightarrow Ax = \lambda x$

Note: $\Delta(\lambda) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n$ degree n

$= (s-\lambda_1)^{m_1}(s-\lambda_2)^{m_2}\cdots(s-\lambda_k)^{m_k}, \quad \sum m_i = n$

$\Rightarrow$ A has n eigenvalues (not necessarily distinct)

<u>Plan</u>: 1) use the eigenvectors of A q_i as a basis for $\mathbb{R}^n$

eigenvectors are n lin ind. solutions to $(A-\lambda I)q_i = 0$

2) Find the representation of A wrt this basis

3) This will yield $\bar{A}$ diagonal or block diagonal

$\rightarrow$ we will see this

4) $A = Q\bar{A}Q^{-1}$, $Q = [q_1 \; q_2 \cdots q_n]$

$\boxed{e^{At} = Qe^{\bar{A}t}Q^{-1}}$ $e^{\bar{A}t}$ easier to compute STM

5) use to solve linear system equation

$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \Rightarrow x(t) = \underbrace{e^{A(t-t_0)}}_{= \Phi(t,t_0)} x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$

$\Phi(t,t_0)$ S.T.M.

Case: Eigenvalues of A are distinct

Let

$\lambda_1, \lambda_2, \ldots, \lambda_n$ eigenvalues of $A_{n\times n}$

all distinct

Let q_i be the eigenvector of A associated with λ_i

$$A q_i = \lambda_i q_i \qquad i = 1, \ldots, n$$

q_i is unique solution to $(A - \lambda_i I) q_i = 0$

Then $\{q_1, q_2, \ldots, q_n\}$ is linearly independent and can be used as a basis for $\mathbb{R}^n$.

n distinct eigenvectors

Let $\bar{A}$ be the representation of A wrt this basis.

<u>Question</u> What does $\bar{A}$ look like?

1^{st} column of $\bar{A}$ = representation of Aq_1 wrt $\{q_1, q_2, \ldots q_n\}$

$$Aq_1 = \lambda_1 q_1 \quad \text{by definition}$$

$$= [q_1 \; q_2 \; \cdots \; q_n] \begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ 1^{st} column of $\bar{A}$ is $\begin{bmatrix} \lambda_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$

2^{nd} column of $\bar{A}$ = representation of Aq_2 wrt $\{q_1 \ldots q_n\}$

$$Aq_2 = \lambda_2 q_2$$

$$= [q_1 \; q_2 \cdots \; q_n] \begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\Rightarrow$ 2^{nd} column of $\bar{A}$ is $\begin{bmatrix} 0 \\ \lambda_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ ... etc

Thus

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & 0 & 0 & \lambda_n \end{bmatrix}$$

<u>diagonal matrix</u>

STM = $e^{At} = Q e^{\bar{A}t} Q^{-1}$, $Q = [q_1 \cdots q_n]$

We can conclude:

- Every matrix with distinct eigenvalues has a diagonal matrix representation by using its eigenvectors as a basis.

- Different orderings of eigenvectors will yield different diagonal matrices for the same A.

- $Q := [q_1 \ q_2 \ \cdots \ q_n]$ as before

$\boxed{AQ = Q\bar{A}}$ used to verify representation

Example (Chen p. 56) Distinct Eigenvalues

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$ First find eigenvalues of A:

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 0 \\ -1 & \lambda & -2 \\ 0 & -1 & (\lambda-1) \end{bmatrix}$$

$$= \lambda[\lambda(\lambda-1)-2] = (\lambda-2)(\lambda+1)\lambda$$

$\Rightarrow$ A has eigenvalues 2, −1, 0 distinct

Now find corresponding eigenvectors:

$\underline{\lambda_1 = 2}$: q_1 is any nonzero solution of

$$A q_1 = \lambda_1 q_1$$

$$(A - 2I)\, q_1 = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{13} \end{bmatrix} = 0$$

Choose $q_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

Example (continued)

$\lambda_2 = -1$:

$$A q_2 = \lambda_2 q_2$$

$$(A - (-1) I) q_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix} q_2 = 0$$

Choose $q_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$

$\lambda_3 = 0$:

$$A q_3 = \lambda_3 q_3$$

$$(A - (0) I) q_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} q_3 = 0$$

Choose $q_3 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

Thus, the representation of A wrt $\{q_1, q_2, q_3\}$ is

$$\bar{A} = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{diagonal}$$

Alternate way to compute $\bar{A}$:

$$\bar{A} = Q^{-1} A Q \qquad \text{for } Q = [q_1 \ q_2 \ q_3] = \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Simpler to verify

$$Q \bar{A} = A Q \quad \checkmark$$

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \stackrel{?}{=} \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 2 \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

Then $e^{At} = Q e^{\bar{A}t} Q^{-1}$

Case: Eigenvalues of A are not all distinct.

Definition eigenvalue with multiplicity 1 is a simple eigenvalue.
eigenvalue with multiplicity ≥ 2 is a repeated eigenvalue

Recall If A has only simple eigenvalues,
Then A always has a diagonal representation

But If A has repeated eigenvalues.
Then A may not have a diagonal form representation
But A does have a block-diagonal
and triangular form representation.

Subcase 1: A has one distinct eigenvalue λ with multiplicity n.

Let $n=4$ for simplicity.

Suppose $(A-\lambda I)$ has rank $n-1=3$ rank = # lin. indep columns

or nullity $=1$ [recall nullity $= n - \text{rank}(A-\lambda I)$]

Then

$$(A-\lambda I)q = 0$$

has only 1 independent solution (from above Theorem)

and only 1 eigenvector associated with λ.

$\Rightarrow$ We need $n-1=3$ more lin. indep. vectors to form a basis for $\mathbb{R}^n = \mathbb{R}^4$

Let q_1 be st $(A-\lambda I)q_1 = 0$.

Need to find q_2, q_3, q_4

$\Rightarrow$ We will construct q_2, q_3, q_4 from q_1

Definition A vector v is called a generalized eigenvector of grade n if

$$(A-\lambda I)^n v = 0$$

$$(A-\lambda I)^{n-1} v \neq 0$$

Note for $n=1$, $\left\{ \begin{array}{r} (A-\lambda I)v = 0 \\ v \neq 0 \end{array} \right\}$ v is an ordinary eigenvector

For $n=4$, define

$$v_4 := v$$

$$v_3 \overset{④}{:=} (A-\lambda I) v_4 = (A-\lambda I) v$$

$$v_2 \overset{③}{:=} (A-\lambda I) v_3 = (A-\lambda I)^2 v$$

$$v_1 \overset{②}{:=} (A-\lambda I) v_2 = (A-\lambda I)^3 v$$

chain of generalized eigenvectors of length $4 = n$

These have the properties

$$(A-\lambda I) v_1 = 0 \quad ①$$

$$(A-\lambda I)^2 v_2 = 0$$

$$(A-\lambda I)^3 v_3 = 0$$

$$(A-\lambda I)^4 v_4 = 0$$

$\Rightarrow$ $\{v_1, v_2, v_3, v_4\}$ lin. indep. $\Rightarrow$ use as basis

Find the representation of A w.r.t. $\{v_1, v_2, v_3, v_4\}$:

$A v_1 = \lambda v_1$ from ①

$A v_2 = v_1 + \lambda v_2$ from ②

$A v_3 = v_2 + \lambda v_3$ from ③

$A v_4 = v_3 + \lambda v_4$ from ④

The representation of A wrt $\{v_1, v_2, v_3, v_4\}$ is

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & \lambda \end{bmatrix}$$

Jordan Block of order 4

To see this:

1^{st} column of J is representation of

$$A v_1 = \lambda v_1$$

with respect to $\{v_1, v_2, v_3, v_4\}$

which is $\begin{bmatrix} \lambda \\ 0 \\ 0 \\ 0 \end{bmatrix}$

Then

$$e^{At} = Q e^{Jt} Q^{-1}$$

2^{nd} column of J is representation of

$$A v_2 = v_1 + \lambda v_2$$

wrt $\{v_1, v_2, v_3, v_4\}$

which is $\begin{bmatrix} 1 \\ \lambda \\ 0 \\ 0 \end{bmatrix}$ etc.

Subcase 2: A has 2 lin. indep. eigenvectors (for general n)

Suppose $(A - \lambda I)$ has rank $n-2$

or nullity $= 2$ $[n - \text{rank}(A - \lambda I)]$

Then

$$(A - \lambda I) q = 0$$

has two linearly independent solutions. (from above theorem)

and two linearly independent eigenvectors associated with the two eigenvalues.

$\Rightarrow$ We need $n-2$ more lin. indep. vectors to form a basis for $\mathbb{R}^n$

These will be generalized eigenvectors as above and there will be 2 "chains" of generalized e-vectors

Let the two lin. indep. solutions to

$$(A - \lambda I) q = 0$$

be u_1 and v_1

We then construct two chains of generalized eigenvectors

$$\{u_1, u_2, \ldots, u_l\} \qquad k + l = n$$
$$\{v_1, v_2, \ldots v_k\}$$

Then
$\{v_1, \ldots, v_k, u_1, \ldots u_l\}$ is lin. indep. and can be used as a basis.

Then wrt this basis, the representation of A is a block diagonal matrix

$$\bar{A} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix}$$

where J_1, J_2 are Jordan Blocks of orders k, l, resp.

Example $n = 5$

A 5×5 with eigenvalue λ_1 of multiplicity 4
eigenvalue λ_2 of Multiplicity 1 simple e-value

Then ∃ nonsingular matrix Q st

$$\bar{A} = Q^{-1} A Q$$

Consider now the different forms of $\bar{A}$

$$\bar{A}_1 = \left[\begin{array}{cccc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

nullity $(A - \lambda_1 I) = 1$
⇒ 1 Jordan Block associated with λ_1

$$\bar{A}_2 = \left[\begin{array}{ccc|c|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 1 & 0 & 0 \\ 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

nullity $(A - \lambda_1 I) = 2$
⇒ 2 Jordan Blocks associated with λ_1

One possibility

$$\bar{A}_3 = \left[\begin{array}{cc|cc|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 1 & 0 \\ 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

nullity $(A-\lambda_1 I) = 2$

Second possibility

$$\bar{A}_4 = \left[\begin{array}{cc|c|c|c} \lambda_1 & 1 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

nullity $(A-\lambda_1 I) = 3$

$\Rightarrow$ 3 Jordan Blocks associated with λ_1

$$\bar{A}_5 = \left[\begin{array}{c|c|c|c|c} \lambda_1 & 0 & 0 & 0 & 0 \\ \hline 0 & \lambda_1 & 0 & 0 & 0 \\ \hline 0 & 0 & \lambda_1 & 0 & 0 \\ \hline 0 & 0 & 0 & \lambda_1 & 0 \\ \hline 0 & 0 & 0 & 0 & \lambda_2 \end{array}\right]$$

nullity $(A-\lambda_1 I) = 4$

$\Rightarrow$ 4 Jordan Blocks associated with λ_1

Note: The Jordan Form has the eigenvalues of A on the diagonal and either 0 or 1 on the super diagonal.

Note: $e^{At} = Q e^{Jt} Q^{-1}$, $Q = [q_1 \cdots q_n]$

Fact: # Jordan Blocks associated with an eigenvalue λ is equal to the nullity of $(A-\lambda I)$

Note: Positions of the Jordan Blocks can be changed by changing the order of the basis.

Comment: All of the above representations of A are triangular and block diagonal with Jordan Blocks on the diagonal

$\Rightarrow$ they are in "Jordan Form"

Fact: $\det A = \det(Q\bar{A}Q^{-1}) = \det Q \det \bar{A} \det Q^{-1}$

$= \det \bar{A}$

$=$ product of all eigenvalues of A

$=$ all of the diagonal entries of $\bar{A}$

Fact: A is nonsingular iff it has no zero eigenvalue

$\Longrightarrow$

Example (Chen I, p. 41)

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$ Find the Jordan Form

Step 1 Find the characteristic polynomial of A

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda-1 & -1 & -2 \\ 0 & \lambda-1 & -3 \\ 0 & 0 & \lambda-2 \end{bmatrix} \leftarrow \text{upper diagonal}$$

$$= (\lambda-1)(\lambda-1)(\lambda-2)$$

$$= (\lambda-1)^2(\lambda-2) \qquad \text{roots } 1, 1, 2$$

step 2 Find the eigenvalues of A

$\Delta(\lambda) = 0$ at $\left.\begin{matrix}\lambda_1 = 1 \\ \lambda_2 = 1\end{matrix}\right\}$ with multiplicity 2

$\lambda_3 = 2$ Simple eigenvalue

↑ roots of c.p.

step 3 Compute the corresponding eigenvectors

$$\lambda_3 = 2: \quad (A - \lambda_3 I)\, v_3 = 0 = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_{31} \\ v_{32} \\ v_{33} \end{bmatrix}$$

$$0 = -v_{31} + v_{32} + 2v_{33}$$

$$0 = -v_{32} + 3v_{33}$$

Choose $v_3 = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$

$\lambda_1 = \lambda_2 = 1$: Need to use GEVs

$$(A - \lambda_1 I) = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$ nullity $(A - \lambda_1 I) = 1$

$\Rightarrow$ 1 Jordan Block associated with λ_1

$$(A - \lambda_1 I)^2 = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}$$

Search for a vector v s.t.

$$(A - \lambda_1 I)^2 v = 0$$

$$(A - \lambda_1 I)\, v \neq 0$$

from definition of GEVs

$$(A-\lambda_1 I)v = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} v \neq 0$$

$$(A-\lambda_1 I)^2 v = \begin{bmatrix} 0 & 0 & 5 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix} v = 0$$

Choose $v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ simplest one

$\Rightarrow$ v is a Generalized Eigenvector of Grade 2

Then Define $v_2 := v = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$v_1 := (A-\lambda_1 I)v = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\{v_1, v_2, v_3\}$ are linearly independent

Check: $\det \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \det Q = 1 \neq 0$

Step 4 Use eigenvectors as a basis to compute $\bar{A}$

i^{th} column of $\bar{A}$ is the representation of Av_i with respect to the basis $\{v_1, v_2, v_3\}$

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= v_1 + \lambda_1 v_2 \\ Av_3 &= \lambda_3 v_3 \end{aligned}\right\} \text{ from gen. e-vectors (p. 7.11)}$$

$$\Rightarrow \quad \bar{A} = J = \left[\begin{array}{cc|c} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ \hline 0 & 0 & \lambda_3 \end{array}\right] \quad \text{Jordan Form}$$

$$= \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 1 & 0 \\ \hline 0 & 0 & 2 \end{array}\right]$$

Check $Q\bar{A} = AQ$ for $Q = [v_1 \; v_2 \; v_3]$

Then $e^{At} = Q e^{Jt} Q^{-1}$

Functions of a Square Matrix

Purpose: Easy way to compute state transition matrix
$\Phi(t, t_0) = e^{A(t-t_0)}$
to compute solution, $x(t)$, to the
LTI state-space system of equations

$$x(t) = e^{A(t-t_0)} x_0 + \int_{t_0}^{t} e^{A(\tau - t_0)} B u(\tau) d\tau$$

Simplify computation: by using Jordan Form of A

$$A = Q \bar{A} Q^{-1}$$

$$e^{At} = Q e^{\bar{A}t} Q^{-1}$$

← easier to compute because $\bar{A}$ is block diagonal

Definition

Let $f(\lambda)$ be any function (not necessarily polynomial)
Let A be $n \times n$ matrix with characteristic polynomial

$$\Delta(\lambda) = \prod_{i=1}^{m} (\lambda - \lambda_i)^{n_i}, \quad n = n_1 + n_2 + \cdots + n_m$$

$$= (\lambda - \lambda_1)^{n_1} (\lambda - \lambda_2)^{n_2} \cdots (\lambda - \lambda_m)^{n_m}$$

m eigenvalues with multiplicity n_i

Define $h(\lambda)$ polynomial of degree $n-1$,

$$h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2 + \cdots + \beta_{n-1} \lambda^{n-1}$$

where $\beta_0, \beta_1, \beta_2, \ldots \beta_{n-1}$ are n unknown coefficients

Find the coefficients of $h(\lambda)$ by letting $h(\lambda)$ have the same values as $f(\lambda)$ on the spectrum of A:

$$f^{(\ell)}(\lambda_i) = h^{(\ell)}(\lambda_i) \quad \ell = 0, 1, \ldots, n_i - 1, \quad i = 1, 2, \ldots m$$

where

$$f^{(\ell)}(\lambda_i) := \left. \frac{d^\ell}{d\lambda^\ell} f(\lambda) \right|_{\lambda = \lambda_i}$$

values of f on the spectrum of A

$$h^{(l)}(\lambda_i) = \frac{d^l}{d\lambda^l} h(\lambda)\Big|_{\lambda=\lambda_i}$$ values of h on the spectrum of A

Then $f(A) = h(A)$

and $h(\lambda)$ is said to equal $f(\lambda)$ on the spectrum of A

Procedure

1. Find $\Delta(\lambda)$, $\{\lambda_i\}$, $\{f^{(l)}(\lambda_i)\}$
2. Construct an $(n-1)^{th}$-order polynomial
   $$h(\lambda) = \beta_0 + \beta_1 \lambda + \cdots + \beta_{n-1}\lambda^{n-1}$$
   such that f and h have the same value on the Spectrum of A.
3. Substitute A in $h(\lambda)$ to get $\boxed{f(A) = h(A)}$.

Example (Chen p. 64) Compute A^{100} for $A = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$

We want to compute $f(A)$ where $f(\lambda) = \lambda^{100}$

The characteristic polynomial is

$$\Delta(\lambda) = \det(\lambda I - A) = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda+2 \end{bmatrix} = \lambda(\lambda+2)+1$$
$$= \lambda^2 + 2\lambda + 1$$
$$= (\lambda+1)^2$$

This gives two eigenvalues $\lambda_1 = -1$, $\lambda_2 = -1$

Compute values of f on the spectrum of A:

$$f(-1) = (-1)^{100} = 1$$
$$f'(-1) = 100(-1)^{99} = -100$$

Construct $h(\lambda) = \beta_0 + \beta_1\lambda$

$$h(-1) = \beta_0 - \beta_1$$
$$h'(-1) = \beta_1$$

Equate f and h on the spectrum of A

$$f(-1) = h(-1) \qquad f'(-1) = h'(-1)$$

$$1 = \beta_0 - \beta_1 \qquad -100 = \beta_1$$

$$\beta_0 = -99$$

Thus $h(\lambda) = \beta_0 + \beta_1 \lambda = -99 - 100\lambda$

Substituting A, we get

$$f(A) = h(A) = -99I - 100A$$

$$= -99 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - 100 \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -99 & 0 \\ 0 & -99 \end{bmatrix} - \begin{bmatrix} 0 & 100 \\ -100 & -200 \end{bmatrix}$$

$$= \begin{bmatrix} -99 & -100 \\ 100 & 101 \end{bmatrix} = A^{100}$$

Simpler than multiplying $A \cdot A \cdot A \cdots A$ 100 times!

Example (Chen p. 65) Compute e^{At} for $A = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix}$ S.T.M.

$f(\lambda) = e^{\lambda t}$. Find $f(A)$

Characteristic Polynomial:

$$\Delta(\lambda) = \det(\lambda I - A) = \det \begin{bmatrix} \lambda & 0 & 2 \\ 0 & \lambda-1 & 0 \\ -1 & 0 & \lambda-3 \end{bmatrix}$$

$$= \lambda (-1)^{1+1} \begin{vmatrix} \lambda-1 & 0 \\ 0 & \lambda-3 \end{vmatrix} + (-1)(-1)^{(3+1)} \begin{vmatrix} 0 & 2 \\ \lambda-1 & 0 \end{vmatrix}$$

$$= \lambda(\lambda-1)(\lambda-3) + (-1)(-2(\lambda-1))$$

$$= \lambda(\lambda-1)(\lambda-3) + 2(\lambda-1)$$

$$= (\lambda-1)(\lambda(\lambda-3)+2)$$

$$= (\lambda-1)(\lambda^2 - 3\lambda + 2)$$

$$= (\lambda-1)(\lambda-1)(\lambda-2)$$

$$= (\lambda-1)^2(\lambda-2)$$

This gives eigenvalues $\lambda_1 = 1$, $\lambda_2 = 1$, $\lambda_3 = 2$

Compute f on the spectrum of A

$$f(1) = e^t$$

$$f'(1) = te^t$$

$$f(2) = e^{2t}$$

$n_1 = 2$ multiplicity

$n_2 = 1$

Construct $h(\lambda) = \beta_0 + \beta_1 \lambda + \beta_2 \lambda^2$

$$h(1) = \beta_0 + \beta_1 + \beta_2$$

$$h'(1) = \beta_1 + 2\beta_2$$

$$h(2) = \beta_0 + 2\beta_1 + 4\beta_2$$

Equate and Solve

$f(1) = h(1)$: $\quad e^t = \beta_0 + \beta_1 + \beta_2$ ①

$f'(1) = h(1)$: $\quad te^t = \beta_1 + 2\beta_2$ ②

$f(2) = h(2)$: $\quad e^{2t} = \beta_0 + 2\beta_1 + 4\beta_2$ ③

② $\Rightarrow$ $\beta_1 = te^t - 2\beta_2$ ④

② → ① $\Rightarrow$ $e^t = \beta_0 + (te^t - 2\beta_2) + \beta_2 = \beta_0 + te^t - \beta_2$

$\Rightarrow \beta_0 = e^t - te^t + \beta_2$ ⑤

④, ⑤ → ③ $\Rightarrow$ $e^{2t} = (e^t - te^t + \beta_2) + 2(te^t - 2\beta_2) + 4\beta_2$

$= e^t - te^t + \beta_2 + 2te^t - \cancel{4\beta_2} + \cancel{4\beta_2}$

$= e^t + te^t + \beta_2$

$\Rightarrow \beta_2 = e^{2t} - e^t - te^t$ ⑥

⑥ → ④ $\Rightarrow$ $\beta_1 = te^t - 2(e^{2t} - e^t - te^t)$

$= te^t - 2e^{2t} + 2e^t + 2te^t$

$\beta_1 = 3te^t - 2e^{2t} + 2e^t$ ⑦

⑥ → ⑤ $\Rightarrow$ $\beta_0 = \cancel{e^t} - te^t + (e^{2t} - \cancel{e^t} - te^t)$

$\beta_0 = e^{2t} - 2te^t$ ⑧

Equations ⑥, ⑦, ⑧ give coefficients for $h(\lambda)$

$$h(\lambda) = (e^{2t} - 2te^{t}) + (3te^{t} - 2e^{2t} + 2e^{t})\lambda + (e^{2t} - e^{t} - te^{t})\lambda^2$$

Thus

$$f(A) = h(A) = (e^{2t} - 2te^{t}) I + (3te^{t} - 2e^{2t} + 2e^{t}) A + (e^{2t} - e^{t} - te^{t}) A^2$$

$$A^2 = \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 0 & 0 & -2 \\ 0 & 1 & 0 \\ 1 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & 0 & -6 \\ 0 & 1 & 0 \\ 3 & 0 & 7 \end{bmatrix}$$

$$f(A) = \begin{bmatrix} e^{2t} - 2te^{t} & 0 & 0 \\ 0 & e^{2t} - 2te^{t} & 0 \\ 0 & 0 & e^{2t} - 2te^{t} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & -2(3te^{t} - 2e^{2t} + 2e^{t}) \\ 0 & 3te^{t} - 2e^{2t} + 2e^{t} & 0 \\ 3te^{t} - 2e^{2t} + 2e^{t} & 0 & 3(3te^{t} - 2e^{2t} + 2e^{t}) \end{bmatrix}$$

$$+ \begin{bmatrix} -2(e^{2t} - e^{t} - te^{t}) & 0 & -6(e^{2t} - e^{t} - te^{t}) \\ 0 & e^{2t} - e^{t} - te^{t} & 0 \\ 3(e^{2t} - e^{t} - te^{t}) & 0 & 7(e^{2t} - e^{t} - te^{t}) \end{bmatrix}$$

$$= \begin{bmatrix} (-e^{2t} + 2e^{t}) & 0 & (2e^{2t} + 2e^{t}) \\ 0 & e^{t} & 0 \\ (e^{2t} - e^{t}) & 0 & (2e^{2t} - e^{t}) \end{bmatrix}$$

$= e^{At}$ $\Rightarrow$ use as $\Phi(t, 0)$ S.T.M. for $x(t)$ Solution of LTI System equation