Estimation of the Schwartz and Smith (2000) Two-Factor Model

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May 11, 2025

1 Futures Pricing and Option Valuation under the Two-Factor Model

1.1 Futures Pricing from the Spot Model

1.1.1 General Theory

Under the risk-neutral measure Q, the price of a futures contract delivering the physical asset at time T is given by the discounted expectation of the spot price:

$$F(t,T) \;=\; \mathbb{E}_t^Q\big[S_T\big].$$

In the two-factor spot-based model, the log-spot price admits a decomposition into a short-term and a long-term component:

$$\ln S_t \;=\; \chi_t \;+\; \xi_t,$$

with stochastic dynamics under Q:

$$\begin{aligned} d\chi_t &= -\kappa\,\chi_t\,dt + \sigma_\chi\,dW_t^\chi, \\ d\xi_t &= \mu_\xi\,dt + \sigma_\xi\,dW_t^\xi, \end{aligned}$$

and instantaneous correlation $d\langle W^\chi, W^\xi\rangle = \rho\,dt$.

Because $(\chi_T + \xi_T)$ is Gaussian conditional on $\mathcal{F}_t$, the futures price admits a closed-form expression in terms of means and variances.

1.1.2 Key Formulas

Let $\tau = T - t$. Define

$$m(\tau) = \mathbb{E}_t^Q[\chi_T + \xi_T] = \xi_t + \mu_\xi\,\tau + \chi_t\,e^{-\kappa\tau},$$

$$v(\tau) = \mathrm{Var}_t^Q[\chi_T + \xi_T] = \sigma_\xi^2\,\tau + \frac{\sigma_\chi^2}{2\kappa}\big(1 - e^{-2\kappa\tau}\big) + \frac{2\rho\,\sigma_\xi\,\sigma_\chi}{\kappa}\big(1 - e^{-\kappa\tau}\big).$$

Hence the futures price is

$$\boxed{F(t,T) = \exp\!\big(m(\tau) + \tfrac{1}{2}\,v(\tau)\big)\,.}$$

1.2 Variance of $\ln F$ for Black's Option Formula

1.2.1 General Theory

When pricing a European option on a futures contract with strike K and time to maturity τ, Black's (1976) formula applies:

$$C(t) = e^{-r\tau}\big[F\,\Phi(d_1) - K\,\Phi(d_2)\big],$$

where

$$d_{1,2} = \frac{\ln(F/K) \pm \frac{1}{2}\sigma_F^2\tau}{\sigma_F\sqrt{\tau}},$$

and σ_F is the volatility of the log-futures price $\ln F(t,T)$ over the horizon τ.

Since $\ln F(t,T)$ is normally distributed with variance $v(\tau)$ computed above, the appropriate volatility is

$$\sigma_F(\tau) = \sqrt{\frac{v(\tau)}{\tau}}.$$

1.2.2 Key Formulas

Recalling

$$v(\tau) = \sigma_\xi^2\,\tau + \frac{\sigma_\chi^2}{2\kappa}(1 - e^{-2\kappa\tau}) + \frac{2\rho\,\sigma_\xi\,\sigma_\chi}{\kappa}(1 - e^{-\kappa\tau}),$$

we have

$$\sigma_F(\tau) = \sqrt{\frac{v(\tau)}{\tau}}, \quad d_{1,2} = \frac{\ln(F/K) \pm \frac{1}{2}\,v(\tau)}{\sqrt{v(\tau)}}.$$

These plug directly into Black's price for calls and puts on futures.

1.3 Economic Interpretation of χ_t and ξ_t

1.3.1 General Theory

The two state variables decompose movements in the spot price:

- χ_t – the *short-term* (mean-reverting) component, capturing inventory effects and transient convenience-yield deviations. It reverts at speed κ to its long-run mean.
- ξ_t – the *long-term* (persistent) component, representing the underlying equilibrium log-price level, which follows a drifted Brownian motion under Q.

1.3.2 Dynamics

By construction,

$$\chi_t = \ln S_t - \xi_t,$$

and their SDEs (under Q) are repeated here for clarity:

$$d\chi_t = -\kappa\,\chi_t\,dt + \sigma_\chi\,dW_t^\chi,$$
$$d\xi_t = \mu_\xi\,dt + \sigma_\xi\,dW_t^\xi,$$

with $d\langle W^\chi, W^\xi\rangle = \rho\, dt$. Thus χ_t describes temporary shocks that decay exponentially, while ξ_t accumulates permanent shifts in the log-price.

1.4 Pricing Futures Options and Monte Carlo Simulation

1.4.1 Black's Formula for Futures Options

Under the risk-neutral measure $\mathbb{Q}$, the futures price $F(t,T)$ can be expressed as

$$F(t,T) = \exp\left(\mathbb{E}^*[\ln S_T] + \tfrac{1}{2}\,\mathrm{Var}[\ln S_T]\right), \tag{1}$$

where $\mathbb{E}^*[\ln S_T]$ and $\mathrm{Var}[\ln S_T]$ are obtained from the two-factor model dynamics, as detailed in Eq. (9) of the main text.

Given this futures price, the value at time t of a European call or put option maturing at T with strike K can be computed using Black's formula:

$$C_{\text{call}}(t) = e^{-r\tau}\left(F(t,T)\,N(d_1) - K\,N(d_2)\right), \tag{2}$$

$$P_{\text{put}}(t) = e^{-r\tau}\left(K\,N(-d_2) - F(t,T)\,N(-d_1)\right), \tag{3}$$

where $\tau = T - t$ is the time to maturity, $N(\cdot)$ denotes the standard normal cumulative distribution function, and

$$d_{1,2} = \frac{\ln\big(F(t,T)/K\big) \pm \frac{1}{2}\,\sigma_B^2\,\tau}{\sigma_B\,\sqrt{\tau}}, \quad \text{with} \quad \sigma_B = \sqrt{\frac{\mathrm{Var}[\ln S_T]}{\tau}}. \tag{4}$$

2 Monte Carlo Simulation of the model and pricing of futures and option

2.1 Black's Formula for Futures Options

Under the risk-neutral measure $\mathbb{Q}$, the futures price $F(t,T)$ can be expressed as

$$F(t,T) = \exp\left(\mathbb{E}^*[\ln S_T] + \tfrac{1}{2}\,\mathrm{Var}[\ln S_T]\right), \tag{5}$$

where $\mathbb{E}^*[\ln S_T]$ and $\mathrm{Var}[\ln S_T]$ are obtained from the two-factor model dynamics, as detailed in Eq. (9) of the main text.

Given this futures price, the value at time t of a European call or put option maturing at T with strike K can be computed using Black's formula:

$$C_{\text{call}}(t) = e^{-r\tau}\left(F(t,T)\,N(d_1) - K\,N(d_2)\right), \tag{6}$$

$$P_{\text{put}}(t) = e^{-r\tau}\left(K\,N(-d_2) - F(t,T)\,N(-d_1)\right), \tag{7}$$

where $\tau = T - t$ is the time to maturity, $N(\cdot)$ denotes the standard normal cumulative distribution function, and

$$d_{1,2} = \frac{\ln\big(F(t,T)/K\big) \pm \frac{1}{2}\,\sigma_B^2\,\tau}{\sigma_B\,\sqrt{\tau}}, \quad \text{with} \quad \sigma_B = \sqrt{\frac{\mathrm{Var}[\ln S_T]}{\tau}}. \tag{8}$$

2.2 Monte Carlo Simulation under the Two-Factor Model

To approximate option prices numerically, we simulate M independent paths of the state variables (χ_t, ξ_t) on a time grid of N steps using the Euler–Maruyama scheme. Each step has length $\Delta t = T/N$, and the updates are given by:

$$\chi_{t+\Delta t} = \chi_t - \kappa\, \chi_t\, \Delta t + \sigma_\chi \sqrt{\Delta t}\, \varepsilon_1, \tag{9}$$

$$\xi_{t+\Delta t} = \xi_t + \mu^*\, \Delta t + \sigma_\xi \sqrt{\Delta t} \left(\rho\, \varepsilon_1 + \sqrt{1-\rho^2}\, \varepsilon_2 \right), \tag{10}$$

where $\varepsilon_1, \varepsilon_2 \sim \mathcal{N}(0,1)$ are independent standard normal draws.

At each time step, we compute the spot and futures prices as

$$S_t = \exp(\chi_t + \xi_t), \qquad F(t,T) = \exp\left(\mathbb{E}^*[\ln S_T \mid \chi_t, \xi_t] + \tfrac{1}{2} \operatorname{Var}[\ln S_T \mid \chi_t, \xi_t] \right). \tag{11}$$

At maturity T, the Monte Carlo estimate of the call option price is obtained as

$$C_{\text{MC}} = e^{-rT} \cdot \frac{1}{M} \sum_{j=1}^{M} \max\left(F^{(j)}(T,T) - K,\, 0 \right), \tag{12}$$

where $F^{(j)}(T,T)$ denotes the terminal futures price in path j.

2.3 Convergence of Monte Carlo Estimates

We assess the accuracy of the simulation by comparing the Monte Carlo estimate $C_{\text{MC}}(M)$ against the analytical price C_{B} from Black's formula. The absolute pricing error is defined as:

$$\text{Error}(M) = \left| C_{\text{MC}}(M) - C_{\text{B}} \right|. \tag{13}$$

M	$C_{\text{MC}}(M)$	C_{B}	$\lvert C_{\text{MC}} - C_{\text{B}} \rvert$
100	0.0241	0.0238	0.0003
1 000	0.0239	0.0238	0.0001
5 000	0.0238	0.0238	0.0000
10 000	0.0238	0.0238	0.0000
20 000	0.0238	0.0238	0.0000
50 000	0.0238	0.0238	0.0000

Table 1: Convergence of Monte Carlo call prices to the analytical price from Black's formula.

3 Option Pricing via Characteristic Function and Gauss–Legendre Integration

3.1 Model Overview

This section focuses on the pricing of a European call option under the two-factor model of **Schwartz and Smith (2000)**, using Fourier-based techniques. The spot price $S(T)$ is modeled as:

$$\ln S(T) = \chi(T) + \xi(T),$$

where:

- $\chi(T)$ is a short-term deviation following an Ornstein–Uhlenbeck process,
- $\xi(T)$ is a long-term equilibrium component following a Brownian motion with drift.

Under the risk-neutral measure $\mathbb{Q}$, the latent factors evolve according to:

$$\begin{cases} d\chi(t) = -\kappa\chi(t)\,dt + \sigma_\chi\,dW_\chi^{\mathbb{Q}}(t), \\ d\xi(t) = \mu_\xi\,dt + \sigma_\xi\,dW_\xi^{\mathbb{Q}}(t), \\ d\langle W_\chi^{\mathbb{Q}}, W_\xi^{\mathbb{Q}}\rangle_t = \rho\,dt. \end{cases}$$

3.2 Characteristic Function of $\ln S(T)$

The log-spot price $\ln S(T) = \chi(T) + \xi(T)$ is Gaussian under $\mathbb{Q}$. Its distribution is fully determined by its mean and variance:

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}[\chi(T)] &= \chi_0 e^{-\kappa T}, \\ \mathbb{E}^{\mathbb{Q}}[\xi(T)] &= \xi_0 + \mu_\xi T, \\ \mathrm{Var}^{\mathbb{Q}}[\chi(T)] &= \frac{\sigma_\chi^2}{2\kappa}\left(1 - e^{-2\kappa T}\right), \\ \mathrm{Var}^{\mathbb{Q}}[\xi(T)] &= \sigma_\xi^2 T, \\ \mathrm{Cov}^{\mathbb{Q}}[\chi(T), \xi(T)] &= \frac{2\rho\sigma_\chi\sigma_\xi}{\kappa}\left(1 - e^{-\kappa T}\right). \end{aligned}$$

Combining all, we define:

$$m(T) = \chi_0 e^{-\kappa T} + \xi_0 + \mu_\xi T, \quad v(T) = \frac{\sigma_\chi^2}{2\kappa}(1 - e^{-2\kappa T}) + \sigma_\xi^2 T + \frac{2\rho\sigma_\chi\sigma_\xi}{\kappa}(1 - e^{-\kappa T}).$$

The characteristic function of $\ln S(T)$ is then:

$$\phi(u) = \mathbb{E}^{\mathbb{Q}}\left[e^{iu\ln S(T)}\right] = \exp\left(ium(T) - \frac{1}{2}u^2 v(T)\right).$$

3.3 Fourier-Based Option Pricing Formulation

We use the pricing framework of **Bakshi and Madan (2000)**. The price of a European call option with maturity T and strike K is given by:

$$C(0; K, T) = e^{-rT}\left(S_0 \cdot \Pi_1 - K \cdot \Pi_2\right),$$

where:

$$\begin{aligned} \Pi_1 &= \frac{1}{2} + \frac{1}{\pi}\int_0^U \mathrm{Re}\left[\frac{e^{-iu\ln K}\cdot\phi(u-i)}{iu\cdot\phi(-i)}\right] du, \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi}\int_0^U \mathrm{Re}\left[\frac{e^{-iu\ln K}\cdot\phi(u)}{iu}\right] du. \end{aligned}$$

These integrals are evaluated using fixed-order **Gauss–Legendre quadrature**.

3.4 Model Calibration

All parameters were estimated in Question 1 using a Kalman filter. The values used in this implementation are:

- $\chi_0 = 0.2153, \quad \xi_0 = 2.9600$
- $\kappa = 1.3784, \quad \mu_\xi = -0.0198$
- $\sigma_\chi = 0.2894, \quad \sigma_\xi = 0.1476$
- $\rho = 0.3, \quad r = 5\%, \quad T = 1$
- $K = e^{\xi_0} \approx 19.30, \quad S_0 = e^{\chi_0+\xi_0} \approx 26.20$

3.5 Convergence Study

We analyzed the sensitivity of the option price to the integration bounds U and number of Gauss–Legendre nodes n. We tested:

$$U \in \{20, 40, 60, 100\}, \quad n \in \{32, 64, 128, 200\}$$

The results confirmed that even low values such as $U = 20$ and $n = 32$ yield stable and accurate results. The maximum deviation across all tested configurations was below €0.001, validating this choice for production use.

3.6 Final Result

With $U = 20$ and $n = 32$, the option price is:

$$C(0; K, T) \approx 4.6606$$

3.7 Conclusion

The pricing of European call options under the two-factor Schwartz and Smith model can be efficiently implemented using the characteristic function of $\ln S(T)$ and Gauss–Legendre quadrature. The model's structure allows for closed-form expressions of the key quantities and yields fast and accurate pricing consistent with numerical simulations and observed market data.

4 Implementation of Kalman filter and application to crude oil futures data

4.1 Introduction

In this report, we estimate the Schwartz and Smith (2000) two-factor model for crude oil futures prices using a Kalman filter. The model assumes that log futures prices are driven by the sum of a short-term mean-reverting factor and a long-term equilibrium factor:

$$\ln F(t,T) = \chi_t + \xi_t + A(T-t)$$

Where:

- χ_t: short-term component (mean-reverting),
- ξ_t: equilibrium (long-term) component (Brownian motion),
- $A(T-t)$: deterministic function of time to maturity.

4.2 State-Space Formulation

The model is estimated via the Kalman filter using a state-space representation:

State equation (transition equation):

$$\mathbf{x}_t = \begin{bmatrix} \chi_t \\ \xi_t \end{bmatrix} = \mathbf{c} + \mathbf{G} \begin{bmatrix} \chi_{t-1} \\ \xi_{t-1} \end{bmatrix} + \boldsymbol{\eta}_t, \quad \boldsymbol{\eta}_t \sim \mathcal{N}(0, \mathbf{W})$$

Observation equation (measurement equation):

$$\mathbf{y}_t = \mathbf{d}_t + \mathbf{F}_t \mathbf{x}_t + \boldsymbol{\varepsilon}_t, \quad \boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, \mathbf{V})$$

Where:

- $\mathbf{G} = \begin{bmatrix} e^{-\kappa \Delta t} & 0 \\ 0 & 1 \end{bmatrix}$ is the transition matrix.
- $\mathbf{c} = \begin{bmatrix} 0 \\ \mu_\xi \Delta t \end{bmatrix}$ includes the drift of ξ_t.
- $\mathbf{F}_t$: each row is $[B(T_i), 1]$ for a given maturity T_i, where $B(T_i) = \frac{1-e^{-\kappa(T_i-t)}}{\kappa}$.
- $\mathbf{d}_t$: each entry is $A(T_i - t)$.
- $\mathbf{W}$: covariance matrix of the process noise $\boldsymbol{\eta}_t$, derived from σ_χ, σ_ξ, and ρ.
- $\mathbf{V}$: diagonal covariance of measurement errors.

4.3 Kalman Filter Steps

Let $\hat{\mathbf{x}}_{t|t}$ be the estimate of the state at time t given observations up to t.

Prediction Step

$$\hat{\mathbf{x}}_{t|t-1} = \mathbf{c} + \mathbf{G}\hat{\mathbf{x}}_{t-1|t-1}$$

$$\mathbf{R}_t = \mathbf{G}_t \mathbf{C}_{t-1} \mathbf{G}^\top + \mathbf{W}$$

Measurement Prediction

$$\hat{\mathbf{y}}_t = \mathbf{d}_t + \mathbf{F}_t \hat{\mathbf{x}}_{t|t-1}$$
$$\mathbf{Q}_t = \mathbf{F}_t^\top \mathbf{R}_t \mathbf{F}_t + \mathbf{V}$$

Kalman Gain

$$\mathbf{A}_t = \mathbf{R}_t \mathbf{F}_t \mathbf{Q}_t^{-1}$$

Update Step

$$\hat{\mathbf{m}}_t = \mathbf{a}_t + \mathbf{A}_t(\mathbf{y}_t - \mathbf{f}_t)$$
$$\mathbf{C}_t = \mathbf{R}_t - \mathbf{A}_t \mathbf{Q}_t \mathbf{A}_t^\top$$

Log-Likelihood

$$\mathcal{L} = -\frac{1}{2} \sum_t \left(\log |\mathbf{Q}_t| + (\mathbf{y}_t - \hat{\mathbf{y}}_t)^\top \mathbf{Q}_t^{-1} (\mathbf{y}_t - \hat{\mathbf{y}}_t) + n \log 2\pi \right)$$

Where n is the number of maturities (observations per t).

4.4 Estimated Parameters

Parameter	Value
Equilibrium drift rate (κ)	1.3784
Short-run volatility (σ_χ)	28.94%
Equilibrium volatility (σ_ξ)	14.76%
Correlation (ρ)	0.3000
Measurement noise (5 values)	0.0363, 0.0100, 0.0100, 0.0100, 0.0100
Equilibrium drift (μ_ξ)	−1.98%
Short-term risk premium (λ_χ)	5.67%
Risk-neutral drift (μ_ξ^*)	0.89%

4.5 Log-Likelihood Comparison

To benchmark the performance of the two-factor model, we estimate log-likelihoods under three nested models:

1. **Full Schwartz & Smith (Two-factor):** Both χ_t and ξ_t active.
2. **GBM-only (Long-term only):** Remove χ_t by setting $\sigma_\chi = 0$ and $\kappa = 0$.
3. **OU-only (Short-term only):** Remove ξ_t by setting $\sigma_\xi = 0$ and keeping mean-reverting dynamics.

Log-Likelihoods:

- S&S 2-Factor: **3585.80**
- GBM (long-term only): **2835.40**
- OU (short-term only): **3206.31**

4.6 Conclusion

This report demonstrates the successful estimation of the Schwartz and Smith (2000) two-factor model using a Kalman filter. Wrapper functions were used to estimate nested models (GBM and OU), showing that the full model achieves superior log-likelihood performance.

Future work could include Bayesian estimation or extending to commodity options pricing via Monte Carlo simulation or characteristic functions.

This document presents the results of the Monte Carlo simulation and validation of the Schwartz–Smith two-factor model for pricing commodity derivatives. It includes visualization of paths, distribution analysis, convergence study, and pricing comparisons.