Class 3

Lecture 5

Solutions to LTV Systems

Lecture 6

Solutions to LTI Systems

Outline

Fundamental Theorem of Differential Equations

State Transition Matrix

Solutions of D.E.'s – LTV Systems

Homogeneous Solution $(u(t) \equiv 0)$

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) \\ x(t_0) = x_0 \end{cases}$$

Nonhomogeneous Solution

$$\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ x(t_0) = x_0 \end{cases}$$

Fundamental Theorem of Differential Equations

Given the differential equation

$$\begin{cases} \dot{x}(t) = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

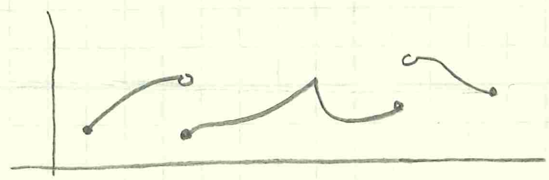
Let $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ satisfy

(a) $\forall x \in \mathbb{R}^n$, $t \longmapsto f(x,t)$ is piecewise continuous with its set of discontinuity points $= D$ may be zero

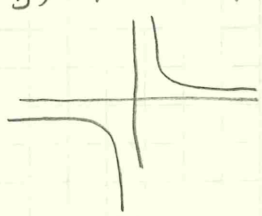
Note piecewise continuous

A function is piecewise continuous if it consists of a finite number of continuous pieces with finite discontinuities between them.

e.g.,



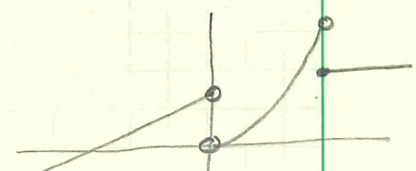
e.g., $f(x) = 1/x$ is not p.c.



b/c discontinuity between two continuous pieces is not finite

e.g.

$$f(x) = \begin{cases} x+4 & x < 0 \\ x^2 & 0 < x < 5 \\ 7 & x \geq 5 \end{cases}$$



(b) $f(x,t)$ satisfies the Lipschitz Condition:

$\forall t \in \mathbb{R}^+$, $\forall \xi, \xi'$, $\exists\, k(t)$ piecewise continuous fct

s.t. $\| f(\xi, t) - f(\xi', t) \| \leq k(t) \| \xi - \xi' \|$

norm = length of vector

e.g., $x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$ $\quad \|x\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$

Then

If (1) Existence

$\forall x_0 \in \mathbb{R}^n$, $\forall t_0 \in \mathbb{R}^+$, $\exists\ \phi : \mathbb{R}^+ \to \mathbb{R}^n$ continuous s.t.

$$\begin{cases} \dot{\phi}(t) = f(\phi(t), t) & \forall t \in \mathbb{R}^+, t \notin D \\ \phi(t_0) = x_0 \end{cases}$$

(2) Uniqueness

ϕ is unique

(3) ϕ depends on x_0 continuously

Then $\Rightarrow$ ϕ is called the solution of the differential equation.

Remark Importance of Fundamental Theorem

☆ Gives a large class of problems for which we can speak of the unique solution

☆ We rely on the Theorem to prove equalities

Kailath p. 597

e.g. to show two functions are equal

$$L(t) = R(t) \quad \forall t \in \mathbb{R}^+$$

i.e., We may observe that

(i) $L(t_0) = R(t_0)$

(ii) $L(\cdot) = R(\cdot)$ are solutions to the same d.e., which itself satisfies the conditions of the Fundamental Theorem

Sketch of Proof

Proof of Existence (Picard Iteration Scheme) (by construction)

On $[0,\infty)$, construct a sequence of functions

$$x_{m+1}(t) = x_0 + \int_{t_0}^{t} f(x_m(t'), t')\,dt' \quad \forall t \in [0,\infty)$$

Show that on every finite interval, $[t_0, t_1]$ say, $\{x_i\}_0^\infty$ converges uniformly to a continuous function that satisfies the d.e. wherever $t \notin D$.

Proof of Uniqueness (by contradiction)

Suppose $\exists$ two solutions $\phi(t)$ and $\psi(t)$ satisfying

$$\begin{cases} \dot{x} = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

By integrating the d.e. and subtracting,

$$\phi(t) - \psi(t) = \int_{t_0}^{t} [f(\phi(\tau),\tau) - f(\psi(\tau),\tau)]\,d\tau \quad \forall t \in \mathbb{R}^+$$

Restricting our attention to a finite interval $[t_0, t_1]$,

$$\|\phi(t) - \psi(t)\| \le \int_{t_0}^{t} \|f(\phi(\tau),\tau) - f(\psi(\tau),\tau)\|\,d\tau$$

Lipschitz Condition $\le \bar{k} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\|\,d\tau \quad \forall t \in [t_0, t_1]$

Then, for any $c_1 \ge 0$,

$$\|\phi(t) - \psi(t)\| \le c_1 + \bar{k} \int_{t_0}^{t} \|\phi(\tau) - \psi(\tau)\|\,d\tau \qquad \bar{k} \text{ constant}$$

Bellman-Gronwall Lemma $\le c_1 e^{\bar{k}(t-t_0)} \quad \forall t \in [t_0, t_1]$

Holds $\forall c_1 \ge 0$. Thus, for $c_1 = 0$, we must have

$$\|\phi(t) - \psi(t)\| = 0 \quad \forall t \in [t_0, t_1]$$

$\Rightarrow \phi(t) = \psi(t) \quad \forall t \in [t_0, t_1]$

$\Rightarrow$ Solution is unique on $[t_0, t_1]$

Extend domain of uniqueness step-by-step to all $t \in \mathbb{R}^+$

∎

Bellman-Gronwall Lemma

$c_1 \ge 0$ constant.

$u(\cdot)$, $k(\cdot)$ real-valued p.c. fcts on $\mathbb{R}^+$, $k(\cdot)$ non-negative.

If $u(t) \le c_1 + \int_{t_0}^{t} k(\tau) u(\tau)\,d\tau$

Then $u(t) \le c_1 \exp \int_{t_0}^{t} k(\tau)\,d\tau \quad \forall t \in \mathbb{R}^+$

Def. Dynamical System

A model of a physical system is a dynamical system iff we can associate with it a quintuple (list of 5 objects)

$$D = (\mathcal{U}, \Sigma, Y, s, r)$$

satisfying two axioms.

Let $\mathcal{U} = \{u(t),\ t \in T \subset \mathbb{R}\}$ — input set

$\Sigma = \{x(t),\ t \in T\}$ — state set

$Y = \{y(t),\ t \in T\}$ — output set

s = state transition function

$$x(t_1) = s(t_1, t_0, x_0, u) \qquad t_1 \geq t_0,\ x_0 \in \Sigma,\ t_0, t_1 \in T$$

"the state at t_1 reached from x_0 at t_0 as a result of u"

r = read-out function

$$y(t) = r(t, x(t), u(t)) \qquad t \in T,\ x \in \Sigma,\ u \in \mathcal{U}$$

"the response of the system at t when it is in state $x(t)$ at t and the input is $u(t)$"

State Transition Axiom $\forall\, t_0, t_1 \in T,\ \forall\, x_0 \in \Sigma$

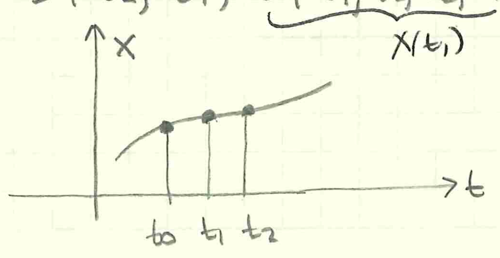
If $u, \bar{u} \in \mathcal{U}$ and $u(t) = \bar{u}(t)\ \forall\, t \in [t_0, t_1] \cap T$

Then $s(t_1, t_0, x_0, u) = s(t_1, t_0, x_0, \bar{u}) = x(t_1)$

i.e., $x(t_1)$ depends only on $u_{[t_0, t_1]}$

State Composition Axiom $\forall\, t_0 \leq t_1 \leq t_2 \in T,\ \forall\, x_0 \in \Sigma,\ \forall\, u \in \mathcal{U}$

$$s(t_2, t_1, \underbrace{s(t_1, t_0, x_0, u)}_{x(t_1)}, u) = s(t_2, t_0, x_0, u) = x(t_2)$$



aka Semi-group Axiom

LTV System

$$\textcircled{1}\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) & x \in \mathbb{R}^n, \quad u \in \mathbb{R}^k \\ y(t) = C(t)\,x(t) + D(t)\,u(t) & y \in \mathbb{R}^m \end{cases}$$

$A(\cdot)$, $B(\cdot)$, $C(\cdot)$, $D(\cdot)$ matrices w/ elements piecewise cts
$n\times n$ $n\times k$ $m\times n$ $m\times k$ real-valued fcts defined on $\mathbb{R}_+$

Refer to ① as the system representation $R = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$

Recall: a given model of a physical system may have many representations $\Rightarrow$ carefully distinguish between properties of the model & of its representation.

Theorem $R = [A(\cdot), B(\cdot), C(\cdot), D(\cdot)]$ represents a linear dynamical system.

Proof (sketch)

Let $\mathcal{U}$ = set of piecewise cts functions $: \mathbb{R}_+ \to \mathbb{R}^k$
$\mathcal{Y}$ = " $: \mathbb{R}_+ \to \mathbb{R}^m$
$\Sigma = \mathbb{R}^n$

From the Fund. Thm. of d.e., R represents a dynamical system.

Show 3 Properties are satisfied:

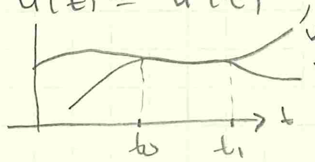
1) Unique Solution

$\forall u(\cdot)$, ① has a unique solution denoted by

$s(t, t_0, x_0, u)$
$$\begin{cases} \frac{d}{dt} s(t, t_0, x_0, u) = A(t)\, s(t, t_0, x_0, u) + B(t)\, u(t) \\ s(t_0, t_0, x_0, u) = x_0 \end{cases}$$

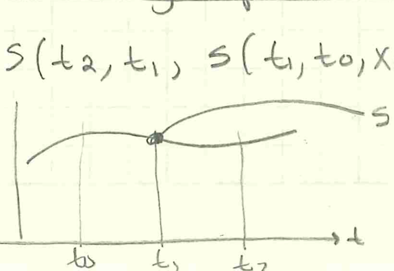
2) State Transition Axiom

$u(t) = \bar{u}(t)$, $t \in [t_0, t_1] \Rightarrow s(t_1, t_0, x_0, u) = s(t_1, t_0, x_0, \bar{u})$



3) Semi-group Axiom

$s(t_2, t_1, s(t_1, t_0, x_0, u_{[t_0,t_1]}), u_{[t_2,t_1]}) = s(t_2, t_0, x_0, u_{[t_0,t_2]})$



Note $u_{[t_0,t_2]} = u_{[t_0,t_1]} \cup u_{[t_1,t_2]}$

Solution to Homogeneous LTV System

$u(t) \equiv 0$

D.E. $\begin{cases} \dot{x}(t) = A(t)\,x(t) & x \in \mathbb{R}^n \\ y(t) = C(t)\,x(t) \\ x(t_0) = x_0 \end{cases}$

Find solution to D.E.

$$s(t, t_0, x_0, u) = \phi(t, t_0, x_0) \qquad \text{(no } u(t)\text{)}$$

which will satisfy the D.E.

$$\begin{cases} \dfrac{d}{dt}\,\phi(t, t_0, x_0) = A(t)\,\phi(t, t_0, x_0) \\ \phi(t_0, t_0, x_0) = x_0 \end{cases}$$

Note ϕ is a <u>linear</u> <u>map</u> of x_0 to the solution of the D.E.

$$\begin{aligned} \phi(t, t_0, \cdot) : \mathbb{R}^n &\longrightarrow \mathbb{R}^n \\ x_0 &\longrightarrow x(t) = \phi(t, t_0, x_0) \end{aligned}$$

Definition: Basis

A set of vectors $\{v_1, v_2, \ldots, v_n\}$ is a basis of a linear space $V \subset \mathbb{R}^n$ iff

1) $\{v_1, v_2, \ldots, v_n\}$ is a linearly independent set

$$\text{so, } \sum_{i=1}^{n} \alpha_i v_i = 0 \Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

2) $\text{span}\{v_1, v_2, \ldots, v_n\} = V$

Span is the set of all linear combinations of elements in V

Thus, if $x \in V$ and $\{v_1, v_2, \ldots, v_n\}$ is a basis of V

then $\exists!$ set of scalars $\{\alpha_1, \ldots \alpha_n\}$ s.t.

$$x = \sum_{i=1}^{n} \alpha_i v_i$$

$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$ is the "representation" of x w.r.t. the basis $\{v_1, v_2, \ldots v_n\}$

Remark: A basis is not unique.

Example $\mathbb{R}_3[s]$ real polynomials of degree < 3

Basis #1

$\{v_1, v_2, v_3\} = \{1, s, s^2\}$

Basis #2

$\{\bar{v}_1, \bar{v}_2, \bar{v}_3\} = \{1, s+1, s^2+s\}$

$$x = 5s^2 + 4s + 7 = [v_1 \ v_2 \ v_3] \begin{bmatrix} 7 \\ 4 \\ 5 \end{bmatrix} \leftarrow \alpha$$

$$= [\bar{v}_1 \ \bar{v}_2 \ \bar{v}_3] \begin{bmatrix} 8 \\ -1 \\ 5 \end{bmatrix} \leftarrow \bar{\alpha}$$

Fundamental Matrix of Homogeneous LTV System

Let the Fundamental Matrix of the homogeneous LTV D.E. be defined as the matrix representation of the linear map $\phi(t, t_0, x_0)$ w.r.t. the basis $\{v_i\}$

$$x_0 = x(t_0) = \sum_{i=1}^{n} \alpha_i v_i \qquad \text{initial state}$$

The Fundamental Matrix is

$$\Phi(t) : x_0 \longmapsto x(t) = \phi(t, t_0, x_0) = \begin{bmatrix} \phi_1(t, t_0, x_0) \\ \vdots \\ \phi_n(t, t_0, x_0) \end{bmatrix}$$

or

$$x(t) = \Phi(t)\, x_0$$

$$= \begin{bmatrix} \phi(t_1, t_0, v_1) & \phi(t_1, t_0, v_2) & \cdots & \phi(t, t_0, v_n) \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Properties of F.M.:

1) $\Phi(t_0) = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ nonsingular b/c $\{v_i\}$ is a basis

2) $\begin{cases} \dot{\Phi}(t) = A(t)\,\Phi(t) \\ \Phi(t_0) = \Phi_0 \end{cases}$ Satisfies D.E.

Remarks $\Phi(t)$ is not unique

$\Phi(t)$ is nonsingular.

State Transition Matrix

Choose $\{v_i\}$ as the standard basis $\{e_i\}$.

Then $\Phi(t,t_0) \triangleq \left[\phi(t,t_0,e_1) \,\middle|\, \phi(t,t_0,e_2) \,\middle|\, \cdots \,\middle|\, \phi(t,t_0,e_n) \right]$

is called the State Transition Matrix. Unique

Properties of the STM

1) By definition, the unique solution to the d.e. $\begin{cases} \dot{x} = A(t)x(t) \\ x(t_0) = x_0 \end{cases}$

is $\phi(t,t_0,x_0) = \boxed{x(t) = \Phi(t,t_0)\,x_0}$ ZIR

$y = C(t)x(t)$

2) $\Phi(t,t_0)$ is the unique solution to

$$\begin{cases} \frac{d}{dt}\Phi(t,t_0) = A(t)\Phi(t,t_0) \\ \Phi(t_0,t_0) = I \end{cases}$$ Matrix d.e.

3) $\Phi(t,t_0)$ is nonsingular $\forall t$

4) $\Phi(t,t_0) = \left[\Phi(t_0,t)\right]^{-1}$ invertible

5) Semi-group Property

$\Phi(t,t_0) = \Phi(t,t_1)\,\Phi(t_1,t_0)$

How to Compute STM

1) i^{th} column of $\Phi(t,t_0)$ is the unique Solution to

$\begin{cases} \dot{x} = Ax \\ x(t_0) = e_i \end{cases}$ $\{e_i\}$ standard basis

or

2) $\Phi(t,t_0) = X(t)\,X^{-1}(t_0)$ X = Fundamental Matrix of $\dot{x} = Ax$ Homogeneous System

Example (Chen p. 107)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \dot{x}(t) = \begin{bmatrix}0 & 0\\ t & 0\end{bmatrix} x(t) \quad = A(t)\,x(t)$$

For every initial state $x_i(t_0)$, $i=1,2$ $\exists!$ solution $x_i(t)$, $i=1,2$.

Arrange n solutions as columns of $X(t) = [x_1 \,\vdots\, x_2]$

Then we will have $\dot{X}(t) = A(t)X(t)$

$$\dot{x}_1(t) = 0 \xrightarrow{\text{solution}} \text{for } t_0 = 0 \quad x_1(t) = x_1(0)$$

$$\dot{x}_2(t) = t\,x_1(t) \longrightarrow x_2(t) = \int_0^t \tau\, x_1(0)\, d\tau + x_2(0)$$

$$= \tfrac{1}{2}t^2 x_1(0) + x_2(0)$$

Look at two initial conditions (arbitrary)

$$x(0) = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix} \Rightarrow x(t) = \begin{bmatrix}1\\ \tfrac{1}{2}t^2\end{bmatrix}$$

linearly independent

$$x(0) = \begin{bmatrix}1\\ 2\end{bmatrix} \Rightarrow x(t) = \begin{bmatrix}1\\ \tfrac{1}{2}t^2+2\end{bmatrix}$$

$$\Rightarrow X(t) = \begin{bmatrix}1 & 1\\ \tfrac{1}{2}t^2 & \tfrac{1}{2}t^2+2\end{bmatrix}$$ A Fundamental Matrix (nonsingular)

To find STM:

$$[X(t)]^{-1} = \begin{bmatrix}\tfrac{1}{4}t^2+1 & -\tfrac{1}{2}\\ -\tfrac{1}{4}t^2 & \tfrac{1}{2}\end{bmatrix}$$

$$\Rightarrow \Phi(t,t_0) = X(t)\,X^{-1}(t_0) = \begin{bmatrix}1 & 0\\ \tfrac{1}{2}(t^2-t_0^2) & 1\end{bmatrix} \quad 2\times 2$$

$\Rightarrow$ $\boxed{x(t) = \Phi(t,t_0)\,x_0}$ is the unique solution to the d.e. $\begin{cases}\dot{x} = Ax\\ x(t_0) = x_0\end{cases}$ for this x_0

Solution to Nonhomogeneous Linear System

$$\text{d.e.}\begin{cases} \dot{x}(t) = A(t)\,x(t) + B(t)\,u(t) \\ x(t_0) = x_0 \\ y(t) = C(t)\,x(t) + D(t)\,u(t) \end{cases}$$

The solution to this d.e. = ZIR + ZSR "complete Solution"

$$x(t) = \underbrace{\Phi(t,t_0)\,x_0}_{ZIR} + \underbrace{\int_{t_0}^{t} \Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau}_{ZSR}$$

To see this, can check

$x(t_0) = x_0$ ✓

$$\dot{x}(t) = \frac{d}{dt}\Phi(t,t_0)\,x_0 + \frac{d}{dt}\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$\frac{d}{dt}\Phi = A\Phi$

$$\frac{d}{dt}\int f(t,s)\,ds = f(t,t) + \int \frac{\partial f}{\partial t}(t,s)\,ds$$

$$= A(t)\,\Phi(t,t_0)\,x_0 + \overbrace{\Phi(t,t)}^{=I}\,B(t)\,u(t) + \int_{t_0}^{t}\frac{d}{dt}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$= A(t)\,\Phi(t,t_0)\,x_0 + B(t)\,u(t) + A(t)\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau$$

$$= A(t)\left[\Phi(t,t_0)\,x_0 + \int_{t_0}^{t}\Phi(t,t_0)\,B(\tau)\,u(\tau)\,d\tau\right] + B(t)\,u(t)$$

$$= A(t)\,x(t) + B(t)\,u(t) \quad ✓ \quad \text{unique}$$

Connection Between I/O & State-Space Descriptions

ZSR $\quad y(t) = C(t)\int_{t_0}^{t}\Phi(t,\tau)\,B(\tau)\,u(\tau)\,d\tau + D(t)\,u(t)$

$$= \int_{t_0}^{t}\left[C(t)\,\Phi(t,\tau)\,B(\tau) + D\,\delta(t-\tau)\right]u(\tau)\,d\tau$$

$$= \int_{t_0}^{t} G(t,\tau)\,u(\tau)\,d\tau \qquad \text{Impulse Response Matrix } G$$

$$\Rightarrow G(t,\tau) = C(t)\,\Phi(t,\tau)\,B(\tau) + D(t)\,\delta(t-\tau)$$

$$= C(t)\,X(t)\,X^{-1}(\tau)\,B(\tau) + D(t)\,\delta(t-\tau)$$

How to solve d.e. (Chen p. 110)

Solving for

$$x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)\,d\tau$$

and $y(t) = C(t)x(t) + D(t)u(t)$

rely on solving

$$\dot{x}(t) = A(t)x(t) \quad \text{or} \quad \frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0)$$

$\Rightarrow$ very difficult!

Special Cases:

1) If $A(t)$ is triangular

$$\dot{x}(t) = \begin{bmatrix} a_{11}(t) & 0 \\ a_{21}(t) & a_{22}(t) \end{bmatrix} x$$

solve $\dot{x}_1(t) = a_{11}(t)x(t)$ — Saw this in above example

& substitute into $\dot{x}_2(t) = a_{22}(t)x_2(t) + a_{21}(t)x_1(t)$

2) If $A(t)$ is diagonal (or constant) and has the commutative property

$$A(t)\left(\int_{t_0}^{t} A(\tau)\,d\tau\right) = \left(\int_{t_0}^{t} A(\tau)\,d\tau\right)A(t) \quad \forall t, t_0$$

Then the solution to $\begin{cases} \frac{\partial}{\partial t}\Phi(t,t_0) = A(t)\Phi(t,t_0) \\ \Phi(t_0,t_0) = I \end{cases}$

is $\Phi(t,t_0) = \exp \int_{t_0}^{t} A(\tau)\,d\tau$

Note – We want A to be in a "nice" form

– will see how to do this

Outline

Solution to LTI System

State Transition Matrix for LTI

Matrix Exponential & Properties

Cayley-Hamilton Theorem

Characteristic Polynomial

Lecture 6

LTI System Representation

Consider the LTI System:

$$
① \begin{cases} \dot{x}(t) = Ax(t) + Bu(t) & x \in \mathbb{R}^n \quad u \in \mathbb{R}^q \\ x(t_0) = x_0 & x_0 \in \mathbb{R}^n \\ y(t) = Cx(t) + Du(t) & y \in \mathbb{R}^n \end{cases}
$$

System representation $R = [A, B, C, D]$

Solutions to Homogeneous LTI System $u(t) \equiv 0$

$$
② \begin{cases} \dot{x}(t) = Ax(t) & t \geq 0 \\ x(t_0) = x_0 \\ y(t) = Cx(t) \end{cases}
$$

The unique solution to ② is given by the State Transition Matrix:

$$\boxed{x(t) = \Phi(t, t_0)\, x_0} \qquad \text{Solution}$$

where

$$\Phi(t, t_0) = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k$$

$$= e^{A(t-t_0)}$$

Thus, $\boxed{x(t) = e^{A(t-t_0)} x_0}$ Solution to ②

Proof

The STM satisfies the d.e. ②

$$
\begin{cases} \frac{\partial}{\partial t} \Phi(t, t_0) = A\Phi(t, t_0) & t \geq 0 \\ \Phi(t_0, t_0) = I \end{cases}
$$

Integrating, we get

$$\Phi(t, t_0) = I + \int_{t_0}^{t} A\,\Phi(\tau, t_0)\, d\tau$$

Use Picard Iteration:

$\Phi_0(t,t_0) = I$

$\Phi_1(t,t_0) = I + \int_{t_0}^{t} A\,\Phi_0(\tau,t_0)\,d\tau = I + A(t-t_0)$

$\Phi_2(t,t_0) = I + \int_0^t A\,I_1(\tau,t_0)\,d\tau$

$= I + \int_0^t A\,(I + A(\tau - t_0))\,d\tau$

$= I + At + \frac{A^2t^2}{2!}$

$\vdots$

$\Phi(t,t_0) = I + At + \frac{A^2t^2}{2!} + \frac{A^3t^3}{3!} + \cdots = \sum_{k=0}^{\infty} \frac{(t-t_0)^k}{k!} A^k$

$= e^{A(t-t_0)}$ by definition (Taylor series expansion of exponential)

This is called the Matrix Exponential

$$e^M := \sum_{k=0}^{\infty} \frac{1}{k!} M^k$$

Solution to Nonhomogeneous LTI System

$$\begin{cases} \dot{x}(t) = A\,x(t) + B\,u(t) & x \in \mathbb{R}^n,\ u \in \mathbb{R}^2 \\ x(t_0) = x_0 & x_0 \in \mathbb{R}^n \\ y(t) = C\,x(t) + D\,u(t) & y \in \mathbb{R}^m \end{cases}$$

As in LTV case, we have the solution

$x(t) = \Phi(t,t_0)\,x_0 + \int_{t_0}^{t} \Phi(t,\tau)\,B\,u(\tau)\,d\tau$

$$\Rightarrow \boxed{\begin{aligned} x(t) &= e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}\,B\,u(t)\,d\tau \quad \text{unique solution} \\ y(t) &= C\,x(t) + D\,u(t) \quad \text{response} \end{aligned}}$$

* Note $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ Taylor Series expansion

Properties of the Matrix Exponential (STM)

$$e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!}(At)^k = I + At + A^2\frac{t^2}{2!} + \frac{A^3t^3}{3!} + \cdots$$

1) e^{At} is the unique solution to

$$\begin{cases} \frac{d e^{At}}{dt} = A e^{At} & t \geq 0 \\ e^{A \cdot 0} = I \end{cases}$$

b/c it is the STM

2) i^{th} column of e^{At} is unique solution to

$$\begin{cases} \dot{x}(t) = A x(t) & t \geq 0 \\ x(0) = e_i \end{cases}$$

e_i = i^{th} vector of standard basis for $\mathbb{R}^n$ $\quad i \rightarrow \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

3) $e^{At}e^{A\tau} = e^{A(t+\tau)} \quad \forall t, \tau \in \mathbb{R}$

4) e^{At} is nonsingular $\forall t \in \mathbb{R}$

$(e^{At})^{-1} = e^{-At}$

5) If $AB = BA$, $e^{A+B} = e^A e^B$

6) $e^{At} = \mathcal{L}^{-1}[(sI - A)^{-1}]$ ★ useful formula

7) $A e^{At} = e^{At} \cdot A$

8) $e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i \quad \forall t \in \mathbb{R}$

α_i, $i = 0, \ldots n-1$ scalar functions

e.g., $n = 2$, $e^{At} = \alpha_0(t) I + \alpha_1(t) A$

Cayley - Hamilton Theorem

Used to prove property #8 above

Useful to compute e^{At}, hence STM, hence solution $x(t)$.

<u>Theorem</u>: For every $n \times n$ matrix A,

$$\Delta(A) = A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I = 0_{n\times n}$$

where $\Delta(s) = s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n = \det(sI - A)$

is the characteristic polynomial of A.

<u>Proof</u> (by construction)

$$\begin{cases} \frac{\partial}{\partial t}\Phi(t) = A\Phi(t) \\ \Phi(0) = I \end{cases}$$

STM satisfies d.e.

Let $t_0 = 0$ b/c time-invariant

response only depends on $t - t_0$

Take Laplace Transform:

$$s\hat{\Phi}(s) - I = A\hat{\Phi}(s)$$

$$\hat{\Phi}(s) = (sI - A)^{-1}$$

$$= \frac{1}{\det(sI-A)}\left[\operatorname{adj}(sI-A)\right]'$$

adjoint matrix of $(sI-A)$ (cofactors)

$o(n)$ polynomial

$o(n-1)$ or less polynomial

$$(sI-A)^{-1} = \frac{1}{s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n}\left[s^{n-1}\beta_0 + s^{n-2}\beta_1 + \cdots + s\beta_{n-2} + \beta_{n-1}\right]$$

$$\underbrace{(s^n + \alpha_1 s^{n-1} + \cdots + \alpha_n)}_{\Delta(s)} I = \left[s^{n-1}\beta_0 + s^{n-2}\beta_1 + \cdots + \beta_{n-1}\right](sI - A)$$

$$= s^n\beta_0 + s^{n-1}(\beta_1 - \beta_0 A) + \cdots + \beta_{n-1}A$$

Assuming we know $\Delta(s)$ characteristic polynomial, we can calculate $\{\beta_i\}$ by equating like powers

$$\begin{cases} \beta_0 = I \\ \beta_1 = \beta_0 A + \alpha_1 I \\ \beta_2 = \beta_1 A + \alpha_2 I \\ \vdots \\ \beta_k = \beta_{k-1} A + \alpha_k I \\ \vdots \\ 0_{n\times n} = \beta_{n-1} A + \alpha_n I \end{cases}$$

Then substitute

$$0_{n\times n} = \beta_{n-1}A + \alpha_n I = (\beta_{n-2}A + \alpha_{n-1}I)A + \alpha_n I$$

$$= \beta_{n-2}A^2 + \alpha_{n-1}A + \alpha_n I$$

$$\vdots$$

$$= A^n + \alpha_1 A^{n-1} + \cdots + \alpha_n I$$

$$= \Delta(A)$$

$\boxplus$

No. 937 811E Engineer's Computation Pad

Example (Chen p. 89)

$$\dot{x}(t) = \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Find $x(t)$ solution for $t_0 = 0$

$$x(t) = e^{At} x(0) + \int_0^t e^{A(t-\tau)} B\, u(\tau)\, d\tau$$

$$e^{At} = \mathcal{L}^{-1}\left[(sI - A)^{-1} \right]$$

$$(sI-A)^{-1} = \left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & -1 \\ 1 & -2 \end{bmatrix} \right)^{-1}$$

$$= \begin{bmatrix} s & 1 \\ -1 & s+2 \end{bmatrix}^{-1} = \frac{1}{s(s+2)+1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$= \frac{1}{s^2+2s+1} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix} = \frac{1}{(s+1)^2} \begin{bmatrix} s+2 & -1 \\ 1 & s \end{bmatrix}$$

$$= \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix}$$

$$\mathcal{L}^{-1}\left[(sI-A)^{-1} \right] = \mathcal{L}^{-1} \begin{bmatrix} \frac{s+2}{(s+1)^2} & \frac{-1}{(s+1)^2} \\ \frac{1}{(s+1)^2} & \frac{s}{(s+1)^2} \end{bmatrix} \overset{\star}{=} \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix}$$

Then

$$x(t) = \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} (1+t-\tau)e^{-(t-\tau)} & -(t-\tau)e^{-(t-\tau)} \\ (t-\tau)e^{-(t-\tau)} & (1-(t-\tau))e^{-(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)\, d\tau$$

$$= \begin{bmatrix} (1+t)e^{-t} & -te^{-t} \\ te^{-t} & (1-t)e^{-t} \end{bmatrix} x(0) + \begin{bmatrix} -\int_0^t (t-\tau)e^{-(t-\tau)} u(\tau)\, d\tau \\ \int_0^t (1-(t-\tau))e^{-(t-\tau)} u(\tau)\, d\tau \end{bmatrix}$$

$\star$ **Note**

see Handout 3

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = 1 \qquad \mathcal{L}^{-1}\left[\frac{n!}{s^{n+1}}\right] = t^n$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t \qquad \mathcal{L}^{-1}\left[\frac{n!}{(s-k)^{n+1}}\right] = t^n e^{kt}$$

$$\mathcal{L}^{-1}\left[\frac{1}{s-a}\right] = e^{at}$$

SYSTEM ⟶ Solution

LTV

$$\begin{cases} \dot{x} = A(t)x + B(t)u \\ y = C(t)x + D(t)u \\ x(t_0) = x_0 \end{cases} \xrightarrow[\Phi]{X} \begin{cases} x(t) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,\tau)B(\tau)u(\tau)d\tau \\ y(t) = C(t)\Phi(t,t_0)x_0 + C(t)\int_{t_0}^{t}\Phi(t,\tau)B(\tau)u(\tau)d\tau + D(t)u(t) \end{cases}$$

$\downarrow$ ZSR, $x_0 = 0$

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau \xrightarrow{\text{realization}} G(t,\tau) = C(t)X(t)X^{-1}(\tau)B(\tau) + D(t)\delta(t-\tau)$$

convolution

$$y(t) = \int_{t_0}^{t} G(t,\tau)u(\tau)d\tau \xrightarrow{\text{discretize}} y(k\Delta) = \sum_{m=k_0}^{k} G(k\Delta, m\Delta)\,u(m\Delta)\,\Delta$$

Δ ↑ step size

compute numerically on computer

LTI

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \\ x(t_0) = x_0 = 0 \end{cases} \xrightarrow{\mathcal{L}} \begin{cases} \hat{x}(s) = (sI-A)^{-1}B\hat{u}(s) \\ \hat{y}(s) = C(sI-A)^{-1}B\hat{u}(s) + D\hat{u}(s) \end{cases} \xrightarrow{\mathcal{L}^{-1}} y(t) = \mathcal{L}^{-1}[\hat{Y}(s)]$$

$$\hat{Y}(s) = \hat{G}(s)\hat{u}(s) \longrightarrow y(t) = \mathcal{L}^{-1}[\hat{G}(s)\hat{u}(s)]$$

Solve with Matlab

but hard if multiple poles

Xfer Fct

$$\hat{G}(s) = \frac{\text{num}(s)}{\text{den}(s)} \xrightarrow{\text{realization}} \text{LTI state-space model} \longrightarrow y(t) = Ce^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau + Du(t)$$

$$\hat{G}(s) = C(sI-A)^{-1}B \xrightarrow{\text{realization}} \text{LTI state-space model}$$

STM ↓ ↙ matrix exponential: $\Phi(t,t_0) = \mathcal{L}^{-1}[(sI-A)^{-1}]$

Solve directly in time domain