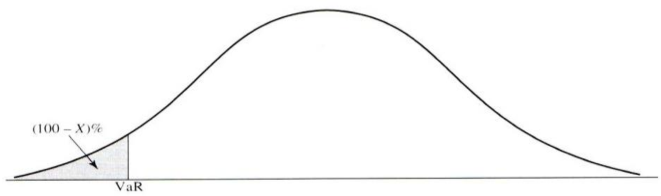
Chapter 3

Risk Measure

Financial institutions will be usually managing hundreds, or even thousands, of portfolios of derivatives everyday. It is important to have a single number that summarizes the total risk in a portfolio of financial assets and measures the total risk to which the financial institution is exposed. **Value at Risk** (VaR) is an attempt to provide such a measure. It was pioneered by J.P. Morgan and is used by the Basel Committee in setting capital requirements for banks throughout the world. Closely related to VaR is the **Expected Shortfall** (ES). Definition and properties of these risk measures will be introduced. Other related issues such as Back-testing, Historical simulation and Extreme value theory will also be discussed.

3.1 Value at Risk (VaR) and Expected Shortfall (ES)

Suppose we have a portfolio *Q* and we say that 1-day *VaR(x)* of *Q* is *V* means that we are *x* percent certain that we will not lose more than *$V* in next day. In other words, *VaR(x)* is the ***(1-x)-percentile*** of the distribution of the change in *Q* (denoted by ΔQ). Mathematically speaking; let f be the density of ΔQ, or *F* be the CDF of ΔQ, then $VaR(x) = V \Rightarrow \int_{-\infty}^{V} f(t)dt = F(V) = 1 - x$ or $V = F^{-1}(1-x)$.

Loss  Profit

Note that N-day *VaR(x)*, denoted by $VaR(N, x) = \sqrt{N}\, VaR(x)$ assuming that the daily changes are i.i.d.. Usually we work with the loss distribution of *Q* (i.e. -ΔQ) so that the left tail becomes the right tail.

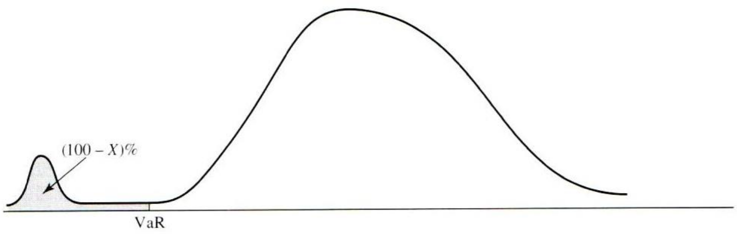
Example 3.1

Assume that the daily loss distribution of a portfolio is normal with mean *0* and s.d. *$20* million. Find the *10*-day *95% VaR* and *252*-day *99% VaR*.

Since 1-day $VaR(x) = \sigma\, \Phi^{-1}(x)$ at any confidence level *x*, therefore 1-day *VaR*(0.95)=($20m)(1.645)=32.9m and 1-day *VaR*(0.99)=($20m)(2.326)=$46.52m.

Then the 10-day *VaR(0.95)* is $(\sqrt{10})(\$32.9m) = \$104.04$ m and 252-day *VaR(0.99)* is $(\sqrt{252})(\$46.52m)$ = $738.48 m.

VaR asks how bad the loss will be while **Expected Shortfall** (**ES**) asks: "if things do get bad, what is the expected loss?"

Loss  Profit

Mathematically speaking; let *L* be the loss of *Q*, $ES(x) = E[L \mid L > VaR(x)]$. ES is also known as Conditional VaR (**CVaR**) for this reason; for the continuous underlying distribution, it is equivalent to the tail conditional expectation, i.e., with the equality in the conditional event.

Example 3.2

Consider a *$10* million 1-year loan which has a *1.25%* chance of defaulting. If the loan default, the recovery of the loan principal is equally likely from *0%* to *100%*. Find the 1-year *99% VaR* and the 1-year *99% ES* respectively.

If the loan is defaulted (with prob *0.0125*), let *x* be the percentage of loan principal recovered. Then *(0.0125)(x)=0.01* or *x=0.8*. That is, *80%* of *$10m = $8m* will be recovered with prob. *0.01*; or *VaR(0.99)=$10m - $8m = $2m*.
In other words, the prob. of loss greater than *$2m* is *80%* of *1.25% = 1%*.

1-year *99% ES* is the expected loss given the loss>*$2m*. Since the loss is uniformly distributed between *$2m* to *$10m*, the mean is *$6m*.

Example 3.3

Suppose we have a portfolio consisting of two $10 million 1-year loan as in Example 3.2. For simplicity, we further assume that if one loan defaults then it is certain the other loan will not default. If the loan does not default, a profit of *$0.2* million is made. Find the 1-year *99% VaR* and 1-year *99% ES* of this portfolio.

Each loan defaults with prob. 1.25% and they never default together. Therefore a default occurs with prob. *2.5%*. Let *x* be the percentage of loan principal recovered. Then *(0.025)(x)=0.01* or *x=0.4*. That is, *40%* of *$10m = $4m* will be recovered with prob. *0.01*. However there is a profit of $0.2m is made on the other loan so that the 1-year *99% VaR* is *$10m - $4m - $0.2m = $5.8m*.

1-year *99%* ES of this portfolio is the expected loss given the loss>*\$5.8*. Since the loss is uniformly distributed between *\$5.8m* to *\$9.8m*, the mean is *\$7.8m*.

3.2 Basel Accord and properties of risk measures

A risk measure is used for specifying capital requirement such as cash (or capital) that must be added to the bank or financial institution to provide a buffer for the underlying risk being acceptable to the regulators. Supervisory authorities for Belgium, Canada, France, Germany, Italy, Japan, Luxembourg, Netherlands, Sweden, Switzerland, UK and US form a Committee on Banking Supervision, known as Bank for International Settlements (BIS). They meet regularly in Basel, Switzerland and in 1996 Basel Committee issued an amendment to the 1998 Accord which was then sometimes referred as “BIS 98”. Banks are required to calculate the *10-day 99% VaR* measure. This means that it focuses on the revaluation of loss over a *10-day* period that expects to happen with prob. no more than *1%*. The Basel committee requires the bank to hold *k* times this *VaR* measure. This regulatory multiplier *k* is chosen on a bank-by-bank basis and must be at least *3*. For a bank with excellent well-tested *VaR* model, it is likely that *k=3*. For bank’s *VaR* model is not perform well during the last *250* days, *k* may be as high as *4*.

There are some desirable properties that the risk measures should have:

1. Monotonic: If a portfolio has lower returns than another portfolio for all possible outcomes, its risk measure should be greater. [$R_A \le R_B \Rightarrow \rho(R_A) \ge \rho(R_B)$]
2. Translation invariance: If we add \$K cash to a portfolio, its risk measure should go down by \$K. [$\rho(R_A + K) = \rho(R_A) - K$]
3. Homogeneity: Changing the size of a portfolio by a factor k should result in the risk measure being multiplied by k. [$\rho(kR_A) = k\rho(R_A)$]
4. Subadditivity: The risk measure for two portfolios after they have been merged should not be greater than the sum of their individual risk measures before they were merged. [$\rho(R_A + R_B) \le \rho(R_A) + \rho(R_B)$]

The first three conditions are straightforward while the fourth condition means diversification helps to reduce risks. Risk measure satisfying all these four conditions are said to be **coherent**. Let us recall the *VaR* and *ES* in examples 3.2 and 3.3. From example 3.2 and 3.3, the 1-year *99% VaR* of the two loans separately is *\$2m+\$2m=\$4m* which is less than *\$5.8m*, the *VaR* of the portfolio. This implies that *VaR* does not satisfy the subadditivity condition in general; yet it is for elliptical distribution.

The 1-year *99% ES* of the two loans separately is *\$6m+\$6m=\$12m* which is greater than *\$7.8m*, the *ES* of the portfolio; indeed, ES generally fulfills the subadditivity condition.

Although *VaR* is not coherent, it is easier to understand and use in the back-testing than ES. Therefore, *VaR* has become the most popular risk measure among the regulators and risk managers. In fact, coherent risk measure is a rather commonly used concept within the academics, though it is not as important in the actual practice.

3.3 Historical simulation (nonparametric) approach

In the previous simplified examples, we calculate the *VaR* by assuming that the return of portfolio follows a uniform or normal distribution for simplicity. There is another approach to calculate *VaR* based on the historical data. Suppose that today is day n and we define v_i as the value of a market variable (stock price or index) Then the value tomorrow estimated based on the *i-th* scenario is $\hat{v}(i) = v_n \times v_i / v_{i-1}$ for *i=1,...,n*. From these, the portfolio values of these *n* scenarios are computed and the *VaR* can be obtained. Let us illustrate this by the *stock.csv* example. Suppose we spend $40,000 on buying HSBC, $30,000 on CLP and $30,000 on CK respectively on 31/12/2002. The following R codes compute the 1-day *VaR* of this portfolio using historical simulation.

```
d<-read.csv("stock.csv")     # read in data
x<-as.matrix(d)              # change to matrix
n<-nrow(x)                   # no. of obs

xn<-as.vector(x[n,])         # select the last obs
w<-c(40000,30000,30000)      # amount on each stock
p0<-sum(w)                   # total amount
ws<-w/xn                     # no. of shares bought at day n
ns<-n-1                      # no.of scernarios
hsim<-NULL                   # initialize hsim
for (i in 1:ns) {
  t<-xn*(x[i+1,]/x[i,])      # scenario i
  hsim<-rbind(hsim,t)        # append t to hsim
}

hsim<-as.matrix(hsim)        # change to matrix
ws<-as.matrix(ws)
ps<-as.vector(hsim%*%ws)     # compute portfolio value
loss<-p0-ps                  # compute loss
(VaRs<-quantile(loss,0.99))  # compute and display 1-day 99% VaR
3535.733
```

Note that the cost of the portfolio is $100,000 based on the closing price on 31/12/2002. Then we compute the stock prices of these *n* scenarios and save them in *hsim*. Then, we compute the portfolio value *ps* and the losses of these *n* scenarios and finally the 1-day 99% *VaR* is obtained from the 99th percentile, *VARs=$3535.733*.

The above historical simulation method puts equal weight to each historical data. This method can be modified by putting different weight depending on the ratio the *(n+1)-th* and *i-th* volatilities. From Chapter 2, as we assume $\frac{1}{\sigma_i}\frac{v_i - v_{i-1}}{v_{i-1}} \sim N(0,1)$ for all i, the value of v_{n+1} can be estimated by the *i-th* scenario as $\frac{1}{\sigma_{n+1}}\frac{\hat{v}(i)-v_n}{v_n}=\frac{1}{\sigma_i}\frac{v_i-v_{i-1}}{v_i}$, which gives,

$\hat{v}(i)=v_n\times\frac{v_{i-1}+(v_i-v_{i-1})\,\sigma_{n+1}/\sigma_i}{v_{i-1}}$ for *i=1,...,n* ;

where σ_i is the estimated volatility using the *EWMA* or *GARCH(1,1)* model mentioned in Chapter 2.

3.4 Model building approach

Another approach of calculating *VaR* is based on the assumption that the relative return of the p market variables $u_{i,j}=(v_{i,j}-v_{i,j-1})/v_{i,j-1}$ at time j and $i=1,...,p$ follows *p-variate* normal distribution $N_p(0,\Sigma)$. Assume that we have $\$w=(w_1,\ldots,w_p)$ invested on the p market variables. We bought, for $i=1,...,p$, w_i/v_{i0} shares of the i-th market variable at time 0. When the market price changes from v_{i0} to v_{i1}, the value of each share is w_iv_{i1}/v_{i0} and the change is $w_iv_{i1}/v_{i0}-w_i=w_i(v_1-v_0)/v_0=w_iu_1$. Hence the change in the portfolio value is $\Delta P=w_1u_1+\ldots+w_pu_p$. We estimate the standard deviation of ΔP by $\sqrt{w'Sw}$ where S is the sample covariance matrix of $u=(u_1,...,u_p)'$. Since we assume the mean of u is zero, therefore the 1-day 99% *VaR* is $z_{0.99}\times\sqrt{w'Sw}=2.32635\times\sqrt{w'Sw}$. Let us illustrate this by the *stock.csv* example again with the following R codes.

```
t1<-as.ts(d$HSBC)          # change to time series
t2<-as.ts(d$CLP)
t3<-as.ts(d$CK)
u1<-(lag(t1)-t1)/t1        # compute u
u2<-(lag(t2)-t2)/t2
u3<-(lag(t3)-t3)/t3
u<-cbind(u1,u2,u3)         # form matrix u
S<-var(u)                  # sample cov. matrix
dp<-as.vector(u%*%w)       # Delta P
sdp<-sd(dp)                # sd of portfolio (same as sqrt(w%*%S%*%w))
(VaRn<-qnorm(0.99)*sdp)    # compute and display 1-day 99% VaR
3062.165
```

Note that in the above R codes, we directly compute the sample standard deviation of ΔP which is the same as $\sqrt{w'Sw}$. The 1-day 99% *VaR* using normal model is *VaRn=$3062.165* which is less than *VaRs=$3535.733*. The normality assumption may not be valid since most returns have a fatter tail than that of the normal distribution. Hence the *VaRn* is over-optimistic. Recall that we can model the return by a student's t distribution in Section 1.5. Let us compute the sample excess kurtosis of ΔP, $\hat{\varsigma}_2$, and estimate the degree of freedom by $\nu=round(6/\hat{\varsigma}_2+4)$.

```
ku<-sum((dp/sdp)^4)/length(dp)-3       # sample excess kurtosis
v<-round(6/ku+4)                       # degree of freedom
VaRt<-qt(0.99,v)*sdp                   # 1-day 99% VaR
VaRt
4136.686
```

3.5 Approach via Extreme value theory (EVT)

In modeling empirical data, one useful law is the **Power Law**. This law states that the probability of a random variable *V*>*x*, $\Pr\{V > x\} \propto x^{-\alpha}$ for some $\alpha > 0$. This law provides a good model for estimating the tail probability of rare events.

Let *F(x)* be the CDF of X, i.e. $\Pr\{X \le u\} = F(u)$ or $\Pr\{X > u\} = 1 - F(u)$.

$\Pr\{u < X \le u + y\} = F(u+y) - F(u)$.

Now given *X*>*u*, the conditional probability of $u < X \le u + y$ is

$$F_u(y) = \Pr\{u < X \le u + y \mid X > u\} = \frac{F(u+y) - F(u)}{1 - F(u)}.$$

Gnedenko (1943) proved that $F_u(y) \to G_{\xi,\beta}(y) = 1 - (1 + \xi\, y/\beta)^{-1/\xi}$ as $u \to \infty$. $G_{\xi,\beta}(y)$ is called the generalized Pareto distribution with shape ξ and scale β. For most financial data, $\xi > 0$ and $\beta \in [0.1, 0.4]$.

Suppose we sort the *n* observations x_i in decreasing order and found that there are n_u observations $x_i > u$. We can estimate $\Pr\{X > u + y \mid X > u\}$ by $1 - G_{\xi,\beta}(y)$ and $1 - F(u)$ by n_u / n. Hence the unconditional probability:

$$\Pr\{X > u + y\} \approx [1 - F(u)][1 - G_{\xi,\beta}(y)] = \frac{n_u}{n}\left(1 + \xi\frac{y}{\beta}\right)^{-1/\xi}$$

$$\Rightarrow \Pr\{X > x\} \approx \frac{n_u}{n}\left(1 + \xi\frac{x-u}{\beta}\right)^{-1/\xi} \quad \Rightarrow \quad \Pr\{X \le x\} \approx 1 - \frac{n_u}{n}\left(1 + \xi\frac{x-u}{\beta}\right)^{-1/\xi}.$$

Note that if we take $u = \beta/\xi$, then $\Pr\{X > x\} \approx (n_u/n)(\xi/\beta)^{-1/\xi} x^{-1/\xi}$ which satisfy the Power law. From the above equation, we can calculate 1-day $\varepsilon \times 100\%$ VaR_ε.

$$\varepsilon = 1 - \frac{n_u}{n}\left(1 + \xi\frac{VaR_\varepsilon - u}{\beta}\right)^{\frac{1}{\xi}} \Rightarrow VaR_\varepsilon = u + \frac{\beta}{\xi}\left[\left(\frac{n(1-\varepsilon)}{n_u}\right)^{-\xi} - 1\right].$$

Since the density of the generalized Pareto distribution is

$g_{\xi,\beta}(y) = dG_{\xi,\beta}(y)/dy = (1/\beta)\,(1 + \xi\, y/\beta)^{-1/\xi - 1}$, the log-likelihood function is

$$l = \sum_{i=1}^{n_u}\left\{-\left(\frac{1}{\xi} + 1\right)\ln\left[1 + \frac{\xi(x_i - u)}{\beta}\right] - \ln\beta\right\} = -\left\{n_u \ln\beta + \left(\frac{1}{\xi} + 1\right)\sum_{i=1}^{n_u}\ln\left[1 + \frac{\xi(x_i - u)}{\beta}\right]\right\}.$$

We can find the MLE $\hat{\xi}$ and $\hat{\beta}$ by maximizing the log-likelihood function and then calculate the 1-day $(1-\varepsilon)100\%$ *VaR* by $VaR_\varepsilon = u + (\hat{\beta}/\hat{\xi})\{[n(1-\varepsilon)/n_u]^{-\hat{\xi}} - 1\}$.

The only question remains is how to choose the threshold value *u*. In practice, we should first apply the standardize transformation $z_i = (x_i - \bar{x})/s$ and find the VaR_ε of z using some threshold values between 2.7 to 3.2. Finally the VaR_ε of x will be $\bar{x} + s\,VaR_\varepsilon$.

Let us illustrate this by the previous example. Note that we use the R built-in function *optim()* to minimize the –log-likelihood function (See *help(optim)* for details).

```
# EVT
u<-3.2              # threshold value
m<-mean(loss)       # mean loss
s<-sd(loss)         # sd loss
z<-(loss-m)/s       # standardize loss
zx<-z[z>u]          # select z>u
nu<-length(zx)      # no. of zx
```

```
# define -log_likelihood function
log_lik<-function(p,dat) {          # parameter vector p=(xi,beta)
  length(dat)*log(p[2])+(1/p[1]+1)*sum(log((1+p[1]*dat/p[2])))
}

p0<-c(0.2,0.01)                     # initial p0=(xi,beta)
res<-optim(p0,log_lik,dat=(zx-u))   # min -log_lik
(p<-res$par)                        # MLE p=(xi,beta)
[1] 0.6755755 0.3117039
> -res$value                        # max value
[1] -3.058536

q<-0.99
(VaR<-u+(p[2]/p[1])*((length(z)*(1-q)/nu)^(-p[1])-1))
[1] 3.056386

(VaRe<-m+VaR*s)                     # 1-day 99% VaR using EVT
[1] 4000.848
```

Now the 1-day 99% *VaR* using normal-model is *VaRn=$3535.733*; using t-model is *VaRt=$4136.686* and using extreme value theory is *VaRe=$4000.848* and. A natural question is which *VaR* is more suitable? In general, the extreme value theory (EVT) is theoretically the most rigorous, and should be preferred if there is abundant data, as EVT uses only data at the tail of the loss distribution. For small dataset, the *t*-model is preferred over the normal-model as loss distributions are more likely to be fat tail. We can actually test these *VaR* using the previous data to see which one is more reasonable. This is called back-testing and will be introduced in the next section.

3.6 Back Testing

VaR calculation is based on the knowledge of the loss distribution. It is important to test how accurate the *VaR* estimate really is. An important reality check is back testing. Let 1-day *X% VaR* be *$V.* We called an exception occur if the portfolio value is less than $\$V$ on a given day. If the *VaR* model is accurate, the prob. of the portfolio value less than $\$V$ on any given day is $p=1-X$. Suppose we look at a total of n days and we observe that m exceptions occur in these n days, where $m/n>p$. Should we reject the model for producing a low *VaR* value? This can be tested formally by using the following binomial test:

H_0: Probability of an exception on any given day is p.

H_1: Probability of an exception on any given day is greater than p.

The prob. of having at least m exceptions out of n days is $p_0 = \sum_{k=m}^{n} \binom{n}{k} p^k (1-p)^{n-k}$ under H_0. We reject H_0 if $p_0 < \alpha$, say 0.05.

This probability can be computed using R function: *1-pbinom(m,n,p)* or using EXCEL function: *1-binomdist(m,n,p,TRUE)*. Let us set $n=250$ days, $X=0.99$ so that $p=0.01$. We compute p_0 for $m=0,\ldots, 10$.

```
> m<-0:10
> round(1-pbinom(m,250,0.01),4)
0.9189 0.7142 0.4568 0.2419 0.1078 0.0412 0.0137 0.0040 0.0011 0.0003 0.0001
```

It is clear that $p_0 > 0.05$ if $m<5$. The 1986 BIS Amendment requires *VaR* models to be back tested. Banks should look at the number of exceptions m during the previous *250* days. If $m<5$, then the regulatory multiplier is set at its minimum value ($k=3$). If m is *5, 6, 7, 8, or 9*, k is set at *3.4, 3.5, 3.65, 3.75,* and *3.85* respectively. If m is *10* or more, k will be set to *4*.

Now let us continue with the *stock.csv* and back test the 1-day 99% *VaR* using historical simulation and normal model.

```
n<-nrow(d)-1                   # no. of obs. of u
n1<-n-250+1                    # starting index for 250 days before n
x<-as.matrix(d[n1:n,])         # select the most recent 250 days
ps<-as.vector(x%*%ws)          # compute portfolio value
ps<-c(ps,sum(w))               # add total amount at the end
loss<-ps[1:250]-ps[2:251]      # compute daily loss
sum(loss>VaRs)                 # count the no of exceptions
0
sum(loss>VaRn)
0
sum(loss>VaRt)
0
sum(loss>VaRe)
0
```

From the output, we found that the numbers of exceptions in the past 250 days for all approach are less than 5. According to 1986BIS, if the exceptions found in the past 250 days less than 5, the regulatory multiplier should be set to *3*.

3.7 Estimation of Expected Shortfall

Recall that the Expected Shortfall $ES(x)=E[L\,|\,L>VaR(x)]$, where L is the loss. Let $x=q_\varepsilon$, where q_ε is the $(1-\varepsilon)100\%$ from N(0,1), i.e. if $\varepsilon=0.01$, $q_\varepsilon=2.3262$.

1. If we assume $L\sim N(\mu,\sigma^2)$, then $ES(q_\varepsilon)=\dfrac{\sigma}{\varepsilon\sqrt{2\pi}}\exp(\dfrac{-q_\varepsilon^2}{2})+\mu=\dfrac{\sigma}{\varepsilon}\phi(q_\varepsilon)+\mu$.

Proof. For simplicity, we first assume that $L\sim N(0,1)$. The conditional density of L given $L>q_\varepsilon$ is $\dfrac{1}{\varepsilon\sqrt{2\pi}}\exp(-x^2/2)$ for $q_\varepsilon<x<\infty$. Then

$$ES(q_\varepsilon)=\int_{q_\varepsilon}^{\infty}\frac{1}{\varepsilon\sqrt{2\pi}}\,x\exp(-\frac{x^2}{2})\,dx=\frac{1}{\varepsilon\sqrt{2\pi}}\int_{q_\varepsilon}^{\infty}d(-\exp(\frac{-x^2}{2}))$$

$$=\frac{-1}{\varepsilon\sqrt{2\pi}}\exp(\frac{-x^2}{2})\Big|_{q_\varepsilon}^{\infty}=\frac{1}{\varepsilon\sqrt{2\pi}}\exp(\frac{-q_\varepsilon^2}{2})=\frac{1}{\varepsilon}\phi(q_\varepsilon).$$

In general, when $L\sim N(\mu,\sigma^2)$, $ES(q_\varepsilon)=\dfrac{\sigma}{\varepsilon\sqrt{2\pi}}\exp(\dfrac{-q_\varepsilon^2}{2})+\mu=\dfrac{\sigma}{\varepsilon}\phi(q_\varepsilon)+\mu$.

For $\varepsilon=0.01$, $q_\varepsilon=2.3262$, $ES(q_\varepsilon)\approx 2.6652\sigma+\mu$.

2. If we have enough data, the expected shortfall can be more accurately determined by the Pareto Distribution. Its cumulative probability function

$$G_{\xi,\beta}(x):=1-[1+\frac{\xi}{\beta}(x-u)]^{-1/\xi}$$

is the conditional probability $\Pr(u<X<u+x\,|\,X>u)$, for threshold u. The associated conditional probability density function is

$$g_{\xi,\beta}(x):=\frac{\partial G_{\xi,\beta}(x)}{\partial x}=\frac{1}{\beta}[1+\frac{\xi}{\beta}(x-u)]^{-1/\xi-1},$$

$$E[X\mid X>u]=\int_u^\infty[xg_{\xi,\beta}(x)]dx$$

$$=\{x[G_{\xi,\beta}(x)-1]\}_u^\infty-\int_u^\infty[G_{\xi,\beta}(x)-1]dx$$

$$=\left\{-x[1+\frac{\xi}{\beta}(x-u)]^{-1/\xi}-\frac{\beta}{1-\xi}[1+\frac{\xi}{\beta}(x-u)]^{1-1/\xi}\right\}_u^\infty$$

$$=\left\{[1+\frac{\xi}{\beta}(x-u)]^{1-1/\xi}[\frac{\beta x}{\beta+\xi(x-u)}+\frac{\beta}{1-\xi}]\right\}_\infty^u$$

If $\xi<1$, which is generally the case, $\lim\limits_{x\to\infty}[1-\dfrac{\xi}{\beta}(x-u)]^{1-1/\xi}=0$, hence

$$E[X\mid X>u]=u+\frac{\beta}{1-\xi}$$

Setting the threshold $u = VaR_\varepsilon$, we get the expected shortfall

$$ES = E[X \mid X > VaR_\varepsilon] = VaR_\varepsilon + \frac{\beta}{1-\xi}$$

Given enough data with loss greater than VaR_ε, the parameters ξ and β can be accurately estimated.

3. If we do not want to impose any distributional assumption on *L*, we can estimate $ES(q_\varepsilon)$ by sorting the loss: $L_{(1)} \geq \ldots \geq L_{(n)}$ in descending order. Then

$$ES(q_\varepsilon) = \frac{1}{\varepsilon n}\sum_{k=1}^{K-1} L_{(k)} + (1 - \frac{K-1}{\varepsilon n})L_K$$, where *K* is the largest integer less than $n\varepsilon$.

These calculations can be easily done by using R or EXCEL.

3.8 EXCEL

In EXCEL, computing *VaRs*, *VaRn* and *VaRt* are rather simple and straight forward. EXCEL has built-in function *NORMSINV()* and *TINV()* to return the percentile points of normal and *t* distribution respectively.

1. In the **hist-sim tab**, A2:C1044 are the closing prices of HSBC, CLP and CK. Column E2:E1043 is the scenario number.
2. Enter the formula =A$1044*A3/A2 in F2 for computing the future closing price of HSBC based on the first scenario. Copy this formula to F2:H1043 for other stocks and scenarios.
3. Enter 40000, 30000 and 30000 in N3:P3 to represent the amount of money spent on each stock and Q3 is the total amount. Enter =N3/A$1044 in N4 to compute the number of shares for HSBC. Copy it to O4:P4 for other stocks.
4. Enter the formula =SUMPRODUCT(F2:H2,$N$4:$P$4) in I2 and =$Q$3-I2 in J2 to compute the portfolio value and loss based on the first scenario. Copy them downward to I1043:J1043 for other scenarios.
5. K2:K1043 contains the loss sorted in descending order. N6 is the 1-day 99% VaRs and is obtained by entering =PERCENTILE(K2:K1043,0.99).

6. In the **model tab**, Columns A, B and C contains the closing prices and Columns E, F and G contains the corresponding returns. K2:M2 contains the amount spent on each stock.
7. Enter =SUMPRODUCT(E2:G2,$K$2:$M$2) in H2 to compute the change in portfolio value ΔP. Copy it downward to H1043 to form the distribution of ΔP.
8. Use the built-in covariance function to compute the Sample covariance matrix of *u* and store it in K5:M7.
9. Enter the formula =SUMPRODUCT(MMULT(K2:M2,K5:M7),K2:M2) in K10 to compute the sample variance of ΔP and =STDEV(H2:H1043) to compute the sample s.d. of ΔP.
10. Enter =NORMSINV(0.99) in K14 to return the 99% percentile of standard normal and =K14*K12 in K15 to give 1-day 99% *VaRn*.

11. For the student's t model, enter =KURT(H2:H1043) in K17 to compute the excess kurtosis of ΔP and =ROUND(6/K17+4,0) in K18 to estimate the degree of freedom. Finally enter =TINV(0.02,6) in K20 to return the 99% percentile of t distribution and enter =K20*K12 in K21 to give 1-day 99% *VaRt*.

12. In the **EVT tab**, u1, u2, u3, ΔP and Loss ($= -\Delta P$) are stored in columns E to I respectively. The mean and sd of Loss are computed and stored in O3 and O4. Then we compute the standardized score of Loss by entering =(I2-$O$3)/$O$4 in J2 and copy it down to J1043. Column K are the sorted value of Z.
13. Suppose we choose u=3.2 in Q4. Then we can create the Loss>u in L2 to L7. Enter some initial values for ξ and β in O7 and P7 , say 0.5 and 0.2.
14. Enter the formula in M2 =(-1/$N$7-1)*LN(1+$N$7*(K2-$Q$4)/$O$7)-LN($O$7) which used to compute the log-likelihood value.
15. We compute the log-likelihood value in P9 by =SUM(M2:M7). Use solver() to maximize P9 with variable cells O7 and P7 to obtain MLE.
16. Variance of the Z in computed in P11 and the VaRe is computed in P12.
17. In the **back-test** tab, columns A, B and C contain the closing prices. K2:M2 contain the amount spent on each stock. K3:M3 contains the number of shares of each stock.
18. Enter =SUMPRODUCT(A2:C2,$K$3:$M$3) in D2 to compute the portfolio value and enter =D2-D3 in E2 to compute the loss. Copy it downward to E1044 to give the portfolio value and loss on each day. Copy *VaRs*, *VaRn*, *VaRt* and *VaRe* in K6:K9.
19. Finally we can find out the number of exceptions in the past 250 days by entering =($E794>$K$6)+0 in F794, copy it down to F1043 and =SUM(F794:F1043) in F1096. Similarly for columns G and I.
20. In the ES tab, the change in portfolio value ΔP is sorted in column I. The sd and mean of ΔP is given in L10 and L11 respectively. The formula in L14 =-NORMDIST(NORMINV(0.99,0,1),0,1,FALSE)/0.01*L10+L11
gives the *ES* using normal model.
21. There are *n=1042* observations, $n\varepsilon = 10.42$ and hence *K=10*. The formula in L8 =(SUM(I2:I11)/L4+(L5-10/L4)*I12)/L5
estimates the *ES* from the data.

22. Finally in the **ES tab**, the setup is similar. ΔP, Loss and sorted Loss are in columns H, I and J respectively. M4:6 are the $n, \varepsilon, n\varepsilon$. In M8, enter =(SUM(J2:J11)/M4+(M5-10/M4)*J12)/M5 to obtain the ES without the normality assumption.
23. M10 and M11 are the mean and sd of Loss in column I. In M14, enter =M11*NORMDIST(NORMINV(1-M5,0,1),0,1,FALSE)/M5+M10 to obtain the ES under the normal model.

Reference

Chapter 8 of Risk Management and Financial Institutions, by John Hull, Wiley.