Almost Ready for Prime Time: Weak Versions of the Prime Number Theorem

What did the drowning number theorist say? "Log log log log log..."

1. Prove the inequality in the meme: For all $t \in \mathbf{R}$, we have $3 + 4\cos(t) + \cos(2t) \geq 0$.

   Cos and Effect. As it turns out, this leads to a proof that $\zeta(1 + it) \neq 0$ for all $t \in \mathbf{R}^\times$, which is a key ingredient in analytic proofs of the Prime Number Theorem.

2. Show that the following statements are equivalent to the Prime Number Theorem:

   (a) The number of primes up to x is asymptotically
   $$\pi(x) \sim \mathrm{li}(x).$$

   (b) For all $B \in \mathbf{R}$, we have
   $$\pi(x) \sim \frac{x}{\log x + B}.$$
   In other words, Legendre's estimate is not that bad after all!

   (c) Let p_n denote the nth prime, e.g., $p_1 = 2, p_2 = 3, p_3 = 5, \ldots$ Then $p_n \sim n \log n$.

3. Deduce from the version of Chebyshev's theorem proved in the slides that for sufficiently large x, there exists a prime in the interval $(x, 125x)$.

4. (Sylvester 1892, Schur 1929, Erdős 1934) By tweaking the proof of Bertrand's postulate in the slides, show that for every $n \geq 1$ and $k \geq 2n$, the binomial coefficient $\binom{k}{n}$ has a prime factor larger than n.

5. The number 30 has the property that if $1 < m \leq 30$, and $\gcd(m, 30) = 1$, then m is in fact prime. In other words, when $\mathbf{U}_{30}$ is represented by its least residue system, all its elements are either 1 or prime.

   (a) Find all other $n \leq 30$ with this property.

   (b) Suppose n has this property. Consider the least prime p_k such that $p_k \nmid n$. Then $p_k^2 > \prod_{j=1}^{k-1} p_j$.

   (c) Use Bertrand's postulate to show that the above inequality does not hold for $k > 4$. Deduce that 30 is the largest such positive integer with the desired property.