EE 547 (PMP): Linear Systems Theory

Practice Problems

Problem 1 (**Linearization and State-space Representation of a Dynamical System**) Consider the following system of nonlinear differential equations:

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} (1+x_1)x_1 + x_2^2 x_3 + u\cos(x_1) \\ x_1^3 + x_2\sin(x_1) + x_3^2 + u\sin(x_2) \\ x_1 + x_2 + x_3^2 + u \end{bmatrix}$$

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1^2 + x_2 + x_3^2 \\ x_3^2 + \sin(x_3) \end{bmatrix}$$

(a) Linearize the given system about the equilibrium point $\mathbf{x}^{eq} = \begin{bmatrix} x_1^{eq} \\ x_2^{eq} \\ x_3^{eq} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, $u^{eq} = 0$ to find the Jacobian matrices $A = \left(\frac{\partial f}{\partial x}\right)\Big|_{(x^{eq},u^{eq})}$, $B = \left(\frac{\partial f}{\partial u}\right)\Big|_{(x^{eq},u^{eq})}$, $C = \left(\frac{\partial g}{\partial x}\right)\Big|_{(x^{eq},u^{eq})}$ and $D = \left(\frac{\partial g}{\partial u}\right)\Big|_{(x^{eq},u^{eq})}$.

(b) Represent the linearized system in the state-space form.

Problem 2 (**Equivalent Representations of a Linear System**) Consider the following two systems:

$$\dot{x}(t) = \begin{bmatrix} 1 & 0 \\ -3 & 2 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + 0 \cdot u(t)$$

$$\dot{\bar{x}}(t) = \begin{bmatrix} -2 & -5 \\ 0 & -1 \end{bmatrix} \bar{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \bar{x}(t) + 0 \cdot u(t)$$

(a) Are these systems zero-state equivalent?

(b) Are these systems algebraically equivalent?

Problem 3 (**Transfer Function and Time Response of an LTI System**) Consider a linear time-invariant system:

$$\dot{x} = \begin{bmatrix} 3 & 5 \\ 5 & 3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} x$$

(a) Find the transfer function of the given system. Is the obtained transfer function a proper rational function?

(b) Assume $t_0 = 0$ and compute the state transition matrix, $\Phi(0, t)$ of the given system.

(c) Given the initial state $x(0) = \begin{bmatrix} 1 & 1 \end{bmatrix}$, compute the zero-input response of the given system.

(d) Given the unit-step input $u(t) = 1,\ \ t \geq 0$, compute the zero-state response of the given system.

(e) Write down the complete response of the given system.

Problem 4 (**State Transition Matrix of an LTV System**) Consider the system:

$$\dot{x} = \begin{bmatrix} 3 & 0 \\ t & 0 \end{bmatrix} x + \begin{bmatrix} t \\ 0 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

(a) Compute by hand the fundamental matrix of the given systems.

(b) Compute by hand the state transition matrix of the given system.

Problem 5 (**Functions of a Square Matrix**) Consider matrices:

$$A = \begin{bmatrix} -1 & -3 & -7 \\ 0 & -4 & -2 \\ 0 & 0 & -5 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 0 & -1 \\ 5 & 0 & 3 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$$

(a) Find the characteristic polynomials $\Delta(\lambda)$ of matrices A and B.
(b) Find the eigenvalues of matrices A and B.
(c) Find matrix powers A^{10} and B^{15}.

Problem 6 (**Jordan Decomposition of Matrices**) Consider matrices:

$$A_1 = \begin{bmatrix} 2 & 2 & 2 \\ 4 & 0 & -4 \\ -1 & -1 & -1 \end{bmatrix} \qquad \text{and} \qquad A_2 = \begin{bmatrix} -2 & 0 & 0 & 0 \\ -3 & 0 & 0 & 0 \\ -3 & 1 & -4 & 0 \\ -2 & 3 & 2 & 0 \end{bmatrix}$$

(a) Find the characteristic polynomial and eigenvalues of matrices A_1 and A_2.
(b) Find the Jordan form representation of matrices A_1 and A_2.

Problem 7 (**Stability of Linear Time-invariant Systems**)
(a) Consider a system in Problem 2. Is the given system BIBO stable? Explain your response.
(b) Consider matrices A_1 and A_2 given in Problem 4. Assume the matrices A_1 and A_2 define the following continuous-time homogeneous LTI systems:

$$\dot{x}_1 = A_1 x_1$$
$$\dot{x}_2 = A_2 x_2$$

Are continuous-time systems defined by matrices A_1 and A_2 asymptotically stable? Are they marginally stable? Explain your response.

Problem 8 (**Stability of Discrete-time LTI Systems**) Consider the following discrete-time system:

$$x[k+1] = \begin{bmatrix} -1 & 0 & -3 \\ 2 & 0.5 & 2 \\ 0 & 0 & -0.25 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

Is the given system asymptotically stable? Is it marginally stable? Explain your answer.

Problem 9 (**Lyapunov Test for Stability of Linear Time-invariant Systems**) Consider the following continuous system:

$$\dot{x} = \begin{bmatrix} -1 & 0 \\ 2 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Use the Lyapunov test for stability to check if the given system is asymptotically stable. Use positive definite matrix $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in your computations.