EE547 (PMP) Midterm - Winter 2015

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Click here to see the midterm code.

Contents

Initialization

```
function midterm()
```

```
opengl('save', 'software')
format shortG
set(0, 'defaultTextInterpreter', 'latex');
numerical_precision = 1e-9;
syms s x x_1 x_2 x_3 u t t_0
```

Problem 1

a) Linearize the system around equilibrium points

```
x = [x_1; x_2; x_3];
f = [-9*x_1 - 4*x_2 - (1+x_3)*x_3 + sin(x_3) + sin(u);...
    (x_2*x_3 - 4)*x_1 - 10*sin(x_2) + 3*cos(x_3) + x_3^2*sin(u);...
    9*x_1 + (x_1^2 - 4)*x_3 - 10*x_2 + u];
g = [x_1 + x_2*x_3 + sin(u);...
    x_2 + x_1*x_3 + u^2;...
    x_3 + x_2*x_3 + cos(u)];
render_latex(['f = ' latex(f)], 12, 0.7)
render_latex(['g = ' latex(g)], 12, 0.7)
```

$$f = \left[\begin{array}{c} \sin(u) - 4\,x_2 - 9\,x_1 + \sin(x_3) - x_3\,(x_3+1) \\ 3\,\cos(x_3) - 10\,\sin(x_2) + {x_3}^2\,\sin(u) + x_1\,(x_2\,x_3 - 4) \\ u + 9\,x_1 - 10\,x_2 + x_3\,\left({x_1}^2 - 4\right) \end{array}\right]$$

$$g = \left[\begin{array}{c} x_1 + \sin(u) + x_2\,x_3 \\ u^2 + x_2 + x_1\,x_3 \\ x_3 + \cos(u) + x_2\,x_3 \end{array}\right]$$

The state-space matrices are found using

$$\mathbf{A} = \frac{\partial f}{\partial \mathbf{x}}\bigg|_{x^{eq},u^{eq}} \quad \mathbf{B} = \frac{\partial f}{\partial u}\bigg|_{x^{eq},u^{eq}} \quad \mathbf{C} = \frac{\partial g}{\partial \mathbf{x}}\bigg|_{x^{eq},u^{eq}} \quad \mathbf{D} = \frac{\partial g}{\partial u}\bigg|_{x^{eq},u^{eq}}$$

where,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

```
xeq = [-0.1 0.1 -0.2];
ueq = 0;
A = subs(jacobian(f, x), [x_1, x_2, x_3, u], [xeq, ueq]);
B = subs(jacobian(f, u), [x_1, x_2, x_3, u], [xeq, ueq]);
C = subs(jacobian(g, x), [x_1, x_2, x_3, u], [xeq, ueq]);
D = subs(jacobian(g, u), [x_1, x_2, x_3, u], [xeq, ueq]);
render_latex(['\mathbf{A} = ' latex(simplify(A))], 12, 0.75)
render_latex(['\mathbf{B} = ' latex(simplify(B))], 12, 0.75)
render_latex(['\mathbf{C} = ' latex(simplify(C))], 12, 0.75)
render_latex(['\mathbf{D} = ' latex(simplify(D))], 12, 0.75)
A = double(A);
B = double(B);
C = double(C);
D = double(D);
```

$$\mathbf{A} = \begin{bmatrix} -9 & -4 & \cos\left(\frac{1}{5}\right) - \frac{3}{5} \\ -\frac{201}{50} & \frac{1}{50} - 10\cos\left(\frac{1}{10}\right) & 3\sin\left(\frac{1}{5}\right) - \frac{1}{100} \\ \frac{226}{25} & -10 & -\frac{399}{100} \end{bmatrix}$$

$$\mathbf{B} = \begin{bmatrix} 1 \\ \frac{1}{25} \\ 1 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} 1 & -\frac{1}{5} & \frac{1}{10} \\ -\frac{1}{5} & 1 & -\frac{1}{10} \\ 0 & -\frac{1}{5} & \frac{11}{10} \end{bmatrix}$$

$$\mathbf{D} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

b) Find the transfer functions of this system. Are these transfer functions proper rational functions?

Given the state-space matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. The transer function is found using

$$\hat{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

NOTE since the symbolic representation of G using

```
sI_A = s*eye(size(A)) - A;
G = C*inv(sI_A)*B;
```

results in unreadable expressions, use *ss2tf* instead.

```
[num, den] = ss2tf(A, B, C, D)
```

```
num =

           1        24.012        168.73        384.04
           0         -0.26       -8.4994       -34.742
           0         1.092         30.91        219.17


den =

           1         22.92        151.25        316.96
```

Proper rational? The degree of the numerators are, at most, equal to the degree of the denominator. Therefore, the transfer functions are proper rational.

d) Evaluate the Jordan form of the matrix A and corresponding transformation matrix Q.

Find the eignevalues of A

NOTE use *eig* to get more accurate results

```
lambda = eig(A)
```

```
lambda =

      -13.396 +          0i
      -4.7621 +    0.99194i
      -4.7621 -    0.99194i
```

Since the eigenvalues of A are distinct, the Jordan form is simply the eigenvalues of A along the diagonal. The tranformation matrix, Q, is found by finding a solution to the homogenuous equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\, q_i = \mathbf{0}$$

```
J = diag(lambda);
Q = zeros(size(A));
for i = 1:length(lambda)
    Q(:, i) = null(A - lambda(i)*eye(size(A)));
end
J_ = array2table(J)
Q_ = array2table(Q)
```

```
J_ =

      J1              J2                    J3
    _______    ________________    ________________

    -13.396         0+0i                 0+0i
          0    -4.7621+0.99194i          0+0i
          0         0+0i           -4.7621-0.99194i


Q_ =

      Q1               Q2                     Q3
    ________    __________________    __________________

    -0.65801     0.050572+0i           0.050572+0i
    -0.73756    -0.025398-0.1026i     -0.025398+0.1026i
    -0.15173      0.29661-0.94778i      0.29661+0.94778i
```

e) Find the state transition matrix from the Jordan form.

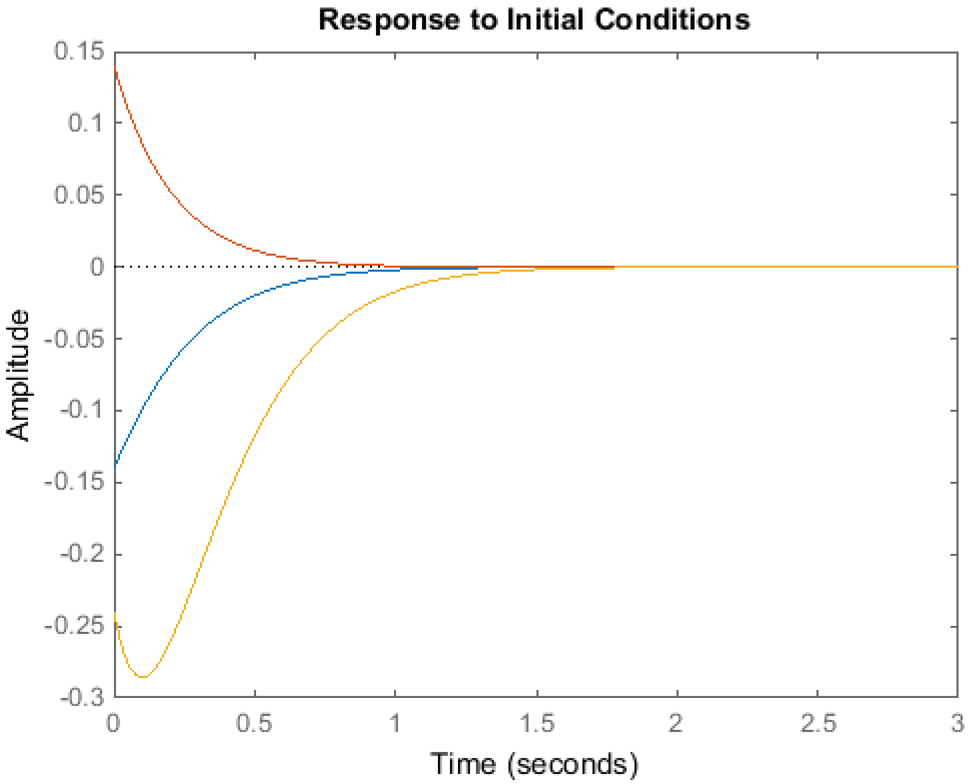
For an LTI system, the State Transition Matrix $\Phi(t,t_0)$ using the Jordan form is given by

$$\Phi(t,t_0) = \mathbf{Q}e^{\mathbf{J}(t-t_0)}\mathbf{Q}^{-1}$$

```
Phi = Q*expm(J*(t))*inv(Q);
% render_latex(['\Phi(t-t_0) = ' latex(simplify(Phi))], 16, 1.2)
```

f) Given the initial state evaluate and plot zero-input responses of the system.

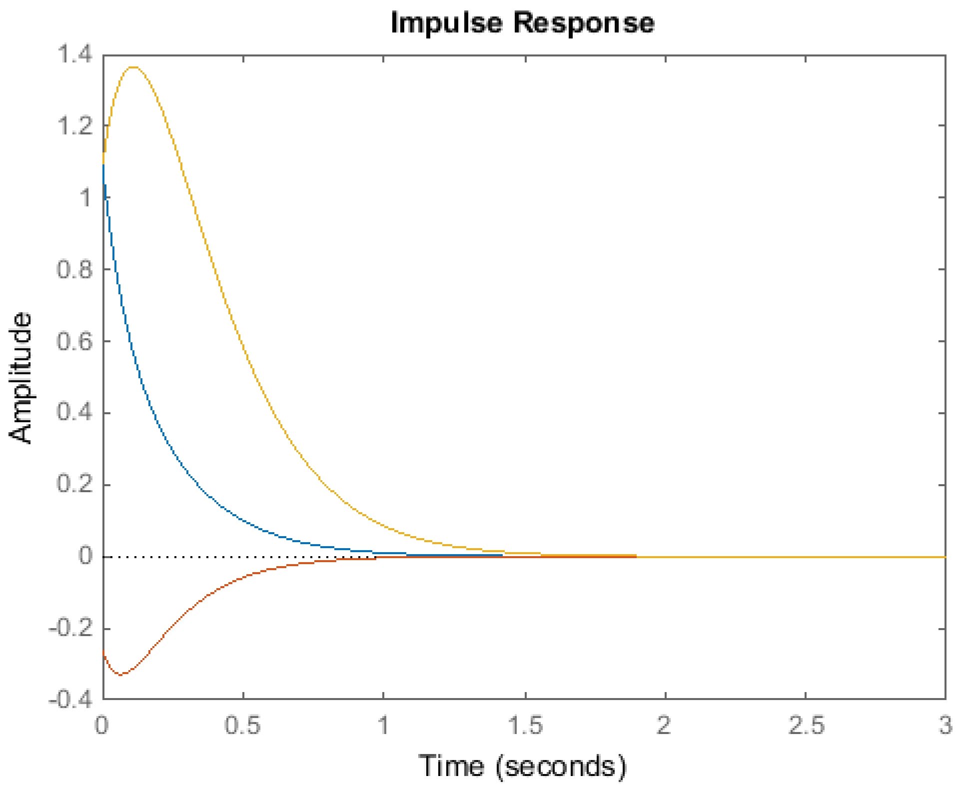
```
x0 = [-0.1; 0.1; -0.2];
sys = ss(A, B, C, D);
figure,
initial(sys(1), sys(2), sys(3), x0, 3)
[y_zir, t, x_zir] = initial(sys, x0, 3);
```

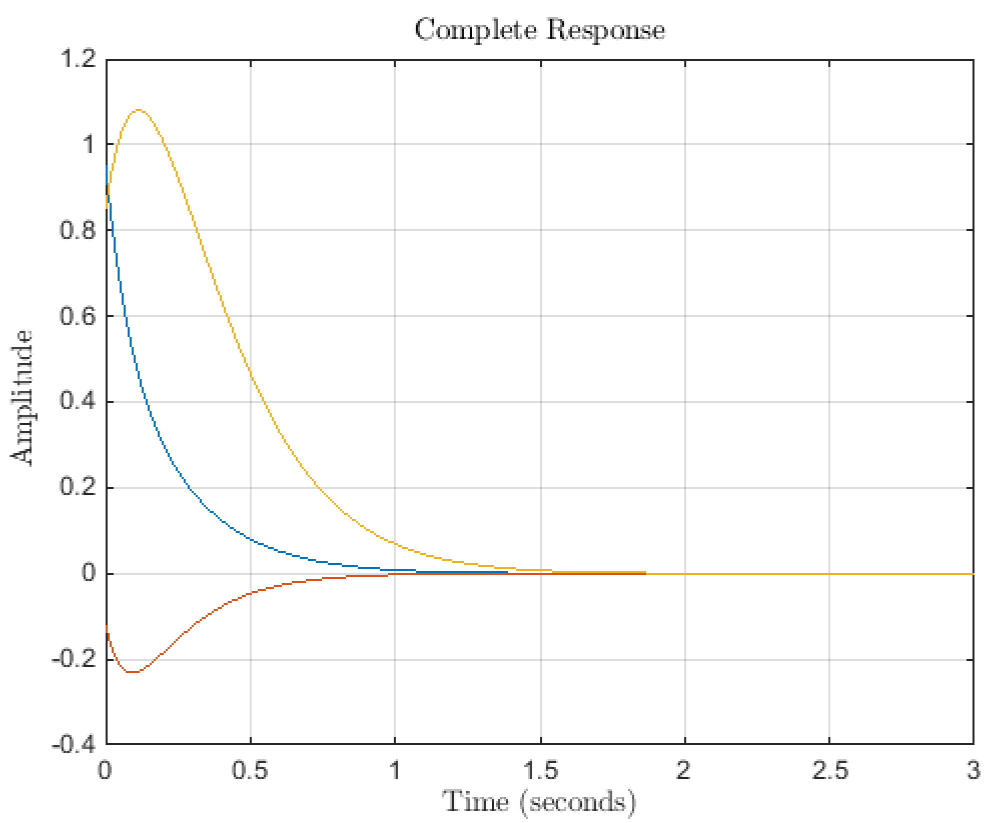


g) Evaluate the complete impulse response of this system.

```
figure,
impulse(sys(1), sys(2), sys(3), 3);
[y_zsr, ~, x_zsr] = impulse(sys, 3);

figure,
plot(t, y_zir + y_zsr)
title('Complete Response')
xlabel('Time (seconds)'); ylabel('Amplitude')
```





Problem 2

Find Transfer Function for System 1

Given the state-space matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ and $\mathbf{D}$. The transer function is found using

$$\hat{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$$

```
A = [-5 -9 4; 2 -9 2; 9 -10 -8];
B = [-1; 2; -3];
C = eye(size(A));
D = zeros(3, 1);
sI_A = s*eye(size(A)) - A;
G = C*inv(sI_A)*B;
render_latex(['\hat{G_1}(s) = ' latex(simplify(G))], 16, 1.2)
```

$$\hat{G_1}(s) = \left[\begin{array}{c} -\frac{s^2+47\,s+370}{s^3+22\,s^2+159\,s+522}\\ \frac{2\,\left(s^2+9\,s-40\right)}{s^3+22\,s^2+159\,s+522}\\ -\frac{3\,s^2+71\,s+512}{s^3+22\,s^2+159\,s+522} \end{array}\right]$$

Find Transfer Function for System 2

```
A_ = [-82 -8 54; -174 -33 130.5; -138 -16 93];
B_ = [2; -7; 0];
C_ = [-4 -1 3.5; 1 0 -0.5; 2 1 -2];
D_ = zeros(3, 1);
sI_A = s*eye(size(A_)) - A_;
G_ = C_*inv(sI_A)*B_;
render_latex(['\hat{G_2}(s) = ' latex(simplify(G_))], 16, 1.2)
```

$$\hat{G_2}(s) = \left[\begin{array}{c} -\frac{s^2+47\,s+370}{s^3+22\,s^2+159\,s+522}\\ \frac{2\,\left(s^2+9\,s-40\right)}{s^3+22\,s^2+159\,s+522}\\ -\frac{3\,s^2+71\,s+512}{s^3+22\,s^2+159\,s+522} \end{array}\right]$$

Verify equality using *ss2tf*

```
G = ss2tf(A, B, C, D);
G_ = ss2tf(A_, B_, C_, D_);
if max(max(G - G_)) > numerical_precision
    disp('Systems are not zero-state equivalent')
else
    disp('Systems realize the same transfer function, therefore, they are zero-state equivalent.')
end
```

```
Systems realize the same transfer function, therefore, they are zero-state equivalent.
```

Problem 3

```
A = [-4 3 -1 10 3;...
    -6 -9 5 7 -6;...
    -8 -5 2 -9 2;...
    -4 -5 6 -1 2;...
    -6 8 -8 -4 2];
```

a) Evaluate the eigenvalues and characteristic polynomial of A

The characteristic polynomial of $\mathbf{A}$ is given by

$$\Delta(\lambda) = \det s\mathbf{I} - \mathbf{A}$$

And the eigenvalues of $\mathbf{A}$ are the roots of the characteristic polynomial

```
sI_A = s*eye(size(A)) - A;
CharPoly = det(sI_A);
render_latex(['\Delta(\lambda) = ' latex(CharPoly)], 12, 0.5)
lambda = roots(sym2poly(CharPoly))
```

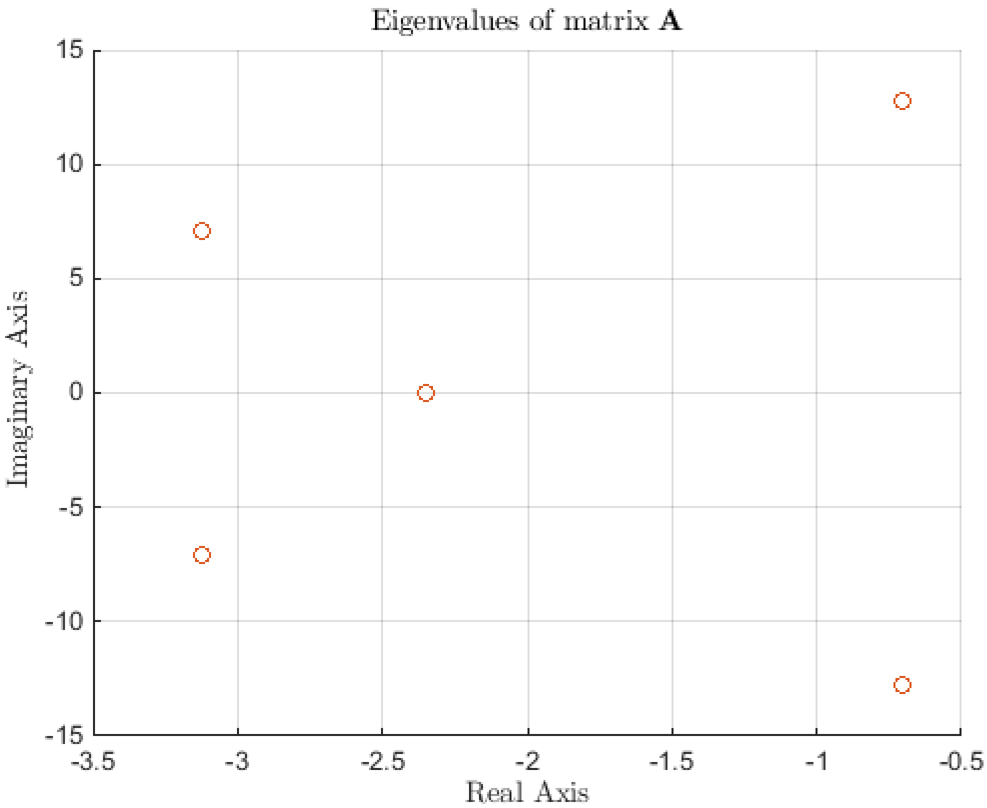
$$\Delta(\lambda) = s^5 + 10\,s^4 + 251\,s^3 + 1658\,s^2 + 12462\,s + 23160$$

```
lambda =

    -0.70175 +     12.798i
    -0.70175 -     12.798i
     -3.1229 +     7.0865i
     -3.1229 -     7.0865i
     -2.3508 +          0i
```

Verify by using *eig* and displaying

```
scatter(real(lambda), imag(lambda)); hold on
scatter(real(eig(A)), imag(eig(A))); hold off
xlabel('Real Axis'); ylabel('Imaginary Axis');
title('Eigenvalues of matrix $\mathbf{A}$')
```



b) Evaluate Jordon form and transformation matrix Q of matrix A

Since the eigenvalues of A are distinct, the Jordan form is simply the eigenvalues of A along the diagonal. The tranformation matrix, Q, is found by finding a solution to the homogenuous equation

$$(\mathbf{A} - \lambda_i \mathbf{I})\, q_i = \mathbf{0}$$

```
J = diag(lambda);
Q = zeros(size(A));
for i = 1:length(lambda)
    Q(:, i) = null(A - lambda(i)*eye(size(A)));
end
J_ = array2table(J)
Q_ = array2table(Q)
```

J_ =

J1	J2	J3	J4	J5
-0.70175+12.798i	0+0i	0+0i	0+0i	0
0+0i	-0.70175-12.798i	0+0i	0+0i	0
0+0i	0+0i	-3.1229+7.0865i	0+0i	0
0+0i	0+0i	0+0i	-3.1229-7.0865i	0
0+0i	0+0i	0+0i	0+0i	-2.3508

Q_ =

```
        Q1                   Q2                   Q3                   Q4                 Q5
  ________________     ________________     ________________     ________________     ________

   0.33956+0i           0.33956+0i            0.4554+0i            0.4554+0i          -0.066064
   0.30505+0.32656i     0.30505-0.32656i    -0.31672+0.56509i    -0.31672-0.56509i      0.70522
  -0.35779+0.49983i    -0.35779-0.49983i    -0.21487+0.35643i    -0.21487-0.35643i      0.64957
  0.069349+0.43875i    0.069349-0.43875i    0.007789+0.11375i    0.007789-0.11375i    -0.077968
  -0.28216-0.1739i     -0.28216+0.1739i      0.35229+0.25029i     0.35229-0.25029i     -0.26512
```

Verify transformation

```
if max(max(A*Q - Q*J)) > numerical_precision
    disp('The transformation is not valid.')
else
    disp('The transformation is valid.')
end
```

```
The transformation is valid.
```

c) Evaluate function

$f_2(A) = A^5 + 10A^4 + 251A^3 + 1658A^2 + 12462^A + 23160I_{5x5}$

The solution equates $f(\mathbf{A}) = f(\lambda)$ to $h(\lambda)$ where $h(\lambda) = \beta_0 + \beta_1\lambda + ... + \beta_{n-1}\lambda^{n-1}$

However, I discovered the hard way that solving linear equations with this $\mathbf{A}$ given the numeric precision available in Matlab leads to badly scaled matrices which factor with error.

Instead, use Jordan form of $\mathbf{A}$, then $\mathbf{A}^k = \mathbf{Q}^{-1}\mathbf{J}^k\mathbf{Q}$.

Since the eigenvalues of $\mathbf{A}$ are distinct, $\mathbf{J}^k = \lambda^k\mathbf{I}_5$, where $k = 5, 4, 3, 2, 1, 0$

```
k = 5:-1:0;
b = [1 10 251 1658 12462 23160];
f2 = zeros(size(A));
f2_ = zeros(size(A));
for i = 1:length(k)
    A_to_the_k = inv(Q)*(diag(lambda.^k(i)))*Q;
    f2 = f2 + A_to_the_k*b(i);
    f2_ = f2_ + A^k(i); % the direct solution
end
array2table(f2)
```

```
ans = 

             f21                       f22                       f23                       f24
f25
    ______________________    ______________________    ______________________    
______________________    ______________________
```

```
     6.1846e-11+5.4624e-11i    -3.8569e-12-6.6762e-11i     8.0036e-11+1.4584e-10i    -7.5331e-
11+5.4035e-13i     7.3724e-11-1.1642e-10i
     7.7464e-11-1.7868e-11i    -2.1828e-10+2.2976e-12i     1.2775e-10+8.4419e-11i    -1.0593e-
10+1.4391e-10i    -5.9815e-11-1.2904e-10i
    -2.4738e-10+1.6371e-11i     -1.199e-10+3.7706e-10i    -6.1846e-11-3.3809e-11i     9.2016e-
11+3.7964e-10i    -2.6242e-10+4.4941e-11i
     3.2048e-11+3.0495e-10i     2.0485e-10+9.2502e-11i    -1.1043e-10+1.1513e-10i     7.6398e-
11+7.276e-12i      1.9142e-10+2.1442e-10i
     2.1219e-12+1.1408e-10i     1.7893e-10+1.7273e-10i     3.2475e-11+1.7093e-10i     9.1654e-
11+1.8122e-10i     -1.055e-10+3.7602e-11i
```

Verify Results using explicit A^k

```
if max(max(f2_ - f2)) > numerical_precision
    disp('The transformation is not valid.')
else
    disp('The transformation is valid.')
end
```

```
The transformation is valid.
```

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