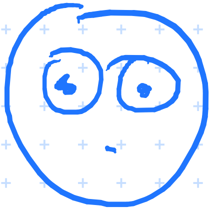
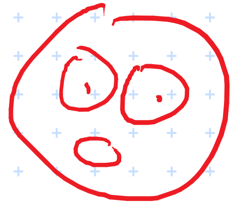


πr^2?

NO, PIE ARE ROUND!



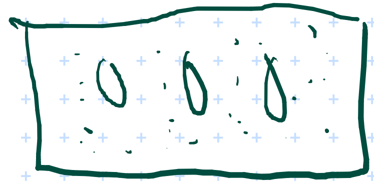
The WORST way to find the area of a circle

On the Gauss circle problem and related questions

Paco Adajar

University of Georgia

THIS IS A PIE.



THIS IS A LIE.

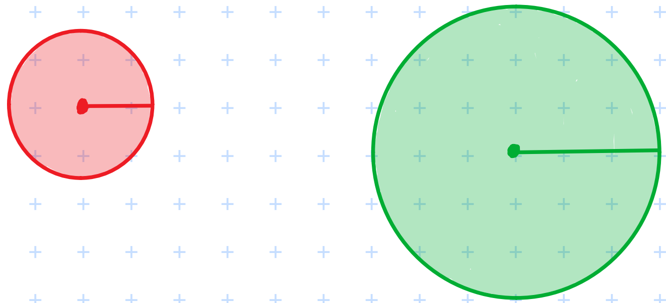
Overview

Suppose one day you forgot the formula for the area of a circle of radius r. oh no!

But you do know that area:

- scales quadratically, i.e. the area $A(r)$ of a circle of radius r is given by

$$A(r) = Cr^2$$

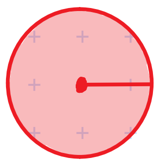


for some absolute constant C.

green radius $= 2 \times$ red radius

green area $= 4 \times$ red area

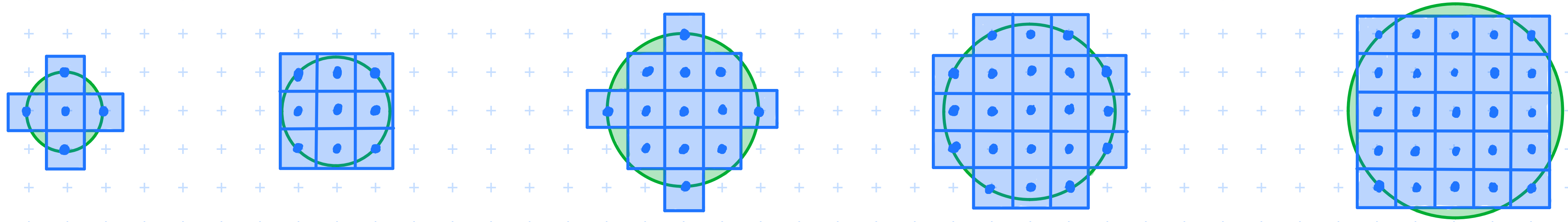
- is translation-invariant, i.e. independent of where the center is.



still has the same area!

The Strategy: Point-Counting

So we draw a circle of radius r on a grid or lattice, with the center on a point, and count the number of points $N(r)$ inside or on the circle:



$r=1$, $N(r)=5$, $\frac{N(r)}{r^2}=5$

$r=\sqrt{2}$, $N(r)=9$, $\frac{N(r)}{r^2}=4.5$

$r=2$, $N(r)=13$, $\frac{N(r)}{r^2}=3.25$

$r=\sqrt{5}$, $N(r)=21$, $\frac{N(r)}{r^2}=4.2$

$r=\sqrt{8}$, $N(r)=25$, $\frac{N(r)}{r^2}=3.125$

Each lattice point is the center of a unit square, so the number of points $N(r)$ can be viewed as an area estimate:

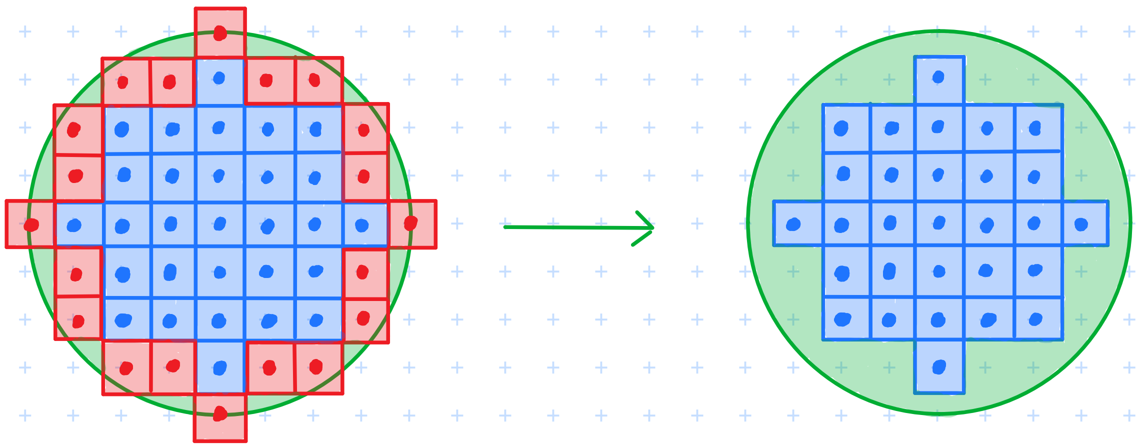
$$N(r) = A(r) + E(r)$$

for some error term $E(r)$.

Initial Error Bounds

Just by looking at the circle, it's not always obvious that $N(r)$ is an overestimate or underestimate.

But, for sure, if you remove all the "outermost points" you get an underestimate:

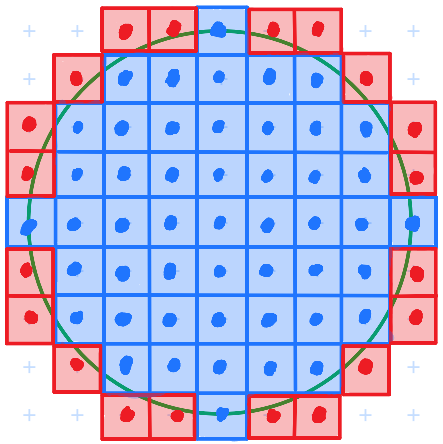


The removed points scale with the number of rows / columns and are thus at most some constant multiple of r:

$N(r) - c_1 r < A(r)$ for some absolute constant c_1.

Initial Error Bounds

In a similar vein, adding points around the perimeter will guarantee an overestimate:



The added points also scale with the number of rows / columns and are thus at most some constant multiple of r:

$N(r) + c_2 r > A(r)$ for some absolute constant c_2.

Initial Error Bounds

Putting it together, we get

$$-c_2 r < E(r) = N(r) - A(r) < c_1 r$$

$$|E(r)| < c_0(r) \quad \text{where } c_0 = \max\{c_1, c_2\}$$

That is, $|E(r)|$ is bounded above by a fixed constant times r.

To save us effort and notation, we write

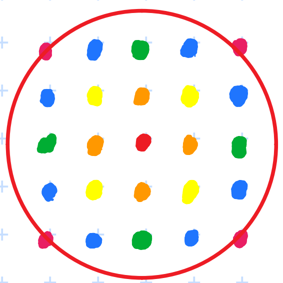
$$E(r) = O(r) \quad \text{or} \quad N(r) = A(r) + O(r).$$

Counting Points by Distance

To make point-counting easier, we assign a coordinate system.
Take the center to be $(0,0)$.

Let the distance between two neighboring lattice points be 1 unit.
In particular, all lattice points have integer coordinates.

By the distance formula, every lattice point in the circle is $\sqrt{n}$ units away from the center for some integer n.



Points by distance $\sqrt{n}$ from center

n	count
0	1
1	4
2	4
3	0
4	4

n	count
5	8
6	0
7	0
8	4

$$\text{Total} = 1+4+4+0+4+8+0+0+1$$
$$= 25 = N(\sqrt{8}).$$

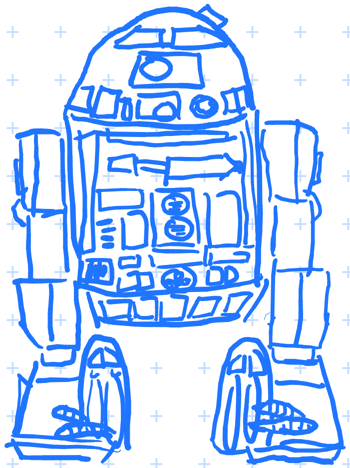
Counting Points by Distance

More precisely: A point (x,y) has distance D from the center center if and only if $x^2+y^2=D^2$.

Thus, the number of points distance $\sqrt{n}$ from the center is exactly the number of ordered pairs of integers (x,y) satisfying $x^2+y^2=n$.

We denote this number by $r_2(n)$.

Or, if you prefer, $r_2(D^2)$.



BEEP BEEP BOOP
WHIIIIIRRRR
BLEEP BLOOP

Counting Points by Distance

The points inside (or on) the circle are precisely the points (x,y) satisfying

$$\sqrt{x^2+y^2} \le r \quad \text{or} \quad x^2+y^2 \le r^2.$$

Thus, we have the formula

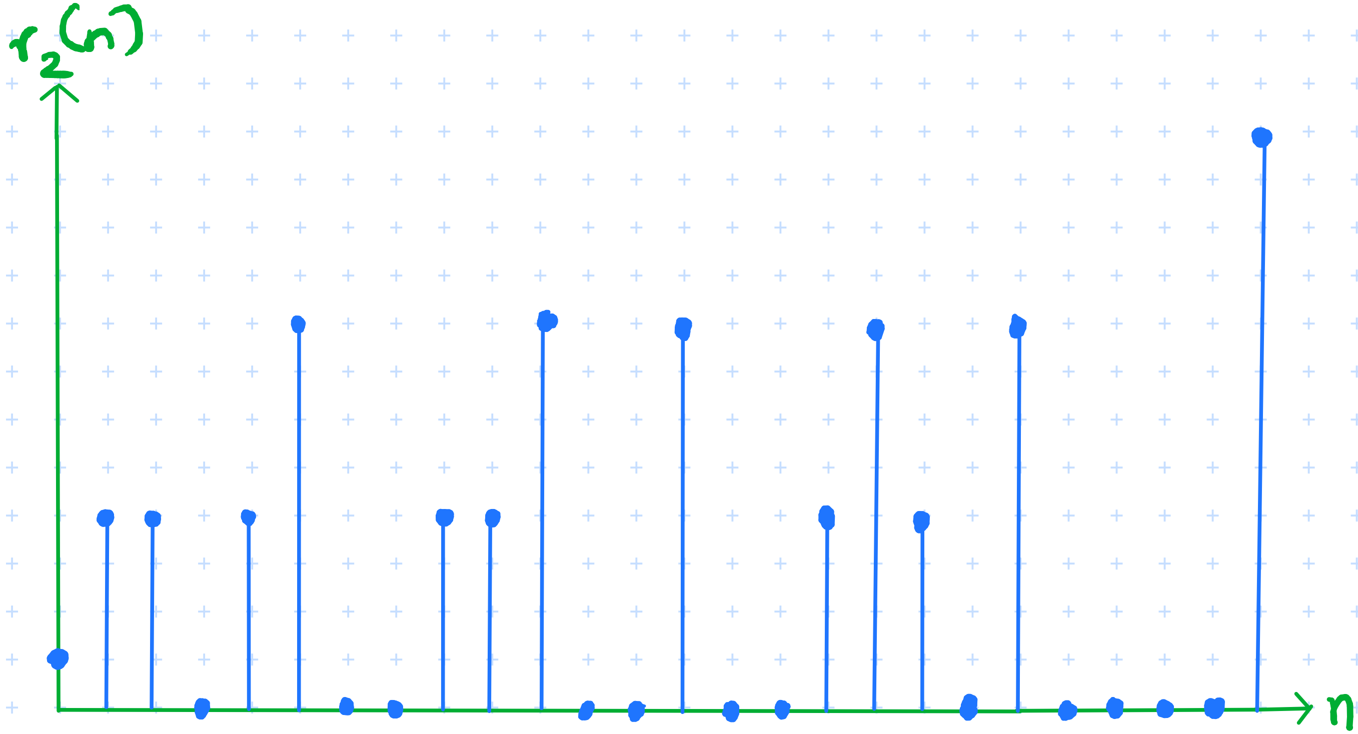
$$N(r) = \sum_{n=0}^{\lfloor r^2 \rfloor} r_2(n)$$

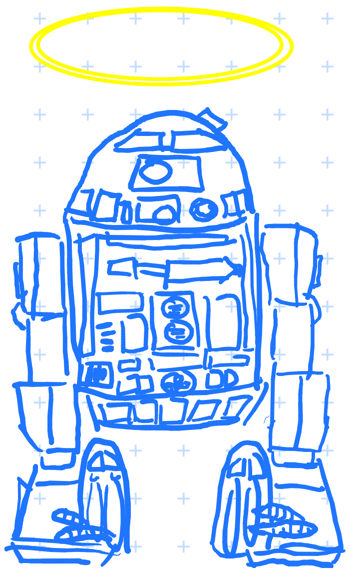
$$= 1 + \sum_{n=1}^{\lfloor r^2 \rfloor} r_2(n).$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x.

Counting Points by Distance

Unfortunately for us, r_2 is pretty badly behaved.
(No, not you, R2D2, you're good.)





We need a better strategy.

A Formula for $r_2(n)$

Fortunately for us, we have a useful formula from number theory.

Let $d_1(n)$ denote the number of divisors of d congruent to 1 mod 4.

Similarly, let $d_3(n)$ denote the number of divisors of d congruent to 3 mod 4.

We have, for $n > 0$,

$$r_2(n) = 4(d_1(n) - d_3(n)).$$

A Formula for $r_2(n)$

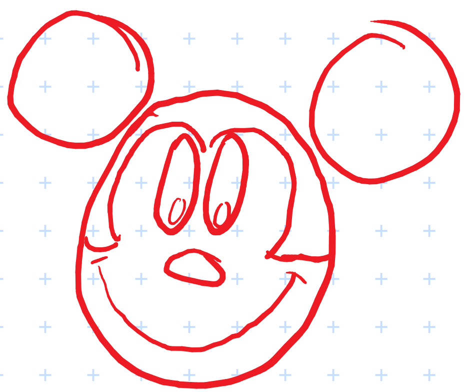
For example: $n = 25$

$$d_1(25) = 3 \quad (\{1, 5, 25\}) \ ; \quad d_3(25) = 0$$

$$r_2(25) = 4(3-0) = 12.$$

This doesn't look particularly usable at first glance.

IT'S A SURPRISE TOOL
THAT WILL HELP US LATER.



A Formula for $r_2(n)$

However, it lets us turn the sum around: instead of counting divisors for each n, it lets us count how many times some divisor d divides some $n \leq r^2$.

As it turns out, this is a lot simpler!

Any positive integer d appears as a divisor of some n from 1 to r^2 exactly $\left\lfloor \frac{r^2}{d} \right\rfloor$ times.

This is true even for $d > r^2$.

A Formula for $r_2(n)$

Thus, we have:

$$N(r) = 1 + 4\sum_{n=1}^{\lfloor r^2 \rfloor} \left(d_1(n) - d_3(n)\right)$$

$$= 1 + 4\left(\sum_{\substack{d \geq 1 \\ d \equiv 1 \pmod 4}} \left\lfloor \frac{r^2}{d} \right\rfloor - \sum_{\substack{d' \geq 1 \\ d' \equiv 3 \pmod 4}} \left\lfloor \frac{r^2}{d'} \right\rfloor\right)$$

$$= 1 + 4\sum_{k=0}^{\infty} \left(\left\lfloor \frac{r^2}{4k+1} \right\rfloor - \left\lfloor \frac{r^2}{4k+3} \right\rfloor\right)$$

$$= 1 + 4\sum_{j=0}^{\infty} (-1)^j \left\lfloor \frac{r^2}{2j+1} \right\rfloor.$$

Finishing Off The Sum

We now analyze the error generated by removing the floor brackets from

$$\sum_{j=0}^{\infty} (-1)^j \left\lfloor \frac{r^2}{2j+1} \right\rfloor = \sum_{j=0}^{\left\lfloor \frac{r+1}{2} \right\rfloor} (-1)^j \left\lfloor \frac{r^2}{2j+1} \right\rfloor + \sum_{j=\left\lfloor \frac{r+1}{2} \right\rfloor+1}^{\infty} (-1)^j \left\lfloor \frac{r^2}{2j+1} \right\rfloor$$

The first sum has $O(r)$ terms and removing the brackets from each term generates an error less than 1, so the total error generated is $O(r)$.

The second sum generates an error of $O(r)$ by the alternating series test.

Finishing Off The Sum

Thus, the total error generated is $O(r)$, i.e.,

$$\sum_{j=0}^{\infty} (-1)^j \left\lfloor \frac{r^2}{2j+1} \right\rfloor = \sum_{j=0}^{\infty} (-1)^j \frac{r^2}{2j+1} + O(r)$$

$$= r^2 \sum_{j=0}^{\infty} \frac{(-1)^j}{2j+1} + O(r)$$

$$= \frac{\pi}{4} r^2 + O(r)$$

by the Leibniz series for π.

Finishing Off The Sum

Returning to the original sum, we get

$$N(r) = 1 + 4\sum_{j=0}^{\infty} \left\lfloor \frac{j^2}{2j+1} \right\rfloor$$

$$= 1 + 4\left(\frac{\pi}{4}r^2 + O(r)\right)$$

$$= \pi r^2 + O(r).$$

By the squeeze theorem,

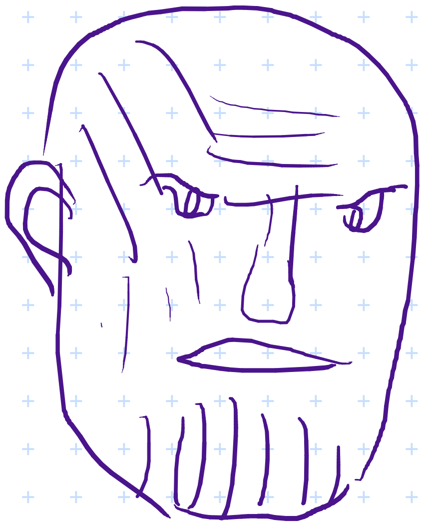
$$\lim_{r\to\infty} \frac{N(r)}{r^2} = \pi.$$

Finishing Off The Sum

But recall that $N(r) = Cr^2 + O(r)$ for some constant C, so that similarly

$$\lim_{r\to\infty} \frac{N(r)}{r^2} = C.$$

Thus, $C = \pi$, and so $A(r) = \pi r^2$.



ALL THAT FOR A DROP OF BLOOD...

Finishing Off The Sum

The value

$$\frac{N(r)}{r^2} = \frac{1}{r^2}\left(1 + \sum_{n=1}^{\lfloor r^2 \rfloor} r_2(n)\right)$$

is more or less the average value of r_2 on $[0, r^2]$.
In a sense, r_2 has an average value of π.
Averaging often makes badly behaved arithmetic functions easier to work with!

Of course, we want to know, where this magical formula for $r_2(n)$ came from.

One approach is to use tools from analysis...

Exponential Sums and Fourier Analysis

For f a 1-periodic function, (i.e. $f(x) = f(x+1)$) we define the nth Fourier coefficient $c_n(f)$ of f by

$$c_n(f) = \int_0^1 f(x)\, e^{-2\pi i n x}\, dx,$$

The series

$$\sum_{n \in \mathbb{Z}} c_n(f)\, e^{2\pi i n x}$$

is called the Fourier series of f.

Exponential Sums and Fourier Analysis

Let $f: \mathbb{R} \to \mathbb{C}$ be an integrable (usually in $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$). We define the Fourier transform $\hat{f}$ of f (alternatively $\mathcal{F}(f)$) by

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i \xi x} \, dx .$$

Under suitable conditions, we can recover f from $\hat{f}$ using the Fourier inversion formula:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} \, d\xi .$$

The above gives $\mathcal{F}^2(f)(x) = f(-x)$, so $\mathcal{F}^4(f)(x) = f(x)$. Fourier transform, indeed!

Exponential Sums and Fourier Analysis

Now we present (in my opinion) the most important theorem in analytic number theory, the Poisson summation formula:

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k).$$

Exponential Sums and Fourier Analysis

We also present the following fact: the Gaussian function $g(x) = e^{-\pi x^2}$ is its own Fourier transform:

$$\hat{g}(\xi) = e^{-\pi \xi^2}.$$

A corollary of this, using the appropriate substitution: If $g_t(x) = e^{-\pi x^2 t}$, then

$$\hat{g}_t(\xi) = \frac{1}{\sqrt{t}} e^{-\frac{\pi \xi^2}{t}}.$$

I CAN
FIX HIM

$e^{-\pi x^2}$

Automorphic / Modular Forms

Consider the action of $\mathbb{C}$ on itself by

$$c \cdot z = z + c$$ (not to be confused w/ scalar multiplication)

Let f be a 1-periodic function. Then f is invariant under the action of the discrete subgroup $\mathbb{Z}$ of $\mathbb{C}$:

$$f(n \cdot z) = f(z) \quad \text{for } n \in \mathbb{Z}$$

Automorphic / Modular Forms

The notion of automorphic forms generalizes this:
A "nice" function f is an automorphic form on a discrete group G if

$$f(g \cdot z) = j_g(z) f(z)$$

for some holomorphic function j_g, called the factor of automorphy.

Often we impose some "growth conditions" on these.

Automorphic / Modular Forms

Consider the action of $\Gamma := SL_2(\mathbb{Z})$, the multiplicative group of 2×2 integer matrices with determinant ± 1, on $\mathbb{H} = \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \}$, the complex upper-half plane, defined by:

$$\text{If } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma, \text{ then } \gamma z = \frac{az+b}{cz+d}.$$

Note that $(-\gamma)z = \gamma z$ for all γ and all z, so we can view this as an action of the quotient group

$$PSL_2(\mathbb{Z}) = \Gamma / \{\pm I\}$$

Automorphic / Modular Forms

A modular form is an automorphic form on Γ (or some subgroup thereof):

(i) holomorphic on $\mathbb{H}$;

(ii) with automorphic factor $j_\gamma(z) = (cz+d)^k$ for some positive integer k; that is, for $\gamma \in \Gamma$,

$$f(\gamma z) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z);$$

The number k is said to be the weight of the modular form f.

Automorphic / Modular Forms

(iii) is "holomorphic at the cusp", i.e., bounded as $z \to i\infty$.

If $f(z) \to 0$ as $z \to i\infty$, we refer to f as a cusp form.

Automorphic / Modular Forms

Note that:

- the set $M_k(\Gamma)$ of modular forms of weight k on Γ is a $\mathbb{C}$-vector space.

- If f is a modular form of weight k and g is a modular form of weight ℓ, then their product fg is a modular form of weight $k+\ell$. Thus, the ring

$$M(\Gamma) = \bigoplus_{k \geq 0} M_k(\Gamma)$$

is in fact a graded ring. In fact, it is a graded Lie algebra with the bracket

$$[f, g] = kfg' - \ell f'g.$$

Automorphic / Modular Forms

Now, the image of Γ in $PSL(2,\mathbb{Z})$ is generated by $T=\begin{pmatrix}1&1\\0&1\end{pmatrix}$ & $S=\begin{pmatrix}0&-1\\1&0\end{pmatrix}$, so to check that f is automorphic it suffices to verify

$$f(Tz)=f(z+1)=f(z) \quad \text{and}$$

$$f(Sz)=f\left(-\frac{1}{z}\right)=z^k f(z).$$

In particular, f is 1-periodic, and thus has a Fourier series.

If f is a cusp form, its Fourier series has constant term 0.

Automorphic / Modular Forms

Also, take $\gamma = -I = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$; if f is a modular form of weight k then

$$f(\gamma z) = f(z) = (-1)^k f(z).$$

This tells us that there are no nontrivial modular forms on Γ for odd k.

Automorphic / Modular Forms

Example: Eisenstein series.

For k a positive integer, consider the function

$$G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(m+n\tau)^{2k}}$$

For $k \geq 2$ this series converges absolutely; we can show that G_{2k} is a modular form on Γ of weight $2k$.

Unfortunately, this fails at $k=1$ due to conditional convergence issues!

Automorphic / Modular Forms

For $k \geq 2$, G_{2k} has Fourier series

$$G_{2k}(\tau) = 2\zeta(2k)\left(1 + c_{2k}\sum_{n=1}^{\infty} \sigma_{2k-1}(n)q^n\right)$$

where:

ζ is the Riemann zeta $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$

$\sigma_m(n) = \sum_{d|n} d^m$ is the divisor power sum function

$q = e^{2\pi i\tau}$

c_{2k} is a constant depending on k.

Automorphic / Modular Forms

It is usual to normalize G_{2k} so that the constant term is equal to 1:

$$E_{2k} := \frac{G_{2k}}{2\zeta(2k)}$$

Some Fourier series:

$$E_4 = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n) q^n$$

$$E_6 = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n$$

$$E_8 = 1 + 480 \sum_{n=1}^{\infty} \sigma_7(n) q^n$$

Automorphic / Modular Forms

Unfortunately for us, G_2 is not absolutely convergent, and fails the automorphy condition.

However, G_2 is still periodic, and so is its normalization $E_2 = \dfrac{G_2}{2\zeta(2)}$; it has Fourier series

$$E_2 = 1 + 24 \sum_{n=1}^{\infty} \sigma(n)$$

Automorphic / Modular Forms

As it turns out:

For each integer k, the space $M_{2k}(\Gamma)$ is a finite-dimensional $\mathbb{C}$-vector space, with a basis consisting of products of E_4 & E_6.

For instance, we have: $E_8 = E_4^2$, $E_{14} = E_8 E_6 = E_4^2 E_6$

The nontrivial cusp form of minimal weight,

$$\Delta(z) = q \prod_{n=1}^{\infty} (1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n) q^n$$

where τ is the Ramanujan tau function:

$$\Delta = E_4^3 - E_6^2 .$$

Congruence Subgroups

A congruence subgroup of $SL_2(\mathbb{Z})$ is a subgroup of finite index defined by some congruence on the entries.

Example: Let $N > 1$ be an integer. The principal congruence subgroup $\Gamma(N)$ is the kernel of the reduction mod N homomorphism from $SL_2(\mathbb{Z})$ to $SL(\mathbb{Z}/N\mathbb{Z})$. That is,

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : \begin{array}{l} a \equiv d \equiv 1 \pmod N \\ b \equiv c \equiv 0 \pmod N \end{array} \right\}$$

Congruence Subgroups

Its index in Γ is given by

$$[\Gamma : \Gamma(N)] = N^3 \prod_{p \mid N} \left(1 - \frac{1}{p^2}\right).$$

A congruence subgrou of Γ is a subgroup Γ' containing $\Gamma(N)$ for some N; the least N is called the level of Γ'.

GET ON MY LEVEL! — $\begin{pmatrix} 1 & 0 \\ 99 & 1 \end{pmatrix}$

Congruence Subgroups

The family of congruence subgroups we care about:

For $N > 1$, define the congruence subgroup $\Gamma_0(N)$ by

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma : c \equiv 0 \pmod{N} \right\}$$

Clearly, it is a congruence subgroup of level N, with index

$$[\Gamma : \Gamma_0(N)] = N \cdot \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

Congruence Subgroups

Recall: $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ generate the image of Γ in $PSL(2, \mathbb{Z})$.

Similarly, for $N > 1$, T and $S_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$

generate the image of $\Gamma_0(N)$ in $PSL(2, \mathbb{R})$.

Dirichlet Characters and Twisted Eisenstein Series

Let $N \geq 1$ be an integer. A Dirichlet character mod N is a homomorphism $\chi: (\mathbb{Z}/n\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, extended to a map $\chi: \mathbb{Z} \to \mathbb{C}$ by setting

$$\chi(n) = \begin{cases} \chi(n \bmod N) & \text{if } (n,N)=1 \\ 0 & \text{otherwise.} \end{cases}$$

The trivial character mod N:

$$\chi(n) = \begin{cases} 1 & \text{if } (n,N)=1 \\ 0 & \text{otherwise.} \end{cases}$$

Dirichlet Characters and Twisted Eisenstein Series

For a Dirichlet character χ we define the Dirichlet L-function

$$L(\chi, s) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}$$

extended by analytic continuation.

Dirichlet Characters and Twisted Eisenstein Series

Nontrivial & relevant example: If $N=4$, take

$$\chi_{-4}(n) = \begin{cases} (-1)^{\frac{n-1}{2}} & \text{if } n \text{ even} \\ 0 & \text{if } n \text{ odd} \end{cases}$$

As seen earlier,

$$L(\chi_{-4}, 1) = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4},$$

Dirichlet Characters and Twisted Eisenstein Series

If χ is a nontrivial character mod N and $\chi(-1) = (-1)^k$, the Eisenstein series $G_{k,\chi}$ given by the Fourier series

$$G_{k,\chi}(\tau) = \frac{1}{2} L(1-k, \chi) + \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \chi(d) d^{k-1} \right) q^n$$

is a modular form of weight k on $\Gamma_0(N)$, with

$$G_{k,\chi}(\tau) = \chi(a)(cz+d)^k G_{k,\chi}(z).$$

Putting It Together: The Jacobi Theta Function

Define the function $\theta : \mathbb{H} \longrightarrow \mathbb{C}$ by

$$\theta(\tau) = \sum_{n \in \mathbb{Z}} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2}.$$

This is often referred to as the Jacobi theta function.

It is not hard to show that

$$\theta(\tau)^2 = \sum_{n \in \mathbb{Z}} r_2(n) q^n.$$

Putting It Together: The Jacobi Theta Function

Now, clearly $\theta(\tau+1) = \theta(\tau)$ and

$$\theta\left(-\frac{1}{4\tau}\right) = \sqrt{\frac{2\tau}{i}}\,\theta(\tau)$$

by using Poisson summation and the Fourier transform of the Gaussian.

This implies that $\theta(\tau)^2$ is a weight 1 modular form on $\Gamma_0(4)$.

Putting It Together: The Jacobi Theta Function

As it turns out, $M_1(\Gamma_0(4))$ is a 1-dimensional $\mathbb{C}$-vector space, spanned by

$$G_{1,\chi_{-4}}(\tau) = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\sum_{d \mid n} \chi_{-4}(d) \right) q^n$$

and so $\Theta(t)^2 = C \cdot G_{1,\chi_{-4}}(\tau)$ for some C.

Putting It Together: The Jacobi Theta Function

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and so $\Theta(t)^2 = C \cdot G_{1,\chi_{-4}}(\tau)$ for some C.

Putting It Together: The Jacobi Theta Function

By equating constant terms, $C = 4$.

Thus, $\theta(\tau)^2 = 4\, G_{1,\chi_{-4}}(\tau)$, and equating Fourier coefficients gives

$$r_2(n) = 4 \sum_{d \mid n} \chi(d)$$

$$= 4\left(\sum_{\substack{d \mid n \\ d \equiv 1 \pmod 4}} 1 - \sum_{\substack{d' \mid n \\ d' \equiv 3 \pmod 4}} 1 \right)$$

$$= 4\left(d_1(n) - d_3(n)\right)$$

as claimed. ∎

The Error Term

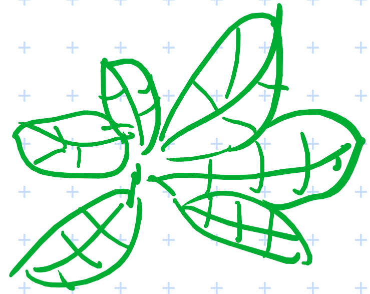
So we have an exact formula for $N(r)$, and we've shown in two different ways that

$$N(r) = \pi r^2 + O(r).$$

Can we do better?

Sure we can, by using similar tools, albeit more advanced and in a more careful manner.

Bessel Functions (of the first kind)



BASIL FUNCTION

For $\alpha \in \mathbb{C}$ (usually positive integer / half integer) $J_\alpha(x)$ is given by

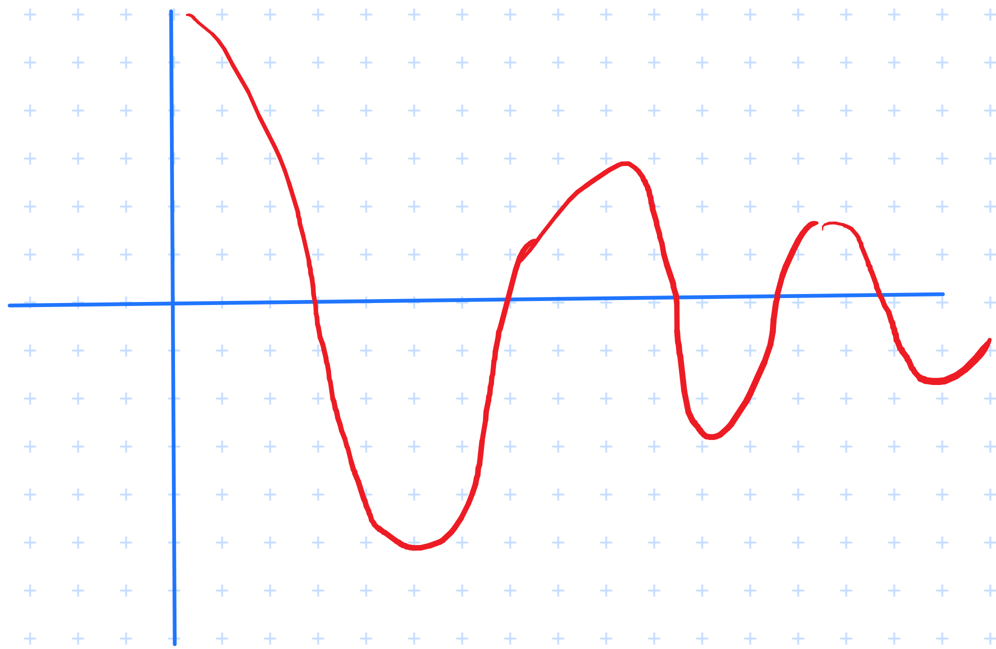
$$J_\alpha(x) = \sum_{m=1}^{\infty} \frac{(-1)^m}{m!\,\Gamma(m+\alpha+1)} \left(\frac{x}{2}\right)^{2m+\alpha},$$

They are solutions to Bessel's equation

$$x^2 \frac{d^2 y}{dx^2} + x\frac{dy}{dx} + (x^2 - \alpha^2) y = 0.$$

Bessel Functions (of the first kind)

They're quite like exponential / sine functions!
For example, $J_0(x)$ looks like a damped cosine wave.



Bessel Functions (of the first kind)

Just like exponentials, Bessel functions have a knack for showing up in things where there is radial symmetry. For example, they appear in solutions to Laplace's equation

$$\nabla^2 f = \nabla \cdot \nabla f = 0.$$

in cylindrical coordinates.

This equation shows up as an equidistribution condition in the general sum of squares problem.

Bessel Functions (of the first kind)

To continue the analogy with exponential sums, we have given α, ξ, and a function f, the Hankel transform

$$F_\alpha(\xi) = \int_0^\infty f(x) J_\alpha(\xi x)\, dx$$

and an analogous inversion formula

$$f(x) = \int_0^\infty F_\alpha(\xi) J_\alpha(\xi x)\, d\xi .$$

Two-ier transform!

Bessel Functions (of the first kind)

In analogy, we have the Voronoï summation formula for what are known as Maass forms, basically modular forms but with the holomorphicity requirement relaxed:

$$\sum_{a \le n \le b}{}' r_2(n) f(n) = \sum_{n=0}^{\infty}{}' r_2(n) \int_a^b f(x)\, \pi J_0(2\pi\sqrt{nx})\, dx.$$

Bessel Functions (of the first kind)

Voronoï found a similar formula for the divisor function, and Sierpiński used it to show that the error term in the circle problem is $O(n^{2/3})$.

In the opposite direction, Hardy & Landau independently showed that the error term is not $o(n^{1/2})$:

$$\lim_{r\to\infty} \frac{E(r)}{r^{1/2}} \neq 0.$$

Bessel Functions (of the first kind)

The conjecture: $E(r) = O(r^{1/2+\varepsilon})$ for all $\varepsilon > 0$.

Attack: use the asymptotic

$$J_\alpha(x) = \left(\frac{2}{\pi x}\right)^{1/2} \cos\left(x - \frac{\alpha}{2}\pi - \frac{\pi}{4}\right) + O\left(x^{-3/2}\right).$$

Current record:

$$E(r) = O\left(r^{131/208}\right)$$

(Huxley, 2003).