Preguntas de los viernes. 09/2024

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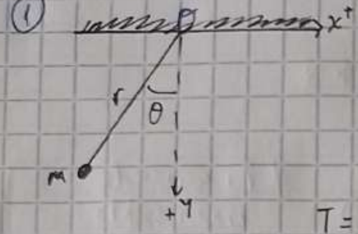
①




$$m\begin{cases} x = r\sin\theta \\ y = r\cos\theta \end{cases} \rightarrow \quad \dot{x} = \dot{r}\sin\theta + r\dot{\theta}\cos\theta \qquad \dot{y} = \dot{r}\cos\theta - r\dot{\theta}\sin\theta$$

$$\dot{x}^2 + \dot{y}^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad \text{la ligadura es } \dot{r} = a$$

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) \quad y \quad V = mgr\cos\theta$$

① $L = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - mgr\cos\theta$ (a) ecuaciones de E-L

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = \frac{\partial L}{\partial q} \quad \text{para } q = (r, \theta)$$

$$r: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r} \Rightarrow \frac{d}{dt}(m\dot{r}) = mr\dot{\theta}^2 - mg\cos\theta = m\ddot{r} = mr\dot{\theta}^2 - mg\cos\theta. \ ②$$

$$\theta: \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} \Rightarrow \frac{d}{dt}(mr^2\dot{\theta}) = mgr\sin\theta = 2mr\dot{r}\dot{\theta} + mr^2\ddot{\theta} = mgr\sin\theta \ ③$$

La función energía $\mathcal{H}(q, \dot{q}, t) = \frac{\partial L}{\partial \dot{q}}\dot{q} - L$

$$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \rightarrow \quad m\dot{r}\cdot\dot{r} + mr^2\dot{\theta}\dot{\theta} - \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + mgr\cos\theta$$

$$\mathcal{H} = \frac{1}{2}m(\dot{x}^2 + r^2\dot{\theta}^2) + mgr\cos\theta, \quad \text{como } \dot{r} = a$$

$$E = \frac{1}{2}m(a^2 + r^2\dot{\theta}^2) + mgr\cos\theta \ (b)$$ # Energía total del Sistema

②

$$L = \frac{1}{2}(m\dot{q}^2 - kq^2)e^{\frac{\alpha}{m}t} \ ①$$

a) Esto describe un oscilador ~~[illegible]~~ amortiguado ya que si $t \to \infty$ su amplitud ~~[illegible]~~ varía.

b $Q = e^{\frac{\alpha}{2m}t}q, \quad q = Qe^{\frac{-\alpha}{2m}t}, \quad \dot{q} = \dot{Q}e^{\frac{-\alpha}{2m}t} - \frac{Q\alpha}{2m}e^{\frac{-\alpha}{2m}t}$

# Realizamos un cambio de coordenadas $q \to Q$, $L(q) \to L(Q)$

$$L = \frac{1}{2}\left[m\left(\dot{Q}e^{\frac{-\alpha}{2m}t} - \frac{Q\alpha}{2m}e^{\frac{-\alpha}{2m}t}\right)^2 - kQ^2e^{\frac{-\alpha}{m}t}\right]e^{\frac{\alpha}{m}t}$$

$$L = \frac{1}{2}\left[m\left(\dot{Q}^2e^{\frac{-\alpha}{m}t} - \frac{2Q\dot{Q}\alpha}{m}e^{\frac{-\alpha}{m}t} + \frac{\dot{Q}^2\alpha^2}{4m^2}e^{-\frac{\alpha}{m}t}\right) - k\dot{Q}^2e^{-\frac{\alpha}{m}t}\right]e^{\frac{\alpha}{m}t}$$

$$L = \frac{1}{2}\left[m\left(\dot{Q}^2 - \frac{Q\dot{Q}\alpha}{m} + \frac{\dot{Q}^2\alpha^2}{4m^2}\right) - kQ^2\right]. \ ②$$

ⓒ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}}\right)=\left(\frac{\partial L}{\partial Q}\right)$:

$$\frac{d}{dt}\left[\frac{1}{2}m\left(2\dot{Q}-\frac{Q\alpha}{m}+\frac{2\dot{Q}\alpha^2}{4m^2}\right)\right]=\frac{1}{2}m\left[-\frac{\dot{Q}\alpha}{m}\right]-kQ$$

$$m\ddot{Q}-\frac{\dot{Q}\alpha}{2}+\frac{\ddot{Q}\alpha^2}{4m}=-\frac{\dot{Q}\alpha}{2}-kQ\rightarrow\ddot{Q}\left(m+\frac{\alpha^2}{4m}\right)=-kQ$$

$$\ddot{Q}=\frac{-k}{\left(m+\frac{\alpha^2}{4m}\right)}Q\longrightarrow w^2=\frac{-k}{\left(m+\frac{\alpha^2}{4m}\right)}\longrightarrow\ddot{Q}=-w^2Q$$ ③ # Ecuación de movimiento

ⓓ Para la Lagranciana $L(Q)$ sí hay una conservación de la energía pero en este caso no es una conservación de la energía mecánica, sino de la energía total en el nuevo sistema de coordenadas, ya que

$$\frac{\partial L(Q)}{\partial t}=0$$

e partiendo de ③ si $Q(t)=e^{mt}$, $\dot{Q}=me^{mt}$, $\ddot{Q}=m^2e^{mt}$

reemplazamos $(m^2+w^2)e^{mt}=0\rightarrow m=\pm wi$ de la cual sabemos la solución

$Q(t)=A\cos(wt+\phi)$ siendo $\phi$ la fase, como $Q=qe^{\frac{\alpha}{2m}t}$

$qe^{\frac{\alpha}{2m}t}=A\cos(wt+\phi)\rightarrow q(t)=A\cos(wt+\phi)e^{\frac{-\alpha}{2m}t}$ # a medida que pasa el tiempo las oscilaciones tienen menor amplitud

③ el lagrangiano en coordenadas esféricas

$$L=\frac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2+r^2\sin^2\theta\dot{\phi}^2\right)+Fr$$

a) las transformaciones que dejan invariante al lagrangiano son

1) → sumar alguna constante que se cancela al derivar las eq de movimiento

2) Adición de una función $\frac{d}{dt}\left[\Lambda(q_i,t)\right]$ que se cancela al sacar las eq de movimiento

$$L_2=L_1+\frac{d\Lambda}{dt}\Rightarrow\frac{d}{dt}\left(\frac{\partial L_2}{\partial \dot{q}_i}\right)-\frac{\partial L_2}{\partial q_i}=\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i}\right)-\frac{\partial L_1}{\partial q_i}+\cancel{\frac{\partial^2\Lambda}{\partial q\partial t}}-\cancel{\frac{\partial^2\Lambda}{\partial q\partial t}}$$

b) las cantidades conservadas son aquellas que $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)=0$

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)=mr\dot{\theta}^2+mr\sin^2\theta\dot{\phi}^2+F$, no se conserva (r)

$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right)=mr^2\sin\theta\cos\theta\dot{\phi}^2$, no se conserva (θ), $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right)=0$. El momento $P_\phi$ se conserva

$$mr^2\sin^2\theta\dot{\phi}=k$$

a $\mathbf{n} = a\hat{x} + b\hat{y} + c\hat{z}$ , $\vec{V} = \dot{x}\hat{x} + \dot{y}\hat{y} + \dot{z}\hat{z}$ y $\boldsymbol{\ell} = \mathbf{r} \times m\mathbf{v}$

$$\boldsymbol{\ell} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x & y & z \\ \dot{x} & \dot{y} & \dot{z} \end{vmatrix} \begin{matrix} \hat{x} \\ x \\ \dot{x} \end{matrix} \quad (\dot{z}y - \dot{y}z)\hat{x} + (x\dot{z} - z\dot{x})\hat{y} + (x\dot{y} - y\dot{x})\hat{z}$$

$$V(\mathbf{r}, \mathbf{v}) = U(\mathbf{r}) + \mathbf{n}\cdot\boldsymbol{\ell} = U(r) + a(\dot{z}y - \dot{y}z) + b(x\dot{z} - z\dot{x}) + c(x\dot{y} - y\dot{x})$$

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - U(r) - a(\dot{z}y - \dot{y}z) - b(x\dot{z} - z\dot{x}) + c(x\dot{y} - y\dot{x})$$

Ecuaciones de movimiento

$$x: \frac{d}{dt}(m\dot{x} + bz + cy) = -\frac{\partial U}{\partial x} - b\dot{z} - c\dot{y}$$

$$m\ddot{x} + b\dot{z} + c\dot{y} = -\frac{\partial U}{\partial x} - b\dot{z} - c\dot{y} \Rightarrow m\ddot{x} = -\frac{\partial U}{\partial x} - 2b\dot{z} - 2c\dot{y}$$

$$y: \frac{d}{dt}(m\dot{y} + az - cx) = -\frac{\partial U}{\partial y} - a\dot{z} + c\dot{x} \qquad \ddot{x} = \frac{-1}{m}\left(\frac{\partial U}{\partial x} + 2b\dot{z} + 2c\dot{y}\right)$$

$$m\ddot{y} + a\dot{z} - c\dot{x} = -\frac{\partial U}{\partial y} - a\dot{z} + c\dot{x} \rightarrow m\ddot{y} = -\frac{\partial U}{\partial y} - 2a\dot{z} + 2c\dot{x}$$

$$z: \frac{d}{dt}(m\dot{z} - ay - bx) = -\frac{\partial U}{\partial z} + a\dot{y} + b\dot{x} \rightarrow \ddot{y} = \frac{-1}{m}\left(\frac{\partial U}{\partial y} + 2a\dot{z} - 2c\dot{x}\right)$$

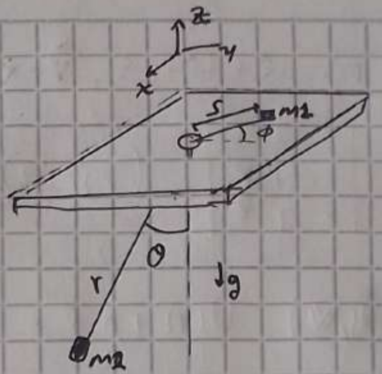
punto b

$$m\ddot{z} - a\dot{y} - b\dot{x} = -\frac{\partial U}{\partial z} + a\dot{y} + b\dot{x} \rightarrow m\ddot{z} = -\frac{\partial U}{\partial z} + 2a\dot{y} + 2b\dot{x}$$

$$\ddot{z} = \frac{-1}{m}\left(\frac{\partial U}{\partial z} - 2a\dot{y} - 2b\dot{x}\right)$$

c Cantidades conservadas

$\frac{\partial L}{\partial t} = 0$ , se conserva la energía del sistema




⑤ ligaduras:

a) $m_1$ en $z=$ constante
$m_2$ se mueve verticalmente

$r+s=\ell$ y $\dot r=-\dot s$ , $\dot r^2=\dot s^2$

grados de libertad y coordenadas

$3N-f=n \Rightarrow 6-3=\boxed{3}$ coordenadas $\rightarrow (r,\theta,\phi)$

el lagrangiano: $T-U$

$$T = T_1+T_2 = \frac{1}{2}m_1(\dot s^2+s^2\dot\phi^2)+\frac{1}{2}m_2(\dot r^2+r^2\dot\theta^2)$$

$$= \frac{1}{2}m(\dot r^2+(\ell-r)^2\dot\phi^2)+\frac{1}{2}m_2(\dot r^2+r^2\dot\theta^2)$$

solo $m_2$ tiene $U=+m_2 g r\cos\theta$

$$L = \frac{1}{2}m_1(\dot r^2+(\ell-r)^2\dot\phi^2)+\frac{1}{2}m_2(\dot r^2+r^2\dot\theta^2)+m_2 g r\cos\theta$$

Ecuaciones de movimiento

$$r: \frac{d}{dt}(m_1\dot r+m_2\dot r) = -m_1(\ell-r)\dot\phi^2+m_2 r\dot\theta^2+mg\cos\theta$$

$$(m_1+m_2)\ddot r = -m_1(\ell-r)\dot\phi^2+m_2 r\dot\theta^2-m_2 g\cos\theta$$

$$\theta: \frac{d}{dt}(m_2 r^2\dot\theta) = m_2 g r\sin\theta = m_2(2r\dot r\dot\theta+r^2\ddot\theta) = m_2 g r\sin\theta$$

$$\phi: \frac{d}{dt}(m_1(\ell-r)^2\dot\phi) = 0 \rightarrow -m_1(\ell-r)\dot\phi+m_1(\ell-r)^2\ddot\phi=0$$

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Rta: $$\ddot r = \frac{1}{(m_1+m_2)}\left(-m_1(\ell-r)\dot\phi^2+m_2 r\dot\theta^2-m_2 g\cos\theta\right)$$ a.

$$\ddot\theta = \frac{g}{r}\sin\theta-\frac{2\dot r\dot\theta}{r} \quad \wedge \quad \ddot\phi = \frac{\dot\phi}{\ell-r}$$

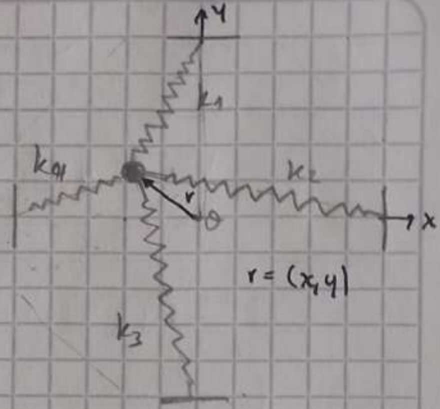
b: se conserva el $p_\phi \rightarrow \dot p_\phi=0$ y la energía del sistema $\frac{\partial L}{\partial t}=0$

c: Equilibrio se da cuando $\dot r=-\dot s=0$ y que $\ddot r=0$, este equilibrio depende de la velocidad angular de $m_1$

usando $\ddot r \rightarrow$, $0=-m_1(\ell-r)\dot\phi^2+m_2 r\dot\theta^2-m_2 g\cos\theta$

$$\dot\phi^2 = \frac{m_2}{m_1(\ell-r)}(r\dot\theta^2-g\cos\theta)$$ , $\dot\phi$ es inversamente proporcional a su radio de giro y directamente proporcional a la diferencia entre la aceleración centrípeta del péndulo y el peso de $m_2$.

## (6) Oscilador armónico anisotrópico




Nuestros grados de libertad

$2N = 2$ 2 grados de libertad $(x, y)$

en principio supondremos cada $k_i$ distinta, como caso general, el lagrangiano entonces es

$$T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) \quad \wedge \quad V(r) = \sum_{i=1}^{4} \frac{1}{2} k_i \Delta r_i^2 \qquad i = 1,2,3,4$$

$$L = \frac{1}{2}m(\dot{x}^2+\dot{y}^2) - \frac{1}{2}\Big[\overset{k_y^+}{k_1(x^2+(y+a)^2)} + \overset{k_x^+}{k_2((x+a)^2+y^2)} + \overset{k_y^-}{k_3(x^2+(y-a)^2)} + \overset{k_x^-}{k_4((x-a)^2+y^2)}\Big]$$

a

$$x: \; m\ddot{x} = -\frac{1}{2}\left[2k_1 x + 2k_2(x+a) + 2k_3 x + 2k_4(x-a)\right]$$

$$x: \; m\ddot{x} = -x(k_1+k_2+k_3+k_4) + a(k_2-k_4) = \Lambda_x L$$

$$y: \; m\ddot{y} = -\frac{1}{2}\left[2k_1(y+a) + 2k_2 y + 2k_3(y-a) + 2k_4 y\right]$$

$$m\ddot{y} = -y\left[k_1+k_2+k_3+k_4\right] + a(k_1-k_3) = \Lambda_y L$$

Ecuaciones de movimiento

b: Caso isotrópico $k_1 = k_2 = k_3 = k_4 = k$

$$m\ddot{x} = -x(4k) \Rightarrow \ddot{x} = -\frac{4k}{m}x \quad y \quad \ddot{y} = -\frac{4k}{m}x$$

Caso anisotrópico $k_1 = k_3 \neq k_2 = k_4$

$$m\ddot{x} = -x(2k_1+2k_2) \quad \wedge \quad m\ddot{y} = -y(2k_1+2k_2)$$