Homework Assignment N°2

BML36
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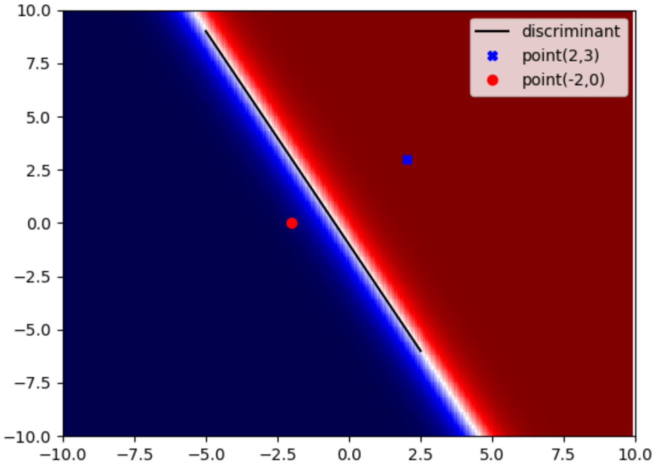
Contents

1 Exercise 1: Perceptron Learning Rule

1.1 Part a

To have an idea of what our perceptron looks like, let's plot its discriminant line and the two data points.



We can also compute the prediction of the model for the two datapoints:

$$\text{prediction}_1 = sign(w^\top x_1) = sign(8) = 1 \neq -1$$

$$\text{prediction}_2 = sign(w^\top x_2) = sign(-3) = -1 \neq 1$$

So our two datapoints are missclassified, now we can say that on the plot below area in red (upper right of the plot) represents the area classified as class 1 by our model and blue (lower left part of the plot) is for class -1

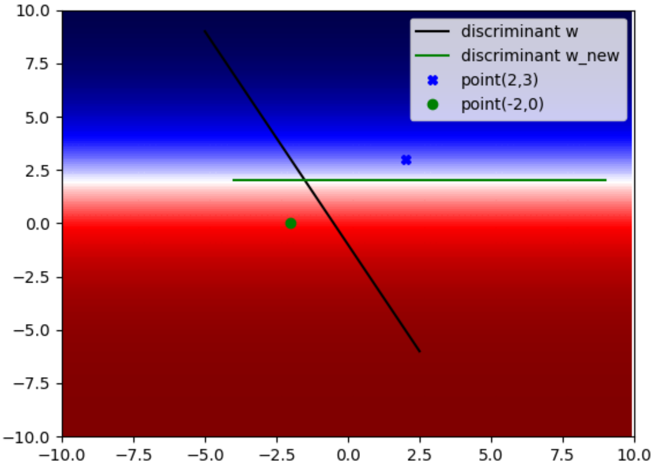
1.2 Part b

After one iteration of a batch learning algorithm we get the following new discriminant:

$$w_{new} = w_{old} + \eta \sum_{i=1}^{N} x_i c_i$$

$$w_{new} = \begin{bmatrix} 1 & 0 & -0.5 \end{bmatrix}$$

And now, both data points are acorrectly classified by the new model:



1.3 Part c

Learning is an iterative process. In a batch version learning, at each iteration we compute a new model based on the whole dataset whereas in a stochastic version, at each iteration we pick up a single data in the dataset and we base our computation on this data point only.
When the dataset becomes too big, the computation time of a batch iteration is going to need ressources proportional to the size of the dataset and a stochastic iteration will always use the same amount of ressources independantly of the size of the dataset.

1.4 Part d

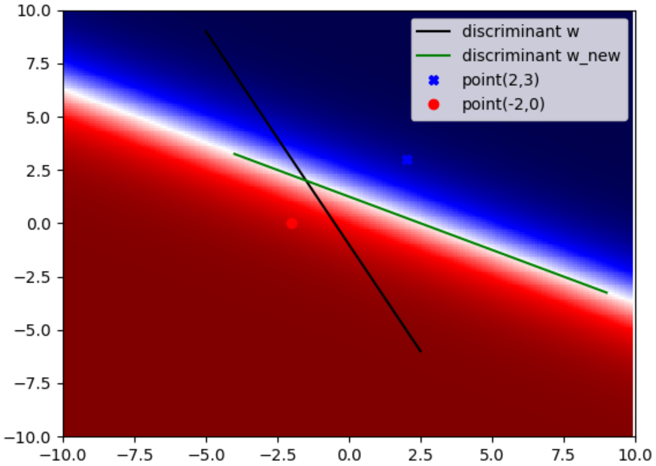
The formula to compute the new discriminant with a stochastic algorithm is the following:

$$w_{new} = w_{old} + \eta x_i c_i$$

We repeat it for each data in the dataeset, updating the discriminant at each iteration.
When applying the stochastic algorithm, we get the following (different) result:

$$w_{new} = \begin{bmatrix} 1 & -0.4 & -0.8 \end{bmatrix}$$

And again, both points are correctly classified after the complete sweep:



2 Exercise 2: Logistic classification & discrimination

From now we will use $\sigma(x) = \frac{1}{1+e^{-x}}$

2.1 Part a

- Initialize w_0 ?

  1. Some fixed w_0 like $\begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$
  2. The result of computation around the dataset like the mean: $w_0 = \frac{1}{N}\sum_{i=1}^{N} x_i$, concatenated with a constant.
  3. A random vector

  Any vector except the null vector and the multiples of $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$ is suitable to initialize w_0,

- How to learn: for batch learning use this equation at each step

$$w_{n+1} = w_n - \eta \nabla E(w_n) = w_n - \eta \sum_{n=1}^{N} (y(n) - t_n)\, x_n$$

- How to stop the iterative process ?

  1. Stop when the norm of the difference vector is low: $\Delta_n = \frac{\|w_{n+1} - w_n\|}{\|w_n\|} < \epsilon$
     This is a commonly used criterion that stops the process when the steps we take are getting small compared to our current result.
  2. Stop after fixed number of iteration
     This ensures we won't enter in a infinite non-convergent process.

3. Stop when a threshold error is reached: $E(w_n) < \epsilon$
   This is actually a bad idea because most of the time we can't be certain it is possible to reach such threshold on the error. It would result in an infinite process.

In a batch version, we can use criteria 1 and 2 together and stop whenever one of the criteria is reached.
In a stochastic version, criterium 1 is not applicable because it would stop the learning process whenever a well classified data is picked for an iteration.

Our algorithm goes as follows:

1. Chose ϵ, N and η respectivelly for precision, maximum number of iterations and speed convergency.

2. Set current error Δ to $+\infty$ and n to 0

3. Chose the initial discriminant: $w_{current} = \begin{bmatrix} 0 & 0 & \cdots & 1 \end{bmatrix}$.

4. While $\Delta > \epsilon \wedge n < N$ do

   (a) Compute and store next discriminant w_{next}:

   $$w_{next} = w_{current} - \eta \sum_{n=1}^{N} \left(\sigma({w_{current}}^\top x_n) - t_n \right) x_n$$

   (b) Compute and store the new error Δ:

   $$\Delta = \frac{\|w_{next} - w_{current}\|}{\|w_{current}\|}$$

   (c) Prepare for next iteration: store w_{next} in place of $w_{current}$ and increment n

5. If $\Delta > \epsilon$, it means we have not converged enough towards the limit. We should consider increasing N OR using another algorithm for convergence (eg. Newton-Raphson)

6. Result is stored in $w_{current}$, number of steps in n.

2.2 Part b

First important thing to notice is that the point $x = \begin{bmatrix} -1 & 1 \end{bmatrix}$ is missclassified.
We define $\bar{x} = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$ thus it means that when we compute $w^\top \bar{x}$ the sign of the result is incorrect.
In our case, $w^\top \bar{x} = 1 > 0$, thus the real class of x is 0.
The formula to update the weights is the following:

$$w_{new} = w - \eta\sigma(w^\top \bar{x})\bar{x}$$

We get the following result:

$$w_{new} \approx \begin{bmatrix} 0.5614 & 2.4386 & 1.5614 \end{bmatrix}$$

3 Exercise 3: Error function & Gradient descent

3.1 Part a

If $y_n = \sigma(w^\top x_n)$ then:

$$E(w) = \sum_{n=1}^{N} (t_n - y_n)^4$$

$$\nabla_w E(w) = -4 \sum_{n=1}^{N} (t_n - y_n)^3 (y_n(1 - y_n)) x_n$$

3.2 Part b

$$E(w) = \sum_{n=1}^{N} |t_n - y_n|$$

Computing the gradient of an absolute value implies to derivate an absolute value function which is not C^1. We won't be able to assign a value to the gradient if the value inside the absolute function is 0.

$$\nabla_w E(w) = \sum_{n=1}^{N} y_n(1 - y_n) x_n \times \left\{ \begin{array}{ll} -1 & \text{if } t_n - y_n > 0 \\ 1 & \text{if } t_n - y_n < 0 \end{array} \right.$$

4 Exercise 4: MCQ

4.1 First MCQ

This first question tests if the candidate knows how to compute the new discriminant from the previous one, using gradient descent.
Q: Which of the following equations can be used to update the discriminant w using the method of gradient descent ?

1. $w_{n+1} - w_n = \eta \nabla E(w_n)$
2. $w_{n+1} = w_n - \eta \nabla E(w_n)$
3. $w_n + \eta \nabla E(w_{n-1}) = w_{n-1}$
4. $w_n = \eta w_n - \nabla E(w_{n+1})$

4.2 Second MCQ

The second question verifies the student understood well what is implied behind the omnipresent formula $w^\top x$ and each of its components.
Q: When learning in a k-dimensions space, knowing that we compute the class of x by the following formula $C(x) = h(w^\top x + w_0)$ where h is the heaviside function. What are the ranges of w and x ?

1. $range(w) = k \quad range(x) = k$
2. $range(w) = k + 1 \quad range(x) = k$

3. $range(w) = k \qquad range(x) = k+1$

4. $range(w) = k+1 \quad range(x) = k+1$