On odd near-perfect and deficient-perfect numbers with k distinct prime divisors

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The Search for Odd Perfect Numbers

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2. Odd perfect numbers, however, have proven to be more elusive.

The Search for Odd Perfect Numbers

Theorem (Ochem and Rao, 2012)

There are no odd perfect numbers less than 10^{1500}.

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Theorem (Ochem and Rao, 2012)

Any odd perfect number must have at least 101 prime factors in total (counting multiplicities).

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Remark

This implies, in particular, that an odd near-perfect number with k distinct prime factors is less than 2^{4^k}.

The Search for Odd Perfect Numbers

Theorem (Pollack, 2011)

There are at most 4^{k^2} odd perfect numbers with k distinct prime factors.

Near-Perfect Numbers

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- d is known as the **redundant divisor** of n.

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- 12 is a near-perfect number because $12 = 1 + 2 + 3 + 6$, and $1, 2, 3, 4, 6$ are the proper divisors of 12. In this case, 4 is the redundant divisor.

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See sequence A181595 of the OEIS for more discussion.

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See sequence A271816 of the OEIS for more discussion.

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- Like with perfect numbers, odd near-perfect and deficient-perfect numbers are much rarer.
- Tang, Ren, Li showed that any odd near-perfect number must have at least four odd distinct prime factors.
- Tang and Feng showed the same for any odd deficient-perfect number.

Odd Near-Perfect and Deficient-Perfect Numbers

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- In 2012, Donovan Johnson found the smallest odd near-perfect number: $173369889 = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$.
- In February 2016, the authors found the smallest deficient-perfect number: $9018009 = 3^2 \cdot 7^2 \cdot 11^2 \cdot 13^2$.

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1. *The set*

$$\{\sigma_{-1}(p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}) \geq M : r_1, r_2, \ldots, r_k \in 2\mathbb{N}\},$$

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ii. *The set*

$$\{\sigma_{-1}(p_1^{r_1} p_2^{r_2} \ldots p_k^{r_k}) \leq M : r_1, r_2, \ldots, r_k \in 2\mathbb{N} \cup \{\infty\}\},$$

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2. In the near-perfect case, this inductive proof can be adapted into a recursive algorithm.
3. In particular, given a set of odd primes $p_1, p_2, \ldots, p_k$, by using i. and setting $M = 2$, we can find a minimal lower bound $M_k > 2$ for $\sigma_{-1}(n)$ over all odd abundant squares n with prime factors $p_1, p_2, \ldots, p_k$. Since no odd perfect squares can be perfect, we must have $M_k > 2$.

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2. For each eligible r_1, calculate m_{r_1}. Stop as soon as $m_{r_1} = m_\infty$.
3. Find the minimum of all computed values of $\sigma_{-1}(p_1^{r_1})m_{r_1}$. This is the desired minimal element.

A Lemma

1. Recall: if n is near-perfect, we have $\sigma_{-1}(n) = \frac{2\frac{n}{d}+1}{\frac{n}{d}}$. By noting that $\frac{n}{d}$ must have all its prime factors in $\{p_1, p_2, \ldots, p_k\}$, we can limit the number of cases we need to check.

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2. Thus, if $\sigma_{-1}(n) \geq M_k$, we have $\dfrac{n}{d} \leq \dfrac{1}{M_k - 2}$.
3. Moreover, we have, from Nielsen's earlier result,

$$n < \left(1 + \frac{n}{d}\right)^{4^k}.$$

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2. Thus, if $\sigma_{-1}(n) \geq M_k$, we have $\dfrac{n}{d} \leq \dfrac{1}{M_k - 2}$.
3. Moreover, we have, from Nielsen's earlier result,
$$n < \left(1 + \frac{n}{d}\right)^{4^k}.$$
4. These and usual number theoretic arguments allow us to check for near-perfect numbers with a given set of prime factors.

Main Result

In fact, using this lemma and Nielsen's result, we obtain:

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Remark

In a way, this is analogous to Dickson's result for odd perfect numbers.

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- Note that $3 \mid n$; otherwise, n is deficient.
- Similarly, at least one of 5 and 7 divides n.

Odd Near-Perfect Numbers of the Form $3^w 7^x p^y q^z$

- If $n = 3^w 7^x p^y q^z$ is abundant, then we need $\frac{3}{2} \cdot \frac{7}{6} \cdot \frac{p}{p-1} \cdot \frac{q}{q-1} > 2$. This rearranges to $(p-8)(q-8) < 56$.

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- Therefore, $(p, q) \in \{(11, 13), (11, 17), (11, 19), (11, 23), (13, 17), (13, 19)\}$.

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- Therefore, $(p, q) \in \{(11, 13), (11, 17), (11, 19), (11, 23), (13, 17), (13, 19)\}$.
- Individually checking each case with computer assistance yields one solution: $n = 3^4 \cdot 7^2 \cdot 11^2 \cdot 19^2$, as found by Donovan Johnson in 2012.

Odd Near-Perfect Numbers of the Form $3^w 5^x p^y q^z$

- If $n = 3^w 5^x p^y q^z$ is abundant, then we need $\frac{3}{2} \cdot \frac{5}{4} \cdot \frac{p}{p-1} \cdot \frac{q}{q-1} > 2$. This rearranges to $(p-16)(q-16) < 240$.

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- Therefore, $p \leq 31$.
  - If $p = 17$, then $q \leq 251$.
  - If $p = 19$, then $q \leq 89$.
  - If $p = 23$, then $q \leq 47$.
  - If $p = 29$, then $q = 31$.
- If $p = 7, 11, 13$, there are no restrictions on q. We need more.

Odd Near-Perfect Numbers of the Form $3^w 5^x 7^y q^z$

- If $n = 3^w 5^x 7^y q^z$ is near-perfect, then unconditionally on q, we get that $\sigma_{-1}(n) > \sigma_{-1}(3^2 5^2 7^2) > 2 + \frac{1}{12}$ and so $\frac{n}{d} < 12$.

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- We note that $7 \mid \sigma(n)$ and so $q \neq 13$ (since $\mathrm{ord}_7(13) = 2$).
- We now consider two cases: $w = 2$ and $w \geq 4$.
- $w = 2$ is impossible, since we need $13 \mid 2\frac{n}{d} + 1$ but $\frac{n}{d} < 12$ and is odd.

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- If $w \geq 4$, then we have $\frac{n}{d} \in \{3, 5\}$.
- This means that $v_3(\sigma(n)) \geq 3$. However, we cannot have $v_3(\sigma(7^y)) \geq 2$; or else we must have $37 \cdot 1063 \mid \sigma(n)$ which is not possible. Hence, $q \equiv 1 \pmod 3$ and $q \neq 11$.

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- We also need $5 \mid \sigma(q^z)$ and $7 \mid \sigma(q^z)$; this implies that $q \geq 151$.

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- We also need $5 \mid \sigma(q^z)$ and $7 \mid \sigma(q^z)$; this implies that $q \geq 151$.
- Moreover, we show that $w = 4$ is impossible: we must have $121 \mid \sigma(n)$ and so $121 \mid 2\frac{n}{d} + 1$ (since $q \neq 11$). Hence $w \geq 6$.

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- Moreover, we show that $w = 4$ is impossible: we must have $121 \mid \sigma(n)$ and so $121 \mid 2\frac{n}{d} + 1$ (since $q \neq 11$). Hence $w \geq 6$.
- It follows that $z \equiv -1 \pmod{3^4}$ and $z \equiv -1 \pmod 5$, so $z \geq 404$.

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- We also need $5 \mid \sigma(q^z)$ and $7 \mid \sigma(q^z)$; this implies that $q \geq 151$.
- Moreover, we show that $w = 4$ is impossible: we must have $121 \mid \sigma(n)$ and so $121 \mid 2\frac{n}{d} + 1$ (since $q \neq 11$). Hence $w \geq 6$.
- It follows that $z \equiv -1 \pmod{3^4}$ and $z \equiv -1 \pmod 5$, so $z \geq 404$.
- Hence, $n > q^z > 151^{404} > 6^{256}$. This is a contradiction.

Odd Near-Perfect Numbers of the Form $3^w 5^x 11^y q^z$

The proof for $p = 11$ requires a few additional preliminaries, namely two lemmas from Nielsen:

Lemma (Nielsen, 2006)

Let p be an odd prime and let q be 3 or 5. If $q^{p-1} \equiv 1 \pmod{p^2}$ then either $(q, p) = (3, 11)$ or $q^{\mathrm{ord}_p(q)} - 1$ has a prime divisor greater than 10^{13}.

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Lemma (Nielsen, 2006)

Let p and q be primes with $p \in (10^2, 10^{11})$ and $q = 7, 11$ or 13. If $q^{p-1} \equiv 1 \pmod{p^2}$ then $\sigma(q^{\mathrm{ord}_p(q)-1})$ is divisible by two primes greater than 10^{11}.

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- We can start, here, by considering two cases: $w = 2$ and $w \geq 4$.
- If $w = 2$, then $q \leq 139$. With some computer-assisted casework as in the others, there are no near-perfect odd numbers of this form.

Odd Near-Perfect Numbers of the Form $3^w 5^x 11^y q^z$

- In the case $w \geq 4$, unconditionally, we have $\frac{n}{d} \leq 27$. A little work improves this bound to $\frac{n}{d} \leq 15$.

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- We start by discounting all $q < 100$.
- Nielsen's two lemmas imply that $q \notin (10^2, 10^{11})$.

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- We have $\log_{11} q > 10$, and so this chain of inequalities:

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\frac{w+x+y+3}{10} &\geq \log_q(\sigma(3^w 5^x 11^y)) \\
&\geq v_q(\sigma(3^w)) + v_q(\sigma(5^x)) + v_q(\sigma(11^y)) \\
&\geq z \\
&\geq 3^{w-2}5^{x-1} - 1 \\
&\geq 15(w-2)(x-1) - 1.
\end{aligned}$$

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- Given that $w \geq 4$ and $x \geq 2$, this means that $y \geq 282$.

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Odd Near-Perfect Numbers of the Form $3^w 5^x 11^y q^z$

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- By similar arguments as to the case $p = 7$, we have $z \geq 44$.
- Hence, $n > 11^{282} \cdot 10^{484} > 15^{256}$.
- The proof for $q = 13$ is almost the same.

Odd Near-Perfect Numbers of the Form $3^w 5^x 11^y q^z$

- By similar arguments as to the case $p = 7$, we have $z \geq 44$.
- Hence, $n > 11^{282} \cdot 10^{484} > 15^{256}$.
- The proof for $q = 13$ is almost the same.
- There are no odd near-perfect numbers of the form $3^w 5^x p^y q^z$.

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