Ehrhart Theory

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Pick's Theorem

A lattice polygon is a polygon whose vertices are lattice points, i.e., points with integer coordinates.

Counting lattice points in a lattice polygon gives you a way to find its area!

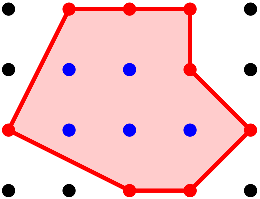
Theorem (Pick 1899)

Suppose $\mathcal{P} \subseteq \mathbf{R}^2$ is a lattice polygon with B lattice points on its boundary and I lattice points in its interior. Then its area A is given by

$$A = I + \frac{1}{2}B - 1.$$

Pick's Theorem

Example:

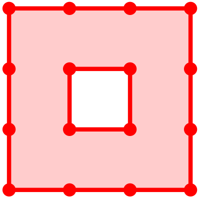


We have

$$A = 5 + \frac{1}{2} \cdot 8 - 1 = 8.$$

Pick's Theorem

As it turns out, Pick's theorem isn't *quite* right as stated.



For the rest of the talk, we'll assume convexity to get rid of counterexamples like this.

Anyway. What can be said in higher dimensions?

An aside on polytopes

Loosely speaking, a **polytope** is a generalization of a polygon to higher dimensions.

An **integer polytope** is the convex hull of a finite set of integer points.

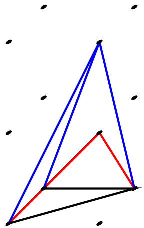
Counting lattice points in integer polytopes is a problem that shows up in many areas of mathematics:

- Combinatorics
- Geometry
- Number theory
- Linear programming
- Coding theory

Back to Pick's Theorem

Let $r \in \mathbf{Z}^+$. Consider the Reeve tetrahedron $\mathcal{R}_r \subseteq \mathbf{R}^3$, the integer polytope with vertices $(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, r)$.

Then $\mathcal{R}_r$ in fact only contains these four lattice points.



However, $\mathcal{R}_r$ has volume $\frac{1}{6}r$.

So it doesn't seem that Pick's theorem generalizes to higher dimensions.

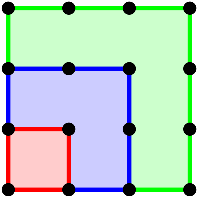
The Ehrhart polynomial

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be a convex polytope. For $t \in \mathbf{Z}_{\geq 0}$, we write $t\mathcal{P}$ for the polytope obtained when $\mathcal{P}$ is dilated by a factor of t.

We write $L_{\mathcal{P}}(t)$ for the number of lattice points contained in $t\mathcal{P}$.

Example: Let $\mathcal{Q}_d \subseteq \mathbf{R}^d$ be the d-dimensional unit hypercube, i.e., the convex hull of the 2^d points of the form $(\epsilon_1, \ldots, \epsilon_d)$ with $\epsilon_i \in \{0, 1\}$. In other words, $\mathcal{Q}_d = [0, 1]^d$.

We have $L_{\mathcal{Q}_d}(t) = (t + 1)^d$.

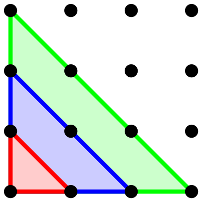


The Ehrhart polynomial

Example: Let $\Delta_d \subseteq \mathbf{R}^d$ be the d-dimensional standard simplex, i.e., the integer polytope whose vertices are $\vec{0}$ and the points $\vec{e_i} = (0, \ldots, 1, \ldots, 0)$ where all coordinates are 0 except for 1 in the ith position.

Another characterization: it is the set of all points $(x_1, \ldots, x_d)$ with $x_i \geq 0$ for all i and $x_1 + \cdots + x_d \leq 1$.

We have $L_{\Delta_d}(t) = \binom{t+d}{d} = \frac{1}{d!} t(t-1)\ldots(t-d+1)$.



The Ehrhart polynomial

In both examples, $L_{\mathcal{P}}(t)$ was a polynomial of degree d. As it turns out, this is indeed the case in general!

Theorem (Ehrhart 1962)

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope of full dimension. Then for all $t \in \mathbf{Z}_{\geq 0}$, $L_{\mathcal{P}}(t)$ agrees with a polynomial in t of degree d.

We thus refer to $L_{\mathcal{P}}(t)$ as the **Ehrhart polynomial** of $\mathcal{P}$.

The Ehrhart polynomial

Suppose $\mathcal{P}$ is an integer polytope with Ehrhart polynomial $L_{\mathcal{P}}(t) = \sum_{i=0}^{d} a_d t^d$. Then we know the following about its coefficients:

- The leading coefficient a_d gives the d-dimensional volume (Lebesgue measure) of $\mathcal{P}$.
- The second coefficient a_{d-1} is half the weighted sum of the $(d-1)$-dimensional volumes of the faces of $\mathcal{P}$.
- For the constant term a_0, we have $L_{\mathcal{P}}(0) = a_0 = 1$.

The Ehrhart polynomial

Recall: $L_{\mathcal{Q}_d}(t) = (t+1)^d$.

Now, if we compute the Ehrhart polynomial $L_{\mathcal{Q}_d^\circ}$ of the *interior* $\mathcal{Q}_d^\circ$ of $\mathcal{Q}_d$, we have

$$\begin{aligned} L_{\mathcal{Q}_d^\circ}(t) &= (t-1)^d \\ &= (-1)^d(-t+1)^d \\ &= (-1)^d L_{\mathcal{Q}_d}(-t). \end{aligned}$$

The Ehrhart polynomial

As it turns out, this is true in general!

Theorem (Ehrhart 1962, Macdonald 1971)

(Ehrhart-Macdonald reciprocity) Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope, and let $\mathcal{P}^\circ$ be its interior. Then we have

$$L_{\mathcal{P}^\circ}(t) = (-1)^d L_{\mathcal{P}}(-t).$$

The Ehrhart polynomial

Let $\mathcal{P} \subseteq \mathbf{R}^2$ be an integer polygon with I interior points and B boundary points. We have

$$L_{\mathcal{P}}(1) = I + B$$
$$L_{\mathcal{P}}(0) = 1$$

and by Ehrhart-Macdonald reciprocity, we also have

$$L_{\mathcal{P}}(-1) = I.$$

These three values uniquely determine $L_{\mathcal{P}}(t)$; we obtain

$$L_{\mathcal{P}}(t) = \left(I + \frac{1}{2}B - 1\right) t^2 + \frac{1}{2}Bt + 1.$$

This recovers Pick's theorem!

The Ehrhart polynomial

Let $\mathcal{R}_r \subseteq \mathbf{R}^3$ be the Reeve tetrahedron from earlier. We have

$$\begin{aligned} L_{\mathcal{R}_r}(1) &= 4 \\ L_{\mathcal{R}_r}(0) &= 1 \\ L_{\mathcal{R}_r}(-1) &= 0 \end{aligned}$$

So if we compute $L_{\mathcal{R}_r}(t)$ at one more value of t, we can deduce $L_{\mathcal{R}_r}(t)$, and thus its volume!

For convenience, $L_{\mathcal{R}_r}(2)$ is probably the best choice.

The Ehrhart polynomial

By the way, this method works for any 3D polyhedron–not just the Reeve tetrahedra.

And this works in higher dimensions, too!

Theorem (generalization of Reeve 1957)

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope. Then there exists a formula for the volume of the polytope $\mathcal{P}$ in terms of $(d+1)$ distinct values of $L_{\mathcal{P}}(t)$.

Ehrhart series

As usual, let $\mathcal{P} \subseteq \mathbf{R}^d$ be a convex polytope. We define the **Ehrhart series** of $\mathcal{P}$ by

$$\mathrm{Ehr}_{\mathcal{P}}(z) := \sum_{t=0}^{\infty} L_{\mathcal{P}}(t) z^t.$$

In other words, the Ehrhart series of $\mathcal{P}$ is the generating function of $L_{\mathcal{P}}(t)$.

Ehrhart series

Theorem (Ehrhart 1962)

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope. Then $\mathrm{Ehr}_{\mathcal{P}}(z)$ is a rational function of z. In particular, we have

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1-z)^{d+1}}$$

where $h_1(z) \in \mathbf{C}[z]$ is a polynomial of degree at most d, and with $h(1) \neq 0$.

Ehrhart series

The Ehrhart series of the unit square $\mathcal{Q}_2$ is given by

$$\begin{aligned}\mathrm{Ehr}_{\mathcal{Q}_2}(z) &= \sum_{t=0}^{\infty}(t+1)^2 z^t \\ &= \frac{1+z}{(1-z)^3}.\end{aligned}$$

Ehrhart series

For the Ehrhart series of the standard simplex Δ_d, we have

$$\begin{aligned}\mathrm{Ehr}_{\Delta_d}(z) &= \sum_{t=0}^{\infty} \binom{t+d}{d} z^t \\ &= \frac{1}{(1-z)^{d+1}}.\end{aligned}$$

Ehrhart series

As it turns out, the coefficients of the numerator $h(z)$ are all non-negative integers!

Theorem (Stanley 1980)

(Stanley's non-negativity theorem) Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope with Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1-z)^{d+1}},$$

with $h(z) = \sum_{j=0}^{d} h_j^\star z^j$. Then $h_j^\star \in \mathbf{Z}_{\geq 0}$ for each j.

What more can be said about the coefficients of h?

Reflexive polytopes

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope. We define its **(polar) dual** $\mathcal{P}^\star \subseteq \mathbf{R}^d$ by

$$\mathcal{P}^\star := \{\vec{y} \in \mathbf{R}^d : \vec{x} \cdot \vec{y} \leq 1 \text{ for all } \vec{x} \in \mathcal{P}\}.$$

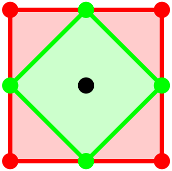
If $\mathcal{P}^\star$ is also an integer polytope, then $\mathcal{P}$ is said to be **reflexive**.

Reflexive polytopes

Example: $\mathcal{P} \subseteq \mathbf{R}^2$, $\mathcal{P} = \mathrm{conv}\{(\pm 1, \pm 1)\}$.

Then $\mathcal{P}^\star = \mathrm{conv}\{(0, \pm 1), (\pm 1, 0)\}$ and $(\mathcal{P}^\star)^\star = \mathcal{P}$.

So both $\mathcal{P}$ and $\mathcal{P}^\star$ are reflexive.



Theorem (Lagarias-Ziegler 1991)

Fix d. There are only finitely many reflexive polytopes in $\mathbf{R}^d$ (up to equivalence).

Reflexive polytopes

Now we compute the Ehrhart series of $\mathcal{P}$ and $\mathcal{P}^\star$.

For the Ehrhart polynomials, we have

$$L_{\mathcal{P}}(t) = (2t+1)^2$$
$$L_{\mathcal{P}^\star}(t) = 2t^2 + 2t + 1$$

and so we get for the Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \sum_{t=0}^{\infty} (2t+1)^2 z^t = \frac{z^2+6z+1}{(1-z)^3}$$
$$\mathrm{Ehr}_{\mathcal{P}^\star}(z) = \sum_{t=0}^{\infty} (2t^2+2t+1) z^t = \frac{z^2+2z+1}{(1-z)^3}.$$

Reflexive polytopes

Notice the numerators $h(z)$ are palindromic and of largest possible degree.

As it turns out, this is true in general.

Theorem (Hibi 1990)

(Hibi palindromicity) Let $\mathcal{P} \subseteq \mathbf{R}^d$ with $\vec{0} \in \mathcal{P}^\circ$ have Ehrhart series

$$\mathrm{Ehr}_{\mathcal{P}}(z) = \frac{h(z)}{(1-z)^{d+1}}$$

with $h(z) = \sum_{j=0}^{d} h_j^\star z^j$. Then $\mathcal{P}$ is reflexive if and only if $h_j^\star = h_{d-j}^\star$ for all $0 \leq j \leq d/2$.

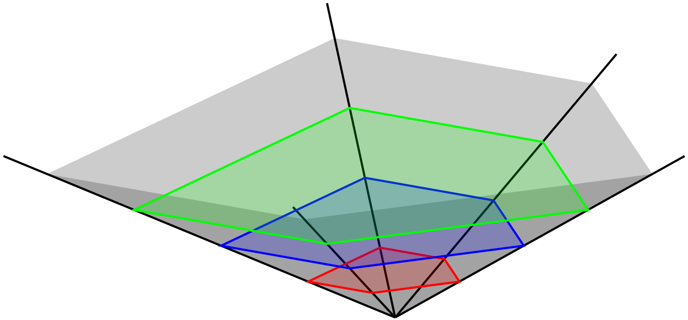
A view from commutative algebra

Let $\mathcal{P} \subseteq \mathbf{R}^d$ be an integer polytope with vertices $\vec{v}_1, \ldots, \vec{v}_n$. For each i, we define $\vec{w}_i \in \mathbf{R}^{d+1}$ by $\vec{w}_i = (\vec{v}_i, 1)$.

We then define the **cone** over $\mathcal{P}$ as

$$\text{cone}(\mathcal{P}) := \left\{ \sum_{i=1}^{n} \lambda_i \vec{w}_i : \lambda_i \geq 0 \text{ for all } i \right\}$$

A view from commutative algebra



A view from commutative algebra

Now we define the **Ehrhart algebra** $A_{\mathcal{P}}$ of $\mathcal{P}$ as the $\mathbf{C}$-algebra generated by monomials of the form $\vec{z}^{\vec{m}} = z_1^{m_1} \dots z_{d+1}^{m_{d+1}}$ where $\vec{m} \in \text{cone}(\mathcal{P} \cap \mathbf{Z}^{d+1})$.

Viewed in this way, the Hilbert function and Hilbert series of $A_{\mathcal{P}}$ are precisely the Ehrhart polynomial and Ehrhart series of $\mathcal{P}$!

A view from commutative algebra

One can deduce Ehrhart's theorem from the fact that the Ehrhart algebra $A_{\mathcal{P}}$ of an integer polytope $\mathcal{P}$ is in fact what is known as a Cohen-Macaulay algebra.

Similarly, Stanley's non-negativity theorem also follows from the fact that $A_{\mathcal{P}}$ is Cohen-Macaulay.

In a sense, Ehrhart theory is the study of certain Cohen-Macaulay algebras.

A view from algebraic geometry

A geometric perspective: each polytope has a projective toric variety associated with it. Examining these toric varieties is one way to study these Ehrhart polynomials and Ehrhart series.

In particular, studying the Todd classes of these toric varieties is one way to study the coefficients of the Ehrhart polynomial. Tools such as the Grothendieck-Riemann-Roch theorem and Fourier analysis have been used.

Other (probably) open questions and directions

What can be said about the roots of the Ehrhart polynomial?

When do the coefficients of $h(z)$, the numerator of $\mathrm{Ehr}_{\mathcal{P}}(z)$, form a unimodal sequence? Known for many "nice" families of polytopes, but still very much open.

How about Ehrhart functions of rational polytopes? The Ehrhart functions are quasipolynomial instead of polynomial, but the Ehrhart series are still rational, and generalizations of Ehrhart reciprocity and others do hold.

End

THANK YOU!

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