STAT3006 TUTORIAL5

Basic Concepts in Markov Chain and Examples

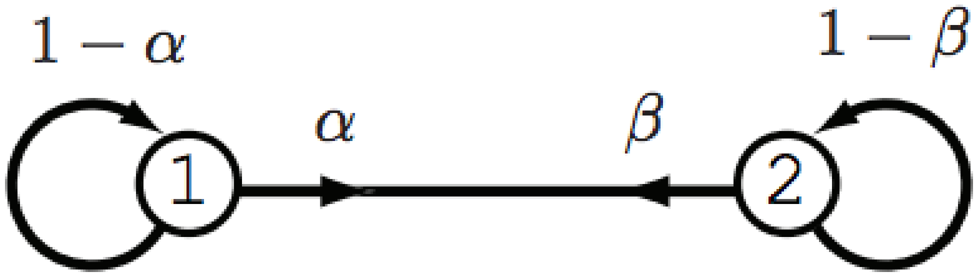
More information on Markov chain:
http://u.math.biu.ac.il/~amirgi/CLN.pdf

MARKOV CHAIN

- Let S be a countable set .
- Each i ∊ S is called a state and S is called state space.
- A sequence of random variables taking values in S is called (time-homogeneous) Markov chain if $P(X_{n+1} \ = j \,|X_n = i, X_{n-1} = i_{n-1}, \dots, X_1 = i_1, X_0 = i_0) = P(X_{n+1} \ =$

MARKOV CHAIN

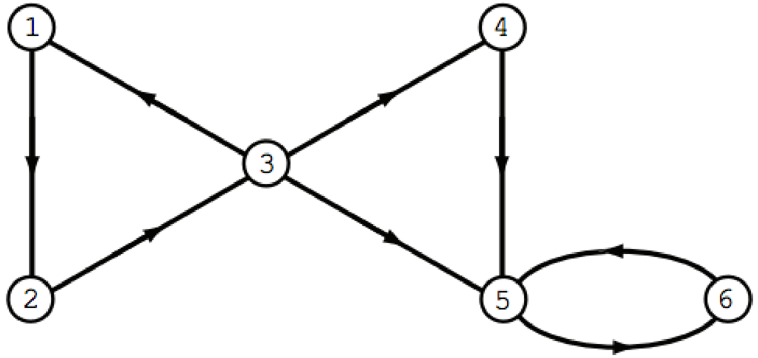
- Example (Two state Markov chain):



- Calculate n-step transition matrix of this Markov chain.
  - Transition matrix P is decomposed into the product of a block diagonal matrix J and orthogonal matrix U(Jordan normal form).
    - $P = UJU^{-1}$
  - The diagonals of J is eigenvalues of P.
  - $P^{(n)} = P^n = UJ^nU^{-1}$

MARKOV CHAIN

- In the state space S, i and j ∈ S, if $\exists n,\ s.t. P_{ij}^{(n)} > 0$, then we say i leads to j, written by $i \ \rightarrow j$.
- If i leads to j and j leads to i, we say i communicates with j, written by $i \ \leftrightarrow j$.
- " $\leftrightarrow$ " is an equivalent relation.
  - For a given i, the equivalent relation induces a communicating class, $\{j: j \leftrightarrow i\}$.
  - The state space is partitioned by these communicating classes.



Three communicating classes : {1,2,3} {4} {5,6} in the figure.

For a Markov chain, if its state space has only one communicating class, the Markov chain is called irreducible.

MARKOV CHAIN

- Define $T_i = \inf\{n \geq 1; X_n = i\}$= first passage time on i.
- Define $V_i = \sum_{n=0}^{\infty} I(X_n = i)$ = number of visits to i.
- Define $f_i = P_i(T_i < \infty)$ = return probability to i from the initial state X_0=i.
  - $f_i = \sum_{k\geq 2} P_i(\mathrm{V_i} = \mathrm{k}) = \mathrm{P_i}(\mathrm{V_i} \geq 2)$
- If $f_i = 1$, we say state i is recurrent; otherwise, i is transient.
- Lemma: $P_i(V_i \geq k+1) = (f_i)^k$.
- Theorem: $f_i = 1 \Leftrightarrow P_i(V_i = \infty) = 1$ and $f_i < 1 \Leftrightarrow P_i(V_i = \infty) = 0$
  - $P_i(V_i = \infty)$ is either 1 or 0.
- Theorem: $f_i = 1 \Leftrightarrow \sum_{n=0}^{\infty} p_{ii}^{(n)} = \infty$ (*) and $f_i < 1 \Leftrightarrow \sum_{n=0}^{\infty} p_{ii}^{(n)} < \infty$
- E.g. two-state Markov chain in the example is recurrent based on (*).

MARKOV CHAIN

- If a Markov chain is irreducible,
- One state is recurrent, all states are recurrent.
- One state is transient, all states are transient.

MARKOV CHAIN

- Simple random walk on integers:
  - Starting from 0.
  - $P(Y_n = 1) = \frac{1}{2} = p, P(Y_n = -1) = \frac{1}{2} = q$
  - $X_n = \sum_{m=1}^{n} Y_m$
  - Irreducible

$$p_{00}^{(2n)} = \binom{2n}{n} p^n q^n. \qquad \sum_{n=N}^{\infty} p_{00}^{(2n)} \geq \frac{1}{2A} \sum_{n=N}^{\infty} \frac{1}{\sqrt{n}} = \infty$$

  - Recurrent

- Simple random walk on 2D integers :
  - Starting from (0,0)
  - $P(Y_n = (1,0)) = \frac{1}{4}, P(Y_n = (-1,0)) = \frac{1}{4},\ P(Y_n = (0,1)) = \frac{1}{4}, P(Y_n = (0,-1)) = \frac{1}{4}$
  - $X_n = \sum_{m=1}^{n} Y_m$
  - Irreducible $$p_{00}^{(2n)} = \left(\binom{2n}{n} \left(\frac{1}{2}\right)^{2n} \right)^2 \sim \frac{2}{A^2 n} \quad \text{as } n \to \infty$$

  - Recurrent

MARKOV CHAIN

- For simple random walk on 3D integers

$$p_{00}^{(2n)} \leq \binom{2n}{n}\left(\frac{1}{2}\right)^{2n}\binom{n}{m\ m\ m}\left(\frac{1}{3}\right)^{n} \sim \frac{1}{2A^3}\left(\frac{6}{n}\right)^{3/2} \quad \text{as } n \to \infty.$$

- Transient!

- *A drunk man will find his way home, but a drunk bird may get lost forever.* (Shizuo Kakutani)