Distances in isotropic Gaussian distributions are Chi-distributed

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In this paper we prove that the distribution of distances in a multivariate isotropic Gaussian distribution follows a generalized Chi distribution. We show that the generalized Chi distribution can be used to calculate distribution of intensity displacement following Gaussian filtering and the distribution of Brownian particle displacement in n dimensions.

Theorem 1. *Let $Y \in \mathbb{R}^n$ be any random variable following an isotropic multivariate normal distribution such that it's probability density function (PDF) is:*

$$f(\mathbf{y}) = \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \exp\left(-\frac{||\boldsymbol{y}||^2}{2\sigma^2}\right) \tag{1}$$

The distribution of distances $X = \sqrt{||Y||^2}$ is the generalized Chi distribution with PDF and cumulative density function (CDF):

$$\begin{aligned} f(x) &= \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \\ P(X < x) &= \frac{\gamma(\frac{n}{2}, \frac{x^2}{2\sigma^2})}{\Gamma(\frac{n}{2})} \end{aligned}$$

Proof. We will start with the multivariate Normal distribution (eq.2) and modify it to obtain the multivariate isotropic Gaussian distribution (eq.1).

$$f(\mathbf{x} : \mu, \Sigma) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} \exp\left(-\frac{1}{2}(\mathbf{x} - \mu)^T \Sigma^{-1} (\mathbf{x} - \mu)\right) \tag{2}$$

Where $\mu \in \mathbb{R}^n$ is the mean, Σ is a square covariance matrix, n is the number of dimensions and $|\Sigma|$ is the determinant of the covariance matrix.

Using an average of $\mu = 0$ removes the term and does not restrict us since we want to calculate our CDF from the center of the distribution. To get an isotropic normal density, we reduce the covariance matrix to a diagonal matrix with value σ^2 on the diagonal. This leads to

$$\mathbf{x}^T \Sigma^{-1} \mathbf{x} = \frac{||\mathbf{x}||^2}{\sigma^2}$$

and

$$|\Sigma| = (\sigma^2)^n.$$

We then obtain:

$$f(\mathbf{x} : \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-||x||^2/2\sigma^2}, \tag{3}$$

Which is eq.1 from the theorem.

In eq.3, $\mathbf{x}$ is a vector, but since we only use it's norm we can use the square root of the norm as a variable of the function without losing generality. This gives us an univariate function of the radial distance r instead of a multivariate one:

$$f(r : \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-r^2/2\sigma^2}, \tag{4}$$

where $||\mathbf{x}||^2 = r^2$.

We now want to find the CDF of the distribution X of distances from the center of an isotropic multivariate Gaussian distribution. It will be following this form $P(X < r) = f(n, \sigma)$: the probability of choosing a point at random from a Normal distribution with a distance from the center of r or less. We will count the number of point at every distance r and calculate their combined probability density function. To calculate this we will construct an hypersphere with a gaussian density and calculate it's mass. The n-dimensional multivariate Gaussian density defines iso-contours of equiprobable points. In the case of an isotropic multivariate Gaussian density, these lie on the surface of a hypersphere in $\mathbb{R}^n$.

The mass of the n-dimensional hypersphere can be calculated with the following formula:

$$M(r) = \int_0^r density(r)\ dV$$

For the hypersphere density we will use eq.4. Which results in the CDF:

$$F(r) = P(X < r) = \int_0^r \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{x^2}{2\sigma^2}} dV \tag{5}$$

where V is the n-dimensional volume (content) of the hypersphere. Let S_n be the hyper-surface area of an n-dimensional hypersphere of unit radius. From the literature[1]:

$$S_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

and

$$V = \int_0^r S_n x^{n-1} dx = \frac{S_n r^n}{n}.$$

Which we use to calculate dV:

$$\begin{aligned} V &= \frac{2\pi^{n/2} r^n}{n\Gamma(\frac{n}{2})} \\ dV &= \frac{V}{dr} = \frac{2\pi^{n/2} r^{n-1}}{\Gamma(\frac{n}{2})} \end{aligned} \tag{6}$$

[1] `https://mathworld.wolfram.com/Hypersphere.html`

We can now combine eq.5 and 6 to calculate the CDF:

$$P(X < r) = \int_0^r \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1}\ dx \tag{7}$$

We move the terms we can out of the integral.

$$= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^r e^{-x^2/2\sigma^2} x^{n-1}\ dx$$

We will modify the terms of the integral in order to obtain the lower gamma function γ starting with a variable substitution:

$$\frac{x^2}{2\sigma^2} = y \rightarrow \begin{cases} \frac{x}{\sigma^2} dx = dy \\ x = \sqrt{2y\sigma^2} \end{cases}$$

The upper limit of the integral goes from r to $\frac{r^2}{2\sigma^2}$.

$$= \frac{2\sigma^2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^r e^{-x^2/2\sigma^2} x^{n-2} \frac{x}{\sigma^2}\ dx = \frac{2\sigma^2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^{\frac{r^2}{2\sigma^2}} e^{-y} (\sqrt{y 2\sigma^2})^{n-2}\ dy$$

$$= \frac{2^{n/2}\sigma^n}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^{\frac{r^2}{2\sigma^2}} e^{-y} y^{n/2-1}\ dy$$

By definition of the γ function we obtain:

$$= \frac{1}{\Gamma(\frac{n}{2})}\ \gamma(\frac{n}{2}, \frac{r^2}{2\sigma^2})$$

$$P(X < r) = \frac{\gamma(\frac{n}{2}, \frac{r^2}{2\sigma^2})}{\Gamma(\frac{n}{2})} \tag{8}$$

You can change variable r to $x\sigma$, where x is the mahalanobis distance:

$$P(X < x) = \frac{\gamma(\frac{n}{2}, \frac{x^2}{2})}{\Gamma(n/2)} \tag{9}$$

Eq.9 is the CDF of the chi distribution. It is related to the chi-square distribution, which is the distribution of the sum of n squared standard normal distribution. We will work with eq.8 instead to keep generality. We call the distribution having this CDF the generalized Chi distribution.

We can obtain the probability density function from eq.7 by deriving it:

$$f(x) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \tag{10}$$

We now concluded that the distribution of distances to the center in a Gaussian filter follow a generalized chi distribution. $\square$

1 Generalized Chi distribution properties

Here we look a the generalized chi distribution properties. We call it generalized because the chi distribution is a sum of Normal distributions with $\sigma = 1$. We choose instead to work with Normal distributions with any sigma value.

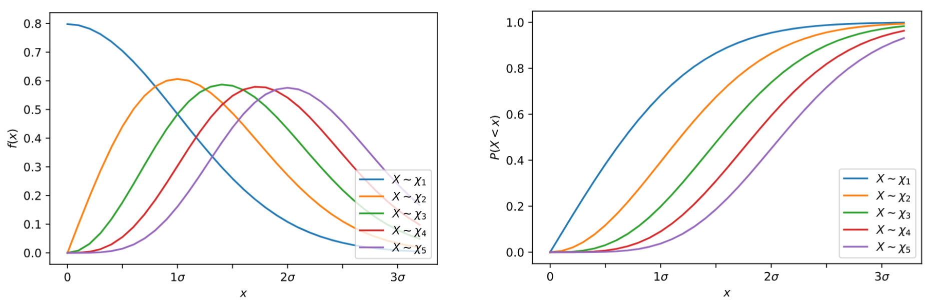
1.1 PDF and CDF

The PDF and CDF of the Generalized Chi distribution are

$$f(x) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \tag{11}$$

$$P(X < x) = \frac{\gamma(\frac{n}{2}, \frac{x^2}{2\sigma^2})}{\Gamma(\frac{n}{2})} \tag{12}$$

with $x \geq 0$. Those 2 equations are plotted for different dimensions in Figure 1 a and b.



(a) PDF

(b) CDF

Figure 1: Generalized Chi Distribution PDF and CDF

The dimension 1 PDF curve is really similar to the 1D normal distribution. The differences are that $x \geq 0$ and it's value is double. We can see that in 1 dimension, points closer to the center have the highest probability. This is not the case for higher dimensions. While the single point with the highest probability will always be at a distance 0, the higher number of points away from the center grants them a higher probability per distance. Thus, the distance of highest probability increases with the dimension.

1.2 Moments

We start by calculating moments i of a random variable X_n following the generalized chi distribution.

$$
\begin{aligned}
E[X_n^i] &= \int_0^\infty x^i \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} e^{-x^2/2\sigma^2} x^{n-1}\ dx \\
&= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-x^2/2\sigma^2} x^{n+i-1}\ dx
\end{aligned}
$$

We will modify the terms of the integral to obtain the Gamma function Γ starting with a variable substitution:

$$
\begin{aligned}
&\frac{x^2}{2\sigma^2} = y \rightarrow \begin{cases} \frac{x}{\sigma^2} dx = dy \\ x = \sqrt{2y\sigma^2} \end{cases} \\
&= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-x^2/2\sigma^2} x^{n+i-2} \sigma^2 \frac{x}{\sigma^2}\ dx = \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-y} (\sqrt{y2\sigma^2})^{n+i-2}\ \sigma^2 dy \\
&= \frac{2^{(n+i)/2}\sigma^{n+i}}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-y} y^{(n+i)/2-1}\ dy
\end{aligned}
$$

By definition of the Γ function we obtain:

$$
E[X_n^i] = 2^{i/2}\sigma^i \frac{\Gamma(\frac{n+i}{2})}{\Gamma(\frac{n}{2})}
$$

Below are values of the Γ function related to our case:

$$
\begin{aligned}
\Gamma(\frac{1}{2}) &= \sqrt{\pi} \\
\Gamma(n + \frac{1}{2}) &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi}
\end{aligned}
$$

The moments of the generalized chi distribution follow eq.13

$$
E[X_n^i] = 2^{i/2}\sigma^i \frac{\Gamma(\frac{n+i}{2})}{\Gamma(\frac{n}{2})} \tag{13}
$$

It simplifies to

$$
E[X_n^2] = n\sigma^2
$$

for the second moment.

The first and second moments are shown in figure 2 and table 1 contains some values for $E[X_n^1]$ and $E[X_n^2]$.

Raw moments of the generalized Chi distribution are more useful than central moments to characterize the isotropic multivariate Gaussian distribution since it gives information on the variation of distance with the center of the distribution as a reference. While the reference point of the central moments are offset by the first moment, which is nothing concrete in our case.

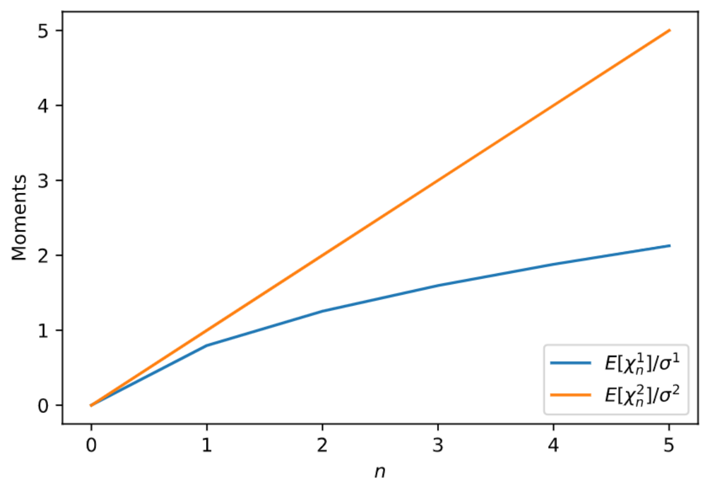


Figure 2: Generalized Chi Distribution Moments

n	$E[X_n^1]$	$E[X_n^2]$
1	0.7979σ	σ^2
2	1.2533σ	$2\sigma^2$
3	1.5957σ	$3\sigma^2$

Table 1: Expected values for different dimensions

We can see that the 2nd moment for X_1 is σ^2, which is equal to the variance of the 1D Normal distribution. This is not the case for distributions of higher dimensions. This is because we are looking at the 2nd moment of the distance to the center, while the parameter σ^2 of the normal multivariate isotropic distribution represents the variance along any axis. In 2D, for example, the variance on the x axis is σ^2 and so is the variance on the y axis. But the 2nd moment of the distance to the center of the distribution is $2\sigma^2$.

2 Chi Distribution in Computer Vision

In computer vision, the isotropic Gaussian filter is often defined in terms of a radial distance $r > 0$ as follows [2]:

$$f(r : \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-r^2/2\sigma^2} \tag{14}$$

This equation accurately defines the contribution of N-d points to the filter at any distance r from the center. However it is not a probability density for any dimension. This can be seen by verifying it in 1D:

$$\int_0^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^1} e^{-r^2/2\sigma^2} dr = 0.5 \tag{15}$$

And noticing that an increase in n will result in eq.15 not being equal to 1. So probability properties such as CDF and moments cannot be calculated from eq.14. To calculate such properties we can use the isotropic multivariate normal distribution with the following PDF:

$$f(\mathbf{x} : \sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-||\mathbf{x}||^2/2\sigma^2} \tag{16}$$

Where $x \in \mathbb{R}^n$. Eq.16 is helpful in calculating the probability of a particular point. But the CDF of this equation covers an hyperrectangle in the Gaussian filter starting from $-\infty_n$, which is hard to apply in any concrete computer vision context. It is a lot more convenient to have our reference point on the center of the Gaussian filter instead. The Chi distribution enables us to do that.

This distribution enables the use of probability tools to analyse how Gaussian filters displace information, the same way we are used to using the univariate Normal distribution.

2.1 Compact proof that 2D Gaussian filter intensity displacement follows the Chi distribution

Let's say we want to know the **sum of all coefficients of a 2D Gaussian Filter that are closer to the center then the distance** r. You can calculate that value by multiplying the density of the Gaussian filter by the area occupied by pixels with that density:

$$P(X < r) = \int \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{x^2}{2\sigma^2}}\ dA \tag{17}$$

In the 2D case, dA is the derivative of the area of a circle $(2\pi r)$. This gives us:

$$\begin{aligned} P(X < r) &= \int_0^r \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{x^2}{2\sigma^2}} 2\pi r\ dr \\ &= 1 - e^{-\frac{r^2}{2\sigma^2}} \end{aligned} \tag{18}$$

Eq.18 is the CDF of the generalized Chi distribution with 2 degrees of freedom. See figure 3 to visualise the 2D Chi distribution along a Gaussian filter.

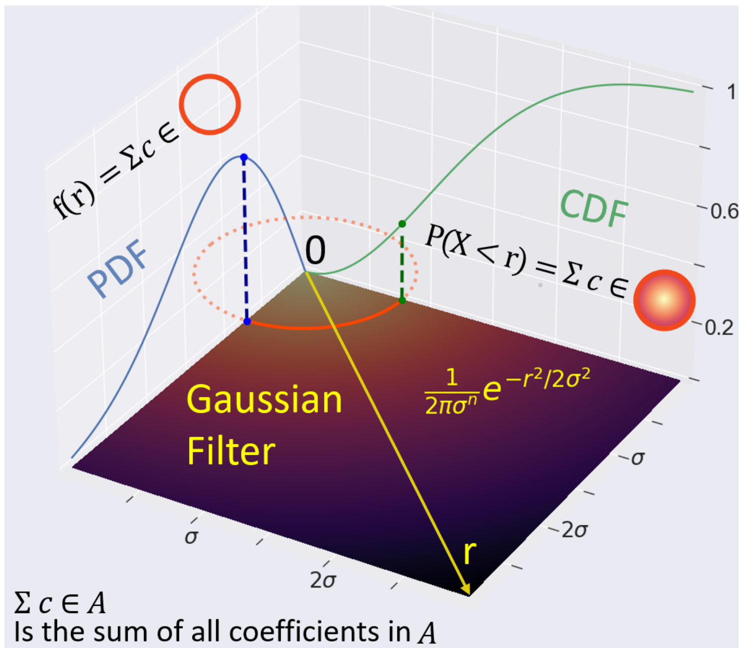


Figure 3: 2D Chi distribution visualisation
The 2D gaussian filter is shown on the XY plane at z=0. $P(X < r)$ can be seen as the sum of all pixels in a Gaussian circle with radius r. $f(r)$ is the product of the value of the Gaussian filter at distance r times the circumference of a circle with radius r. Hence the PDF gets bigger with higher circumference, which is balanced by the Gaussian coefficient getting lower for higher distances.

3 Brownian motion

The Brownian motion is the random motion of particles suspended in a medium. It can be described by the path the particles take, but also by the displacement of the particle. Einstein calculated the mean squared displacement of such particles in 1905 [1].

The mean squared displacement of of a Brownian particle is $2Dt$, where D is the Diffusion constant and t is the time. It can be calculated from the heat equation:

$$P(x,t) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right) \tag{19}$$

Which states that the probability of finding the particle at position x and time t is Gaussian. To convert eq.19 to the form we use in this paper (eq.1) we can use the following substitution:

$$\sigma^2 = 2Dt$$

The mean squared displacement is then equal to σ^2. This is the 2nd moment of the generalized Chi distribution. This substitution is also valid in N-dimension. The mean squared displacement in n dimension is equal to $2nDt$, which is equivalent to the 2nd moment of the Chi distribution of $n\sigma^2$.

To calculate the probability that a N-dimension Brownian particle is situated at position $\mathbf{x} \in \mathbb{R}^n$ we can use the following equation:

$$P(\mathbf{x},t) = \frac{1}{\sqrt{(4\pi Dt)^n}} \exp\left(-\frac{||\mathbf{x}||^2}{4Dt}\right) \tag{20}$$

Again we can perform the same substitution and obtain:

$$P(\mathbf{x}) = \frac{1}{\sqrt{(2\pi\sigma^2)^n}} \exp\left(-\frac{||\mathbf{x}||^2}{2\sigma^2}\right) \tag{21}$$

Which is equal to eq.1 that from our theorem.

This would imply that a Brownian particle displacement follows the generalized Chi distribution. The generalized Chi distribution could be used to answer more complex questions than the mean squared displacement can, such as:

- What's the most probable displacement?
- What's the probability of a particle having a motion of X units or less?
- What's the average displacement?
- ...

A probability distribution provides more insight then a mean squared average on how a variable varies.

References

[1] Albert Einstein. On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat. *Annals of Physics*, 17:549–560, 1905.

[2] Tony Lindeberg. Provably scale-covariant continuous hierarchical networks based on scale-normalized differential expressions coupled in cascade. *Journal of Mathematical Imaging and Vision*, 62(1):120–148, 2020.