Homework Assignment N°4

BML36
Thibault Douzon
Rajavarman Mathivanan

October 3rd, 2018

Contents

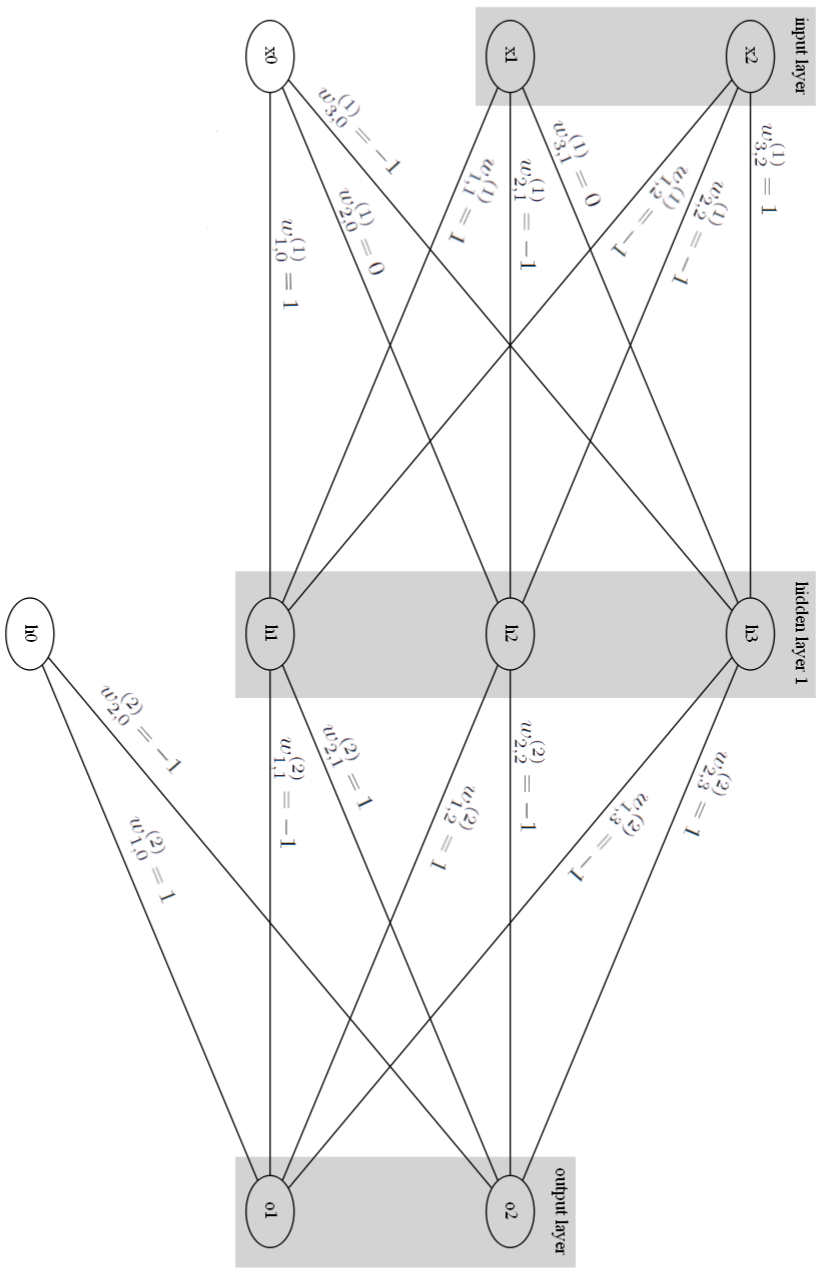
1 Exercise 1: NN 1-D output

1.1 Part a

2 Exercise 2: NN 2-D output

2.1 Part a

The neural network described in this exercise could be represented like this.



It is structured in 3 parts: the input layer first (denoted $x1$, $x2$), then the hidden layer (denoted $h1$, $h2$ and $h3$) and then the output layer (denoted $o1$ and $o2$). Nodes with indice 0 represent the bias introducted into the model.
Each arc carries the value of the weight (denoted $w_{k,j}^{(l)}$ where l is the layer, k is the destination node and j is the origin node which lies in the layer $l-1$).

To compute the output of the neural network, we need to propagate through the network the values of the input. Hidden layer applies a sigmoïd activation function and output neurons have a linear activation function (identity function).
Thus we can first compute every $a_k^{(1)}$, the signal recieved by each node in the hidden layer. It is easier to make this computation under matrix representation, let's introduce the input vector X and the weights of the first layer $W^{(1)}$:

$$X = [a_i^{(0)}] = [x_i] = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$W^{(1)} = [w_{j,i}^{(1)}] = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -1 & 0 \\ -1 & -1 & 1 \end{bmatrix}$$

The signal recieved by the hidden layer is given by the following formula:

$$H_{j \neq 0} = [a_j^{(1)}]_{j \neq 0} = X^\top W^{(1)} = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$$

Before repeating the same procedure, we need to apply the activation function to each recieved signal and then concatenate the bias.
The hidden layer uses the sigmoïd function as activation:

$$\sigma(H_{j \neq 0}) = [\sigma(a_j(1))] = \begin{bmatrix} \sigma(1) \\ \sigma(-2) \\ \sigma(0) \end{bmatrix} \approx \begin{bmatrix} 0.731 \\ 0.119 \\ 0.5 \end{bmatrix}$$

Now we can concatenate the bias at the beginning with a fixed value of 1:

$$H_{out} \approx \begin{bmatrix} 1 \\ 0.731 \\ 0.119 \\ 0.5 \end{bmatrix}$$

The vector H_{out} is the signed emitted by the hidden layer.
We can now repeat the same procedure with the second layer with H_{out} as input

and use the weights of the output layer.
The weights of the second layer are the following:

$$W^{(2)} = [w_{k,j}^{(2)}] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix}$$

And we can compute the signal recieved by the output layer:

$$O = [a_k^{(2)}] = {H_{out}}^\top W^{(2)} \approx \begin{bmatrix} -0.112 \\ 0.112 \end{bmatrix}$$

This is the output of our neural network as the activation fonction of the output neuron is the identity function.

2.2 Part b

The formula to compute $\delta_i^{(l)}$ is the following:

$$\delta_i^{(l)} = \frac{\partial E_n}{\partial a_i^{(l)}}$$

Where l is the layer and i is the index of the neuron.
But to compute the value of this expression for different values of l and i, we need to know the expression of E_n first. E_n is the error of the network on the data sample number n. It is defined as follows:

$$E_n(w) = \frac{1}{2}\sum_{k=1}^{D}(y_k(x_n, w) - t_{n,k})^2$$

Where N is the number of output neurons (in our case 2), x_n the input data and t_n the target related to x_n, w the current weigths of the neural network and y_k the function that computes the k^{th} coordinate of the NN's prediction based on the input data.
In our case, we can rewrite this expression to this:

$$E_n(w) = \frac{1}{2}\sum_{k=1}^{2}(y_k(x_n, w) - t_{n,k})^2$$

And because the activation function of the output layer is the identity function, we also have the following equation:

$$y_k(x_n, w) = h(a_k^{(2)}) = a_k^{(2)}$$

The analytical expressions of $\delta_1^{(2)}$ and $\delta_2^{(2)}$ directly follows:

$$\delta_i^{(2)} = \frac{\partial E_n}{\partial a_i^{(2)}} = \frac{1}{2}\frac{\partial \sum_{k=1}^{2}(a_k^{(2)} - t_{n,k})^2}{\partial a_i^{(2)}} = a_i^{(2)} - t_{n,i}$$

We finally get the following numerical values:

$$\delta_1^{(2)} = a_1^{(2)} - t_{n,1} = -0.112 - 1 = -1.112$$

$$\delta_2^{(2)} = a_2^{(2)} - t_{n,2} = 0.112 - (-1) = 1.112$$

2.3 Part c

The formula to update a weight is