RMSC4002 Tutorial 5

Chapter 3-4

October 25, 2017

1 Back testing

```
n<-nrow(d)-1                # no. of obs. of u
n1<-n-250+1                 # starting index for 250 days before n
x<-as.matrix(d[n1:n,])      # select the most recent 250 days
ps<-as.vector(x%*%ws)       # compute portfolio value
ps<-c(ps,sum(w))            # add total amount at the end
loss<-ps[1:250]-ps[2:251]   # compute daily loss
sum(loss>VaRs)              # count the no of exceptions
0
sum(loss>VaRn)
0
sum(loss>VaRt)
0
sum(loss>VaRe)
0
```

2 Review on Eigenvalues and Eigenvectors

2.1 Basic Concept

Definition A nonzero vector $\mathbf{x}$ is an *eigenvector* of a square matrix $\mathbf{A}$ if there exists a scalar λ, called an *eigenvalue*, such that $\mathbf{Ax} = \lambda\mathbf{x}$.

To find eigenvalues and eigenvectors for a matrix $\mathbf{A}$, first solve the characteristic equation,

$$det(\mathbf{A} - \lambda\mathbf{I}) = 0$$

for the eigenvalues λ, and then for each eigenvalue solve

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{x} = \mathbf{0}$$

for the corresponding eigenvectors.

2.2 Properties

Suppose $\lambda_1, \lambda_2, \cdots, \lambda_n$ are eigenvalues w.r.t. eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ of $\mathbf{A}$, then

$$trace(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i \qquad det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$$

and the eigenvectors from different eigenvalues are orthogonal.

2.3 Orthogonal Matrix

Definition In linear algebra, an *orthogonal matrix* is a square matrix with real entries whose columns and rows are orthonormal vectors, i.e.

$$\mathbf{A}'\mathbf{A} = \mathbf{A}\mathbf{A}' = \mathbf{I}$$

An orthogonal matrix $\mathbf{A}$ is necessarily invertible (with inverse $\mathbf{A}^{-1} = \mathbf{A}'$).

2.4 Eigen Decomposition

Assume $\mathbf{A}$ has non degenerate eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$ and corresponding linearly independent eigenvectors $\mathbf{X}_1, \mathbf{X}_2, ..., \mathbf{X}_n$.

Define the matrices composed of eigenvectors

$$\mathbf{P} = \begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & ... & \mathbf{X}_n \end{bmatrix}$$

and diagonal matrix

$$diag(\mathbf{D})' = \begin{bmatrix} \lambda_1 & \lambda_2 & ... & \lambda_n \end{bmatrix}$$

Then,

$$\begin{aligned}
\mathbf{AP} &= \mathbf{A}\begin{bmatrix} \mathbf{X}_1 & \mathbf{X}_2 & ... & \mathbf{X}_n \end{bmatrix} && (1)\\
&= \begin{bmatrix} \mathbf{AX}_1 & \mathbf{AX}_2 & ... & \mathbf{AX}_n \end{bmatrix} && (2)\\
&= \begin{bmatrix} \lambda_1\mathbf{X}_1 & \lambda_2\mathbf{X}_2 & ... & \lambda_n\mathbf{X}_n \end{bmatrix} && (3)\\
&= \mathbf{P}\begin{bmatrix} \lambda_1 & 0 & ... & 0 \\ 0 & \lambda_2 & ... & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & ... & \lambda_n \end{bmatrix} && (4)\\
&= \mathbf{PD} && (5)
\end{aligned}$$

Thus we can decompose $\mathbf{A}$ into $\mathbf{PDP}^{-1}$

Furthermore, squaring $\mathbf{A}$ we have

$$\begin{aligned}
\mathbf{A}^2 &= (\mathbf{PDP}^-1)(\mathbf{PDP}^-1) && (6)\\
&= \mathbf{PD}(\mathbf{P}^{-1}\mathbf{P})\mathbf{DP}^{-1} && (7)\\
&= \mathbf{PD}^2\mathbf{P}^{-1} && (8)
\end{aligned}$$

and by induction,

$$\mathbf{A}^k = \mathbf{PD}^k\mathbf{P}^{-1}$$

We now define $\mathbf{y}_i = \frac{\mathbf{X}_i}{\sqrt{\langle \mathbf{X}_i, \mathbf{X}_i \rangle}}$ to make all $\mathbf{X}_i$ as unit vectors, and $\mathbf{Q} = \begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & ... & \mathbf{y}_n \end{bmatrix}$, then by the property of orthogonal matrix, we have

$$\mathbf{A} = \mathbf{QDQ}^{-1}$$

3 Principle Component Analysis

```
> d<-read.csv("us-rate.csv")  # read in data
> label<-c("1m","3m","6m","9m","12m","18","2y","3y","4y","5y","7y","10y","15y")
> names(d)<-label             # apply labels
> options(digits=2)           # display the number using 2 digits
> cor(d)                      # compute correlation matrix

      1m   3m   6m   9m  12m  18m   2y   3y   4y   5y   7y  10y  15y
1m  1.00 0.99 0.99 0.99 0.98 0.98 0.97 0.96 0.95 0.95 0.94 0.92 0.91
3m  0.99 1.00 1.00 0.99 0.99 0.99 0.98 0.97 0.96 0.96 0.95 0.94 0.92
6m  0.99 1.00 1.00 1.00 1.00 0.99 0.99 0.98 0.97 0.96 0.96 0.94 0.93
9m  0.99 0.99 1.00 1.00 1.00 1.00 0.99 0.99 0.98 0.97 0.96 0.95 0.94
12m 0.98 0.99 1.00 1.00 1.00 1.00 1.00 0.99 0.98 0.98 0.97 0.96 0.94
18m 0.98 0.99 0.99 1.00 1.00 1.00 1.00 1.00 0.99 0.99 0.98 0.97 0.96
2y  0.97 0.98 0.99 0.99 1.00 1.00 1.00 1.00 0.99 0.99 0.98 0.98 0.96
3y  0.96 0.97 0.98 0.99 0.99 1.00 1.00 1.00 1.00 1.00 0.99 0.99 0.98
4y  0.95 0.96 0.97 0.98 0.98 0.99 0.99 1.00 1.00 1.00 1.00 0.99 0.98
5y  0.95 0.96 0.96 0.97 0.98 0.99 0.99 1.00 1.00 1.00 1.00 1.00 0.99
7y  0.94 0.95 0.96 0.96 0.97 0.98 0.98 0.99 1.00 1.00 1.00 1.00 0.99
10y 0.92 0.94 0.94 0.95 0.96 0.97 0.98 0.99 0.99 1.00 1.00 1.00 1.00
15y 0.91 0.92 0.93 0.94 0.94 0.96 0.96 0.98 0.98 0.99 0.99 1.00 1.00
```

Remarks: In order to avoid large variance in some features, we should use correlation matrix to conduct PCA.

```
> pca<-princomp(d,cor=T)  # perform PCA using correlation matrix
                          # and save the result to the object pca.
> pca$loadings[,1:6]      # display the loadings of the first six PCAs

     Comp.1   Comp.2   Comp.3   Comp.4   Comp.5   Comp.6
1m  -0.2732  0.40434  0.59756  0.56592 -0.27195  0.06041
3m  -0.2758  0.34462  0.28225 -0.29408  0.61221 -0.34923
6m  -0.2770  0.30272 -0.02848 -0.36767  0.16994  0.38335
9m  -0.2783  0.23612 -0.17710 -0.23317 -0.21616  0.25111
12m -0.2786  0.20227 -0.25195 -0.16505 -0.41027  0.18186
18  -0.2798  0.09148 -0.27652  0.03453 -0.20085 -0.31179
2y  -0.2799  0.03527 -0.28849  0.13555 -0.09394 -0.56147
3y  -0.2798 -0.09978 -0.22264  0.22118  0.13877 -0.10916
4y  -0.2791 -0.16815 -0.18806  0.26537  0.25584  0.12151
5y  -0.2785 -0.22976 -0.08357  0.20610  0.21373  0.20016
7y  -0.2772 -0.30150  0.03910  0.13771  0.16234  0.29454
10y -0.2754 -0.37538  0.21868 -0.10556 -0.04935  0.08244
15y -0.2729 -0.44384  0.40821 -0.40615 -0.31461 -0.24199
```

```
> pc1<-pca$loadings[,1]    # save the loading of PC1
> pc2<-pca$loadings[,2]    # save the loading of PC2
> pc1 %*% pc1              # compute α'α (should be 1, unit length)
> pc2 %*% pc2              # compute β'β (should be 1)
> pc1 %*% pc2              # compute α'β (should be 0, orthogonal)
```

```
> s<-pca$sdev     # save the s.d. of all PC to s
> s               # display s
  Comp.1   Comp.2   Comp.3   Comp.4   Comp.5   Comp.6   Comp.7   Comp.8
3.567055 0.487587 0.157683 0.086441 0.053838 0.038195 0.031348 0.026454
  Comp.9  Comp.10  Comp.11  Comp.12  Comp.13
0.000880 0.000854 0.000831 0.000798 0.000768

> round(s^2,4)    # display variance
 Comp.1  Comp.2  Comp.3  Comp.4  Comp.5  Comp.6  Comp.7  Comp.8  Comp.9 Comp.10
12.7239  0.2377  0.0249  0.0075  0.0029  0.0015  0.0010  0.0007  0.0000  0.0000
Comp.11 Comp.12 Comp.13
 0.0000  0.0000  0.0000

> t<-sum(s^2)     # compute total variance (should equals 13)
> round(s^2/t,4)  # proportion of variance explained by each PC
 Comp.1  Comp.2  Comp.3  Comp.4  Comp.5  Comp.6  Comp.7  Comp.8  Comp.9 Comp.10
 0.9788  0.0183  0.0019  0.0006  0.0002  0.0001  0.0001  0.0001  0.0000  0.0000
Comp.11 Comp.12 Comp.13
 0.0000  0.0000  0.0000

> cumsum(s^2/t)   # cumulative sum of proportion of variance
 [1] 0.979 0.997 0.999 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000 1.000
[13] 1.000
```

4 Exercise

Exercise (2014-15 final Q2) Let $u = (u_1, u_2, u_3)'$ be the daily relative return in percentage of three stocks: A, B, and C. Suppose the sample mean of u is $\bar{u} = (0.1314, 0.0395, 0.1031)'$ and the last value of u is $u_n = (-0.2801, -0.6536, 1.7301)'$ and the last stock price is $(89.25, 30.6, 72.25)'$. Let S be the covariance matrix of u and C be the Cholesky decomposition of S, $C'C = S$.

$$S = \begin{bmatrix} 2.5404 & 0.3257 & 1.6883 \\ 0.3257 & 1.4266 & 0.5675 \\ 1.6883 & 0.5675 & 4.1031 \end{bmatrix}, \quad C = \begin{bmatrix} 1.5939 & 0.2043 & 1.0592 \\ 0 & 1.1769 & 0.2986 \\ 0 & 0 & 1.7006 \end{bmatrix}.$$

Let the first and the second largest eigenvalues and their normalized eigenvectors of S be $\lambda_1 = 5.29$, $h_1 = (0.5298, 0.1667, 0.8316)'$, and $\lambda_2 = 1.468$, $h_2 = (0.8444, -0.1958, -0.4987)'$.

- Find the smallest eigenvalue and its normalized eigenvector of S.
- Find a and b so that $(a'Sa)/(b'Sb)$ is maximum, where $a'a = b'b = 1$. What is this maximum value?
- For the vectors a and b, find $a'Sb$.