$$\text{MFES/CSI} \text{ — } \textsc{Relation Calculus Refernece Sheet}$$

Relational composition

Pointwise def.
$$B \xleftarrow{R} A \xleftarrow{S} C \quad b(R \cdot S)c \;\equiv\; \langle \exists\, a \,:\, b\,R\,a \,:\, a\,S\,c \rangle \qquad (5.11)$$
$$\underset{R\cdot S}{}$$

Associativity
$$R \cdot (S \cdot P) = (R \cdot S) \cdot P \qquad (5.12)$$

Identity
$$\begin{cases} R = R \cdot id_A \\ R = id_B \cdot R \end{cases} \qquad (5.13)$$

Converse

Pointwise def.
$$b\,R\,a \quad \Leftrightarrow \quad a\,R^\circ\,b \qquad (5.14)$$

Universal-$^\circ$
$$X^\circ \subseteq Y \;\equiv\; X \subseteq Y^\circ \qquad (5.136)$$

Involution
$$(R^\circ)^\circ = R \qquad (5.15)$$

Contravariance
$$(R \cdot S)^\circ = S^\circ \cdot R^\circ \qquad (5.16)$$

Isomorphism
$$R \subseteq S \equiv R^\circ \subseteq S^\circ \qquad (5.137)$$

"Guardanapo"
$$b(f^\circ \cdot R \cdot g)a \;\equiv\; (f\,b)R(g\,a) \qquad (5.17)$$

Relation inclusion

Pointwise
$$R \subseteq S \quad \textit{iff} \quad \langle \forall\, a,b \,::\, b\,R\,a \Rightarrow b\,S\,a \rangle \qquad (5.19)$$

Reflexion
$$R \subseteq R \qquad (5.21)$$

Transitivity
$$R \subseteq S \wedge S \subseteq T \;\Rightarrow\; R \subseteq T \qquad (5.22)$$

Top and bottom
$$\bot \subseteq R \subseteq \top \qquad (5.25)$$

Absorption
$$R \cdot \bot = \bot \cdot R = \bot \qquad (5.26)$$

Relation equality

Pointwise
$$R = S \quad \textit{iff} \quad \langle \forall\, a,b \,:\, a \in A \wedge b \in B \,:\, b\,R\,a \Leftrightarrow b\,S\,a \rangle \qquad (5.18)$$

Indirect equality
$$\begin{aligned} R = S \;&\equiv\; \langle \forall\, X \,::\, (X \subseteq R \Leftrightarrow X \subseteq S) \rangle \\ &\equiv\; \langle \forall\, X \,::\, (R \subseteq X \Leftrightarrow S \subseteq X) \rangle \end{aligned} \qquad (5.24)$$

"Ping-pong"
$$R = S \;\equiv\; R \subseteq S \wedge S \subseteq R \qquad (5.20)$$

RELATION TAXONOMY

Kernel
$$\text{ker } R = R^\circ \cdot R \qquad (5.32)$$

Image
$$\text{img } R \stackrel{\text{def}}{=} R \cdot R^\circ \qquad (5.33)$$

Duality
$$\begin{aligned} \text{ker } (R^\circ) &= \text{img } R \\ \text{img } (R^\circ) &= \text{ker } R \end{aligned} \qquad (5.35)$$

Criteria

	Reflexive	Coreflexive
ker R	entire R	injective R
img R	surjective R	simple R

(5.36)

FUNCTIONS

Shunting rules
$$\begin{aligned} f \cdot R \subseteq S &\equiv R \subseteq f^\circ \cdot S \\ R \cdot f^\circ \subseteq S &\equiv R \subseteq S \cdot f \end{aligned} \qquad (5.47)$$

Equality
$$f \subseteq g \equiv f = g \equiv f \supseteq g \qquad (5.48)$$
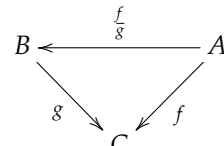
CONSTANT FUNCTIONS

Natural property
$$\underline{c} \cdot R \subseteq \underline{c} \qquad (5.39)$$

Corollary
$$\text{ker } ! = \frac{!}{!} = \top \qquad (5.\text{—})$$

Truth functions
$$\begin{aligned} true &= \underline{\text{True}} \\ false &= \underline{\text{False}} \end{aligned} \qquad (5.40)$$

FUNCTION DIVISION

Definition
$$\frac{f}{g} = g^\circ \cdot f \quad cf. \qquad (5.49)$$

Properties
$$\begin{aligned} \frac{f}{id} &= f \\ \left(\frac{f}{g}\right)^\circ &= \frac{g}{f} \\ \frac{f \cdot h}{g \cdot k} &= k^\circ \cdot \frac{f}{g} \cdot h \\ \frac{f}{f} &= \text{ker } f \\ a \neq b &\Leftrightarrow \frac{a}{\underline{b}} = \bot \end{aligned} \qquad (5.54)$$

RELATION UNION

Pointwise definition
$$b\,(R \cup S)\,a \equiv b\,R\,a \vee b\,S\,a \qquad (5.57)$$

Universal property
$$R \cup S \subseteq X \equiv R \subseteq X \wedge S \subseteq X \qquad (5.59)$$

Right linearity
$$R \cdot (S \cup T) = (R \cdot S) \cup (R \cdot T) \qquad (5.60)$$

Left linearity	$(S \cup T) \cdot R \ = \ (S \cdot R) \cup (T \cdot R)$	(5.61)
Converse-$\cup$	$(R \cup S)^\circ = R^\circ \cup S^\circ$	(5.65)
Top-$\cup$	$R \cup \top = \top$	(5.68)
Bottom-$\cup$	$R \cup \bot = R$	(5.69)
Union simplicity	$M \cup N$ is simple $\equiv$ M, N are simple and $M \cdot N^\circ \subseteq id$	(5.70)

RELATION INTERSECTION

Pointwise definition	$b\,(R \cap S)\,a \ \equiv \ b\,R\,a \wedge b\,S\,a$	(5.56)
Universal property	$X \subseteq R \cap S \ \equiv \ X \subseteq R \wedge X \subseteq S$	(5.58)
Distribution (1)	$(S \cap Q) \cdot R = (S \cdot R) \cap (Q \cdot R) \ \Leftarrow \ \begin{cases} Q \cdot \text{img } R \subseteq Q \\ \vee \\ S \cdot \text{img } R \subseteq S \end{cases}$	(5.62)
Distribution (2)	$R \cdot (Q \cap S) = (R \cdot Q) \cap (R \cdot S) \ \Leftarrow \ \begin{cases} (\text{ker } R) \cdot Q \subseteq Q \\ \vee \\ (\text{ker } R) \cdot S \subseteq S \end{cases}$	(5.63)
Distribution (3)	$g^\circ \cdot (R \cap S) \cdot f \ = \ g^\circ \cdot R \cdot f \cap g^\circ \cdot S \cdot f$	(5.71)
Converse-$\cap$	$(R \cap S)^\circ = R^\circ \cap S^\circ$	(5.64)
Bottom-$\cap$	$R \cap \bot = \bot$	(5.66)
Top-$\cap$	$R \cap \top = R$	(5.67)
Misc.	$k^\circ \cdot (f \cup g) = \frac{f}{k} \cup \frac{g}{k}$, $k^\circ \cdot (f \cap g) = \frac{f}{k} \cap \frac{g}{k}$	(5.73)
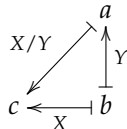
RELATION DIVISION

Universal-$/$	$Z \cdot Y \subseteq X \equiv Z \subseteq X/Y$	(5.157)
Pointwise-$/$	$c\,(X/Y)\,a \ \equiv \ \langle \forall\, b \,:\, a\,Y\,b \,:\, c\,X\,b \rangle$	(5.158)
Universal-$\backslash$	$X \cdot Z \subseteq Y \Leftrightarrow Z \subseteq X \backslash Y$	(5.159)
Pointwise-$\backslash$	$a(X \backslash Y)c \ \equiv \ \langle \forall\, b \,:\, b\,X\,a \,:\, b\,Y\,c \rangle$	(5.160)
Misc.	$X \cdot f \ = \ X/f^\circ$	(5.162)
	$f \backslash X \ = \ f^\circ \cdot X$	(5.163)
	$X/\bot \ = \ \top$	(5.164)

$$X / id \;=\; X \tag{5.165}$$

$$R \setminus (f^\circ \cdot S) \;=\; f \cdot R \setminus S \tag{5.166}$$

$$R \setminus \top \cdot S \;=\; !\cdot R \setminus !\cdot S \tag{5.167}$$

$$R / (S \cup P) \;=\; R / S \cap R / P \tag{5.168}$$

RELATION DIFFERENCE, IMPLICATION AND NEGATION

Universal-$(-)$	$X - R \subseteq Y \;\equiv\; X \subseteq Y \cup R$	(5.138)
Pointwise-$\Rightarrow$	$b(R \Rightarrow S)a \;\equiv\; (b\,R\,a) \Rightarrow (b\,S\,a)$	(5.147)
Universal-$\Rightarrow$	$R \cap X \subseteq Y \;\equiv\; X \subseteq (R \Rightarrow Y)$	(5.148)
Distribution	$f^\circ \cdot (R \Rightarrow S) \cdot g \;=\; (f^\circ \cdot R \cdot g) \Rightarrow (f^\circ \cdot S \cdot g)$	(5.154)
Definition-$\neg$	$\neg R = (R \Rightarrow \bot)$	(5.—-)
Pointwise-$\neg$	$b\,(\neg R)\,a \;\Leftrightarrow\; \neg\,(b\,R\,a)$	(5.—-)
Complementation	$R \cup \neg R = \top$	(5.—-)
Difference versus implication	$\top - R \;\subseteq\; R \Rightarrow \bot$	(5.153)
de Morgan	$\neg(R \cup S) = (\neg R) \cap (\neg S)$	(5.150)
Schröder's rule	$\neg Q \cdot S^\circ \;\subseteq\; \neg R \;\Leftrightarrow\; R^\circ \cdot \neg Q \;\subseteq\; \neg S$	(5.151)

MONOTONICITY

Composition	$R \subseteq S \wedge T \subseteq U \;\Rightarrow\; R \cdot T \subseteq S \cdot U$	(5.78)
Converse	$R \subseteq S \;\Rightarrow\; R^\circ \subseteq S^\circ$	(5.79)
Intersection	$R \subseteq S \wedge U \subseteq V \;\Rightarrow\; R \cap U \subseteq S \cap V$	(5.80)
Union	$R \subseteq S \wedge U \subseteq V \;\Rightarrow\; R \cup U \subseteq S \cup V$	(5.81)

ENDO-RELATION TAXONOMY

Reflexive	$id \subseteq R$	(5.84)
Coreflexive	$R \subseteq id$	(5.85)
Transitive	$R \cdot R \subseteq R$	(5.86)
Symmetric	$R \subseteq R^\circ (\equiv R = R^\circ)$	(5.87)

Anti-symmetric	$R \cap R^{\circ} \subseteq id$	(5.88)
Irreflexive	$R \cap id = \bot$	(5.89)
Connected	$R \cup R^{\circ} = \top$	(5.90)

PAIRING ("SPLITS")

Pairing-pointwise	$(a,b) \langle R,S \rangle c \Leftrightarrow a \, R \, c \wedge b \, S \, c$	(5.101)
Pairing-def	$\langle R,S \rangle = \pi_1^{\circ} \cdot R \cap \pi_2^{\circ} \cdot S$	(5.102)
Universal-pairing	$X \subseteq \langle R,S \rangle \quad \Leftrightarrow \quad \begin{cases} \pi_1 \cdot X \subseteq R \\ \pi_2 \cdot X \subseteq S \end{cases}$	(5.103)
Pairing-fusion	$\langle R,S \rangle \cdot T = \langle R \cdot T, S \cdot T \rangle \\ \quad \Leftarrow R \cdot (\text{img } T) \subseteq R \vee S \cdot (\text{img } T) \subseteq S$	(5.104)
Pairing-fusion (functions)	$\langle R,S \rangle \cdot f = \langle R \cdot f, S \cdot f \rangle$	(5.105)
Pairing and converse	$\langle R,S \rangle^{\circ} \cdot \langle X,Y \rangle = (R^{\circ} \cdot X) \cap (S^{\circ} \cdot Y)$	(5.108)
Kernel of pairing	$\ker \langle R,S \rangle = \ker R \cap \ker S$	(5.111)
Functor-$\times$ def	$R \times S = \langle R \cdot \pi_1, S \cdot \pi_2 \rangle$	(5.107)
Functor-$\times$-id	$id \times id = id$	(5.112)
Functor-$\times$-composition	$(R \times S) \cdot (P \times Q) = (R \cdot P) \times (S \cdot Q)$	(5.113)
Functor-$\times$ absorption	$(R \times S) \cdot \langle P,Q \rangle = \langle R \cdot P, S \cdot Q \rangle$	(5.106)
Functor-$\times$-division	$\frac{f}{g} \times \frac{h}{k} = \frac{f \times h}{g \times k}$	(5.125)
Pairing division	$\frac{f}{g} \cap \frac{h}{k} = \frac{\langle f,h \rangle}{\langle g,k \rangle}$	(5.109)

COPRODUCTS

Universal	$X = [R,S] \quad \Leftrightarrow \quad \begin{cases} X \cdot i_1 = R \\ X \cdot i_2 = S \end{cases}$	(5.114)
Definition	$[R,S] = R \cdot i_1^{\circ} \cup S \cdot i_2^{\circ}$	(5.117)
Reflexion	$\text{img } i_1 \cup \text{img } i_2 = id$	(5.115)
Disjointness	$i_1^{\circ} \cdot i_2 = \bot$	(5.116)
Either and converse	$[R,S] \cdot [T,U]^{\circ} = (R \cdot T^{\circ}) \cup (S \cdot U^{\circ})$	(5.121)
Image of either	$\text{img } [R,S] = \text{img } R \cup \text{img } S$	(5.124)
Exchange law	$[\langle R,S \rangle, \langle T,V \rangle] = \langle [R,T],[S,V] \rangle$	(5.122)

Functor-+-def	$R + S = [i_1 \cdot R , i_2 \cdot S]$	(5.119)
Functor-+-converse	$(R + S)^\circ = R^\circ + S^\circ$	(5.123)
Functor-+-division	$\frac{f}{g} + \frac{h}{k} = \frac{f+h}{g+k}$	(5.126)

INJECTIVITY PREORDER

Definition	$R \leqslant S \;\equiv\; \text{ker } S \subseteq \text{ker } R$	(5.230)
Join	$\langle R, S \rangle \leqslant X \;\equiv\; R \leqslant X \wedge S \leqslant X$	(5.231)
Cancellation	$R \leqslant \langle R, S \rangle \;\text{ and }\; S \leqslant \langle R, S \rangle.$	(5.232)
Shunting	$R \cdot g \leqslant S \;\equiv\; R \leqslant S \cdot g^\circ$	(5.233)
Meet	$X \leqslant [R^\circ , S^\circ]^\circ \;\Leftrightarrow\; X \leqslant R \wedge X \leqslant S$	(5.234)
Either-injective	$id \leqslant [R, S] \;\Leftrightarrow\; id \leqslant R \wedge id \leqslant S \wedge R^\circ \cdot S = \bot$	(5.235)

THUMB RULES

A function f is a bijection iff *its converse f° is a function g*	(5.38)
- converse of *injective* is *simple* (and vice-versa)	(5.43)
- converse of *entire* is *surjective* (and vice-versa)	(5.44)
- *smaller* than injective (simple) is injective (simple)	(5.82)
- *larger* than entire (surjective) is entire (surjective)	(5.83)
$\langle R, id \rangle$ is always *injective*, for whatever R	(5.111a)

QUANTIFIER CALCULUS

Trading-$\forall$	$\langle \forall k \,:\, R \wedge S \,:\, T \rangle \;=\; \langle \forall k \,:\, R \,:\, S \Rightarrow T \rangle$	(A.1)
Trading-$\exists$	$\langle \exists k \,:\, R \wedge S \,:\, T \rangle \;=\; \langle \exists k \,:\, R \,:\, S \wedge T \rangle$	(A.2)
de Morgan	$\neg\langle \forall k \,:\, R \,:\, T \rangle \;=\; \langle \exists k \,:\, R \,:\, \neg T \rangle$	(A.3)
de Morgan	$\neg\langle \exists k \,:\, R \,:\, T \rangle \;=\; \langle \forall k \,:\, R \,:\, \neg T \rangle$	(A.4)
One-point-$\forall$	$\langle \forall k \,:\, k = e \,:\, T \rangle \;=\; T[k := e]$	(A.5)
One-point-$\exists$	$\langle \exists k \,:\, k = e \,:\, T \rangle \;=\; T[k := e]$	(A.6)
Nesting-$\forall$	$\langle \forall a, b \,:\, R \wedge S \,:\, T \rangle \;=\; \langle \forall a \,:\, R \,:\, \langle \forall b \,:\, S \,:\, T \rangle \rangle$	(A.7)

Nesting-∃ $\langle \exists\, a,b \,:\, R \wedge S \,:\, T \rangle \;\; = \;\; \langle \exists\, a \,:\, R \,:\, \langle \exists\, b \,:\, S \,:\, T \rangle \rangle$ (A.8)

Rearranging-∀ $\langle \forall\, k \,:\, R \vee S \,:\, T \rangle \;\; = \;\; \langle \forall\, k \,:\, R \,:\, T \rangle \wedge \langle \forall\, k \,:\, S \,:\, T \rangle$ (A.9)

Rearranging-∀ $\langle \forall\, k \,:\, R \,:\, T \wedge S \rangle \;\; = \;\; \langle \forall\, k \,:\, R \,:\, T \rangle \wedge \langle \forall\, k \,:\, R \,:\, S \rangle$ (A.10)

Rearranging-∃ $\langle \exists\, k \,:\, R \,:\, T \vee S \rangle \;\; = \;\; \langle \exists\, k \,:\, R \,:\, T \rangle \vee \langle \exists\, k \,:\, R \,:\, S \rangle$ (A.11)

Rearranging-∃ $\langle \exists\, k \,:\, R \vee S \,:\, T \rangle \;\; = \;\; \langle \exists\, k \,:\, R \,:\, T \rangle \vee \langle \exists\, k \,:\, S \,:\, T \rangle$ (A.12)

Splitting-∀ $\langle \forall\, j \,:\, R \,:\, \langle \forall\, k \,:\, S \,:\, T \rangle \rangle \;\; = \;\; \langle \forall\, k \,:\, \langle \exists\, j \,:\, R \,:\, S \rangle \,:\, T \rangle$ (A.13)

Splitting-∃ $\langle \exists\, j \,:\, R \,:\, \langle \exists\, k \,:\, S \,:\, T \rangle \rangle \;\; = \;\; \langle \exists\, k \,:\, \langle \exists\, j \,:\, R \,:\, S \rangle \,:\, T \rangle$ (A.14)