Gaussin Filter Intensity Displacement is Chi-distributed

Étienne Pepin

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In computer vision, the Gaussian filter is often defined this way:

$$f(r:\sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-r^2/2\sigma^2}. \tag{1}$$

This equation accurately defines the contribution of N-d points to the filter at any distance r from the center. But it is not the PDF of a probability distribution for any dimension except 1D. You can confirm it by veryfing it in 1D:

$$\int_{-\infty}^{\infty} \frac{1}{(\sqrt{2\pi}\sigma)^1} e^{-r^2/2\sigma^2} dr = 1 \tag{2}$$

And noticing that an increase in n will result in equation 2 not being equal to 1. So probability properties such as CDF and moments cannot be calculated from equation 1. To calculate such properties we can use the isotropic multivariate normal distribution with the following PDF:

$$f(\mathbf{x}:\sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-||x||^2/2\sigma^2} \tag{3}$$

Where $x \in \mathbb{R}^n$. Equation 3 is helpful in calculating the probability of a particular point. But the CDF of this equation covers an hyperrectangle in the Gaussian filter starting from $-\infty_n$, which is hard to apply in any concrete computer vision context. It would be a lot more convenient if our reference point were the center of the Gaussian filter instead.

This is why we will use another distribution to calculate probabilistic properties of the Gaussian filter. The Normal Distribution is the distribution of the position of points $x \in \mathbb{R}^n$. We want to use a distribution of distances to the center instead: $\chi_\sigma = \sqrt{||\mathbf{x}_n||^2}$. The case where σ is equal to 1 is called the chi distribution, we will use the notation $X \sim \chi_{n,\sigma}$ to denote a random variable X that follows the Chi distribution with parameter n and σ.

This distribution enables the use of probability tools to analyse how Gaussian filters displace information, the same way we are used to using the univariate Normal distribution.

In the following sections we will start by presenting how we go from a Gaussian filter to the chi distribution by an example, demonstrate some properties of the chi distribution, discuss how those properties can be useful in the context of computer vision and give the full mathematical proof for going from the Gaussian filter to the chi distribution.

1 Gaussian Filter as a Chi distribution with 2 degrees of freedom

Let's say we want to know the **sum of all coefficients of a 2D Gaussian Filter that are closer to the center then the distance** r. You can calculate that value by multiplying the density of the Gaussian filter by the area occupied by pixels with that density:

$$P(X < r) = \int \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{x^2}{2\sigma^2}} \ dA \tag{4}$$

In the 2D case, dA is the derivative of the area of a circle ($2\pi r$). This gives us:

$$\begin{aligned} P(X < r) &= \int_0^r \frac{1}{(\sqrt{2\pi}\sigma)^2} e^{-\frac{x^2}{2\sigma^2}} 2\pi r \ dr \\ &= 1 - e^{-\frac{r^2}{2\sigma^2}} \end{aligned} \tag{5}$$

Equation 5 is the CDF of the chi distribution with 2 degrees of freedom. We prove this fully for any dimensions in section 3.

2 Generalized Chi distribution properties

Here we look a the generalized chi distribution properties. We call it generalized because the chi distribution is a sum of Normal distributions with $\sigma = 1$. We chose instead to work with Normal distribution with any sigma value since it's more applicable in computer vision.

2.1 PDF and CDF

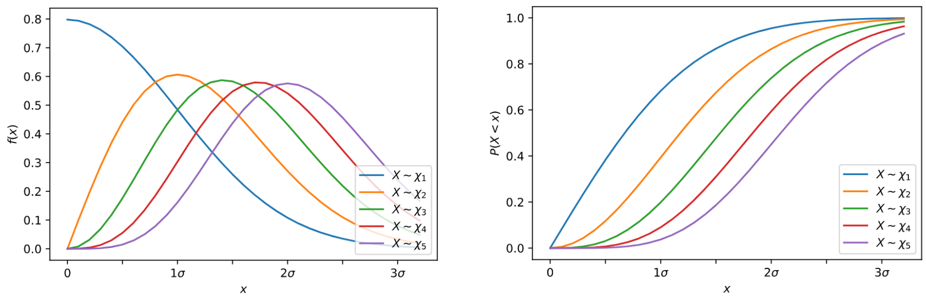
The PDF and CDF of the Generalized Chi distribution are

$$f(x) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \tag{6}$$

$$P(X < x) = \frac{\gamma(\frac{n}{2}, \frac{x^2}{2\sigma^2})}{\Gamma(\frac{n}{2})} \tag{7}$$

with $x \geq 0$. Those 2 equations are plotted for different dimensions in Figure 1 a and b.

The dimension 1 PDF curve is really similar to the 1D normal distribution. The differences are that $x \geq 0$ and it's value is double. We can see that in 1 dimension points closer to the center have the biggest contribution to the Gaussian filter. This is not the case for higher dimensions. While the point with the highest coefficient will always be in the center of the Gaussian filter, the higher number of points away from the center grants them a higher contribution to the Gaussian filter per distance. Thus, the distance of highest contribution increases with the dimension.



(a) PDF (b) CDF

Figure 1: Generalized Chi Distribution PDF and CDF

2.2 Moments

The moments of the generalized chi distribution follow equation 8

$$E[X_n^i] = 2^{i/2}\sigma^i \frac{\Gamma(\frac{n+i}{2})}{\Gamma(\frac{n}{2})} \tag{8}$$

It simplifies to

$$E[X_n^2] = n\sigma^2$$

for the second moment.

The first and second moments are shown in figure 2.

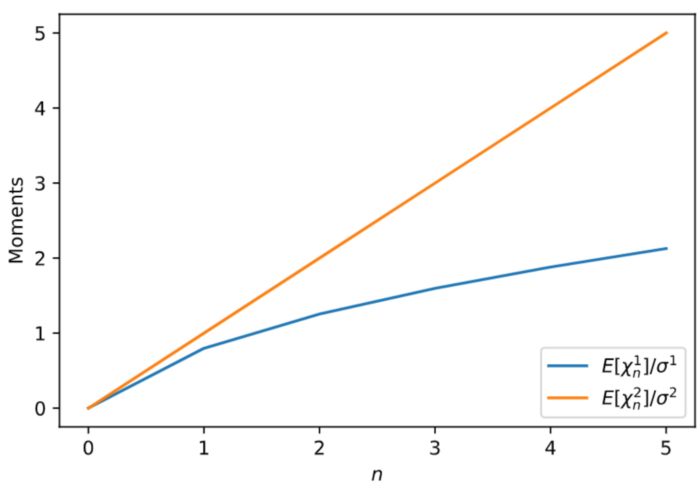


Figure 2: Generalized Chi Distribution Moments

Table 1 contains some values for $E[X_n^1]$ and $E[X_n^2]$.

n	$E[X_n^1]$	$E[X_n^2]$
1	0.7979σ	σ^2
2	1.2533σ	$2\sigma^2$
3	1.5957σ	$3\sigma^2$

Table 1: Expected values for different dimensions

Raw moments of the chi distribution are more useful than central moments to characterize the Gaussian filter since it gives information on the variation of distance with the center of the filter as a reference. While the reference point of the central moments are offset by the first moment, which is nothing concrete in our case.

We can see that the 2nd moment for X_1 is σ^2, which is equal to the variance of the 1D Normal distribution. This is not the case for distributions of higher dimensions. This is because we are looking at the 2nd moment of the distance to the center, while the parameter σ^2 of the normal multivariate isotropic distribution represents the variance along any axis.

In 2D, for example, the variance on the x axis is σ^2 and so is the variance on the y axis. But the 2nd moment of the distance to the center of the distribution is $2\sigma^2$.

In the case of an isotropic distribution, the 2nd moment of the distance to the center seems more appropriate than the variance along an axis as a metric to analyse diffusion of intensity. It also enables comparison between distribution of different dimensions using the same metric, which should not be done with only the σ parameter of Gaussian filters.

3 N-dimension proof

We will start with the multivariate Normal distribution (equation 9) and modify it to obtain the Gaussian filter (equation 1).

$$f(\mathbf{x} : \mu, \Sigma) = \frac{1}{\sqrt{|\Sigma|(2\pi)^n}} \exp\left(-\frac{1}{2}(\mathbf{x}-\mu)^T\Sigma^{-1}(\mathbf{x}-\mu)\right) \tag{9}$$

Where $\mu \in \mathbb{R}^n$ is the mean, Σ is a square covariance matrix, n is the number of dimensions and $|\Sigma|$ is the determinant of the covariance matrix.

Using an average of $\mu = 0$ removes the term and does not restrict us since we want to calculate our CDF from the center of the hypersphere. To get an isotropic normal density, we reduce the covariance matrix to a diagonal matrix with value σ^2 on the diagonal. This leads to

$$\mathbf{x}^T\Sigma^{-1}\mathbf{x} = \frac{||\mathbf{x}||^2}{\sigma^2}$$

and

$$|\Sigma| = (\sigma^2)^n.$$

We then obtain:

$$f(\mathbf{x}:\sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-||x||^2/2\sigma^2} \tag{10}$$

In equation 3, $\mathbf{x}$ is a vector, but since we only use it's norm we can use the square root of the norm as a variable of the function without losing generality. This gives us an univariate function of the radial distance r instead of a multivariate one:

$$f(r:\sigma) = \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-r^2/2\sigma^2}, \tag{11}$$

where $||\mathbf{x}||^2 = r^2$. This is the same equation (eq.1) we used to define the Gaussian filter.

We now want to find the CDF of the distribution X of distances from the center of a Gaussian filter. It will be following this form $P(X < r) = f(n, \sigma)$: the probability of choosing a point at random from a Normal distribution with a distance from the center of r or less. We will count the number of point at every distance r and calculate their combined probability density function. To calculate this we will construct an hypersphere with a gaussian density and calculate it's mass. The n-dimensional multivariate Gaussian density defines iso-contours of equiprobable points. In the case of an isotropic multivariate Gaussian density, these lie on the surface of a hypersphere in $\mathbb{R}^n$.

The mass of the n-dimensional hypersphere can be calculated with the following formula:

$$M(r) = \int_0^r density(r)\ dV$$

For the hypersphere density we will use eq.11. Which results in the CDF:

$$F(r) = P(X < r) = \int_0^r \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-\frac{x^2}{2\sigma^2}} dV \tag{12}$$

where V is the n-dimensional volume (content) of the hypersphere. Let S_n be the hyper-surface area of an n-dimensional hypersphere of unit radius. From the literature[1]:

$$S_n = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})}$$

and

$$V = \int_0^r S_n x^{n-1} dx = \frac{S_n r^n}{n}.$$

Which we use to calculate dV:

$$\begin{aligned} V &= \frac{2\pi^{n/2} r^n}{n\Gamma(\frac{n}{2})} \\ dV &= \frac{V}{dr} = \frac{2\pi^{n/2} r^{n-1}}{\Gamma(\frac{n}{2})} \end{aligned} \tag{13}$$

[1] `https://mathworld.wolfram.com/Hypersphere.html`

We can now combine equation 12 and 13 to calculate the CDF:

$$P(X < r) = \int_0^r \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \ dx \tag{14}$$

We move the terms we can out of the integral.

$$= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^r e^{-x^2/2\sigma^2} x^{n-1} \ dx$$

We will modify the terms of the integral in order to obtain the lower gamma function γ starting with a variable substitution:

$$\frac{x^2}{2\sigma^2} = y \rightarrow \begin{cases} \frac{x}{\sigma^2} dx = dy \\ x = \sqrt{2y\sigma^2} \end{cases}$$

The upper limit of the integral goes from r to $\frac{r^2}{2\sigma^2}$.

$$= \frac{2\sigma^2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^r e^{-x^2/2\sigma^2} x^{n-2} \frac{x}{\sigma^2} \ dx = \frac{2\sigma^2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^{\frac{r^2}{2\sigma^2}} e^{-y} (\sqrt{y2\sigma^2})^{n-2} \ dy$$

$$= \frac{2^{n/2}\sigma^n}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^{\frac{r^2}{2\sigma^2}} e^{-y} y^{n/2-1} \ dy$$

By definition of the γ function we obtain:

$$= \frac{1}{\Gamma(\frac{n}{2})} \ \gamma(\frac{n}{2}, \frac{r^2}{2\sigma^2})$$

$$P(X < r) = \frac{\gamma(\frac{n}{2}, \frac{r^2}{2\sigma^2})}{\Gamma(\frac{n}{2})} \tag{15}$$

You can change variable r to $x\sigma$, where x is the mahalanobis distance:

$$P(X < x) = \frac{\gamma(\frac{n}{2}, \frac{x^2}{2})}{\Gamma(n/2)} \tag{16}$$

Equation 16 is the CDF of the chi distribution. It is related to the chi-square distribution, which is the distribution of the sum of n squared standard normal distribution. We will work with eq.15 instead, since Gaussian filters often don't have $\sigma = 1$.

We can obtain the probability density function from equation 14 by deriving it.

$$f(x) = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \frac{1}{(\sqrt{2\pi}\sigma)^n} e^{-x^2/2\sigma^2} x^{n-1} \tag{17}$$

We now concluded that the distribution of distances to the center in a Gaussian filter follow a generalized chi distribution.

3.1 Moments

With the PDF we can calculate moments i of a random variable X_n following the generalized chi distribution.

$$
\begin{aligned}
E[X_n^i] &= \int_0^\infty x^i \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} e^{-x^2/2\sigma^2} x^{n-1}\ dx \\
&= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-x^2/2\sigma^2} x^{n+i-1}\ dx
\end{aligned}
$$

We will modify the terms of the integral to obtain the Gamma function Γ starting with a variable substitution:

$$
\begin{aligned}
&\frac{x^2}{2\sigma^2} = y \rightarrow \begin{cases} \dfrac{x}{\sigma^2} dx = dy \\ x = \sqrt{2y\sigma^2} \end{cases} \\
&= \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-x^2/2\sigma^2} x^{n+i-2} \sigma^2 \frac{x}{\sigma^2}\ dx = \frac{2}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-y} (\sqrt{y 2\sigma^2})^{n+i-2}\ \sigma^2 dy \\
&= \frac{2^{(n+i)/2}\sigma^{n+i}}{2^{n/2}\Gamma(\frac{n}{2})\sigma^n} \int_0^\infty e^{-y} y^{(n+i)/2-1}\ dy
\end{aligned}
$$

By definition of the Γ function we obtain:

$$
E[X_n^i] = 2^{i/2}\sigma^i \frac{\Gamma(\frac{n+i}{2})}{\Gamma(\frac{n}{2})}
$$

Below are values of the Γ function related to our case:

$$
\begin{aligned}
\Gamma(\frac{1}{2}) &= \sqrt{\pi} \\
\Gamma(n + \frac{1}{2}) &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^n} \sqrt{\pi}
\end{aligned}
$$