# Tutorial 2

**Exercise 1:**

$$C_{12} = \frac{a_{21}}{\sqrt{a_{11}}} = \frac{0.2599}{\sqrt{1}} = 0.2599;$$

$$C_{13} = \frac{a_{31}}{\sqrt{a_{11}}} = \frac{0.4981}{\sqrt{1}} = 0.4981.$$

**Exercise 2:**

① $D = \begin{pmatrix} 2.5404 & 0 & 0 \\ 0 & 1.4266 & 0 \\ 0 & 0 & 4.1031 \end{pmatrix} \Rightarrow D^{-\frac{1}{2}} = \begin{pmatrix} \frac{1}{1.5939} & 0 & 0 \\ 0 & \frac{1}{1.1944} & 0 \\ 0 & 0 & \frac{1}{2.0256} \end{pmatrix}$

$$R = D^{-\frac{1}{2}} S D^{-\frac{1}{2}} = \begin{pmatrix} 1 & 0.1711 & 0.5229 \\ 0.1711 & 1 & 0.2347 \\ 0.5229 & 0.2347 & 1 \end{pmatrix}$$

②

$$CD^{-1/2} = \begin{pmatrix} 1.594 & 0.2043 & 1.0592 \\ 0 & 1.1768 & 0.2984 \\ 0 & 0 & 1.7006 \end{pmatrix} \begin{pmatrix} \frac{1}{1.5939} & 0 & 0 \\ 0 & \frac{1}{1.1944} & 0 \\ 0 & 0 & \frac{1}{2.0256} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0.1710 & 0.5229 \\ 0 & 0.9853 & 0.1473 \\ 0 & 0 & 0.8396 \end{pmatrix}$$

③ $x = \bar{\mu} + C'z = \begin{pmatrix} 0.1314 \\ 0.0395 \\ 0.1031 \end{pmatrix} + \begin{pmatrix} 1.594 & 0 & 0 \\ 0.2043 & 1.1768 & 0 \\ 1.0592 & 0.2984 & 1.7006 \end{pmatrix} \begin{pmatrix} -0.021 \\ -0.312 \\ 0.958 \end{pmatrix}$

$= \cdots$

## Tutorial 3

Exercise :

(1) $u_n = \dfrac{S_{n+1} - S_n}{S_n} = \dfrac{78.1 - 79.7}{79.7} = -0.02008$

$I_{n-1} = 1$ if $u_{n-1} < 0$ and $I_{n-1} = 0$ otherwise

$$\sigma_{n+1}^2 = \hat{w} + \hat{\beta}\sigma_n^2 + \hat{\alpha} u_n^2 + \hat{\theta} I_{n-1} u_n^2$$

$$= 0.0000104 + 0.9170 \times 0.00022 + 0.0222 \times (0.02008)^2 + 0.0397 \times (0.02008)^2$$

$$= 0.00024$$

(2) $\underline{\sigma_{n+1}^2 = \lambda \sigma_n^2 + (1-\lambda) u_n^2}$

$$= \lambda \cdot 0.00022 + (1-\lambda)(0.02008)^2$$

$$= 0.0004 + \lambda (0.00022 - 0.02008^2) = 0.00024$$

$$\Rightarrow \lambda = 0.8733$$

(3) $\sigma_n^2 = (1-\lambda)\sum_{i=1}^{n}\lambda^{i-1}u_{n-i}^2 + \lambda^m\sigma_{n-m}^2$

$$Var(\sigma_n^2) = (1-\lambda)^2\sum_{i=1}^{n}\lambda^{2i-2}Var(u_{n-i}^2) + \lambda^{2m}Var(\sigma_0^2)$$

$$= (1-\lambda)^2\sum_{i=1}^{n}\lambda^{2i-2}\gamma^2$$

## Tutorial 5

(1) $W = (200\times 79.9,\ 600\times 63.1,\ 200\times 119.9)'$

$= (15980,\ 37860,\ 23980)'$

$E(\Delta P) = W'\bar{u} = (15980,\ 37860,\ 23980)'\begin{pmatrix}0.00050\\0.00087\\0.00138\end{pmatrix} = 73.969$

$SD(\Delta P) = \sqrt{W'SW} = 624.623$

Expected value of the portfolio P on Dec. 31, 2012:

$15980 + 37800 + 23980 + E(\Delta P) = 77833.97$

(2) $Z_{0.99}\times SD(\Delta P) = 2.32635\times 624.623 = 1453.092.$

(3) Proof: See Chapter 3. P9

$$ES(q_\varepsilon) = \sigma\phi(q_\varepsilon)/\varepsilon + \mu$$

$$= \frac{624.623\times\phi(2.32635)}{0.01} + 73.969$$

$$\approx 2.6652\times 624.623 + 73.969$$

$$= 1738.714$$

# Tutorial 6

Exercise:

(1) $tr(S) = \lambda_1 + \lambda_2 + \lambda_3 \Rightarrow \lambda_3 = 2.5404 + 1.4266 + 4.1031 - 5.29 - 1.468$

$= 1.3121$

$$Sh_3 = \lambda_3 h_3 \Rightarrow \begin{bmatrix} 2.5404 & 0.3257 & 1.6883 \\ 0.3257 & 1.4266 & 0.5675 \\ 1.6883 & 0.5675 & 4.1031 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 1.3121 \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$\Rightarrow a = \cdots,\ b = \cdots,\ c = \cdots.$ normalized

(2) Let $S = HDH'$, where $H = [h_1, h_2, h_3]$, $D = diag(\lambda_1, \lambda_2, \lambda_3)$

$$\frac{a'Sa}{b'Sb} = \frac{a'HDH'a}{b'HDH'b} = \frac{u'Du}{v'Dv} = \frac{\sum_{i=1}^{3} \lambda_i u_i^2}{\sum_{i=1}^{3} \lambda_i v_i^2} \le \frac{\lambda_1}{\lambda_3} \frac{\sum_{i=1}^{3} u_i^2}{\sum_{i=1}^{3} v_i^2}$$

where $u = H'a$, $v = H'b$

Since $a'a = 1$, $\sum_{i=1}^{3} u_i^2 = a'HH'a = a'a = 1$, Similarly, $\sum_{i=1}^{3} v_i^2 = 1$

$$\frac{a'Sa}{b'Sb} \le \frac{\lambda_1}{\lambda_3} = \frac{5.29}{1.3121} = 4.032$$

When $a = h_1$, $b = h_3$, $\frac{a'Sa}{b'Sb} = \frac{h_1' S h_1}{h_2' S h_2} = \frac{\lambda_1}{\lambda_3}$

(3) $a'Sb = h_1' S h_3 = \cdots$

Tutorial 7

Exercise:

(1) $4k^2=1 \Rightarrow k=0.5 \Rightarrow h_1=(0.5,0.5,0.5,0.5)'$

$Sh_1=\lambda_1 h_1 \Rightarrow$

$$\begin{bmatrix} 2 & 0.8 & 0.8 & 0.8 \\ 0.8 & 2 & 0.8 & 0.8 \\ 0.8 & 0.8 & 2 & 0.8 \\ 0.8 & 0.8 & 0.8 & 2 \end{bmatrix}\begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} = \lambda_1 \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$\Rightarrow 1+0.4\times 3 = 0.5\lambda_1 \Rightarrow \lambda_1 = 2+2.4 = 4.4$

(2) $\text{trace}(S) = 2+2+2+2 = 4.4+3\bar{\lambda}$

$\Rightarrow \bar{\lambda} = \dfrac{8-4.4}{3} = 1.2$

(3) $\text{Sum}(h_1) = 0.5\times 4 = 2$

let $h_2=h_3=h_4=(\beta_1,\beta_2,\beta_3,\beta_4)'$

$$\begin{bmatrix} 2 & 0.8 & 0.8 & 0.8 \\ 0.8 & 2 & 0.8 & 0.8 \\ 0.8 & 0.8 & 2 & 0.8 \\ 0.8 & 0.8 & 0.8 & 2 \end{bmatrix}\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix} = 1.2\begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \end{bmatrix}$$

$\text{sum}(h_2)=\text{sum}(h_3)$
$=\text{sum}(h_4)=0$
$\Uparrow$

$\Rightarrow 2\beta_1+0.8(\beta_2+\beta_3+\beta_4)=1.2\beta_1 \Rightarrow \beta_1+\beta_2+\beta_3+\beta_4=0$